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A short note on the Bartnik mass*

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Abstract: In this note we use a localized deformation construction for the Einstein constraint equations to obtain a variational condition that must be satisfied by a suitable Bartnik quasi-local mass minimizer in the general non-time-symmetric setting.

1. Introduction

The quasi-local mass problem in general relativity is the search for a suitable definition for the mass-energy of a region, a quantity which measures both the energy content of the physical fields as well the contribution of the gravitational field. The stress-energy tensor $T_{\mu\nu}$, which appears in the Einstein equation $\mathrm{Ric}(\bar{g})_{\mu\nu} - \frac{R(\bar{g})}{2}\bar{g}_{\mu\nu} = 8\pi T_{\mu\nu}$ (in units where $c = 1$ and $G = 1$), gives an infinitesimal measure of the contribution from non-gravitational fields. For isolated systems modeled by asymptotically flat space-times, the ADM mass measures the total mass-energy of the system. There has been intense interest in quasi-local mass quantities, including the classic Hawking mass [11] and the Bartnik mass [5], as well as mass quantities defined by Brown-York [9], Jauregui [17], Liu-Yau [18], and Wang-Yau [29]. The Wang-Yau notion of quasi-local mass and momenta has in particular been shown to enjoy a number of interesting properties [10], and as such is a strong candidate for a satisfactory notion of quasi-local quantities.

Bartnik defines the quasi-local mass of a region as the infimum of the ADM mass over an appropriate class of extensions of the given data on the region. A significant amount of care must be taken in formulating the class of extensions. For one thing, to avoid hiding the region behind the event horizon of a black hole with small mass, Bartnik rules out event horizons in spacetime

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extensions, and apparent horizons in spacelike extensions. From an initial data perspective, then, one would rule out certain marginally outer trapped surfaces, which in the time-symmetric case amounts to ruling out certain minimal surfaces. There are various formulations of this notion starting from the original work of Bartnik, cf. the discussion in the recent work of [3] for example. Furthermore, Bartnik [5, 6] conjectures that a minimizing mass extension of a region should be stationary and vacuum outside of the region, but may fail to be smooth across the boundary. A minimizer is conjectured to be Lipschitz across the boundary, with the dominant energy condition holding distributionally. As Bartnik [5, 6] notes, the dependence of the quasi-local mass on the region should come via the geometry at the boundary. The boundary conditions should ensure that a Positive Mass Theorem with corners (cf. e.g. [22, 27, 26]) holds. We will not go into this further in this short note, but we remark that if only the surface data is given (the metric on the boundary surface and some components of the second fundamental form), one might consider not only asymptotically flat extensions, but also *fill-ins*, cf. [8, 17, 28]. Furthermore, if the boundary were minimal, then there may be no minimizing mass extension [19], consistent with static black hole uniqueness theorems, cf. [24].

The purpose of this note is to discuss a result (Theorem 2.2) related to the conjectural stationarity of a minimizing extension in the general case (cf. Section 3.0.1). In the time-symmetric case, a minimizing mass extension is conjectured to be static vacuum, and the question of the staticity of a minimizer was addressed in [12], and sharpened and clarified in [13, 15, 23], cf. [3]. Bartnik has proposed establishing the stationarity of minimal mass extensions by adapting the analysis in [7] to the setting with appropriate boundary conditions; progress in this direction has been made by McCormick [20, 21]. As in [13], we will not go into the details of the boundary conditions here, as we argue that discerning the interior condition satisfied by a minimizer can be decoupled from the issue of the boundary behavior. Thus we will study initial data on a manifold M with boundary, such as might represent the boundary and exterior of an extension. Of course if one wants to study the existence of a minimizer, one will have to marry the interior and boundary conditions together, and that is a different problem from what we consider here. Progress in this direction has been made, mostly in the time-symmetric case, cf. e.g. [2, 3, 4], though we also point out the recent progress by An [1], establishing the ellipticity of the stationary equations with Bartnik boundary conditions, in an appropriate gauge.

