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Quasi-local mass at null infinity in Bondi-Sachs coordinates

Po-Ning Chen, Mu-Tao Wang, Ye-Kai Wang,
and Shing-Tung Yau

Abstract: There are two chief statements regarding the Bondi-Trautman mass [3, 29, 37, 33, 34] at null infinity: one is the positivity [30, 20], and the other is the mass loss formula [3], which are both global in nature. In this note, we compute the limit of the Wang-Yau quasi-local mass on unit spheres at null infinity of an asymptotically flat spacetime in the Bondi-Sachs coordinates. The quasi-local mass leads to a local description of radiation that is purely gravitational at null infinity. In particular, the quasi-local mass is evaluated in terms of the news function of the Bondi-Sachs coordinates.

1. Introduction

An observer of gravitational radiation created by an astronomical event is situated at future null infinity, where light rays emitted from the source approach. The study of the theory of gravitational radiation at null infinity in the last century culminated in a series of papers by Bondi and his collaborators [3, 29, 37, 33, 34], in which the Bondi-Trautman mass and the mass loss formula at null infinity are well understood. In particular, the Bondi-Trautman mass was proved to be positive in the work of Schoen-Yau [30] and Horowitz-Perry [20]. Both the positivity of mass and the mass loss formula are global statements on null infinity: knowledge of the mass aspect is required in *every* direction. For reasons that are both theoretical and experimental, it is highly desirable to have a quasi-local statement of mass/radiation at null infinity.

In [11, 12], we embarked on the evaluation of the Wang-Yau quasi-local mass on surfaces of fixed size near null infinity of a linear gravitational perturbation of the Schwarzschild spacetime. The ideas and technique in [11, 12] were further developed to address the case of the Vaidya spacetime in [15]. The construction of these spheres of unit size at null infinity will be reviewed

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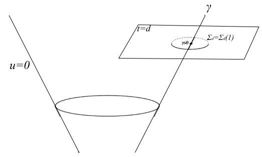
in the next section. In the Vaidya case, we proved in [15] that the quasi-local mass of a unit size sphere at null infinity is directly related to the derivative of the mass aspect function with respect to the retarded time u. In particular, the positivity of the quasi-local mass is implied by the decreasing of the mass aspect function in u. This is in turn a consequence of the positivity of matter density in the Vaidya spacetime. In order to investigate radiation that is purely gravitational, in this article we take on the general case of an asymptotically flat vacuum spacetime described in the Bondi-Sachs coordinates.

A new ingredient in this article is a variational formula (see Theorem 4.1) which facilitates a more straightforward computation of the $O(d^{-2})$ term than the one in [15]. Similar to [15], it is still crucial to compute the $O(d^{-1})$ term of the optimal embedding. This is done in Lemma 5.1 and Lemma 5.2 of the current article. As in Lemma 3.3 of [15], the optimal embedding equation is reduced to two ordinary differential equations. However, it does not seem possible to obtain explicit solutions to the ODE's as in the Vaidya case. The quasi-local mass is then evaluated by combining Theorem 4.1 and the optimal embedding.

The structure of the paper is as follows: in Section 2, we review the general framework of quasi-local mass at null infinity. In Section 3, we compute the geometric quantities on the spheres at null infinity that are necessary to evaluate the quasi-local mass. In Section 4, we derive the formula for the leading order term of the quasi-local mass. In Section 5, we evaluate the quasi-local mass based on the formula derived in Section 4. See Theorem 5.3. In the last section, Section 6, we look at several special examples.

2. General framework of quasilocal mass at null infinity

We consider a null geodesic γ parametrized by an affine parameter d with $d_0 \leq d < \infty$ and a family of surfaces $\Sigma_d(s)$ for $s > 0$ centered at $\gamma(d)$ in the following sense. For each fixed d and s, $\Sigma_d(s)$ is a spacelike surface that bounds a ball $B_d(s)$ with $\partial B_d(s) = \Sigma_d(s)$, such that as $s \to 0$, we have $\lim_{s\to 0} B_d(s) = \lim_{s\to 0} \Sigma_d(s) = \gamma(d)$. We evaluate the quasilocal mass of $\Sigma_d(s)$ as $d \to \infty$. In particular, when $s = 1$, $\lim_{d\to\infty} \Sigma_d(1)$ is the unit sphere limit referred on our previous work. The choice of such a family of spacelike surfaces/balls depends on a timelike direction field along γ. More precisely, one chooses a normal coordinate $\{X^\alpha\}$ centered at $\gamma(d)$ with $\frac{\partial}{\partial X^0}$ pointing to the given timelike direction and defines $\Sigma_d(s) = \{X^0 = 0, (X^1)^2 + (X^2)^2 + (X^3)^2 = s^2\}$. It is, however, technically more convenient to define $\Sigma_d(s)$ coordinate-wise by exploiting a Cartesian coordinate system built from the Bondi-Sachs coordinates.



In practice, such an evaluation is conducted by choosing a family of parametrizations $\mathfrak{F}_d$ from the unit ball B^3, $\mathfrak{F}_d : B^3 \to B_d(1)$ and considering the pull-backs of geometric quantities on $B_d(1)$ as geometric quantities on B^3 that depend on the parameter d. In particular, $\Sigma_d(s)$ is the image of the sphere of radius s in B^3 under $\mathfrak{F}_d$. The unit sphere limit is obtained by setting $s = 1$ and taking the limit as $d \to \infty$.

