

The $\partial\bar{\partial}$ -lemma for general Clemens manifolds

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Dedicated to Sir Simon Donaldson for his 60th birthday

Abstract: We show that the $\partial\bar{\partial}$ -lemma holds for the non-Kähler compact complex manifolds of dimension three with trivial canonical bundle constructed by Clemens as deformations of Calabi-Yau threefolds contracted along smooth rational curves with normal bundle of type $(-1, -1)$, at least on an open dense set in moduli. The proof uses the mixed Hodge structure on the singular fibers and an analysis of the variation of the Hodge filtration for the smooth fibers.

Introduction

Around 1985, Herb Clemens gave a remarkable construction of compact complex manifolds of dimension three and trivial canonical bundle as follows. Let X be a Calabi-Yau threefold, for example a quintic threefold in \mathbb{P}^4 , and let C_1, \dots, C_r be disjoint smooth rational curves in X such that the normal bundle $N_{C_i/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for all i , and such that the classes $[C_1], \dots, [C_r]$ satisfy a linear relation $\sum_i m_i [C_i] = 0$ in $H^4(X; \mathbb{C})$ with all $m_i \neq 0$ and span $H^4(X; \mathbb{C})$. If \bar{X} is the singular compact complex threefold obtained by contracting the C_i to ordinary double points, then \bar{X} is smoothable, and small smoothings of \bar{X} are compact complex manifolds of dimension three with second Betti number $b_2 = 0$ and trivial canonical bundle. We will call any complex manifold obtained in this way a *Clemens manifold*. If for example X is simply connected and the classes $[C_1], \dots, [C_r]$ generate $H^4(X; \mathbb{Z})$, then small smoothings of \bar{X} are diffeomorphic to a connected sum of copies of $S^3 \times S^3$. Moreover, the number r of curves C_i required in the construction can be arbitrarily large, giving examples of an infinite number of topologically different families of Clemens manifolds. Details of Clemens' construction were given in [9], and the construction was subsequently generalized by Tian

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[23], Kawamata [15], and Ran [18], to the case where the classes $[C_i]$ do not necessarily span $H^4(X; \mathbb{C})$.

Given the very simple topological nature of Clemens manifolds, it is tempting to speculate that they play a fundamental role in describing the moduli of Calabi-Yau threefolds, see for example Reid [20]. It is also natural to ask if their cohomology in dimension three carries a polarized weight three Hodge structure. While it is easy to see that the Hodge-de Rham spectral sequence degenerates at E_1 (and we recall this argument in the proof of Theorem 2.7 below), it is not obvious that the resulting filtrations F^\bullet and \bar{F}^\bullet on H^k are k -opposed, or equivalently that the $\partial\bar{\partial}$ -lemma holds (despite the careless statement on p. 107 of [10]). The goal of this paper is to show that indeed the $\partial\bar{\partial}$ -lemma holds for a general Clemens manifold. Here general roughly means that the $\partial\bar{\partial}$ -lemma holds outside of a proper real analytic subvariety, although it seems likely that in fact it holds for all small smoothings of \bar{X} . Unfortunately, the variational methods of this paper do not seem well suited to deciding if the resulting weight three Hodge structures are *polarized*. Of course, it is a general fact that on a compact complex threefold, if $\omega \in H^0(\Omega^3)$ is nonzero, then $\sqrt{-1}\langle\omega, \bar{\omega}\rangle > 0$, where $\langle\cdot, \cdot\rangle$ is the usual pairing on H^3 . But the remaining Hodge-Riemann inequality for Clemens manifolds, that the Hermitian form on $H^{2,1}$ defined by $\sqrt{-1}\langle\eta, \bar{\eta}\rangle$ is negative definite, seems more difficult to establish.

One can also ask if there are good metrics on Clemens manifolds whose existence would imply the existence of a Hodge decomposition, and, even better, the Hodge-Riemann inequalities. Results of Fu-Li-Yau [11] show the existence of *balanced metrics* on Clemens manifolds. These are metrics such that the square of the associated Kähler form is d -closed (in the case of complex dimension three). However, in general the existence of a balanced metric is not sufficient to imply that the $\partial\bar{\partial}$ -lemma holds.

Fine and Panov [7] have constructed a complex structure with trivial canonical bundle on $2(S^3 \times S^3) \# (S^2 \times S^4)$. If X is the corresponding compact complex threefold, they show in addition that there is a nontrivial holomorphic vector field on X , i.e. that $H^0(X; T_X) \neq 0$. It follows easily that the $\partial\bar{\partial}$ -lemma does not hold for X . It would be interesting to know if the deformations of X are obstructed, or if there are small deformations of X to a complex manifold for which the $\partial\bar{\partial}$ -lemma holds.

This paper is organized as follows. In Section 1, we collect some general results about Hodge structures and the $\partial\bar{\partial}$ -lemma. Section 2 deals with the deformation theory of threefolds with ordinary double points and trivial dualizing sheaf, as well as the limiting mixed Hodge structures associated to their smoothings. While all of this material is very well-known to specialists, we

give the arguments in some detail to emphasize that it is enough to assume only that a resolution of the singular fiber satisfies the $\partial\bar{\partial}$ -lemma. We could replace this assumption by the assumption that a resolution of the singular fiber is Kähler, at the cost of increasing the notational complexity of the argument (see Remark 2.3, where we argue that it is enough to consider the case where there is just one smoothing direction in the deformation space). The main calculation is given in Section 3, where we use the nature of the monodromy and the variation of the Hodge filtration to show that the $\partial\bar{\partial}$ -lemma holds on a nonempty open subset of the deformation space of smoothings of the singular fiber. Section 4 deals with a question on deformations of compact complex manifolds satisfying the $\partial\bar{\partial}$ -lemma, which I first learned of from Yau.

1. Some preliminary remarks

We begin with a definition of the statement that the $\partial\bar{\partial}$ -lemma holds for a compact complex manifold V and its link with the existence of a Hodge structure on the cohomology of V .

Definition 1.1. Let V be a compact complex manifold and let $A^k(V)$ denote the space of C^∞ k -forms on V . We say that the $\partial\bar{\partial}$ -lemma holds for V if, for all k , and all $\eta \in A^k(V)$ such that $\partial\eta = \bar{\partial}\eta = 0$, the following property holds: the form η is d -exact, i.e. there exists a form ξ such that $\eta = d\xi \iff$ there exists an $\alpha \in A^{k-2}(V)$ such that $\eta = \partial\bar{\partial}\alpha$.

We then have the following [6, (5.21)] (for the direction (i) \implies (ii), see also [4, (4.3.1)]):

Theorem 1.2. *Let V be a compact complex manifold. Then the following conditions are equivalent:*

- (i) *The Hodge-de Rham spectral sequence for V degenerates at E_1 and, for all k , if F^\bullet is the corresponding filtration on $H^k(V; \mathbb{C})$, then F^\bullet and \overline{F}^\bullet are k -opposed, i.e. for all p , there is an isomorphism*

$$F^p \oplus \overline{F}^{k-p+1} \cong H^k(V; \mathbb{C})$$

induced by the natural inclusions.

- (ii) *The $\partial\bar{\partial}$ -lemma holds for V .* □

If either of the above conditions hold, then we define

$$H^{p,q}(V) = F^p \cap \overline{F}^q \subseteq H^{p+q}(V; \mathbb{C}).$$

Equivalently, by [2, (5.4)(i)], $H^{p,q}(V)$ is the set of $\alpha \in H^{p+q}(V; \mathbb{C})$ such that there exists a de Rham representative for α of type (p, q) . Then we have the usual Hodge decomposition

$$H^k(V; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(V), \quad \text{and } \overline{H^{p,q}(V)} = H^{q,p}(V).$$

Remark 1.3. (i) The conditions that the Hodge-de Rham spectral sequence for V degenerates at E_1 and that, for all k , the filtrations F^\bullet and \overline{F}^\bullet are k -opposed are both open conditions. Hence the condition that the $\partial\bar{\partial}$ -lemma holds for V is an open condition.

(ii) Let V be a compact complex manifold for which the $\partial\bar{\partial}$ -lemma holds. It does not seem to be clear if this property is inherited by a closed holomorphic submanifold Z . However, if Z is a closed holomorphic submanifold of V and the $\partial\bar{\partial}$ -lemma holds for Z , then it is easy to see that the inclusion and the Gysin homomorphism are morphisms of Hodge structures. More generally if Z is a compact complex manifold such that the $\partial\bar{\partial}$ -lemma holds for Z , and $f: Z \rightarrow V$ is a holomorphic map, then f^* and f_* are morphisms of Hodge structures (with the appropriate shift in the case of f_* or the Gysin homomorphism). This follows since clearly $f^*H^{p,q}(V) \subseteq H^{p,q}(Z)$ and because the Poincaré duality isomorphism

$$(H^k(V; \mathbb{Q}))^* \cong H^{2n-k}(V; \mathbb{Q}) \otimes \mathbb{Q}(n)$$

is an isomorphism of Hodge structures.

(iii) Let V be a compact complex manifold for which the $\partial\bar{\partial}$ -lemma holds, and let Z be a submanifold of V for which the $\partial\bar{\partial}$ -lemma also holds. If $\rho: \tilde{V} \rightarrow V$ is the blowup of V along Z , then the $\partial\bar{\partial}$ -lemma holds for \tilde{V} . There have been a number of recent preprints which address this issue [19], [1], [22]. The main point is to show that the Hodge-de Rham spectral sequence degenerates at the E_1 page. This can be done by an examination of the Leray spectral sequence $E_2^{p,q} = H^p(V; R^q \rho_* \Omega_{\tilde{V}}^k) \implies H^{p+q}(\tilde{V}; \Omega_{\tilde{V}}^k)$ and the well-known computation of the Betti numbers of \tilde{V} . In fact, the Leray spectral sequence above additionally degenerates at the E_2 page. The above results then follow easily from a computation due to Gros [13, IV Théorème 1.2.1]; compare also [14, Proposition (3.3)].