2. Statement of the Main Theorem and preliminaries

We recall the constraints operator and the dominant energy condition for an initial data set (M, g, K), comprised of an n-manifold M $(n \geq 3)$ (possibly with boundary), together with a Riemannian metric g and symmetric $(0,2)$-tensor K. Such data would be induced as the first and second fundamental forms of a spacelike hypersurface in a spacetime. The *dominant energy condition* in the spacetime translates into the following inequality on the initial data:

$$\tfrac{1}{2}[R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2] \geq |\mathrm{div}_g(K) - d(\mathrm{tr}_g K)|_g\,,$$

where $R(g)$ is the scalar curvature, $(\mathrm{div}_g(K))_i = g^{j\ell} K_{ij;\ell} = g^{j\ell}\nabla_\ell K_{ij}$ and $\mathrm{tr}_g K = g^{ij}K_{ij}$. Actually it is convenient to re-write the initial data using the *momentum tensor* $\pi^{ij} = K^{ij} - (\mathrm{tr}_g K)g^{ij}$, for which we readily see that $g_{ij}(\mathrm{div}_g \pi)^j = [\mathrm{div}_g(K) - d(\mathrm{tr}_g K)]_i$. We define the *constraints map* as

$$\Phi(g,\pi) = \Big(R(g) - |\pi|_g^2 + \tfrac{1}{n-1}(\mathrm{tr}_g \pi)^2, \mathrm{div}_g \pi\Big) =: (2\mu, J),$$

so that the dominant energy condition $\frac{1}{2}[R(g) - |\pi|_g^2 + \frac{1}{n-1}(\mathrm{tr}_g\pi)^2] \geq |\mathrm{div}_g \pi|_g$ is simply $\mu \geq |J|_g$.

We let $D\Phi_{(g,\pi)}(h,\omega) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Phi(g+\varepsilon h, \pi + \varepsilon\omega)$ define the linearized constraints operator $D\Phi_{(g,\pi)}$, with formal L^2-adjoint $D\Phi^*_{(g,\pi)}$ satisfying the equation

$$\int_M (h,\omega)\cdot_g D\Phi^*_{(g,\pi)}(f,X)\, dv_g = \int_M (f,X)\cdot_g D\Phi_{(g,\pi)}(h,\omega)\, dv_g$$

for all (h,ω) of compact support in the interior of M (i.e. away from the boundary). We recall that at a vacuum initial data set, a nontrivial kernel element of $D\Phi^*_{(g,\pi)}$ corresponds to a spacetime Killing vector for a Cauchy development of the data, and in the case of a timelike Killing vector for a stationary vacuum metric, the kernel element gives the associated lapse function and shift vector, cf. [25].

The main result in this note is stated in terms of a *modified constraint operator* from [14], designed to maintain the dominant energy condition under deformations:

$$\widetilde{\Phi}_{(g,\pi)}(\gamma,\tau) = \Phi(\gamma,\tau) + (0, \tfrac{1}{2}\gamma\cdot_g \mathrm{div}_g\pi),$$

where $(\gamma\cdot_g \mathrm{div}_g\pi)^i = g^{ij}\gamma_{jk}(\mathrm{div}_g\pi)^k$. Observe that

$$\widetilde{\Phi}_{(g,\pi)}(g,\pi) = \Phi(g,\pi) + (0, \tfrac{1}{2}\mathrm{div}_g\pi).$$

The linearized operator $D\widetilde{\Phi}_{(g,\pi)}$ and its formal adjoint $D\widetilde{\Phi}^*_{(g,\pi)}$ are defined just as above. The kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ on an open set $U \subset M \setminus \partial M$ is defined to be the set of all $(f,X) \in H^2_{\mathrm{loc}}(U) \times H^1_{\mathrm{loc}}(U)$ so that $D\widetilde{\Phi}^*_{(g,\pi)}(f,X)$ vanishes weakly on U. If (g,π) is smooth on M, then by elliptic regularity, such a kernel element (f,X) is smooth on U, and in fact (f,X) extends smoothly over $\overline{U}$, with $D\widetilde{\Phi}^*_{(g,\pi)}(f,X)$ vanishing pointwise on $\overline{U}$ [14, Prop. 2.1]. Thus when the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ is trivial on $U = M \setminus \partial M$, we say the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ is trivial on M.

The main technical tool we will employ is a localized deformation for initial data sets which promotes the dominant energy condition, joint work with L.-H. Huang [14]. We state a sufficient version for our purposes. In the following, $\overline{\Omega}$ is a compact, connected, smooth manifold-with-boundary with manifold interior Ω. While for simplicity we work with data (g,π) which is smooth on $\overline{\Omega}$, one can accommodate less regularity, as in [14].