When the spacetime is equipped with a global structure at null infinity that corresponds to limits of null geodesics, these unit sphere limits provide information on gravitational radiation observed at null infinity. We illustrate the construction in the Vaidya case where the spacetime metric takes the simple form:

$$-\left(1 - \frac{M(u)}{r}\right) du^2 - 2dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Let γ be a null geodesic on the null hypsersurface $u = 0$ and with $\theta = \tilde{\theta}, \phi = \tilde{\phi}$, where $\tilde{\theta}$ and $\tilde{\phi}$ are constants. We first consider a global coordinate change from (u, r, θ, ϕ) to (t, y^1, y^2, y^3) with $t = u + r, y^1 = r\sin\theta\sin\phi, y^2 = r\sin\theta\cos\phi$, and $y^3 = r\cos\theta$. In terms of the coordinate system (t, y^1, y^2, y^3), the parametrization $\mathfrak{F}_d$ is then given by

$$\mathfrak{F}_d(s, \hat{\theta}, \hat{\phi}) = (d, d\tilde{d}_1 + s\sin\hat{\theta}\sin\hat{\phi}, d\tilde{d}_2 + s\sin\hat{\theta}\cos\hat{\phi}, d\tilde{d}_3 + s\cos\hat{\theta}),$$

where $(s, \hat{\theta}, \hat{\phi})$ is a coordinate system on B^3 and the constants $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$ satisfy $\tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{d}_3^2 = 1$ and indicate the direction of the null geodesic which is parametrized by $d \mapsto (d, d\tilde{d}_1, d\tilde{d}_2, d\tilde{d}_3)$. In the ball centered at a point on the null geodesic in the direction of $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$, we have

$$\begin{aligned} r &= \sqrt{d^2 + 2sdZ + s^2}, \\ u &= d - \sqrt{d^2 + 2sdZ + s^2}, \\ \frac{y^1}{r} &= \frac{d\tilde{d}_1 + s\sin\hat{\theta}\sin\hat{\phi}}{\sqrt{d^2 + 2sdZ + s^2}}, etc. \end{aligned}$$

where

$$Z = \tilde{d}_1 \sin\hat{\theta}\sin\hat{\phi} + \tilde{d}_2 \sin\hat{\theta}\cos\hat{\phi} + \tilde{d}_3 \cos\hat{\theta}.$$

The pull-back of the global coordinate (u, r, θ, ϕ) under $\mathfrak{F}_d$ defines functions on B^3 depending on d. As $d \to \infty$ we have

$$\lim_{d\to\infty} \mathfrak{F}_d^* u = -sZ, \lim_{d\to\infty} \mathfrak{F}_d^* \theta = \tilde{\theta}, \lim_{d\to\infty} \mathfrak{F}_d^* \phi = \tilde{\phi}, \tag{2.1}$$

where $\tilde{\theta}, \tilde{\phi}$ are related to $\tilde{d}_1, \tilde{d}_2$ and $\tilde{d}_3$ through $\tilde{d}_1 = \sin\tilde{\theta}\sin\tilde{\phi}, \tilde{d}_2 = \sin\tilde{\theta}\cos\tilde{\phi}$, and $\tilde{d}_3 = \cos\tilde{\theta}$. In [15], we evaluate the Wang-Yau quasi-local mass on $\Sigma_d(1)$ as d approaches infinity. In the following, we review the definition of the Wang-Yau quasi-local mass.

Let Σ be a closed spacelike 2-surface in a spacetime N with spacelike mean curvature vector H. Denote the induced metric and connection one-form of Σ by σ and

$$\alpha_H(\cdot) = \left\langle \nabla^N_{(\cdot)} \frac{J}{|H|}, \frac{H}{|H|} \right\rangle \tag{2.2}$$

where J is the reflection of H through the incoming light cone in the normal bundle. Given an isometric embedding $\mathcal{X} : \Sigma \to \mathbb{R}^{3,1}$ and future timelike unit Killing field T_0 in $\mathbb{R}^{3,1}$, we consider the projected embedding $\hat{\mathcal{X}}$ into the orthogonal complement of T_0, and denote the induced metric and the mean curvature of the image surface $\hat{\Sigma}$ by $\hat{\sigma}$ and $\hat{H}$.

The quasi-local energy with respect to $(\mathcal{X}, T_0)$ is

$$\begin{aligned} E(\Sigma, \mathcal{X}, T_0) =& \frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{H} dv_{\hat{\sigma}} \\ &- \frac{1}{8\pi}\int_{\Sigma}\left(\sqrt{1+|\nabla\tau|^2}\cosh\theta|H| - \nabla\tau\cdot\nabla\theta - \alpha_H(\nabla\tau)\right) dv_\sigma, \end{aligned} \tag{2.3}$$

where $\tau = -\langle \mathcal{X}, T_0\rangle$ is considered as a function on the 2-surface, and ∇ and Δ are the gradient and Laplace operator with respect to σ, and

$$\theta = \sinh^{-1}\left(\frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}\right).$$

Moreover, we say that τ solves the *optimal embedding equation* if

$$div_\sigma\left(\rho\nabla\tau - \nabla\left[\sinh^{-1}\left(\frac{\rho\Delta\tau}{|H_0||H|}\right)\right] - \alpha_{H_0} + \alpha_H\right) = 0, \tag{2.4}$$

where H_0 and α_{H_0} are the mean curvature and connection 1-form of $\mathcal{X}(\Sigma)$ and

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}.$$

3. Unit sphere at null infinity in Bondi-Sachs coordinates

The spacetime metric in Bondi-Sachs coordinates is given by

$$\begin{aligned}&-\left(1-\frac{M}{r}+O(r^{-2})\right)du^2 - 2\left(1+O(r^{-2})\right)dudr\\&-2\left(U_A^{(-2)}+O(r^{-1})\right)dudv^A + (r^2\tilde{\sigma}_{AB}+rC_{AB}+O(1))dv^Adv^B,\end{aligned}$$

where $v^A = \theta, \phi$ and $\tilde{\sigma}_{AB}dv^Adv^B = d\theta^2 + \sin^2\theta d\phi^2$ is the standard unit round metric on two-sphere. Each metric coefficient is expanded in inverse integral powers of r, and M, $U_A^{(-2)}$, and C_{AB} depend on u and v^A. Einstein's equation gives further constraints on the metric coefficients. See for example [3, 29]. However, we do not need to use these constraints explicitly. Instead, we will use the vacuum constraint equation on the spacelike hypersurface defined below.