Lemma 1.4. *Let V be a compact complex manifold of dimension d for which the Hodge-de Rham spectral sequence degenerates at E_1 and let F^\bullet be the corresponding filtration on $H^d(V; \mathbb{C})$. Then F^\bullet is isotropic for cup product, in the sense that, for all k , $(F^k)^\perp = F^{d-k+1}$.*

Proof. First, we claim that $F^{d-k+1} \subseteq (F^k)^\perp$. Every element of F^k has a de Rham representative η with $d\eta = 0$ and

$$\eta = \sum_{\ell \geq k} \eta^{\ell, d-\ell}, \quad \eta^{\ell, d-\ell} \in A^{\ell, d-\ell}(V),$$

and similarly for elements of F^{d-k+1} . Thus, if $\xi \in F^k$ and $\xi' \in F^{d-k+1}$, then $\xi \smile \xi'$ has a de Rham representative which is a sum of forms of type $(d+a, d-a)$, $a \geq 1$, and hence is 0, so that $\xi \smile \xi' = 0$.

Since V has dimension d , Kodaira-Serre duality implies that

$$\dim H^i(V; \Omega_V^{d-i}) = \dim H^{d-i}(V; \Omega_V^i).$$

It is then easy to see that F^{d-k+1} and $(F^k)^\perp$ have the same dimension. Since $F^{d-k+1} \subseteq (F^k)^\perp$, we must have $F^{d-k+1} = (F^k)^\perp$. \square

Lemma 1.5. *Let V be a compact complex manifold of dimension d for which the Hodge-de Rham spectral sequence degenerates at E_1 . Then the natural map*

$$F^1 \oplus \bar{F}^d \rightarrow H^d(V; \mathbb{C})$$

is an isomorphism, and hence so is the map $F^d \oplus \bar{F}^1 \rightarrow H^d(V; \mathbb{C})$.

Proof. Since the codimension of F^1 in $H^d(V; \mathbb{C})$ is the dimension of \bar{F}^d , it suffices to show that $F^1 \cap \bar{F}^d = 0$. Let ω be a holomorphic d -form and suppose that $\bar{\omega} \in F^1$. By Lemma 1.4, $F^1 = (F^d)^\perp$ and hence $\int_V \omega \wedge \bar{\omega} = 0$. It follows that $\omega = \bar{\omega} = 0$, and hence that $F^1 \cap \bar{F}^d = 0$ as claimed. \square

Let X be a compact complex manifold of dimension three for which the Hodge-de Rham spectral sequence degenerates at E_1 and let F^\bullet be the corresponding filtration on $H^3(X; \mathbb{C})$. Suppose that $H^i(X; \mathcal{O}_X) = H^0(X; \Omega_X^i) = 0$ for $i = 1, 2$. It follows that, for n odd, $n \neq 3$, $H^n(X; \mathbb{C}) = 0$, and for $n = 2k$ even, the filtration F^\bullet on $H^{2k}(X; \mathbb{C})$ satisfies: $F^p H^{2k}(X; \mathbb{C}) = 0$, $p > k$, and $F^p H^{2k}(X; \mathbb{C}) = H^{2k}(X; \mathbb{C})$, $p \leq k$. Thus trivially the filtrations F^\bullet and \bar{F}^\bullet are $2k$ -opposed and induce a Hodge structure on $H^{2k}(X; \mathbb{C})$ for which $H^{2k}(X; \mathbb{C}) = H^{k,k}(X)$.

Corollary 1.6. *Let X be a compact complex manifold of dimension 3 for which the Hodge-de Rham spectral sequence degenerates at E_1 and let F^\bullet be the corresponding filtration on $H^3(X; \mathbb{C})$. Suppose that $H^i(X; \mathcal{O}_X) = H^0(X; \Omega_X^i) = 0$ for $i = 1, 2$. Then the $\partial\bar{\partial}$ -lemma holds for $X \iff H^3(X; \mathbb{C}) \cong F^2 \oplus \bar{F}^2$.* \square

2. The limiting mixed Hodge structure

2.1. Notation

We fix the following notation for the rest of this section: Let X_0 be a compact complex manifold of dimension 3 with trivial canonical bundle for which the $\partial\bar{\partial}$ -lemma holds. We assume further that $H^i(X_0; \mathcal{O}_{X_0}) = H^0(X_0; \Omega_{X_0}^i) = 0$ for $i = 1, 2$. By the Tian-Todorov theorem, the Kuranishi deformation space for X_0 can be identified with the germ of the origin in $H^1(X_0; T_{X_0}) \cong H^1(X_0; \Omega_{X_0}^2)$, and is thus a smooth germ of dimension h , where

$$h = h^{2,1}(X_0) = \dim H^1(X_0; \Omega_{X_0}^2).$$

Let C_1, \dots, C_r be disjoint smooth curves in X_0 such that, for all i , $C_i \cong \mathbb{P}^1$ and the normal bundle $N_{C_i/X_0} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, i.e. is of type $(-1, -1)$. The C_i can be contracted in X_0 to points p_i , yielding a singular compact threefold \bar{X}_0 . We assume that the cohomology classes $[C_i]$ of the C_i satisfy a linear relation in $H^4(X_0; \mathbb{C})$ of the form

$$\sum_{i=1}^r m_i [C_i] = 0, m_i \in \mathbb{Q},$$

where $m_i \neq 0$ for every i , and that the $[C_i]$ span a subspace of $H^4(X_0; \mathbb{C})$ of dimension $r - 1$, so that no $r - 1$ of the $[C_i]$ are linearly dependent. Note that we can and shall consider the case $r = 1$, in which case the above assumption is simply that $[C_1] = 0$ in $H^4(X_0; \mathbb{C})$.

2.2. The deformation space

To analyze the deformation theory of \bar{X}_0 , let $\mathbb{T}_{\bar{X}_0}^i = \text{Ext}^i(\Omega_{\bar{X}_0}^1, \mathcal{O}_{\bar{X}_0})$ be the objects of Lichtenbaum-Schlessinger theory. Then by [9, Theorem 4.4], there is an exact sequence

$$0 \rightarrow H^1(\bar{X}_0; T_{\bar{X}_0}^0) \rightarrow \mathbb{T}_{\bar{X}_0}^1 \rightarrow \mathbb{C} \rightarrow 0,$$

where the last term \mathbb{C} is identified with the kernel of the fundamental class map

$$\bigoplus_i \mathbb{C}[C_i] \rightarrow H^4(X_0; \mathbb{C}) = H^2(X_0; \Omega_{X_0}^2),$$

$T_{\bar{X}_0}^0$ is the tangent sheaf of \bar{X}_0 , and $H^1(\bar{X}_0; T_{\bar{X}_0}^0) \cong H^1(X_0; T_{X_0})$ by [9, (3.4)] and the following remarks.

The space \bar{X}_0 is smoothable. More precisely, there is the following result due independently to Tian [23], Kawamata [15], and Ran [18]:

Theorem 2.1. *The locally semi-universal deformation space for \bar{X}_0 can be identified with the germ of the origin in $\mathbb{T}_{\bar{X}_0}^1$, and thus is a smooth germ of dimension $h+1$. Moreover, the germ of the hyperplane $H^1(\bar{X}_0; T_{\bar{X}_0}^0)$ corresponds to locally trivial deformations of \bar{X}_0 , which are identified with deformations of X_0 . The points lying over the (germ of) $\mathbb{T}_{\bar{X}_0}^1 - H^1(\bar{X}_0; T_{\bar{X}_0}^0)$ are smooth compact complex manifolds of dimension 3 with trivial canonical bundle. \square*

Remark 2.2. Tian proves the theorem under the assumption that the $\partial\bar{\partial}$ -lemma holds for X_0 . Kawamata's result is stated under the hypothesis that X_0 is projective, but the proof seems to work in much greater generality. Ran's proof apparently only uses the degeneration of the Hodge-de Rham spectral sequence for X_0 . If we make the very stringent assumption that the classes $[C_i]$ span $H^2(X_0; \Omega_{X_0}^2)$ (the main case of interest in this paper), then the above theorem is proved in [9], assuming only that $K_{X_0} \cong \mathcal{O}_{X_0}$.

Remark 2.3. In what follows, to simplify notation, we will use the fact that it is possible to smooth "one dimension at a time." More precisely, suppose that the C_i are smooth rational curves of the type considered, but without the assumption that no $r-1$ of the classes C_i are linearly dependent. We can then reorder the C_i so that no $s-1$ of the classes $[C_1], \dots, [C_s]$ are linearly independent and that $[C_1], \dots, [C_s]$ are linearly dependent. Let X_1 be a general smooth threefold which is a small smoothing of the singular threefold X'_0 with double points obtained by contracting C_1, \dots, C_s . Then K_{X_1} is trivial and (as we shall show) satisfies the $\partial\bar{\partial}$ -lemma. The classes C_{s+1}, \dots, C_r then deform to curves in X_1 , satisfying a linear relation with nonzero coefficients, and we can then repeat the construction.