Theorem 2.1. *Let $0 < \alpha < 1$. Suppose the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ is trivial on Ω. Let Ω_0 be open, with compact closure $\overline{\Omega}_0 \subset \Omega$. There is an $\mathring{\varepsilon} > 0$ and $C > 0$, so that for all ψ which is smooth on $\overline{\Omega}$ with support in $\overline{\Omega}_0$ and with $\|\psi\|_{C^{0,\alpha}} < \mathring{\varepsilon}$, there is (h,ω) which is smooth on $\overline{\Omega}$ with compact support in Ω and with $\|(h,\omega)\|_{C^{2,\alpha}} \le C\|\psi\|_{C^{0,\alpha}}$ and so that with $(\hat{g},\hat{\pi}) := (g+h, \pi+\omega)$ and $(2\hat{\mu},\hat{J}) := \Phi(g+h,\pi+\omega)$, we have $\hat{\mu} - |\hat{J}|_{\hat{g}} \ge \mu - |J|_g + \psi$.*

We will consider *asymptotically flat initial data sets* on a manifold-with-boundary M, which admits a compact set $\mathcal{C} \subset M$ so that $M \setminus \mathcal{C} \subset M \setminus \partial M$ is the union of a finite number of *ends*, each diffeomorphic to the exterior of the unit ball in $\mathbb{R}^n$. As for the initial data (g,π), one generally specifies a level of local regularity (some number of continuous derivatives, possibly in local Hölder spaces) along with decay conditions that can be imposed pointwise, or in weighted Sobolev spaces. As regards the regularity, as noted above we will assume g and π are smooth, though this is not necessary. As regards the asymptotics, the proof will modify the data in the asymptotic region by a simple scaling, which will preserve the asymptotic structure. So for the sake of simplicity, we will operate with the standard decay conditions in appropriate asymptotic coordinates x in each asymptotic end, namely that for some rate $q > \frac{n-2}{2}$, and for multi-indices $|\alpha| \le 2$ and $|\beta| \le 1$,

$$\partial^\alpha_x (g_{ij}(x) - \delta_{ij}) = O(|x|^{-q-|\alpha|})$$

and

$$\partial^\beta_x \pi^{ij}(x) = O(|x|^{-q-1-|\beta|}),$$

or equivalently, $\partial_x^\beta K_{ij}(x) = O(|x|^{-q-1-|\beta|})$. Furthermore for each end to have well-defined ADM energy E and linear momentum P, we assume that $R(g)$ and $\mathrm{div}_g \pi$ are integrable. Recall that when $E \geq |P|$, the ADM mass is defined as

$$m_{ADM}(g, \pi) = \sqrt{E^2 - |P|^2}.$$

We now state the Main Theorem. We note that converging *locally smoothly* is taken to mean converging in C^ℓ on compact subsets, for any $\ell \in \mathbb{Z}_+$.

Theorem 2.2. *Let $n \geq 3$, and let M^n be a connected manifold-with-boundary, with ∂M compact. Suppose (M, g, π) is an asymptotically flat initial data set which satisfies the dominant energy condition $\mu \geq |J|_g$. Assuming that the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ is trivial on M, there is an open set $\mathring{\Omega} \supset \partial M$ and a sequence $\theta_i \searrow 0^+$, along with a sequence (g_i, π_i) of asymptotically flat initial data sets on M, so that with $\Phi(g_i, \pi_i) =: (2\mu_i, J_i)$, we have the following:*

- *$(g_i, \pi_i) = (g, \pi)$ on $\mathring{\Omega}$, $g_i = (1 - \theta_i)^{\frac{4}{n-2}} g$ and $\pi_i = (1 - \theta_i)^{-\frac{6}{n-2}} \pi$ in a fixed neighborhood of infinity in any end, with $\lim_{i\to\infty}(g_i, \pi_i) = (g, \pi)$ locally smoothly on M.*
- *The dominant energy condition holds: $\mu_i \geq |J_i|_{g_i}$, and in fact we have $\mu_i - |J_i|_{g_i} \geq \mu - |J|_g$.*
- *For each end which has timelike future-pointing ADM energy momentum $E > |P|$,*

$$0 < m_{ADM}(g_i, \pi_i) = (1 - \theta_i)^2 \sqrt{E^2 - |P|^2} < m_{ADM}(g, \pi).$$

3. Proof of the Main Theorem

The proof of Theorem 2.2 follows that of an analogous result in the time-symmetric case from [13, Prop. 2.3].