Substituting $u = t - r$, the metric becomes, up to lower order terms,

$$\begin{aligned}&-\left(1-\frac{M}{r}\right)dt^2 + \left(1+\frac{M}{r}\right)dr^2 - \frac{2M}{r}dtdr\\&-2U_A^{(-2)}(dt-dr)dv^A + (r^2\tilde{\sigma}_{AB}+rC_{AB})dv^Adv^B.\end{aligned}$$

The unit timelike normal of $t = d$ slice is given by

$$\vec{n} = \left(1+\frac{M}{r}\right)\partial_t + \frac{M}{r}\partial_r + \frac{U_A^{(-2)}}{r}\frac{\partial_A}{r} + O(r^{-2}).$$

We compute

$$\begin{aligned}\langle\nabla_{\partial_r}\partial_r, \partial_t\rangle &= -\frac{1}{2}\frac{M_u}{r} + O(r^{-2}),\\ \langle\nabla_{\partial_A}\partial_B, \partial_t\rangle &= -\frac{r}{2}(C_{AB})_u + O(1),\end{aligned}$$

to get the second fundamental form of $t = d$ slice

$$k_{rr} = \frac{1}{2}\frac{M_u}{r} + O(r^{-2})$$
$$k_{AB} = \frac{r}{2}(C_{AB})_u + O(1). \tag{3.5}$$

We again consider a global coordinate change from the standard Bondi-Sachs coordinates $(u, r, \theta, \phi) = (u, r, v^A)$ to a Cartesian coordinate system (t, y^1, y^2, y^3) with

$$t = u + r, y^1 = r\sin\theta\sin\phi, y^2 = r\sin\theta\cos\phi, y^3 = r\cos\theta. \tag{3.6}$$

A null geodesic with $u = 0, \theta = \tilde{\theta}, \phi = \tilde{\phi}$ corresponds to points with the new coordinates

$$(t, y^1, y^2, y^3) = (d, d\tilde{d}_1, d\tilde{d}_2, d\tilde{d}_3).$$

Let $d_i = d\tilde{d}_i$. We consider the sphere Σ_d of (Euclidean) radius 1 centered at a point (d, d_1, d_2, d_3) on the null geodesic and the ball B_d bounded by Σ_d in t-slice. Namely,

$$\Sigma_d = \{(t, y^1, y^2, y^3) |\ t = d, \sum_i (y^i - d_i)^2 = 1\}, \tag{3.7}$$

$$\Sigma_d(s) = \{(t, y^1, y^2, y^3) |\ t = d, \sum_i (y^i - d_i)^2 = s^2\}, \tag{3.8}$$

$$B_d = \{(t, y^1, y^2, y^3) |\ t = d, \sum_i (y^i - d_i)^2 \le 1\}. \tag{3.9}$$

In this article, we study the Wang-Yau quasi-local mass of the family of surfaces Σ_d defined in (3.7) as $d \to \infty$ using the frame work outlined in Section 2. Namely, we consider a family of embedding of spheres centered at the point (d, d_1, d_2, d_3):

$$\mathfrak{F}_d(s, \hat{\theta}, \hat{\phi}) = (d, d\tilde{d}_1 + s\sin\hat{\theta}\sin\hat{\phi}, d\tilde{d}_2 + s\sin\hat{\theta}\cos\hat{\phi}, d\tilde{d}_3 + s\cos\hat{\theta}),$$

where (t, y^1, y^2, y^3) is related to the Bondi-Sachs coordinates through (3.6). In particular, $\mathfrak{F}_d$ maps the sphere of radius s, $\Sigma(s)$ in B^3 onto $\Sigma_d(s)$. The pull-backs of M, $U_A^{(-2)}$ and C_{AB} under $\mathfrak{F}_d$ defines tensors on B^3 depending on d. By (2.1), their limits as $d \to \infty$ depend only on sZ. We define the following:

Definition 3.1. We define $F(x)$, $P_{AB}(x)$ and $Q_A(x)$ to be functions of a single variable x such that

$$
\begin{aligned}
F(sZ) &= \lim_{d\to\infty} \mathfrak{F}_d^* M \\
P_{AB}(sZ) &= \lim_{d\to\infty} \mathfrak{F}_d^* C_{AB} \\
Q_A(sZ) &= \lim_{d\to\infty} \mathfrak{F}_d^* U_A^{(-2)}.
\end{aligned}
$$

We use F', P'_{AB} and Q'_A to denote the derivative of these functions with respect to x.

We recall that $Z = \sin\tilde\theta \sin\tilde\phi \sin\hat\theta \sin\hat\phi + \sin\tilde\theta\cos\tilde\phi\sin\hat\theta\cos\hat\phi + \cos\tilde\theta\cos\hat\theta$ is an eigenfunction with eigenvalue -2 on S^2 with respect to the Laplace operator $\tilde\Delta$ the unit round metric $\tilde\sigma = d\hat\theta^2 + \sin^2\hat\theta d\hat\phi^2$.

Together with the functions $(\cos\tilde\theta\cos\tilde\phi)\sin\hat\theta\cos\hat\phi + (\cos\tilde\theta\sin\tilde\phi)\sin\hat\theta\sin\hat\phi - \sin\tilde\theta\cos\hat\theta$ and $-\sin\tilde\phi\sin\hat\theta\cos\hat\phi + \cos\tilde\phi\sin\hat\theta\sin\hat\phi$, which are denoted as Z^A, they form an orthogonal basis of the -2 eigenfunctions on S^2, with respect to the unit round metric $\tilde\sigma$ on S^2.

For simplicity, we denote the coordinate $(\hat\theta, \hat\phi)$ by u^a where $a = 1, 2$ and the unit round metric $\tilde\sigma_{ab} du^a du^b = d\hat\theta^2 + \sin^2\hat\theta d\hat\phi^2$. In terms of Z and Z^A, the transformation formula [15, page 3] gives

$$
\begin{aligned}
dr &= Z ds + s Z_b du^b + O(d^{-1}) \\
dv^A &= (\frac{1}{r} Z^A) ds + (\frac{s}{r} Z_b^A) du^b + O(d^{-2}).
\end{aligned} \tag{3.10}
$$

Let $\bar g$ be the pull-back of the metric on the hypersurface $t = d$ by $\mathfrak{F}_d$. In terms of the coordinate system $\{s, u^a\}$ on B^3, we have

(3.11)

$$
\begin{aligned}
\bar g_{ss} =& 1 + \frac{1}{d}\Big(F(sZ)Z^2 + 2Q_A(sZ)ZZ^A + P_{AB}(sZ)Z^A Z^B\Big) + O(\frac{1}{d^2}) \\
\bar g_{sa} =& \frac{s}{d}\Big(F(sZ)ZZ_a + Q_A(sZ)(ZZ_a^A + Z_a Z^A) + P_{AB}(sZ)Z_a^A Z^B\Big) + O(\frac{1}{d^2}) \\
\bar g_{ab} =& s^2\tilde\sigma_{ab} + \frac{s^2}{d}\Big(F(sZ)Z_a Z_b + Q_A(sZ)(Z_a Z_b^A + Z_b Z_a^A) + P_{AB}(sZ)Z_a^A Z_b^B\Big) \\
&+ O(\frac{1}{d^2}).
\end{aligned}
$$

We first compute geometric data σ, the induced metric, and α_H (2.2) on $\Sigma_d(s)$.