2.3. A normal crossings model

We turn next to semistable models for the deformations of \bar{X}_0 . Let \tilde{X}_0 be the blowup of \bar{X}_0 at the double points, or equivalently of X_0 along the curves C_i . By Remark 1.3(iii), \tilde{X}_0 also satisfies the $\partial\bar{\partial}$ -lemma (as is easy to check directly in this special case). Moreover one easily checks that $H^3(\tilde{X}_0) \cong H^3(X_0)$ is an isomorphism of Hodge structures. The exceptional divisors Q_i over p_i , or C_i , are smooth quadrics. Thus $Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the normal bundle N_{Q_i/\tilde{X}_0} of Q_i in \tilde{X}_0 is $\mathcal{O}_{Q_i}(-1, -1)$ (using the standard notation for line bundles on Q_i). For each i , let E_i be a smooth quadric threefold in \mathbb{P}^4 and identify Q_i with a smooth hyperplane section of E_i , also denoted Q_i , by some

choice of isomorphism. (Since every element in the neutral component of the automorphism group of Q_i is induced by restriction from an automorphism of E_i , the choice of an isomorphism is irrelevant.) Thus $N_{Q_i/E_i} \cong \mathcal{O}_{Q_i}(1, 1)$. Let $\tilde{Y}_0 = \tilde{X}_0 \amalg \coprod_i E_i$ and let

$$Y_0 = \tilde{X}_0 \amalg \coprod_i E_i / \sim ,$$

where the equivalence relation \sim means that we glue $Q_i \subseteq E_i$ to $Q_i \subseteq \tilde{X}_0$ by the choice of an isomorphism above. Note that Y_0 is in the natural way a d -semistable variety with normal crossings in the sense of [8, (1.13)]. Let $\nu: \tilde{Y}_0 \rightarrow Y_0$ be the normalization morphism. We can exhibit a model for the smoothings of Y_0 as follows. Let $\bar{\pi}: \mathcal{X} \rightarrow \bar{S}$ be the germ of the locally semi-universal deformation of \bar{X}_0 , where we can identify \bar{S} with the germ about the origin in $\mathbb{T}_{\bar{X}_0}$. Let $S \rightarrow \bar{S}$ be the double cover of \bar{S} branched along the smooth hypersurface $\bar{S} \cap H^1(\bar{X}_0; T_{\bar{X}_0}^0)$ and let $\bar{\mathcal{Y}} \rightarrow S$ be the pulled back family. If D is the ramification divisor of the cover $S \rightarrow \bar{S}$ or equivalently the inverse image of $H^1(\bar{X}_0; T_{\bar{X}_0}^0)$ in S , then D is the discriminant locus of $\bar{\pi}$, the fibers of $\bar{\mathcal{Y}}$ over D have r ordinary double points corresponding to the singular points and the singularities of the total space $\bar{\mathcal{Y}}$ are locally analytically isomorphic to products of ordinary double points of dimension 4 with D . Blowing up these singular points gives a proper flat morphism $\pi: \mathcal{Y} \rightarrow S$, where \mathcal{Y} is smooth, the discriminant locus of π is D , and the fibers of π over D are locally trivial deformations of the normal crossings varieties Y_0 described above (and hence also have normal crossings). Let $\mathcal{Y}_D = \pi^{-1}(D)$. Thus \mathcal{Y}_D is a divisor with normal crossings in \mathcal{Y} . For $s \notin D$, the fiber $Y_{\bar{s}}$ of π is identified with the corresponding smooth fiber $X_{\bar{s}}$ of $\bar{\pi}$, where $\bar{s} \in \bar{S}$ is the point lying under s .

2.4. A mixed Hodge structure on Y_0

By convention, all cohomology is with \mathbb{C} -coefficients unless otherwise specified. We have the Mayer-Vietoris sequence for Y_0 :

$$0 \rightarrow \mathbb{C}_{Y_0} \rightarrow \nu_* \mathbb{C}_{\tilde{Y}_0} \rightarrow \bigoplus_i (j_i)_* \mathbb{C}_{Q_i} \rightarrow 0,$$

where $j_i: Q_i \rightarrow Y_0$ is the inclusion. Using the fact that $H^1(Q_i) = H^3(Q_i) = 0$, we get an exact sequence

$$\begin{aligned} 0 \rightarrow H^2(Y_0) \rightarrow H^2(\tilde{X}_0) \oplus \bigoplus_i H^2(E_i) &\rightarrow \bigoplus_i H^2(Q_i) \rightarrow \\ &\rightarrow H^3(Y_0) \rightarrow H^3(\tilde{X}_0) = H^3(X_0) \rightarrow 0. \end{aligned}$$

If $b = \dim H^2(X_0)$ is the second Betti number $b_2(X_0)$, then the second Betti number $b_2(\tilde{X}_0)$ of \tilde{X}_0 is $b + r$ and hence $\dim H^2(\tilde{X}_0) \oplus \bigoplus_i H^2(E_i) = b + 2r$. Moreover, $\bigoplus_i H^2(Q_i) \cong \mathbb{C}^{2r}$. In fact, $H^2(Q_i) \cong \mathbb{C}[\sigma_i] \oplus \mathbb{C}[f_i]$, where f_i is a fiber of the morphism $Q_i \rightarrow C_i$, and σ_i is a fiber of the “other ruling” on $Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then, taking the positive generator $[Q_i]$ of $H^2(E_i)$, the homomorphism $H^2(E_i) \rightarrow H^2(Q_i)$ sends $[Q_i]$ to $[\sigma_i] + [f_i]$. The homomorphism $H^2(\tilde{X}_0) \rightarrow H^2(Q_i)$ sends $[Q_i]$ to $-\sigma_i - [f_i]$ and sends a class of the form $\rho^*\xi$, where $\rho: \tilde{X}_0 \rightarrow X_0$ is the blowup morphism, to $(\xi \cdot [C_i])[f_i]$. A brief computation shows the following:

Proposition 2.4. (i) *Let W_2 be the image of $\bigoplus_i H^2(Q_i)$ in $H^3(Y_0) = W_3$. Then W_2 has rank one and $W_3/W_2 \cong H^3(X_0)$.*

(ii) $H^1(Y_0) = H^5(Y_0) = 0$.

(iii) $H^2(Y_0)$ has dimension $b + 1$, and is isomorphic to the following subgroup of $H^2(\tilde{X}_0) \oplus \bigoplus_i H^2(E_i)$

$$\left\{ \rho^*\xi + \sum_i a_i q'_i + \sum_i b_i q''_i : a_i = b_i \text{ and } \xi \cdot [C_i] = 0 \text{ for all } i \right\},$$

where q'_i is the class of Q_i in $H^2(\tilde{X}_0)$ and q''_i is the class of Q_i in $H^2(E_i)$.

(iv) $H^4(Y_0) \cong H^4(X_0) \oplus \mathbb{C}^r$ has dimension $b + r$. □

Part (i) of Proposition 2.4 gives a weight filtration on $H^3(Y_0)$, defined over \mathbb{Q} , with $W_1 = 0$. There are also trivial (increasing) filtrations on $H^k(Y_0)$ for $k \neq 3$: take $W_\ell = H^k(Y_0)$ for $\ell \geq k$ and $W_\ell = 0$ for $\ell < k$. To construct a Hodge filtration, we can use the complex $\Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet$ of [8, (1.5)], where $\Omega_{Y_0}^1$ is the sheaf of Kähler differentials on Y_0 , $\Omega_{Y_0}^\bullet = \bigwedge^\bullet \Omega_{Y_0}^1$, and $\tau_{Y_0}^\bullet$ is the subcomplex of “torsion differentials,” i.e. those supported on $(Y_0)_{\text{sing}}$. By [8, (1.5)], $(\Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet, d)$ is a resolution of the constant sheaf \mathbb{C}_{Y_0} , and there is an exact sequence

$$0 \rightarrow \Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet \rightarrow \nu_* \Omega_{\tilde{Y}_0}^\bullet \rightarrow \bigoplus_i (j_i)_* \Omega_{Q_i}^\bullet \rightarrow 0.$$

Taking hypercohomology gives the Mayer-Vietoris sequence above.

Theorem 2.5. *The spectral sequence with E_1 page*

$$E_1^{p,q} = H^q(Y_0; \Omega_{Y_0}^p / \tau_{Y_0}^p) \implies \mathbb{H}^{p+q}(Y_0; \Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet) = H^{p+q}(Y_0)$$

degenerates at E_1 . The corresponding filtration F^\bullet on $H^k(Y_0)$, together with the weight filtration W_\bullet , give a mixed Hodge structure on $H^n(Y_0)$, which is pure for $n \neq 3$. More precisely,

- (i) $H^n(Y_0) = 0$ for $n = 1, 5$;
- (ii) For $n = 2k$, the mixed Hodge structure on $H^{2k}(Y_0)$ is a pure Hodge structure and $H^{2k}(Y_0) = H^{k,k}(Y_0)$;
- (iii) As mixed Hodge structures over \mathbb{Q} , $H^3(Y_0)$ is an extension of the pure Hodge structure $H^3(X_0)$ by a pure weight two piece $\cong \mathbb{Q}(-1)$.

Proof. Although we have not necessarily assumed that X_0 is Kähler, its cohomology satisfies the $\partial\bar{\partial}$ -lemma and the same is true for the projective varieties E_i and Q_i . Thus all of the terms in the Mayer-Vietoris sequence carry pure Hodge structures and the morphisms are morphisms of Hodge structures. Then the method of proof of [12, (4.2)] shows that there is a mixed Hodge structure on $H^n(Y_0)$, and the usual arguments with mixed Hodge complexes ([5, (8.1.9)] or [17, Theorem 3.18]) show that the above spectral sequence degenerates at E_1 .

