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Introduction

We are happy to present a collection of papers honoring Robert Bartnik's achievements in geometry and general relativity, using his 60th birthday as an excuse.

Bartnik was a PhD student of ST Yau at Princeton University where he learned geometry and general relativity. His thesis was a study of existence of maximal hypersurfaces in general spacetimes with barrier conditions. Since his Master Thesis on the Dirichlet problem for maximal hypersurfaces in Minkowski spacetime, Robert has been regularly making lasting contributions to mathematical general relativity, to become one of the outstanding mathematical relativists of our era. Attending almost any workshop in the field during the last several decades, it is striking how often research which stems from Robert's work has been discussed. His results are widespread and mathematically deep. The influence of his work will extend far into the future.

The years 1986 through 1989 were marked by a particular explosion of Robert's creativity:

His famous paper on the mass of asymptotically flat manifolds appeared in 1986. There he devised a clever and sophisticated method, based on the analysis of Fredholm properties of the Laplacian in weighted Hölder and Sobolev spaces, to show that the Arnowitt-Deser-Misner mass of asymptotically Euclidean metrics is a geometric invariant. While the geometric character of the definition of the mass is an important result in itself, the results of Robert's study of the Laplace operator in this paper have become a classic, used subsequently by many researchers to solve completely unrelated problems.

1988 sees not only one, but three papers by Robert with a lasting imprint on mathematical physics. His discovery, with his undergraduate student John McKinnon, published in Physical Review Letters, of globally regular static solutions of the Einstein-Yang-Mills equations has made him an icon in mathematical relativity. His Acta Mathematica paper on the regularity of area maximising hypersurfaces in Lorentzian spacetimes provided a cornerstone for our understanding of such hypersurfaces. The third paper, published in Communications in Mathematical Physics, builds upon a problem posed by Yau, of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem, which was eventually established by J.-H. Eschenburg. A concrete formulation of this proposal to study the *rigidity* of the Hawking-Penrose singularity theorems is a conjecture of Robert which says that spatially compact globally hyperbolic timelike geodesically complete spacetimes satisfying an

energy condition split as a product. Using his powerful results on the Dirichlet problem for prescribed mean curvature from his Acta Mathematica paper, Robert establishes a criterion for the existence of constant mean curvature (CMC) Cauchy surfaces, which settled the conjecture subject to this condition. The Bartnik splitting conjecture has been the object of studies of many authors, and remains open to this day; the key papers related to this conjecture and concerned with the issues of the existence of CMC Cauchy surfaces are reviewed in Greg Galloway's contribution to this volume.

One of the most mysterious issues in general relativity is whether there is a mathematically meaningful notion of the mass of an object of finite extent. In a letter to Physical Review in 1989, Robert presented an inspired proposal of how to assign a mass to a region with compact boundary. This led him to another brilliant conjecture, namely that of the existence of minimisers for his quasi-local definition of mass under natural geometric conditions.

This conjecture prompted him to discover the construction of "quasi-spherical metrics", which became the first method for constructing solutions of the scalar constraint equation subject to both Dirichlet and Neumann type boundary conditions. The method played the key role in the proof of the positivity of Brown-York mass by Shi-Tam, as well as the proof of the positivity of Liu-Yau and Wang-Yau masses. An improvement of Shi-Tam's result is the subject of the contribution of Yuguang Shi and Luen-Fai Tam to this volume. Robert later generalized this construction to a "null quasi-spherical gauge" which was applied to the study of radiation and numerical solutions of the Einstein equation. Further papers in this volume by Michael Anderson, by Carla Cederbaum, Oliver Rinne and Markus Strehlau, by Justin Corvino, and by Stephen McCormick and Pengzi Miao, are devoted to the understanding of Robert's quasi-local mass.

Needless to say, Robert obtained other important results. Some further favorites of ours include the construction of globally hyperbolic spacetime solutions of Einstein's equations which do not contain spacelike hypersurfaces of constant mean curvature, the proof of the existence of maximal hypersurfaces in a class of asymptotically flat spacetimes together with the associated a priori estimates, and his construction of a Hilbert space structure on the set of solutions of the Einstein constraint equations.

Dear Robert, it has been a pleasure to put together this volume to celebrate your achievements.

Happy Birthday!

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