Lemma 3.2. *On $\Sigma_d(s)$, we have $\sigma_{ab} = s^2\tilde{\sigma}_{ab} + \frac{1}{d}\sigma_{ab}^{(-1)} + O(d^{-2})$ where*

$$\sigma_{ab}^{(-1)} = s^2\Big[F(sZ)Z_aZ_b + Q_A(sZ)(Z_aZ_b^A + Z_bZ_a^A) + P_{AB}(sZ)Z_a^AZ_b^B\Big]$$

and

$$\begin{aligned}&\frac{1}{2}\left[\tilde{\nabla}^a\tilde{\nabla}^b\sigma_{ab}^{(-1)} - tr\sigma^{(-1)} - \tilde{\Delta}(tr\sigma^{(-1)})\right]\\ =&s^2\Big[-\frac{1}{2}sF'(sZ)Z(1-Z^2) - F(sZ)(1-2Z^2)\\ &\qquad + \left(sQ_A'(sZ)Z^2 - sQ_A'(sZ) + 4Q_A(sZ)Z\right)Z^A\\ &\qquad + \left(s^2P_{AB}''(sZ) + sP_{AB}'(sZ)Z + 4P_{AB}(sZ)\right)Z^AZ^B\Big]\end{aligned}$$

Remark 3.3. In the proof we denote functions such as $F(sZ)$, $F'(sZ)$ and $Q_A(sZ)$ by F, F' and Q_A.

Proof. On Σ_d, we have

$$\begin{aligned}\tilde{\nabla}^a\tilde{\nabla}^b\sigma_{ab}^{(-1)} =& F''(1-Z^2)^2 - 7F'Z(1-Z^2)^2 - 3F(1-3Z^2)\\ &+ \left[-2Q_A''Z(1-Z^2) - 6Q_A'(1-Z^2) + 8Q_A'Z^2 + 18Q_AZ\right]Z^A\\ &+ \left[P_{AB}''Z^2 + 7P_{AB}'Z + 9P_{AB}\right]Z^AZ^B\\ \tilde{\Delta}(tr\sigma^{(-1)}) =& F''(1-Z^2)^2 - 6F'Z(1-Z^2)^2 + F(6Z^2-2)\\ &- 2\left[Q_A''Z(1-Z^2) - 6Q_A'Z^2 + 2Q_A' - 6Q_AZ\right]Z^A\\ &- \left[P_{AB}''(1-Z^2) - 6P_{AB}'Z - 6P_{AB}\right]Z^AZ^B\end{aligned}$$

The computation on $\Sigma_d(s)$ is similar. We get a factor of s after each derivative. $\square$

Lemma 3.4. *On Σ_d, we have $(\alpha_H)_a = \frac{1}{d}(\alpha_H^{(-1)})_a + O(d^{-2})$ where*

$$\begin{aligned}(\alpha_H^{(-1)})_a =& -F'ZZ_a + \frac{1}{4}F''(1-Z^2)Z_a + \frac{1}{4}P_{AB}''Z_aZ^AZ^B + \frac{1}{2}P_{AB}'Z_a^AZ^B\\ 2\tilde{\nabla}^a(\alpha_H^{(-1)})_a =& \frac{1}{2}F'''(1-Z^2)^2 - 4F''Z(1-Z^2) - 2F'(1-3Z^2)\\ &+ \left(\frac{1}{2}P_{AB}'''(1-Z^2) - 4P_{AB}''Z - 6P_{AB}'\right)Z^AZ^B.\end{aligned}$$

Proof. The unit normal of Σ_d is $\nu = \partial_s + O(d^{-1})$. By (3.10), we have

$$\begin{aligned}\partial_s &= Z\partial_r + \frac{1}{d}Z^A\partial_A + O(d^{-2}),\\ \partial_a &= Z_a\partial_r + \frac{1}{d}Z_a^A\partial_A + O(d^{-2}).\end{aligned}$$

By (3.5), we get

$$\begin{aligned}-k(\nu,\partial_a) &= \frac{1}{2}\frac{M_u}{d}Z_aZ - \frac{1}{2d}(C_{AB})_uZ_a^AZ^B + O(d^{-2}),\\ \mathrm{tr}_\Sigma k &= -\frac{1}{2}\frac{M_u}{d}(1-Z^2) - \frac{1}{2d}(C_{AB})_uZ^AZ^B + O(d^{-2}).\end{aligned}$$

The assertion follows from $\alpha_H = -k(\nu,\partial_a) + \partial_a\frac{\mathrm{tr}_\Sigma k}{|H|} + O(d^{-2})$. □

4. The expansion of the Wang-Yau quasi-local mass

We consider the Wang-Yau quasi-local mass on the unit sphere constructed in the previous section.

Theorem 4.1. *For $T_0 = (1,0,0,0)$,*

(4.12)

$$\begin{aligned}E(\Sigma_d,X,T_0) =&\frac{1}{8\pi d^2}\Bigg[\int_{B^3}\frac{1}{8}\tilde\sigma^{AD}\tilde\sigma^{BE}(C_{AB})_u(C_{DE})_u - \det(h_0^{(-1)} - h^{(-1)})\\ &+\frac{1}{4}\int_{S^2}(tr_\Sigma k^{(-1)})^2 - \tau^{(-1)}\tilde\Delta(\tilde\Delta+2)\tau^{(-1)}\Bigg] + O(d^{-3})\end{aligned}$$

where $\tau^{(-1)}$ is the solution to the optimal embedding equation

$$\begin{aligned}\tilde\Delta(\tilde\Delta+2)\tau^{(-1)} =& \frac{1}{2}F'''(1-Z^2)^2 - 4F''Z(1-Z^2) - 2F'(1-3Z^2)\\ &+\left(\frac{1}{2}P'''_{AB}(1-Z^2) - 4P''_{AB}Z - 6P'_{AB}\right)Z^AZ^B.\end{aligned}$$

Here $h_0(s)$ and $h(s)$ are the second fundamental forms of $\Sigma_d(s)$ in the slice $\{t=d\}$ and in $\mathbb{R}^3$ (through the isometric embedding) respectively. Also recall that k stands for the second fundamental form of $\{t=d\}$ in the spacetime.