The other statements are proved by explicit calculation. Starting with \mathcal{O}_{Y_0} , we have the usual resolution

$$0 \rightarrow \mathcal{O}_{Y_0} \rightarrow \nu_*(\mathcal{O}_{\tilde{X}_0} \oplus \bigoplus_i \mathcal{O}_{E_i}) \rightarrow \bigoplus_i (j_i)_* \mathcal{O}_{Q_i} \rightarrow 0.$$

It follows that $H^0(Y_0; \mathcal{O}_{Y_0}) \cong \mathbb{C}$, $H^3(Y_0; \mathcal{O}_{Y_0}) \cong H^3(X_0; \mathcal{O}_{X_0}) \cong \mathbb{C}$, and $H^k(Y_0; \mathcal{O}_{Y_0}) = 0$, $k \neq 0, 3$. As for $\Omega_{Y_0}^1/\tau_{Y_0}^1$, beginning with the exact sequence

$$0 \rightarrow \Omega_{Y_0}^1/\tau_{Y_0}^1 \rightarrow \nu_*\Omega_{\tilde{Y}_0}^1 \rightarrow \bigoplus_i (j_i)_*\Omega_{Q_i}^1 \rightarrow 0,$$

we see that $H^0(Y_0; \Omega_{Y_0}^1/\tau_{Y_0}^1) = H^3(Y_0; \Omega_{Y_0}^1/\tau_{Y_0}^1) = 0$, that $H^1(Y_0; \Omega_{Y_0}^1/\tau_{Y_0}^1) \cong H^2(Y_0)$ and that there is an exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow H^2(Y_0; \Omega_{Y_0}^1/\tau_{Y_0}^1) \rightarrow H^2(X_0; \Omega_{X_0}^1) \rightarrow 0.$$

The cases $H^q(Y_0; \Omega_{Y_0}^p/\tau_{Y_0}^p)$, $p = 2, 3$ are analyzed in a similar way. We remark that, by directly checking all possible cases for all k , it follows that

$$\sum_{p+q=k} \dim H^q(Y_0; \Omega_{Y_0}^p/\tau_{Y_0}^p) = \dim H^k(Y_0).$$

Thus we see again that the spectral sequence degenerates at E_1 .

The remaining statements also follow by inspection, using the compatibility of the above exact sequences with the Mayer-Vietoris exact sequence. For example, for the Hodge and weight filtrations on H^3 , there is a surjection from $\bigoplus_i F^2 H^2(Q_i) = 0$ to $F^2 \cap W_2$, so that $F^2 \cap W_2 = 0$, and similarly $F^1 \cap W_2 = W_2$, i.e. W_2 is pure of type $(1, 1)$. □

2.5. The limiting mixed Hodge structure

We begin by constructing the relative log complex. Recall that S is the base of the deformation \mathcal{Y} of Y_0 , with discriminant locus D , and that $\mathcal{Y}_D \rightarrow D$ is the locally trivial part of the deformation of Y_0 . After shrinking, we will assume that S is a polydisk Δ^{h+1} and that D is the divisor $\Delta^h \times \{0\}$. Let $S^* = \Delta^h \times \Delta^*$, where Δ^* is the punctured unit disk, and let $\pi^*: \mathcal{Y}^* \rightarrow S^*$ be the restriction of π to S^* . Thus $R^n(\pi^*)_*\mathbb{C} = \underline{H}^n$ is a local system over S^* .

Define the sheaf $\Omega_{\mathcal{Y}/S}^1(\log \mathcal{Y}_D)$ by the exact sequence

$$0 \rightarrow \pi^*\Omega_S^1(\log D) \rightarrow \Omega_{\mathcal{Y}}^1(\log \mathcal{Y}_D) \rightarrow \Omega_{\mathcal{Y}/S}^1(\log \mathcal{Y}_D) \rightarrow 0.$$

It is a locally free sheaf of rank 3. Define the relative log complex via

$$\Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D) = \bigwedge^\bullet \Omega_{\mathcal{Y}/S}^1(\log \mathcal{Y}_D),$$

with the usual differential. For a fiber Y_s , $s \notin D$, $\Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D)|_{Y_s} \cong \Omega_{Y_s}^\bullet$. For the singular fiber Y_0 , we set $\Lambda_{Y_0}^\bullet = \Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D)|_{Y_0}$. The complex $\Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D)$ is the relative log complex of Deligne-Steenbrink with extra parameters coming from the locally trivial deformations of Y_0 . In fact, if $\Delta \rightarrow S$ is a morphism of the disk to S , transverse to the discriminant locus D , then the pullback of $\Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D)$ to Δ is the usual one parameter relative log complex. The arguments of [21] or [17, Corollary 11.18] show:

Theorem 2.6. *The hypercohomology $\mathbb{H}^n(Y_0; \Lambda_{Y_0}^\bullet)$ is isomorphic to the cohomology $H^n(\mathcal{Y}^* \times_{S^*} \widetilde{S}^*; \mathbb{C})$, where $\widetilde{S}^* = \Delta^h \times \widetilde{\Delta}^*$ is the universal cover of S^* . (Here $\widetilde{\Delta}^* \cong \mathfrak{H}$ is the universal cover of Δ^* .) The sheaf*

$$\overline{\mathcal{H}}^n = \mathbb{R}^n \pi_* \Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D)$$

is locally free and satisfies: $\overline{\mathcal{H}}^n|_{S^}$ is the holomorphic flat vector bundle $\mathcal{H}^n = \underline{H}^n \otimes_{\mathbb{C}} \mathcal{O}_{S^*}$, and $\overline{\mathcal{H}}^n$ is Deligne’s canonical extension of \mathcal{H}^n . \square*

The arguments of [17, Theorem 11.22 and Corollaries 11.23 and 11.24] as well as the method of proof of Theorem 2.5 then show:

Theorem 2.7. (i) *Denote*

$$\mathbb{H}^n(Y_0; \Lambda_{Y_0}^\bullet) \cong H^n(\mathcal{Y}^* \times_{S^*} \widetilde{S}^*; \mathbb{C})$$

by H_{lim}^n . Then there is a mixed Hodge structure on H_{lim}^n , the limiting mixed Hodge structure, some of whose properties we recall below.

(ii) *The spectral sequence with E_1 page*

$$E_1^{p,q} = H^q(Y_0; \Lambda_{Y_0}^p) \implies \mathbb{H}^{p+q}(Y_0; \Lambda_{Y_0}^\bullet) = H_{\text{lim}}^{p+q}$$

degenerates at E_1 and the corresponding filtration on H_{lim}^{p+q} is the Hodge filtration.

(iii) *Possibly after shrinking S , the spectral sequence of coherent sheaves on S whose E_1 page is*

$$E_1^{p,q} = R^q \pi_* \Omega_{\mathcal{Y}/S}^p(\log \mathcal{Y}_D) \implies \mathbb{R}^{p+q} \pi_* \Omega_{\mathcal{Y}/S}^\bullet(\log \mathcal{Y}_D) = \overline{\mathcal{H}}^{p+q}$$

degenerates at E_1 . Thus, for $t \in S^$, the Hodge-de Rham spectral sequence for Y_t degenerates at E_1 . Moreover, the sheaves $R^q \pi_* \Omega_{\mathcal{Y}/S}^p(\log \mathcal{Y}_D)$ are locally free. \square*

In particular, there is a filtration of $\overline{\mathcal{H}}^n$ by holomorphic subbundles F^\bullet , which we will somewhat inaccurately call the *Hodge filtration*. It follows from Lemma 1.4 that, for $n = 3$, this filtration is isotropic over S^* (and in fact over S).

2.6. The monodromy weight filtration

There is an increasing filtration V_\bullet on the complex $\Lambda_{Y_0}^\bullet$. Because Y_0 consists of smooth components meeting transversally along smooth divisors, it takes the following simple form

$$0 \rightarrow V_0 \rightarrow V_1 = \Lambda_{Y_0}^\bullet \rightarrow V_1/V_0 \rightarrow 0.$$

Here $V_0 \cong \Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet$ and $V_1/V_0 \cong \bigoplus_i (j_i)_* \Omega_{Q_i}^{\bullet-1}$, by [8, (3.5)] or [17, 11.2.5]. By [17, Theorem 11.29] (and the discussion prior to the statement), we have

Theorem 2.8. *The homomorphism*

$$\mathbb{H}^n(Y_0; \Omega_{Y_0}^\bullet / \tau_{Y_0}^\bullet) \rightarrow \mathbb{H}^n(Y_0; \Lambda_{Y_0}^\bullet)$$

is the specialization homomorphism $H^n(Y_0; \mathbb{C}) \rightarrow H_{\text{lim}}^n$, and it is a morphism of mixed Hodge structures. \square

Consider now the long exact sequence associated to the short exact sequence $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_1/V_0 \rightarrow 0$. In particular, we get the two exact sequences of mixed Hodge structures (all groups with \mathbb{C} -coefficients)

$$0 \rightarrow H^1(Y_0) \rightarrow H_{\text{lim}}^1 \rightarrow \bigoplus_i H^0(Q_i)(-1) \rightarrow H^2(Y_0) \rightarrow H_{\text{lim}}^2 \rightarrow 0$$

and

$$0 \rightarrow H^3(Y_0) \rightarrow H^3_{\text{lim}} \rightarrow \bigoplus_i H^2(Q_i)(-1) \rightarrow H^4(Y_0) \rightarrow H^4_{\text{lim}} \rightarrow 0.$$

The map $\bigoplus_i H^0(Q_i)(-1) \rightarrow H^2(Y_0)$ is injective, since the composite map

$$\bigoplus_i H^0(Q_i)(-1) \rightarrow H^2(Y_0) \rightarrow H^2(\tilde{X}_0) \oplus \bigoplus_i H^2(E_i)$$

is injective (it restricts to the Gysin map $H^0(Q_i)(-1) \rightarrow H^2(E_i)$ on each summand). Thus H^2_{lim} has dimension $b - r + 1$, and the same must be true for H^4_{lim} . Then, since $H^4(Y_0)$ has dimension $b + r$ and the dimension of H^4_{lim} is $b - r + 1$, the image of $\bigoplus_i H^2(Q_i)(-1)$ in $H^4(Y_0)$ has dimension $2r - 1$ and hence the kernel of this map has dimension one. Explicitly, it is easy to check that the kernel of $\bigoplus_i H^2(Q_i)(-1) \rightarrow H^4(Y_0) \subseteq H^4(\tilde{X}_0) \oplus \bigoplus_i H^4(E_i)$ is identified with

$$\left\{ (m_1([\sigma_1] - [f_1]), \dots, m_r([\sigma_r] - [f_r])) : \sum_i m_i [C_i] = 0 \right\}.$$

Summarizing,

Theorem 2.9. (i) $H^1_{\text{lim}} = H^5_{\text{lim}} = 0$.