Proof. As shown in [14, Prop. 2.1], the condition that $D\widetilde{\Phi}^*_{(g,\pi)}(f, X) = 0$ induces an ODE system along geodesics, and so a kernel element is determined by its 1-jet at any point. As a consequence, there is an *a priori* bound (depending only on n) on the dimension of the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$ on any connected open set, and furthermore, the restriction map between two open connected subsets $U_1 \subset U_2 \subset M$ must be injective on the kernel of $D\widetilde{\Phi}^*_{(g,\pi)}$. Thus by using an exhaustion of M by pre-compact connected open subsets, we can infer that since $D\widetilde{\Phi}^*_{(g,\pi)}$ has trivial kernel on M, there is an open set $\Omega \subset \overline{\Omega} \subset M \setminus \partial M$ on which $D\widetilde{\Phi}^*_{(g,\pi)}$ has trivial kernel, with $\overline{\Omega}$ a compact, connected smooth manifold-with-boundary with manifold interior Ω; we can also easily arrange Ω to contain a region of the form $\{p \in M : r_0 \leq |x(p)| \leq 2r_0\}$ in each asymptotic end, for some $r_0 > 0$. Let $r_0 < r_1 < r_2 < 2r_0$.

We employ Theorem 2.1 with a suitably chosen $\psi \geq 0$, to deform (g, π) to a new initial data set $(\hat{g}, \hat{\pi})$, which from $\hat{\mu} - |\hat{J}|_{\hat{g}} \geq \mu - |J|_g + \psi$ still satisfies the dominant energy condition, with $(g, \pi) = (\hat{g}, \hat{\pi})$ outside Ω, and for some small constant $\mathring{\psi} > 0$, $\hat{\mu} - |\hat{J}|_{\hat{g}} \geq \mathring{\psi} + \mu - |J|_g$ holds in each set of the form $\{p : r_1 \leq |x(p)| \leq r_2\}$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth nonincreasing function with $\varphi(t) = 1$ when $t \leq r_1$, while $\varphi(t) = 0$ for $t \geq r_2$. For $0 \leq \theta < 1$, we define $\varphi_\theta = (1-\theta) + \theta\varphi$, so that $\varphi_\theta(t) = 1$ for $t \leq r_1$ and $\varphi_\theta(t) = 1 - \theta$ for $t \geq r_2$. We let $u_\theta(p) > 0$ be defined for p in an asymptotic end by $u_\theta(p) = \varphi_\theta(|x(p)|)$ for $p \in M \setminus \mathcal{C}$, so that $u_\theta(p) = 1 - \theta$ for $|x(p)| \geq r_2$, while $u_\theta(p) = 1$ for $|x(p)| \leq r_1$. Hence u_θ extends smoothly to all of M by $u_\theta(p) = 1$ for $p \in \mathcal{C}$, and the derivatives of u_θ are $O(\theta)$.

Let $g_\theta = u_\theta^{\frac{4}{n-2}} \hat{g}$ and $\pi_\theta = u_\theta^{-\frac{6}{n-2}} \hat{\pi}$ (as a $(2,0)$-tensor), and we let $c_n = \frac{n-2}{4(n-1)}$, so that

$$R(u_\theta^{\frac{4}{n-2}} \hat{g}) = -c_n^{-1} u_\theta^{-\frac{n+2}{n-2}} (\Delta_{\hat{g}} u_\theta - c_n R(\hat{g}) u_\theta) = u_\theta^{-\frac{4}{n-2}} (-c_n^{-1} u_\theta^{-1} \Delta_{\hat{g}} u_\theta + R(\hat{g})).$$

Since $\mathrm{tr}_{g_\theta} \pi_\theta = u_\theta^{-\frac{2}{n-2}} \mathrm{tr}_{\hat{g}} \hat{\pi}$ and $|\pi_\theta|^2_{g_\theta} = u_\theta^{-\frac{4}{n-2}} |\hat{\pi}|^2_{\hat{g}}$, we have

$$2\mu_\theta := 2\mu(g_\theta) = u_\theta^{-\frac{4}{n-2}} (2\hat{\mu} - c_n^{-1} u_\theta^{-1} \Delta_{\hat{g}} u_\theta) =: 2u_\theta^{-\frac{4}{n-2}} \hat{\mu} + 2\delta_\theta.$$