Proof. We write

$$E(\Sigma_d,X,T_0) = E_{BY}(\Sigma_d) + (E_{LY}(\Sigma_d) - E_{BY}(\Sigma_d)) + (E(\Sigma_d,X,T_0) - E_{LY})$$

where E_{BY} and E_{LY} denote the Brown-York mass and the Liu-Yau mass, respectively. From Lemma 3.1 of [7], we conclude

$$E_{BY} = \frac{1}{8\pi d^2}\int_{B^3} \frac{|k^{(-1)}|^2 - (trk^{(-1)})^2}{2} - \det(h_0^{(-1)} - h^{(-1)}) + O(d^{-3}),$$

where we also use the vacuum constraint equation

$$R = |k|^2 - (trk)^2.$$

It is easy to see that

$$E_{LY} - E_{BY} = \frac{1}{32\pi d^2}\int_{S^2} (tr_\Sigma k^{(-1)})^2 + O(d^{-3}).$$

From the second variation of the Wang-Yau mass in [8, 9], we have

$$E(\Sigma_d, X, T_0) - E_{LY} = \frac{1}{32\pi d^2}\int_{S^2} \tau^{(-1)}\tilde{\Delta}(\tilde{\Delta}+2)\tau^{(-1)} + O(d^{-3}).$$

Finally, we apply (3.5) to evaluate $|k^{(-1)}|$ and $trk^{(-1)}$. □

Remark 4.2. The formula in Theorem 4.1 should be compared with the standard mass loss formula, e.g. (5.102) on page 92 of [14], which states that (in our notations):

$$\frac{\partial M}{\partial u} = -\frac{1}{8}\tilde{\sigma}^{AD}\tilde{\sigma}^{BE}(C_{AB})_u(C_{DE})_u + \frac{1}{4}\tilde{\nabla}^A\tilde{\nabla}^B(C_{AB})_u.$$

On a region $D = \{u_1 < u < u_2\}$ between two level sets $\{u = u_1\}$ and $\{u = u_2\}$ of u, the energy radiated away is thus given by the integral of the density $-\frac{1}{8}\tilde{\sigma}^{AD}\tilde{\sigma}^{BE}(C_{AB})_u(C_{DE})_u$ over D (the second term is a divergence term). The integral over D is a global quantity which requires the information in every direction of null infinity. The first term on the right hand side of (4.1) is the principal term that corresponds to this density integral. We believe that the three other terms on the right hand side of (4.1) are correction terms that are necessary due the quasi-local and BMS invariance nature of the quantity.

5. Evaluating the quasi-local mass

Recall the $O(\frac{1}{d})$ terms of the metric coefficients on B_d

$$\bar{g}_{ss}^{(-1)} = F(sZ)Z^2 + 2Q_A(sZ)ZZ^A + P_{AB}(sZ)Z^AZ^B$$

$$\bar{g}_{as}^{(-1)} = s\left[F(sZ)ZZ_a + Q_A(sZ)(ZZ_a^A + Z_aZ^A) + P_{AB}(sZ)Z_a^AZ^B\right]$$
$$\bar{g}_{ab}^{(-1)} = s^2\left[F(sZ)Z_aZ_b + Q_A(sZ)(Z_aZ_b^A + Z_bZ_a^A) + P_{AB}(sZ)Z_a^AZ_b^B\right]$$

To apply Theorem 4.1, we need to compute $h_0^{(-1)} - h^{(-1)}$ and $\tau^{(-1)}$. We first derive a formula for $h_0^{(-1)} - h^{(-1)}$.

Lemma 5.1. *Let $\mathcal{A}_{AB}(Z,s)$ be a trace-free, symmetric 2-tensor that solves the ODE*

$$\begin{aligned}&\mathcal{A}''_{AB}(Z,s)(1-Z^2) - 6\mathcal{A}'_{AB}(Z,s)Z - 4\mathcal{A}_{AB}(Z,s)\\ &= -\frac{s^3}{2}P''_{AB}(sZ) - \frac{s^2}{2}P'_{AB}(sZ)Z - 2sP_{AB}(sZ),\end{aligned} \tag{5.13}$$

for each $0 < s \leq 1$. Here $\mathcal{A}'_{AB}$ means $\frac{\partial \mathcal{A}_{AB}}{\partial Z}$. Then the difference of second fundamental forms on the sphere of radius s is given by

$$\begin{aligned}&h_0^{(-1)} - h^{(-1)}\\ &= -\mathcal{A}''_{AB}Z_aZ_bZ^AZ^B + (\frac{s^2}{2}P'_{AB}(sZ) - 2\mathcal{A}'_{AB})\left(Z_aZ_b^A + Z_bZ_a^A\right)Z^B\\ &\quad + \left(\mathcal{A}'_{AB}Z + \mathcal{A}_{AB} - \frac{s}{2}P_{AB}(sZ)\right)Z^AZ^B\tilde{\sigma}_{ab}\\ &\quad + \left(sP_{AB}(sZ) - \frac{s^2}{2}P'_{AB}(sZ)Z - 2\mathcal{A}_{AB}\right)Z_a^AZ_b^B.\end{aligned}$$

Proof. We start with $h^{(-1)}$. The unit normal is given by

$$\bar{\nu} = \left(1 - \frac{\bar{g}_{ss}^{(-1)}}{2d}\right)\left(\partial_s - \frac{s^{-2}\tilde{\sigma}^{ab}\bar{g}_{as}^{(-1)}}{d}\partial_b\right) + O(d^{-2}).$$