(ii) H^2_{lim} and H^4_{lim} are pure of weights two and four respectively and dimension $b - r + 1$, with $H^2_{\text{lim}} = H^{1,1}_{\text{lim}}$ and $H^4_{\text{lim}} = H^{2,2}_{\text{lim}}$.

(iii) There is an exact sequence of mixed Hodge structures

$$0 \rightarrow H^3(Y_0) \rightarrow H^3_{\text{lim}} \rightarrow \mathbb{Q}(-2) \rightarrow 0.$$

Thus the weight filtration on H^3_{lim} is given by

$$0 \subseteq W_2 \subseteq W_3 \subseteq W_4 = H^3_{\text{lim}},$$

where $W_3 = H^3(Y_0)$, $W_3/W_2 \cong H^3(X_0)$, $W_2 \cong \mathbb{Q}(-1)$ and $W_4/W_3 \cong \mathbb{Q}(-2)$. □

Remark 2.10. (1) Somewhat more general formulas for $b_2(Y_s)$ and $b_3(Y_s)$ are given in [10, Lemma 8.1] by comparing the Mayer-Vietoris sequences for X_0 and X_t .

(2) In our main case of interest, the classes $[C_i]$ span $H^2(X_0)$ and satisfy one linear relation. Hence $b = r - 1$ and thus $H^2_{\text{lim}} = 0$, i.e. $H^2(Y_t; \mathbb{Z})$ is torsion for $t \notin D$.

An easy argument using Theorem 2.7 then shows:

Corollary 2.11. *If Y_s is a small smoothing of Y_0 , then $H^i(Y_s; \mathcal{O}_{Y_s}) = H^0(Y_s; \Omega_{Y_s}^i) = 0$ for $i = 1, 2$. \square*

We relate the weight filtration to the monodromy filtration on H^3 as follows. Let T be the monodromy of the family acting on H^3 and let $N = T - I$. Thus N is a nilpotent matrix, and in fact $N^2 = 0$. More precisely,

Theorem 2.12. *$\text{Ker } N = W_3 = \text{Im } H^3(Y_0)$ and $\text{Im } N = W_2$.*

Proof. By general theory [17, Theorem 11.28], N is a morphism of mixed Hodge structures of type $(-1, -1)$, and hence $W_3 \subseteq \text{Ker } N$ and $\text{Im } N \subseteq W_2$. Thus N induces a homomorphism of one dimensional \mathbb{Q} -vector spaces $W_4/W_3 \rightarrow W_2$. To see the statement of the theorem, it therefore suffices to prove that $N \neq 0$, or equivalently that $T \neq I$. This follows from Picard-Lefschetz theory: associated to each double point p_i , is a vanishing cycle ξ_i , viewed as an element of cohomology. By assumption, there exists a $\xi \in H^3(Y_t; \mathbb{Z})$ of infinite order such that each ξ_i is a multiple $r_i \xi$ of ξ and the \mathbb{Q} -span of the ξ_i is equal to $\mathbb{Q} \cdot \xi$, so that not all of the r_i can be 0. By the Picard-Lefschetz formula,

$$T(\alpha) = \alpha + \sum_i 2\langle \alpha, \xi_i \rangle \xi_i = \alpha + \left(\sum_i 2r_i^2 \right) \langle \alpha, \xi \rangle \xi,$$

where the 2 reflects the base change of order 2 in the passage from deformations of \bar{X}_0 to deformations of the semistable model Y_0 . Thus there exists a positive rational number r such that

$$T(\alpha) = \alpha + r\langle \alpha, \xi \rangle \xi$$

and so $T \neq I$.

We can give a direct argument that $N: W_4/W_3 \rightarrow W_2$ is an isomorphism as follows. The action of N on the graded pieces $W_4/W_3 \rightarrow W_2$ is calculated in [17, 11.2.5], and one checks (cf. [17, §11.3]) that it is the homomorphism (induced by $\pm \text{Id}: \bigoplus_i H^2(Q_i) \rightarrow \bigoplus_i H^2(Q_i)$):

$$\begin{aligned} & \text{Ker} \left(\bigoplus_i H^2(Q_i) \rightarrow H^4(\tilde{X}_0) \oplus \bigoplus_i H^4(E_i) \right) \\ & \rightarrow \text{Coker} \left(H^2(\tilde{X}_0) \oplus \bigoplus_i H^2(E_i) \rightarrow \bigoplus_i H^2(Q_i) \right). \end{aligned}$$

To see that $N: W_4/W_3 \rightarrow W_2$ is an isomorphism, using the comments before Proposition 2.4 and Theorem 2.9, it suffices to show that, if $(m_1, \dots, m_r) \in \mathbb{Q}^r$ is a nonzero vector such that $\sum_i m_i [C_i] = 0$ in $H^4(X_0)$, then (m_1, \dots, m_r) is not in the subspace

$$I = \{((\xi \cdot [C_1]), \dots, (\xi \cdot [C_r])) : \xi \in H^2(X_0)\}.$$

But (m_1, \dots, m_r) is orthogonal to I under the standard inner product on \mathbb{Q}^r , so that $(m_1, \dots, m_r) \in I \implies (m_1, \dots, m_r) = 0$. \square

Proposition 2.13. *With the alternating nondegenerate pairing $\langle \cdot, \cdot \rangle$ on $H_{\text{lim}}^3 \cong H^3(Y_t)$, $W_2^\perp = W_3$. Hence, if ξ is a generator for W_2 and η generates W_4/W_3 , then $\langle \xi, \eta \rangle \neq 0$.*

Proof. The first statement is clear since $N(\alpha) = r\langle \alpha, \xi \rangle \xi$, with $\xi \neq 0$, so that $\text{Im } N = \mathbb{C} \cdot \xi$ and $\text{Ker } N = \xi^\perp$. (It also follows from the fact that $\langle N(\alpha), \beta \rangle = -\langle \alpha, N(\beta) \rangle$.) The final statement follows because $\langle \cdot, \cdot \rangle$ is nondegenerate. \square

2.7. The differential of the period map

The flat vector bundle \mathcal{H}^3 has an integrable connection ∇ and a decreasing filtration F^\bullet by holomorphic subbundles. Moreover, for every $s \in S^*$, the associated graded

$$F_s^p / F_s^{p+1} \cong H^{3-p}(Y_s; \Omega_{Y_s}^p).$$

In any small simply connected open subset U of S^* , or on the universal cover \widetilde{S}^* , the restriction or pullback of \mathcal{H}^3 is canonically trivialized by ∇ . Given such a trivialization, we define the *period map* on U to be the holomorphic map from U to an appropriate flag manifold defined by sending $s \in U$ to the subspaces F_s^p of $H^3(Y_s)$. By openness of versality, the tangent space to S^* at s is identified with $H^1(Y_s; T_{Y_s})$; more precisely, the Kodaira-Spencer map $T_{S^*,s} \rightarrow H^1(Y_s; T_{Y_s})$ is an isomorphism. The standard arguments in the Kähler case (see e.g. [3, Proposition 7.7]) show that the differential of the period map is computed at the point s via the natural homomorphism

$$H^1(Y_s; T_{Y_s}) \rightarrow \bigoplus_p \text{Hom}(H^{3-p}(Y_s; \Omega_{Y_s}^p), H^{3-p+1}(Y_s; \Omega_{Y_s}^{p-1}))$$

given by cup product and contraction. A similar statement holds globally: the differential of the period map is given by the homomorphism induced by cup product:

$$R^1 \pi_* T_{\mathcal{Y}^*/S^*} \rightarrow \bigoplus_p \text{Hom}(R^{3-p} \pi_* \Omega_{\mathcal{Y}^*/S^*}^p, R^{3-p+1} \pi_* \Omega_{\mathcal{Y}^*/S^*}^{p-1}).$$

Since $\Omega_{Y_t}^3 \cong \mathcal{O}_{Y_t}$, the cup product homomorphism

$$H^1(Y_t; T_{Y_t}) \rightarrow \text{Hom}(H^0(Y_t; \Omega_{Y_t}^3), H^1(Y_t; \Omega_{Y_t}^2)) \cong H^1(Y_t; \Omega_{Y_t}^2)$$

is an isomorphism. Similarly, after trivializing the line bundle $R^0\pi_*\Omega_{\mathcal{Y}^*/S^*}^3$, i.e. after choosing an everywhere generating section of $\Omega_{\mathcal{Y}^*/S^*}^3$, the cup product homomorphism

$$R^1\pi_*T_{\mathcal{Y}^*/S^*} \rightarrow \text{Hom}(R^0\pi_*\Omega_{\mathcal{Y}^*/S^*}^3, R^1\pi_*\Omega_{\mathcal{Y}^*/S^*}^2) \cong R^1\pi_*\Omega_{\mathcal{Y}^*/S^*}^2$$

is an isomorphism.