We also compute $J_\theta := \mathrm{div}_{g_\theta} \pi_\theta = u_\theta^{-\frac{6}{n-2}} \mathrm{div}_{\hat{g}} \hat{\pi} + du_\theta * \hat{\pi} = u_\theta^{-\frac{6}{n-2}} \hat{J} + du_\theta * \hat{\pi}$, where $du_\theta * \hat{\pi}$ indicates a linear combination of products of components of du_θ and $\hat{\pi}$, with bounded coefficients. We write $|J_\theta|_{g_\theta} = u_\theta^{-\frac{4}{n-2}} |\hat{J}|_{\hat{g}} + \delta'_\theta$. Note that there is a $C > 0$ so that for all $0 < \theta \leq \frac{1}{2}$, $|\delta_\theta| + |\delta'_\theta| \leq C\theta$, and $|\delta_\theta| + |\delta'_\theta| = 0$ except for $r_1 \leq |x(p)| \leq r_2$.

We claim the dominant energy condition still holds for (g_θ, π_θ). Indeed, as $0 < u_\theta \leq 1$ and $\psi \geq 0$, where u_θ is constant, we have $\mu_\theta - |J_\theta|_{g_\theta} = u_\theta^{-\frac{4}{n-2}} (\hat{\mu} - |\hat{J}|_{\hat{g}}) \geq (\mu - |J|_g + \psi) \geq 0$. On the other hand, for $r_1 \leq |x(p)| \leq r_2$,

$$\begin{aligned} \mu_\theta - |J_\theta|_{g_\theta} &= u_\theta^{-\frac{4}{n-2}} (\hat{\mu} - |\hat{J}|_{\hat{g}}) + \delta_\theta - \delta'_\theta \\ &\geq u_\theta^{-\frac{4}{n-2}} (\mathring{\psi} + \mu - |J|_g) + \delta_\theta - \delta'_\theta \\ &\geq \mu - |J|_g + \mathring{\psi} - C\theta. \end{aligned}$$

Thus for small enough $\theta > 0$, the dominant energy condition holds everywhere, in fact $\mu_\theta - |J_\theta|_{g_\theta} \geq \mu - |J|_g$.

We now indicate the simple computation using the ADM flux integrals to find the energy-momentum (E_θ, P_θ) of any end of $(M, g_\theta, \pi_\theta)$; since an innocuous typo crept into the analogous formula in [13], we show some details here. Asymptotically flat coordinates x (at rate $q > \frac{n-2}{2}$) for g in an end need to be re-scaled to appropriate asymptotically flat coordinates $y = \alpha x$ for g_θ, with $\alpha = (1-\theta)^{\frac{2}{n-2}}$. Then we have $dy^i = \alpha dx^i$, and $\frac{\partial}{\partial y^i} = \alpha^{-1}\frac{\partial}{\partial x^i}$. Thus at y we have

$$g_\theta(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}) = g(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}) = \delta_{ij} + O(|x|^{-q}) = \delta_{ij} + O(|y|^{-q})$$

as desired. We also note that $\frac{\partial}{\partial y^k}(g_\theta(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})) = \alpha^{-1}\frac{\partial}{\partial x^k}(g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}))$, and similarly for second derivatives, while

$$\pi_\theta(dy^i, dy^j) = \alpha^2 u_\theta^{-\frac{6}{n-2}}\pi(dx^i, dx^j) = \alpha^{-1}\pi(dx^i, dx^j),$$

with an analogous formula for the derivatives of π_θ. In particular, we see that the y coordinates are asymptotically flat with rate q for (g_θ, π_θ). We use index notation for g_θ in the y-coordinates, and for g in the x-coordinates, and we let $d\mathring{\sigma}$ be the area measure on the round unit sphere $\mathbb{S}^{n-1}$, with total area $|\mathbb{S}^{n-1}|$. The relevant flux integral over $|y| = r$ for $E(g_\theta)$ is given by (up to the normalizing factor $\frac{1}{2(n-1)|\mathbb{S}^{n-1}|}$, and note the outward Euclidean unit normal to the sphere at $\omega \in \mathbb{S}^{n-1}$ can be identified with ω)

$$\begin{aligned}
&\int_{\omega\in\mathbb{S}^{n-1}} \sum_{i,j=1}^{n} [(g_\theta)_{ij,i} - (g_\theta)_{ii,j}]\Big|_{y=r\omega} \omega^j r^{n-1} d\mathring{\sigma} \\
&\qquad = \alpha^{n-1} \int_{\omega\in\mathbb{S}^{n-1}} \sum_{i,j=1}^{n} \alpha^{-1}[g_{ij,i} - g_{ii,j}]\Big|_{x=\alpha^{-1}r\omega} \omega^j (\alpha^{-1}r)^{n-1} d\mathring{\sigma}.
\end{aligned}$$