We compute

$$\begin{aligned}h_{ab} =& \frac{1}{2}(\langle D_{\partial_a}\bar{\nu}, \partial_b\rangle + \langle D_{\partial_b}\bar{\nu}, \partial_a\rangle)\\ =& s\tilde{\sigma}_{ab} + \frac{1}{d}\left(\frac{1}{2}\partial_s\bar{g}_{ab}^{(-1)} - \frac{\tilde{\nabla}_a\bar{g}_{bs}^{(-1)} + \tilde{\nabla}_b\bar{g}_{as}^{(-1)}}{2} - \frac{\bar{g}_{ss}^{(-1)}}{2}s\tilde{\sigma}_{ab}\right) + O(d^{-2}).\end{aligned}$$

For $h_0^{(-1)}$, we expand the isometric embedding X as

$$X = s\tilde{X} + \frac{1}{d}X^{(-1)} + O(d^{-2})$$

where $\tilde{X}$ denote the unit sphere in $\mathbb{R}^3$. We decompose $X^{(-1)}$ into $X^{(-1)} = \alpha^a \partial_a + \beta\nu$. The linearized isometric embedding equation reads

$$\sigma_{ab}^{(-1)} = s^2(\tilde{\sigma}_{ac}\tilde{\nabla}_b\alpha^c + \tilde{\sigma}_{bc}\tilde{\nabla}_a\alpha^c) + 2\beta s\tilde{\sigma}_{ab}. \tag{5.14}$$

From the computation in [36, pages 938-939], (5.14) implies that

$$h_0^{(-1)} = -\tilde{\nabla}_a\tilde{\nabla}_b\beta - \beta\tilde{\sigma}_{ab} + \frac{1}{s}\sigma_{ab}^{(-1)}. \tag{5.15}$$

Putting these together, we obtain

$$\begin{aligned} h_0^{(-1)} - h^{(-1)} = & -\tilde{\nabla}_a\tilde{\nabla}_b\beta - \beta\tilde{\sigma}_{ab} + \frac{1}{s}\sigma_{ab}^{(-1)} \\ & - \frac{1}{2}(\partial_s\bar{g}_{ab})^{(-1)} + \frac{\tilde{\nabla}_a\bar{g}_{bs}^{(-1)} + \tilde{\nabla}_b\bar{g}_{as}^{(-1)}}{2} + \frac{\bar{g}_{ss}^{(-1)}}{2}s\tilde{\sigma}_{ab}. \end{aligned} \tag{5.16}$$

To solve β, we consider the expansion of the Gauss curvature $K(d,s)$ of $\Sigma_d(s)$. Let

$$K(d,s) = \frac{1}{s^2} + \frac{1}{d}K^{(-1)} + O(d^{-2})$$

On the one hand, from the metric expansion, we get

$$K^{(-1)} = \frac{1}{s^2}\left(-\tilde{\nabla}^a\tilde{\nabla}^b\sigma_{ab}^{(-1)} + tr_{S^2}\sigma^{(-1)} + \tilde{\Delta}tr_{S^2}\sigma^{(-1)}\right).$$

On the other hand, combining (5.15) and the Gauss equation, we conclude that

$$K^{(-1)} = \frac{2}{s}(\tilde{\Delta} + 2)\beta$$

As a result, β is the solution of

$$2s(\tilde{\Delta} + 2)\beta = -\tilde{\nabla}^a\tilde{\nabla}^b\sigma_{ab}^{(-1)} + tr_{S^2}\sigma^{(-1)} + \tilde{\Delta}tr_{S^2}\sigma^{(-1)}. \tag{5.17}$$

For the right hand side, we compute

$$\begin{aligned} & -\tilde{\nabla}^a\tilde{\nabla}^b\sigma_{ab}^{(-1)} + tr_{S^2}\sigma^{(-1)} + \tilde{\Delta}tr_{S^2}\sigma^{(-1)} \\ = & s^3F'(sZ)Z(1 - Z^2) + s^2F(2 - 4Z^2) \\ & + s^3Q_A'(sZ)(2 - 2Z^2)Z^A - 8s^2Q_A(sZ)ZZ^A \\ & + \left(-s^4P_{AB}''(sZ) - s^3P_{AB}'(sZ) - 4s^2P_{AB}(sZ)\right)Z^AZ^B \end{aligned}$$

On the other hand, let $\mathcal{F}$ and $\mathcal{Q}_A$ be an antiderivative of F and Q_A respectively, and $\mathcal{A}_{AB}$ satisfy (5.13). One verifies that

$$\beta = \frac{\mathcal{F}(sZ)}{2} Z + \mathcal{Q}_A(sZ)Z^A + \mathcal{A}_{AB}(Z,s)Z^A Z^B \tag{5.18}$$

solves the linearized isometric embedding equation (5.17) since, for a trace-free, symmetric 2-tensor $\mathcal{A}_{AB}(Z,s)$,

$$\begin{aligned}&(\tilde{\Delta}+2)\left(\mathcal{A}_{AB}(Z,s)Z^A Z^B\right)\\ =&\left(\mathcal{A}''_{AB}(Z,s)(1-Z^2) - 6\mathcal{A}'_{AB}(Z,s)Z - 4\mathcal{A}_{AB}(Z,s)\right) Z^A Z^B.\end{aligned}$$

We are ready to compute (5.16) where β is given in (5.18). We have

$$\begin{aligned}-\tilde{\nabla}_a\tilde{\nabla}_b\beta - \beta\tilde{\sigma}_{ab} =& \frac{s^2}{2}F'ZZ_aZ_b + \frac{s}{2}FZ^2\tilde{\sigma}_{ab} - s^2Q'_AZ_aZ_bZ^A\\ &+ sQ_AZZ^A\tilde{\sigma}_{ab} - \mathcal{A}''_{AB}Z_aZ_bZ^AZ^B - \frac{1}{s}\sigma^{(-1)}_{ab}\\ &- 2\mathcal{A}'_{AB}(Z_aZ^A_b + Z_bZ^A_a)Z^B + (sP_{AB} - 2\mathcal{A}_{AB})Z^A_aZ^B_b\\ &+ (\mathcal{A}'_{AB}Z + \mathcal{A}_{AB})Z^AZ^B\tilde{\sigma}_{ab},\end{aligned}$$