3. The variational argument

3.1. The basic setup

We begin by abstracting the situation of §2. Let H be a vector space with a nondegenerate alternating bilinear form $\langle \cdot, \cdot \rangle$ and a standard symplectic basis $e_0, \dots, e_{h+1}, f_0, \dots, f_{h+1}$ (i.e. $\langle e_i, f_j \rangle = \delta_{ij}$, and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ for all i, j). We assume that H is in fact defined over \mathbb{Q} , i.e. is the complexification of a \mathbb{Q} -vector space $H_{\mathbb{Q}}$, and that the above basis is a \mathbb{Q} -basis. In particular, H is defined over \mathbb{R} so that complex conjugation is defined on H . Let $N: H \rightarrow H$ be the rational linear map defined by: $N(e_i) = 0$ for all i , $N(f_i) = 0$ for $i \neq h+1$, and $N(f_{h+1}) = e_{h+1}$. Then

$$\langle N(\alpha), \beta \rangle + \langle \alpha, N(\beta) \rangle = 0$$

for all $\alpha, \beta \in H$. Define

$$W_2 = \mathbb{C}e_{h+1} \subseteq W_3 = \text{span} \{e_0, \dots, e_{h+1}, f_0, \dots, f_h\} \subseteq W_4 = H.$$

Let $S = \Delta^h \times \Delta$, with coordinates t_1, \dots, t_h, q , let $S^* = \Delta^h \times \Delta^* \subseteq S$, and let $D = \Delta^h \times \{0\}$. We shall abbreviate (t_1, \dots, t_h, q) by (t, q) . Write $q = e^{2\pi\sqrt{-1}z}$, where z is the usual coordinate on the upper half plane $\mathfrak{H} = \widetilde{\Delta^*}$; equivalently,

$$z = \frac{\log q}{2\pi\sqrt{-1}}.$$

Let $\varphi: \widetilde{S^*} = \Delta^h \times \mathfrak{H} \rightarrow S^*$ be the universal cover map. Setting $T = \exp N$ defines an action of $\pi_1(S^*) \cong \mathbb{Z}$ on H , where 1 acts as T , and hence a local system \underline{H} over S^* . Let $\mathcal{H} = \underline{H} \otimes_{\mathbb{C}} \mathcal{O}_{S^*}$ be the corresponding holomorphic

vector bundle over S^* and $\overline{\mathcal{H}}$ the canonical extension of \mathcal{H} to S . By [3], we can take $\overline{\mathcal{H}} \cong H \otimes_{\mathbb{C}} \mathcal{O}_S$, the trivial holomorphic vector bundle over S with fiber H , with the meromorphic connection ∇ whose associated connection 1-form is $-\frac{N}{2\pi\sqrt{-1}} \frac{dq}{q}$. The bundle $\varphi^*\mathcal{H}$ is trivialized by ∇ and the fiber at any point of S^* is identified with H . The fiber of \mathcal{H} at any point is identified with H modulo the action of $\{T^k = \exp(kN) : k \in \mathbb{Z}\}$. The fiber of $\overline{\mathcal{H}}$ over $0 \in D$ is identified with H up to the action of the unipotent group $\{\exp(\lambda N) : \lambda \in \mathbb{C}\}$. The local flat sections of \mathcal{H} over S^* are then sections locally of the form $\exp(zN)v$, where $v \in H$. A holomorphic section σ of \mathcal{H} , viewed as a holomorphic section σ of the trivial bundle $\varphi^*\mathcal{H} \cong H \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{S}^*}$ with the invariance property $\sigma(t, z+1) = T\sigma(t, z)$, extends to a holomorphic section of $\overline{\mathcal{H}}$ if and only if the section $\exp(-zN)\varphi^*\sigma$, viewed as a holomorphic section of $\varphi^*\mathcal{H}$, extends to a single-valued holomorphic function from S to H . Given a holomorphic section σ of \mathcal{H} , we denote $\nabla_{\partial/\partial t_i} \sigma$ by $\frac{\partial \sigma}{\partial t_i}$, and similarly for the coordinate q .

Finally, we are given a filtration of $\overline{\mathcal{H}}$ by holomorphic subbundles F^\bullet . It satisfies:

- (i) F^3 is a line bundle, hence $F^3 = \mathcal{O}_S \cdot \tilde{\omega}(t, q)$ for some nowhere vanishing holomorphic function $\tilde{\omega}(t, q)$ with values in H .
- (ii) Over S^* , a basis for $F^2|_{S^*}$ is given by

$$\tilde{\omega}, \frac{\partial \tilde{\omega}}{\partial t_1}, \dots, \frac{\partial \tilde{\omega}}{\partial t_h}, \frac{\partial \tilde{\omega}}{\partial q}.$$

We can also replace the last term $\frac{\partial \tilde{\omega}}{\partial q}$ by $\frac{\partial \tilde{\omega}}{\partial z}$ on $\varphi^*\mathcal{H}$, since

$$\frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial z} = q \frac{\partial}{\partial q}.$$

- (iii) (First Hodge-Riemann bilinear relation) With respect to the form $\langle \cdot, \cdot \rangle$, $(F^3)^\perp = F^1$ and $(F^2)^\perp = F^2$, so that F^2 is a maximal isotropic subbundle.
- (iv) For $s \in D = \Delta^h \times \{0\}$, the filtrations F_s^\bullet and W_\bullet define a mixed Hodge structure on H , with $W_2 \cong \mathbb{Q}(-1)$, $W_4/W_3 \cong \mathbb{Q}(-2)$, and W_3/W_2 is a pure weight three Hodge structure with $h^{3,0} = h^{0,3} = 1$, and hence $h^{2,1} = h^{1,2} = h$.

As a consequence, we record the following facts:

Lemma 3.1. *Under the above assumptions,*

- (i) *The subbundle F^2 has rank $h + 2$.*
- (ii) *For $s \in D$, $F_s^3 \subseteq W_3$ and $F_s^2 + W_3 = W_4$. Equivalently, there exists a $v \in F_s^2$ such that, writing v as a linear combination of the e_i, f_i , the coefficient of f_{h+1} in v is 1.*
- (iii) *For $s \in D$, $F_s^2 \cap W_2 = 0$. □*

3.2. The bundle $\overline{\mathcal{H}}_\#$

By the above, e_{h+1} defines a global holomorphic section of \mathcal{H} and of $\overline{\mathcal{H}}$. We define $\overline{\mathcal{H}}_\# = (e_{h+1})^\perp / \mathcal{O}_S \cdot e_{h+1}$. It is a flat vector bundle canonically isomorphic to $H_\# \otimes_{\mathbb{C}} \mathcal{O}_S$, where

$$H_\# = (e_{h+1})^\perp / \mathbb{C} \cdot e_{h+1} = W_3 / W_2.$$

Here we take $(e_{h+1})^\perp$ inside the vector space H , not the holomorphic bundle $\overline{\mathcal{H}}$. For each $s \in D$, the filtration F_s^\bullet induces a pure weight three Hodge structure on $H_\#$. The bundle $\overline{\mathcal{H}}_\#$ has rank $2h + 2$. There is an induced nondegenerate alternating bilinear form $\langle \cdot, \cdot \rangle$ on $H_\#$ and on $\overline{\mathcal{H}}_\#$.

Define $F_\#^2$ to be the image of $F^2 \cap (e_{h+1})^\perp$ in $\overline{\mathcal{H}}_\#$.

Lemma 3.2. *Possibly after shrinking S , $F_\#^2$ is a holomorphic isotropic subbundle of $\overline{\mathcal{H}}_\#$ of rank $h + 1$, and, as C^∞ bundles, $F_\#^2 \oplus \overline{F}_\#^2 \cong H_\# \otimes_{\mathbb{C}} C_S^\infty$.*

Proof. By (ii) of Lemma 3.1, for all $s \in D$, $\langle e_{h+1}, F_s^2 \rangle \neq 0$. Thus, possibly after shrinking S , we can assume that $F^2 \cap (e_{h+1})^\perp$ is a holomorphic subbundle of $\overline{\mathcal{H}}$ of rank $h + 1$. By (iii) of Lemma 3.1, for all $s \in D$, $F_s^2 \cap \mathbb{C} \cdot e_{h+1} = 0$. Thus, again possibly after shrinking S , we can assume that the projection $F^2 \cap (e_{h+1})^\perp \rightarrow \overline{\mathcal{H}}_\#$ is injective and of maximal rank at every point of S . It follows that $F_\#^2$ is a holomorphic subbundle of $\overline{\mathcal{H}}_\#$ of rank $h + 1$, and it is isotropic (i.e. $\langle F_\#^2, F_\#^2 \rangle = 0$) because F^2 is isotropic.

Finally, for $s \in D$, $(F_\#^2)_s \oplus (\overline{F}_\#^2)_s \cong H_\#$ because, for each $s \in S$, $H_\#$ carries a weight 3 Hodge structure for which $(F_\#^2)_s$ is the corresponding piece of the Hodge filtration. After shrinking S , we can assume that, for all $s \in S$, $(F_\#^2)_s \oplus (\overline{F}_\#^2)_s \cong H_\#$. Hence $F_\#^2 \oplus \overline{F}_\#^2 \cong H_\# \otimes_{\mathbb{C}} C_S^\infty$. □

3.3. Normalizing the holomorphic form

Begin by choosing an arbitrary holomorphic, nowhere vanishing section $\tilde{\omega}$ of the line bundle F^3 . We can write (using the basis of flat sections

$e_0, \dots, e_{h+1}, f_0, \dots, f_{h+1}$ of $\varphi^*\mathcal{H}$)

$$\varphi^*\tilde{\omega}(t, z) = \sum_{i=0}^{h+1} \tilde{A}_i e_i + \sum_{i=0}^{h+1} \tilde{B}_i f_i,$$

where the \tilde{A}_i, \tilde{B}_i are holomorphic in t_1, \dots, t_h, z . The invariance property, that $\tilde{\omega}$ defines a holomorphic section of $\tilde{\mathcal{H}}$, implies that $\tilde{A}_0, \dots, \tilde{A}_h$ and $\tilde{B}_0, \dots, \tilde{B}_{h+1}$ are holomorphic functions of t and q on S , and

$$\tilde{A}_{h+1} = C(t, q) + z\tilde{B}_{h+1}(t, q),$$

where $C(t, q)$ is a holomorphic function of t and q on S . Equivalently, viewed as a holomorphic section of the bundle $\tilde{\mathcal{H}} \cong H \otimes_{\mathbb{C}} \mathcal{O}_S$,

$$\tilde{\omega}(t, z) = \sum_{i=0}^h \tilde{A}_i e_i + C e_{h+1} + \sum_{i=0}^{h+1} \tilde{B}_i f_i.$$