Taking the limit as $r \to \infty$, we see $E(g_\theta) = \alpha^{n-2}E(g) = (1-\theta)^2 E(g)$ (fixing the typo from [13]). Similarly the flux integral for the linear momentum P_θ^ℓ for (g_θ, π_θ) is, up to the factor $\frac{1}{(n-1)|\mathbb{S}^{n-1}|}$,

$$\begin{aligned}
&\int_{\omega\in\mathbb{S}^{n-1}} \sum_{i,j=1}^{n} [(\pi_\theta)^{\ell j}(g_\theta)_{jk}]\Big|_{y=r\omega} \omega^k r^{n-1} d\mathring{\sigma} \\
&\qquad = \alpha^{n-1} \int_{\omega\in\mathbb{S}^{n-1}} \sum_{i,j=1}^{n} \alpha^{-1}[\pi^{\ell j} g_{jk}]\Big|_{x=\alpha^{-1}r\omega} \omega^k (\alpha^{-1}r)^{n-1} d\mathring{\sigma}
\end{aligned}$$

from which we can then conclude $P^\ell_\theta = (1-\theta)^2 P^\ell$. We then have the ADM mass

$$m_{ADM}(g_\theta, \pi_\theta) := \sqrt{E_\theta^2 - |P_\theta|^2} = (1-\theta)^2\sqrt{E^2-|P|^2} < m_{ADM}(g,\pi).$$

To produce the desired family of solutions, for each $0 < \varepsilon \leq 1$, we consider the above procedure with $\varepsilon\psi$ in place of ψ, and as we can choose any θ sufficiently small, we choose a corresponding $\theta(\varepsilon) \in (0, \varepsilon)$ small enough as above. So there is a sequence $\varepsilon_i \searrow 0$, with corresponding $\theta_i := \theta(\varepsilon_i) \searrow 0$, and we then take (g_i, π_i) to be $(g_{\theta_i}, \pi_{\theta_i})$. □

Remark 3.1. Suppose we replace φ above by a smooth nondecreasing function for which $\varphi(t) = 0$ for $t \leq r_1$ and $\varphi(t) = 1$ for $t \geq r_2$, and for $\theta \geq 0$ we let $\varphi_\theta = 1+\theta\varphi$, and we define $u_\theta(p) = \varphi_\theta(|x(p)|)$ for $p \in M \setminus \mathcal{C}$, and $u_\theta(p) = 1$ for $p \in \mathcal{C}$. We see $1 \leq u_\theta \leq 1+\theta$ and $u_\theta(p) = 1+\theta$ for $|x(p)| \geq r_2$. Carrying out the rest of the proof, we see that for $0 < \gamma < 1$, and $\theta > 0$ small enough, we have $\mu_\theta - |J_\theta|_{g_\theta} \geq \gamma(\mu - |J|_g) \geq 0$. We let $(g_{\theta_i}, \pi_{\theta_i})$ for suitable $\theta_i \searrow 0$ define an analogous sequence (g_i, π_i) as above, so that for each end with $E > |P|$,

$$m_{ADM}(g_i, \pi_i) = (1+\theta_i)^2\sqrt{E^2-|P|^2} > m_{ADM}(g,\pi).$$

Remark 3.2. An immediate corollary of the above proof is the following: suppose (M, g, π) is as in Theorem 2.2, and suppose (g_k, π_k) is a sequence of asymptotically flat initial data sets on M satisfying the dominant energy condition and with well-defined ADM mass $m_{ADM}(g_k, \pi_k)$, so that (g_k, π_k) converges to (g, π) locally smoothly (of course as per [14], a finite number of derivatives will suffice for the argument), and so that $m_{ADM}(g_k, \pi_k) \to m_{ADM}(g, \pi)$. If Ω is as in the proof of Theorem 2.2, then by the convergence and [14, Thm. 5.3], we may arrange that for all k (sufficiently large, so without loss of generality), the kernel of $D\widetilde{\Phi}^*_{(g_k,\pi_k)}$ is also trivial on Ω (and hence on M). As the analysis in [14] shows, $\mathring{\varepsilon} > 0$ in Theorem 2.1 (and hence ψ in the proof of Theorem 2.2) can be chosen uniformly for data near (g, π). Since the estimates of δ_θ and δ'_θ are uniform in $\theta \in (0, \frac{1}{2}]$ and for data near (g, π), we can find a suitable $\mathring{\theta} > 0$ so that for all k (sufficiently large, so without loss of generality), the construction in the proof of Theorem 2.2 applied to (g_k, π_k) yields the data $((g_k)_{\mathring{\theta}}, (\pi_k)_{\mathring{\theta}})$ which satisfies the dominant energy condition. If $m_{ADM}(g, \pi) > 0$, then for all k sufficiently large,