$$\begin{aligned}&\frac{1}{s}\sigma^{(-1)}_{ab} - \frac{1}{2}\partial_s\bar{g}^{(-1)}_{ab}\\ &\qquad = -\frac{s^2}{2}\left(F'ZZ_aZ_b + Q'_AZ(Z_aZ^A_b + Z_bZ^A_a) + P'_{AB}ZZ^A_aZ^B_b\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(\tilde{\nabla}_a\bar{g}^{(-1)}_{bs} + \tilde{\nabla}_b\bar{g}^{(-1)}_{as}) =& s^2F'ZZ_aZ_b - sFZ^2\tilde{\sigma}_{ab} + \frac{s^2}{2}Q'_AZ(Z_aZ^A_b + Z_bZ^A_a)\\ &+ s^2Q'_AZ_aZ_bZ^A - 2sQ_AZZ^A\tilde{\sigma}_{ab} + \frac{1}{s}\sigma^{(-1)}_{ab}\\ &+ \frac{s^2}{2}P'_{AB}(Z_aZ^A_b + Z_bZ^A_a)Z^B - sP_{AB}Z^AZ^B\tilde{\sigma}_{ab},\\ \frac{1}{2}\bar{g}^{(-1)}_{ss}s\tilde{\sigma}_{ab} =& s\left(\frac{1}{2}FZ^2 + Q_AZZ^A + \frac{1}{2}P_{AB}Z^AZ^B\right)\tilde{\sigma}_{ab}.\end{aligned}$$

We see that terms involving F, Q_A cancel and the result has the asserted form. □

Next we compute $\tau^{(-1)}$.

Lemma 5.2. *Define the second order differential operator*

$$L\mathcal{G}(Z) = \left[(1-Z^2)\mathcal{G}'\right]'(Z) - 4\mathcal{G}'(Z)Z - 6\mathcal{G}(Z).$$

Let $\mathcal{B}_{AB}(Z)$ *be a traceless, symmetric 2-tensor that solves the ODE*

$$L(L+2)\mathcal{B}_{AB} = \frac{1}{2}P'''_{AB}(Z)(1-Z^2) - 4P''_{AB}(Z)Z - 6P'_{AB}(Z). \tag{5.19}$$

Then

$$\tau^{(-1)} = Z\mathcal{F}(Z) + \mathcal{B}(Z)_{AB}Z^AZ^B$$

solves the leading order of optimal embedding equation

$$\begin{aligned}\tilde{\Delta}(\tilde{\Delta}+2)\tau^{(-1)} = {} & \frac{1}{2}F'''(1-Z^2)^2 - 4F''Z(1-Z^2) - 2F'(1-3Z^2) \\ & + \left(\frac{1}{2}P'''_{AB}(Z)(1-Z^2) - 4P''_{AB}(Z)Z - 6P'_{AB}(Z)\right)Z^AZ^B.\end{aligned}$$

Proof. The equation is linear. We look for $\tau_1^{(-1)}$ and $\tau_2^{(-1)}$ such that

$$\begin{aligned}\tilde{\Delta}(\tilde{\Delta}+2)\tau_1^{(-1)} &= \frac{1}{2}F'''(1-Z^2)^2 - 4F''Z(1-Z^2) - 2F'(1-3Z^2),\\ \tilde{\Delta}(\tilde{\Delta}+2)\tau_2^{(-1)} &= \left(\frac{1}{2}P'''_{AB}(Z)(1-Z^2) - 4P''_{AB}(Z)Z - 6P'_{AB}(Z)\right)Z^AZ^B.\end{aligned}$$

From Lemma 3.3 of [15], $\tau_1^{(-1)} = Z\mathcal{F}(Z)$ solves the first equation

$$\tilde{\Delta}(\tilde{\Delta}+2)(Z\mathcal{F}(Z)) = \frac{1}{2}F'''(1-Z^2)^2 - 4F''Z(1-Z^2) - 2F'(1-3Z^2).$$

It is straightforward to verify that $\tau_2^{(-1)} = \mathcal{B}_{AB}(Z)Z^AZ^B$ solves the second equation if the traceless, symmetric 2-tensor $\mathcal{B}_{AB}(Z)$ solves (5.19). $\square$

We are ready to state the main theorem for the quasi-local mass,

Theorem 5.3. *For* $T_0 = (1,0,0,0)$ *and* X *solves the leading order term of* *the optimal embedding equation, the Wang-Yau quasi-local energy*

$$
\begin{aligned}
&E(\Sigma_d, T_0, X)\\
=&\frac{1}{d^2}\Bigg[\int_{B^3}\frac{1}{8}\sum_{A,B}P'_{AB}(sZ)P'_{AB}(sZ)-\det(h_0^{(-1)}-h^{(-1)})\\
&-\frac{1}{4}\int_{S^2}\mathcal{B}_{DE}Z^DZ^E\left(\frac{1}{2}P'''_{AB}(Z)(1-Z^2)-4P''_{AB}(Z)Z-6P'_{AB}(Z)\right)Z^AZ^B\\
&+\frac{1}{4}\int_{S^2}\frac{1}{4}(P'_{AB}Z^AZ^B)^2\Bigg]+O(d^{-3})
\end{aligned}
$$

where $h_0^{(-1)}-h^{(-1)}$ *is as determined in Lemma 5.1 and* $\mathcal{B}_{AB}$ *is as determined in Lemma 5.2.*

Proof. We start with Theorem 4.1 in which $h_0^{(-1)}-h^{(-1)}$ is as determined in Lemma 5.1 and $\tau^{(-1)}$ is as determined in Lemma 5.2. We simplify the expression

$$
\begin{aligned}
&\int_{S^2}(tr_\Sigma k^{(-1)})^2-\tau^{(-1)}\tilde\Delta(\tilde\Delta+2)\tau^{(-1)}\\
=&\int_{S^2}\frac{1}{4}F^2(1-Z^2)^2-\tau_1^{(-1)}\tilde\Delta(\tilde\Delta+2)\tau_1^{(-1)}\\
&+\int_{S^2}\frac{1}{4}\left(P'_{AB}(Z)Z^AZ^B\right)^2-\tau_2^{(-1)}\tilde\Delta(\tilde\Delta+2)\tau_2^{(-1)}.
\end{aligned}
$$

We have

$$
\int_{S^2}\frac{1}{4}F^2(1-Z^2)^2-\tau_1^{(-1)}\tilde\Delta(\tilde\Delta+2)\tau_1^{(-1)}=0
$$

by [15, (3.6)]. This finishes the proof of the theorem. □

Remark 5.4. By the definition of P_{AB} in Definition 3.1, the first term on the right hand of this formula is a quadratic expression of the news $(C_{AB})_u$ (the first term on the right hand side of (4.1)), which is related to the mass loss formula, see Remark 4.2.