In the limit (i.e. for $s \in D$), $F_s^3 \subseteq W_3$ and hence $\tilde{B}_{h+1}(t, 0) = 0$. Nonetheless:

Lemma 3.3. *The coefficient \tilde{B}_{h+1} is not identically 0.*

Proof. Suppose instead that \tilde{B}_{h+1} is identically 0, so that $\tilde{\omega}$ lies in the (flat) subbundle $(e_{h+1})^\perp$ of $\tilde{\mathcal{H}}$. Then, over S^* , the sections

$$\tilde{\omega}, \frac{\partial \tilde{\omega}}{\partial t_1}, \dots, \frac{\partial \tilde{\omega}}{\partial t_h}, \frac{\partial \tilde{\omega}}{\partial q}$$

all lie in $(e_{h+1})^\perp$. It follows that $F^2|_{S^*}$ lies in $(e_{h+1})^\perp$, and hence so does F^2 . But this contradicts (ii) of Lemma 3.1. \square

We define the normalized meromorphic section ω of F^3 by dividing $\tilde{\omega}$ by \tilde{B}_{h+1} . Thus $\omega = (\tilde{B}_{h+1})^{-1}\tilde{\omega}$ and

$$\varphi^*\omega = \sum_{i=0}^h A_i e_i + (A' + z)e_{h+1} + \sum_{i=0}^h B_i f_i + f_{h+1},$$

where A_i, B_i , and A' are meromorphic functions of t and q on S . We write this as

$$\varphi^*\omega = \psi + z e_{h+1}.$$

Henceforth we shall ignore the φ^* and view ω and its derivatives as meromorphic functions either on \widetilde{S}^* or on S^* . Note that

$$\frac{\partial \omega}{\partial t_i} = (\widetilde{B}_{h+1})^{-1} \frac{\partial \widetilde{\omega}}{\partial t_i} + \frac{\partial}{\partial t_i} (\widetilde{B}_{h+1})^{-1} \cdot \widetilde{\omega},$$

and similarly for the partial derivatives with respect to q or z . Thus, over the nonempty open subset of S^* where $\widetilde{B}_{h+1} \neq 0$, the span of ω and its derivatives with respect to t_1, \dots, t_h, q , or equivalently with respect to t_1, \dots, t_h, z , is the holomorphic subbundle F^2 .

3.4. The main calculation

First, a preliminary definition:

Definition 3.4. A real meromorphic function on S is an element of the field of quotients $\mathcal{K}(S)$ of the ring of (complex valued) real analytic functions on S (which is an integral domain). A nonzero real meromorphic function on S is defined and real analytic on an open dense subset of S . Real meromorphic functions on S^* are defined similarly, and we let $\mathcal{K}(S^*)$ be the field of all such. In particular $\mathcal{K}(S)$ is a subfield of $\mathcal{K}(S^*)$.

The function $\log |q|$ is real analytic on S^* , hence is an element of $\mathcal{K}(S^*)$. However:

Lemma 3.5. *The function $\log |q|$ is not a real meromorphic function on S .*

Proof. Write $q = re^{\sqrt{-1}\cdot\theta}$. Suppose that $\log |q| = \log r$ is of the form F/G where F and G are real analytic functions on S and $G \neq 0$. Choose values of θ and t so that $G(t, re^{\sqrt{-1}\cdot\theta})$ is not identically zero and is convergent at $r = 0$ as a power series in r . Then, for $0 < r \ll 1$, $\log r = f(r)/g(r)$, where $f(r)$, $g(r)$ are convergent power series in r (at $r = 0$) and $g(r)$ is not identically 0. Thus $g(r) = r^a g_0(r)$ for some nonnegative integer a , where $g_0(0) \neq 0$. Hence there exists a nonnegative integer a such that $r^a \log r$ extends to a C^∞ function in some interval around $r = 0$. This is a contradiction, since the a^{th} derivative of $r^a \log r$ is unbounded at 0. □

Our goal now is to prove:

Theorem 3.6. *With ω the normalized meromorphic section of F^3 given above, there exist real meromorphic functions M_1 and M_2 on S with $M_1 \neq 0$, such that, as an element of $\Lambda^{2h+4} H \otimes_{\mathbb{C}} \mathcal{K}(S^*)$,*

$$\begin{aligned} &\omega \wedge \bar{\omega} \wedge \frac{\partial \omega}{\partial z} \wedge \frac{\partial \bar{\omega}}{\partial z} \wedge \frac{\partial \omega}{\partial t_1} \wedge \frac{\partial \bar{\omega}}{\partial t_1} \wedge \dots \wedge \frac{\partial \omega}{\partial t_h} \wedge \frac{\partial \bar{\omega}}{\partial t_h} = \\ &= ((z - \bar{z})M_1 + M_2)(e_0 \wedge \dots \wedge e_{h+1} \wedge f_0 \wedge \dots \wedge f_{h+1}). \end{aligned}$$

First, we show how to apply Theorem 3.6 toward establishing the $\partial\bar{\partial}$ -lemma:

Corollary 3.7. *There exists a nonempty open dense subset of S^* , the complement of a proper real analytic subvariety in S^* , such that, for all $s \in S^*$,*

$$F_s^2 \oplus \overline{F}_s^2 \cong H.$$

Proof. By Theorem 3.6, if F_s^2 and \overline{F}_s^2 do not span H on an open subset where M_1, M_2 , and \tilde{B}_{h+1}^{-1} are defined, then $(z - \bar{z})M_1 + M_2$ is identically 0. We have

$$z - \bar{z} = \frac{1}{\pi\sqrt{-1}} \log r = \frac{1}{\pi\sqrt{-1}} \log |q|.$$

Thus $\log |q| = -\pi\sqrt{-1}M_2/M_1$ is a real meromorphic function on S , contradicting Lemma 3.5.

Hence $(z - \bar{z})M_1 + M_2$ is not identically 0 on S^* . Let U be the nonempty open dense subset of S^* where M_1, M_2 , and \tilde{B}_{h+1}^{-1} are defined, and for which $(z - \bar{z})M_1 + M_2$ does not vanish. Then U is the complement of a proper real analytic subvariety in S^* . For $s \in U$, we have $F_s^2 \oplus \overline{F}_s^2 \cong H$ as claimed. \square

Combining Corollary 3.7 and Corollary 1.6, we obtain:

Corollary 3.8. *Let $\pi: \mathcal{Y} \rightarrow S$ be as in §2.3, §2.5. There exists a nonempty open dense subset of S^* , the complement of a proper real analytic subvariety in S^* , such that, for all $s \in S^*$, the fiber Y_s satisfies the $\partial\bar{\partial}$ -lemma.* \square

Proof of Theorem 3.6. Write $\omega = \psi + ze_{h+1}$, where ψ is a meromorphic section of $\overline{\mathcal{H}}$, i.e. whose coordinates are meromorphic functions of (t, q) , and such that $\langle \psi, e_{h+1} \rangle = 1$, i.e. the coefficient of f_{h+1} in ψ is 1. Then $\bar{\omega} = \bar{\psi} + \bar{z}e_{h+1}$. Thus

$$\omega \wedge \bar{\omega} = \bar{z}(\psi \wedge e_{h+1}) - z(\bar{\psi} \wedge e_{h+1}) + (\psi \wedge \bar{\psi}).$$

Taking derivatives, we have

$$\frac{\partial \omega}{\partial z} = \frac{\partial \psi}{\partial z} + e_{h+1}.$$

Here, since $\frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial z} = q \frac{\partial}{\partial q}, \frac{\partial \psi}{\partial z}$ and $\frac{\partial \omega}{\partial z}$ are meromorphic sections of $\overline{\mathcal{H}}$ (their coefficients are meromorphic functions of (t, q)), and the coefficient of f_{h+1} in each is 0. Similarly

$$\frac{\overline{\partial \omega}}{\partial z} = \frac{\overline{\partial \psi}}{\partial z} + e_{h+1}.$$

Computing, we see that

$$\begin{aligned} \Xi &= \omega \wedge \bar{\omega} \wedge \frac{\partial \omega}{\partial z} \wedge \frac{\partial \bar{\omega}}{\partial z} \\ &= [\omega \wedge \bar{\omega}] \wedge \left(\frac{\partial \psi}{\partial z} \wedge e_{h+1} - \frac{\partial \bar{\psi}}{\partial z} \wedge e_{h+1} + \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right) \\ &= \bar{z} \left(\psi \wedge e_{h+1} \wedge \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right) - z \left(\bar{\psi} \wedge e_{h+1} \wedge \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right) \\ &\quad + \psi \wedge \bar{\psi} \wedge \left(\frac{\partial \psi}{\partial z} \wedge e_{h+1} - \frac{\partial \bar{\psi}}{\partial z} \wedge e_{h+1} + \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right). \end{aligned}$$

Setting

$$\Phi = \psi \wedge \bar{\psi} \wedge \left(\frac{\partial \psi}{\partial z} \wedge e_{h+1} - \frac{\partial \bar{\psi}}{\partial z} \wedge e_{h+1} + \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right),$$

we can write the expression above as

$$(*) \quad \Xi = (z - \bar{z}) \left(\frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \right) \wedge e_{h+1} \wedge f_{h+1} + \Phi + \dots,$$

where the remaining terms do not involve f_{h+1} (but might involve z or \bar{z}).