$$m_{ADM}((g_k)_{\mathring{\theta}}, (\pi_k)_{\mathring{\theta}})) = (1-\mathring{\theta})^2 m_{ADM}(g_k, \pi_k) < m_{ADM}(g, \pi).$$

This remark is made with the following in mind. Suppose (g_k, π_k) and (g, π) are as above, where (g_k, π_k) is a mass-infinimizing sequence of extensions in some appropriate class of competitors, defined in particular with certain asymptotics and boundary conditions imposed, and with some no-horizon condition. The limit (g, π) might in principle leave the class of competitors for the minimization problem. What the preceding paragraph shows is that if (g, π) has positive mass and is a suitable limit of an infimizing sequence of competitor extensions, then if (g, π) has trivial kernel on M for $D\widetilde{\Phi}^*_{(g,\pi)}$, we can produce extensions $((g_k)_{\mathring{\theta}}, (\pi_k)_{\mathring{\theta}})$ with smaller ADM mass. We would arrive at a contradiction to the trivial kernel assumption if we could show that for large k and small $\mathring{\theta} > 0$, $((g_k)_{\mathring{\theta}}, (\pi_k)_{\mathring{\theta}})$ inherits whatever no-horizon condition is satisfied by the competitor data (g_k, π_k), (since the data $((g_k)_{\mathring{\theta}}, (\pi_k)_{\mathring{\theta}})$ agrees with (g_k, π_k) near ∂M, and shares the same asymptotic decay properties).

3.0.1. Concluding remarks Theorem 2.2 and the above remark raise several questions for the Bartnik quasi-local mass. An immediate question is the following: if we impose that the original initial data set (g, π) be void of horizons of a certain class (e.g. a certain class of marginally outer trapped surfaces, say), then for small enough $\theta > 0$ in the proof of Theorem 2.2, is the data (g_θ, π_θ) also free of such horizons? If this holds for (g, π), we will call this initial data set *suitable*. See e.g. [13] and cf. [3] for the time-symmetric case; the variational nature of the minimal surface equation allows for some satisfactory statements to hold. We expect a similar formulation to hold in the general case.

Suppose we have a minimal mass extension, which is itself a suitable member of the class of competitors for the minimization problem, or at least a limit (as in Remark 3.2) of suitable competitors. Assuming we are in a setting where the Positive Mass Theorem holds, then the argument above shows that either the ADM energy-momentum vector is null or $D\widetilde{\Phi}^*_{(g,\pi)}$ has nontrivial kernel in the extension. Bartnik conjectures that the extension is stationary and vacuum. Assuming we can apply the equality case of the Positive Mass Theorem to our extension, see [16], if the ADM energy-momentum of (g, π) is null, then the data is a slice in Minkowski spacetime. If the ADM energy-momentum is timelike, then we obtain a kernel element for $D\widetilde{\Phi}^*_{(g,\pi)}$, and have to figure out what that means. Can we infer that the data corresponds to a stationary spacetime? Can we say something about $\mu - |J|_g$? Must it be vacuum? For comparison, in the time-symmetric case, the metric extension must have nontrivial kernel for the formal adjoint DR^*_g of the linearized scalar curvature operator, which in turn implies constant scalar curvature. Thus the extension must be static vacuum: the scalar curvature must vanish, since it must go to zero at infinity. Furthermore along these lines, one might ask whether more refined deformations (akin to the scalar curvature deformations from [13]) can be achieved in case $D\widetilde{\Phi}^*_{(g,\pi)}$ has trivial kernel: can one decrease the ADM mass and preserve μ, $|J|_g$, or $\mu - |J|_g$?

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