In particular, we observe that the answer depends on the leading order term of the news function on B^3 since both ODEs in Lemma 5.1 and Lemma 5.2 are linear ODEs where the right-hand side depends on P_{AB} and their derivatives. In general, we do not have explicit solutions to these ODEs. In the following section, we compute the quasi-local mass explicitly for a few special examples.

6. Special cases

Writing $E(\Sigma_d, T_0, X) = d^{-2}E^{(-2)} + O(d^{-3})$, we evaluate $E^{(-2)}$ for a few special cases of P_{AB}. Let p_{AB}, q_{AB} be two constant symmetric traceless 2-tensors.

Proposition 6.1. *If $P_{AB}(x) = p_{AB} + q_{AB}x$, $E^{(-2)} = 0$. In particular, $E^{(-2)}$ could vanish even in the presence of nonzero news.*

Proof. One verifies that

$$\begin{aligned} \mathcal{A}_{AB}(Z,s) &= \frac{s}{2}p_{AB} + \frac{s^2 Z}{4}q_{AB} \\ \mathcal{B}_{AB}(Z) &= -\frac{1}{4}q_{AB} \end{aligned}$$

solve (5.13) and (5.19) respectively. Direct computation then shows that $h_0^{(-1)} - h^{(-1)} = 0$. Hence, we get $8\pi E^{(-2)}$ is

$$\frac{1}{8}\sum_{A,B} q_{AB}q_{AB}\cdot\frac{4\pi}{3} + \frac{1}{4}\int_{S^2}\frac{1}{4}\left(q_{AB}Z^AZ^B\right)^2 + \frac{1}{4}q_{DE}Z^DZ^E\cdot(-6q_{AB}Z^AZ^B).$$

Using the identity

$$\int_{S^2} Z^AZ^BZ^DZ^E = \frac{4\pi}{15}(\delta^{AB}\delta^{DE} + \delta^{AD}\delta^{BE} + \delta^{AE}\delta^{BD}), \tag{6.20}$$

we get $E^{(-2)} = 0$. ☐

Proposition 6.2. *If $P_{AB}(x) = p_{AB}x^2$. Then $E^{(-2)} = \frac{1}{20}\sum_{A,B} p_{AB}p_{AB}$.*

Proof. One verifies that

$$\begin{aligned} \mathcal{A}_{AB}(Z,s) &= s^3\left(\frac{(Z)^2}{6} + \frac{1}{3}\right)p_{AB} \\ \mathcal{B}_{AB}(Z) &= -\frac{Z}{6}p_{AB} \end{aligned}$$

solve (5.13) and (5.19) respectively. Direct computation shows that

$$\begin{aligned} h_0^{(-1)} - h^{(-1)} =& \frac{s^3}{3}\Big(Z^AZ^B\tilde{\sigma}_{ab} - Z_aZ_bZ^AZ^B + Z\left(Z_aZ_b^A + Z_bZ_a^A\right)Z^B \\ &- \left((Z)^2 + 2\right)Z_a^AZ_b^B\Big)p_{AB}. \end{aligned}$$

We compute

$$\left|h_0^{(-1)} - h^{(-1)}\right|^2_{\tilde{\sigma}} = \frac{s^2}{9}\Big(9\left(p_{AB}Z^AZ^B\right)^2 + (2(Z)^2-8)\delta^{AD}Z^BZ^Ep_{ABp_{DE}} \\ + ((Z)^2+2)^2\delta^{AD}\delta^{BE}p_{AB}p_{DE}\Big),$$

$$\mathrm{tr}_{\tilde{\sigma}}\left(h_0^{(-1)} - h^{(-1)}\right) = sZ^AZ^Bp_{AB}$$

to get

$$\begin{aligned}
&\det\left(h_0^{(-1)} - h^{(-1)}\right)\\
=&\frac{1}{2}\left(\mathrm{tr}_{\tilde{\sigma}}\left(h_0^{(-1)} - h^{(-1)}\right) - \left|h_0^{(-1)} - h^{(-1)}\right|^2_{\tilde{\sigma}}\right)\\
=&-\frac{s^2}{18}\Big(\left(2(Z)^2-8\right)\delta^{AD}Z^BZ^E + \left((Z)^2+2\right)^2\delta^{AD}\delta^{BE}\Big)p_{AB}p_{DE}.
\end{aligned}$$

Denote $|p|^2 = \sum_{A,B} p_{AB}p_{AB}$. The volume integral contributes

$$\frac{1}{3}\int_{S^2}\left[\frac{(Z)^2}{2}|p|^2 + \frac{1}{18}\left((2(Z)^2-8)\delta^{AD}Z^BZ^Ep_{AB}p_{DE} + ((Z)^2+2)^2|p|^2\right)\right]$$

which is $\frac{4\pi}{9}|p|^2$ and the surface integral contributes

$$\frac{1}{4}\int_{S^2}(Z)^2(p_{AB}Z^AZ^B)^2 - \frac{10}{3}(Z)^2Z^DZ^Ep_{DE}Z^AZ^Bp_{AB} = -\frac{2\pi}{45}|p|^2,$$

where we used the identity $\int_{S^2}(Z)^2Z^AZ^BZ^DZ^E = \frac{4\pi}{105}(\delta^{AB}\delta^{DE} + \delta^{AD}\delta^{BE} + \delta^{AE}\delta^{BD})$. $\square$

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Po-Ning Chen
Department of Mathematics
University of California Riverside
USA
E-mail: poningc@ucr.edu

Mu-Tao Wang
Department of Mathematics
Columbia University
USA
E-mail: mtwang@math.columbia.edu

Ye-Kai Wang
Department of Mathematics
National Cheng Kung University
Taiwan
E-mail: ykwang@mail.ncku.edu.tw

Shing-Tung Yau
Department of Mathematics
Harvard University
USA
E-mail: yau@math.harvard.edu