Consider the wedge product

$$\Psi = \frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z} \wedge \frac{\partial \omega}{\partial t_1} \wedge \frac{\partial \bar{\omega}}{\partial t_1} \wedge \dots \wedge \frac{\partial \omega}{\partial t_h} \wedge \frac{\partial \bar{\omega}}{\partial t_h}.$$

Note that none of the terms in the wedge product involve f_{h+1} . In fact, we have the following:

Lemma 3.9. *There exists a nonzero real meromorphic function M_1 on S such that Ψ is of the form*

$$M_1 e_0 \wedge f_0 \wedge \dots \wedge e_h \wedge f_h + \Omega \wedge e_{h+1},$$

for some $\Omega \in \mathcal{K}(S) \otimes_{\mathbb{C}} \wedge^{2h+1} H$.

Proof. It is enough to prove the corresponding statement for the form Ψ' where, in the definition of Ψ , we replace $\frac{\partial \psi}{\partial z} \wedge \frac{\partial \bar{\psi}}{\partial z}$ with $\frac{\partial \omega}{\partial z} \wedge \frac{\partial \bar{\omega}}{\partial z}$. Clearly, we

can view Ψ or Ψ' as an element of $\mathcal{K}(S) \otimes_{\mathbb{C}} \wedge^{2h+2} H$. Consider the meromorphic sections

$$\frac{\partial\omega}{\partial z}, \frac{\partial\omega}{\partial t_1}, \dots, \frac{\partial\omega}{\partial t_h}$$

of $F^2 \cap (e_{h+1})^\perp$. Over the field of meromorphic functions on S^* , the span of

$$\omega, \frac{\partial\omega}{\partial z}, \frac{\partial\omega}{\partial t_1}, \dots, \frac{\partial\omega}{\partial t_h}$$

is the same as the span of

$$\tilde{\omega}, \frac{\partial\tilde{\omega}}{\partial t_1}, \dots, \frac{\partial\tilde{\omega}}{\partial t_h}, \frac{\partial\tilde{\omega}}{\partial q}.$$

Hence $\frac{\partial\omega}{\partial z}, \frac{\partial\omega}{\partial t_1}, \dots, \frac{\partial\omega}{\partial t_h}$ are linearly independent over the field of meromorphic functions on S^* and hence on S . Since $F^2 \cap (e_{h+1})^\perp \rightarrow \overline{\mathcal{H}}_\#$ is injective and of maximal rank, the above sections remain linearly independent when viewed as meromorphic sections of $F_\#^2$.

Let $\sigma_0, \dots, \sigma_h$ be a basis of holomorphic sections for the holomorphic bundle $F_\#^2$. Then, by Lemma 3.2, there exists a nonzero real analytic function A such that

$$\sigma_0 \wedge \bar{\sigma}_0 \wedge \dots \wedge \sigma_h \wedge \bar{\sigma}_h = Ae_0 \wedge f_0 \wedge \dots \wedge e_h \wedge f_h.$$

Let G be the $(h+1) \times (h+1)$ matrix whose entries are meromorphic functions on S which expresses the images of the meromorphic sections $\frac{\partial\omega}{\partial z}, \frac{\partial\omega}{\partial t_1}, \dots, \frac{\partial\omega}{\partial t_h}$ in $F_\#^2$ as linear combinations of $\sigma_0, \dots, \sigma_h$. Furthermore, $\det G \neq 0$, because $\frac{\partial\omega}{\partial z}, \frac{\partial\omega}{\partial t_1}, \dots, \frac{\partial\omega}{\partial t_h}$ are linearly independent over the field of meromorphic functions on S . Then, working in $H_\#$ and the associated C^∞ bundle (i.e. mod e_{h+1}),

$$\begin{aligned} \Psi \text{ mod } e_{h+1} &= \Psi' \text{ mod } e_{h+1} = \pm |\det G|^2 \sigma_0 \wedge \bar{\sigma}_0 \wedge \dots \wedge \sigma_h \wedge \bar{\sigma}_h \\ &= \pm |\det G|^2 Ae_0 \wedge f_0 \wedge \dots \wedge e_h \wedge f_h. \end{aligned}$$

This says that, for some nonzero real meromorphic function M_1 ,

$$\Psi = M_1 e_0 \wedge f_0 \wedge \dots \wedge e_h \wedge f_h$$

mod e_{h+1} , and thus completes the proof of the lemma. □

To finish the proof of Theorem 3.6, our goal is to calculate

$$\Xi \wedge \Xi' = \omega \wedge \bar{\omega} \wedge \frac{\partial \omega}{\partial z} \wedge \frac{\bar{\partial \omega}}{\partial z} \wedge \frac{\partial \omega}{\partial t_1} \wedge \frac{\bar{\partial \omega}}{\partial t_1} \wedge \cdots \wedge \frac{\partial \omega}{\partial t_h} \wedge \frac{\bar{\partial \omega}}{\partial t_h}$$

which is the wedge product of $\Xi = \omega \wedge \bar{\omega} \wedge \frac{\partial \omega}{\partial z} \wedge \frac{\bar{\partial \omega}}{\partial z}$ with

$$\Xi' = \frac{\partial \omega}{\partial t_1} \wedge \frac{\bar{\partial \omega}}{\partial t_1} \wedge \cdots \wedge \frac{\partial \omega}{\partial t_h} \wedge \frac{\bar{\partial \omega}}{\partial t_h}.$$

In our previous notation,

$$\Psi = \frac{\partial \psi}{\partial z} \wedge \frac{\bar{\partial \psi}}{\partial z} \wedge \Xi'.$$

Since Ξ' does not involve f_{h+1} , any terms of Ξ which do not involve f_{h+1} will drop out of $\Xi \wedge \Xi'$. By the above lemma and Equation (*), $\Xi \wedge \Xi'$ is of the form

$$\begin{aligned} \Xi \wedge \Xi' &= (z - \bar{z})\Psi \wedge e_{h+1} \wedge f_{h+1} + \Phi \wedge \Xi' \\ &= (z - \bar{z})M_1 e_0 \wedge f_0 \wedge \cdots \wedge e_{h+1} \wedge f_{h+1} + \Phi \wedge \Xi', \end{aligned}$$

with $M_1 \neq 0$. Since the coefficients of Φ, Ξ' are real meromorphic functions, $\Phi \wedge \Xi' \in \wedge^{2h+4} H \otimes_{\mathbb{C}} \mathcal{K}(S)$ and we can write

$$\Phi \wedge \Xi' = M_2 e_0 \wedge f_0 \wedge \cdots \wedge e_{h+1} \wedge f_{h+1}$$

for some real meromorphic function M_2 . Thus $\Xi \wedge \Xi'$ is as claimed. □

3.5. The main theorem

We can now prove the main theorem of this paper:

Theorem 3.10. *Let X be a compact complex manifold of dimension 3 with $K_X \cong \mathcal{O}_X$ for which the $\partial\bar{\partial}$ -lemma holds, and such that that $H^i(X; \mathcal{O}_X) = 0$ for $i = 1, 2$ and $H^0(X; \Omega_X^j) = 0$ for $j = 1, 2$. Suppose that C_1, \dots, C_r are disjoint smooth rational curves in X such that $N_{C_i/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Assume that the classes $[C_i]$ of the C_i satisfy a linear relation in $H^4(X; \mathbb{C})$ of the form*

$$\sum_{i=1}^r m_i [C_i] = 0,$$

where $m_i \neq 0$ for every i . Let \bar{X} be the singular compact threefold obtained by contracting the C_i . Then there exist smoothings of \bar{X} for which the $\partial\bar{\partial}$ -lemma holds.

Proof. If s is the smallest positive integer such that there exists a subset of the C_i whose classes are linearly dependent, then, possibly after reordering the C_i , we can assume that there exist $n_1, \dots, n_s \in \mathbb{Q}$, such that $n_i \neq 0$ for all i ,

$$\sum_{i=1}^s n_i [C_i] = 0,$$

and the $[C_i]$ span a subspace of $H^4(X)$ of dimension $s - 1$. Let \bar{X}_1 be the singular threefold obtained by contracting C_1, \dots, C_s and let X_1 be a general small smoothing of \bar{X}_1 . By Corollary 3.8, the $\partial\bar{\partial}$ -lemma holds for X_1 . In particular, we have proved the corollary in case $r = s$, and hence in case $r = 1$. Now assume the result by induction for all positive integers less than r and suppose that $s < r$. The curves C_{s+1}, \dots, C_r deform to disjoint smooth rational curves C'_i in X_1 . Since $H^4(X_1) \cong H^4(X) / \sum_{j=1}^s \mathbb{C} \cdot [C_j]$, $\sum_{i=s+1}^r m_i [C'_i] = 0$ in $H^4(X_1)$. Let \bar{X}_1 be the threefold obtained by contracting C'_{s+1}, \dots, C'_r in X_1 . Then \bar{X}_1 is smoothable, and by induction the $\partial\bar{\partial}$ -lemma holds for general small smoothings X_2 of \bar{X}_1 . Such a smoothing will also be a general small smoothing of \bar{X} , completing the proof of the theorem. \square

4. Concluding remarks

First we recall the following standard definition:

Definition 4.1. Let V_1 and V_2 be two compact complex manifolds. Then V_1 and V_2 are *deformation equivalent* if there exists a proper smooth morphism $\pi: \mathcal{V} \rightarrow S$, where \mathcal{V} and S are connected analytic spaces, and two points $s_1, s_2 \in S$, such that $\pi^{-1}(s_i) \cong V_i$, $i = 1, 2$.

I am grateful to S.-T. Yau for calling my attention to the following question: is every compact complex manifold for which the $\partial\bar{\partial}$ -lemma holds deformation equivalent to a compact complex manifold bimeromorphic to a Kähler manifold (also called *of class \mathcal{C}*)? The answer to this question is no:

Proposition 4.2. *A Clemens manifold is not deformation equivalent to a compact complex manifold bimeromorphic to a Kähler manifold.*

Proof. Since the condition that $b_2 = 0$ is preserved under deformation equivalence, it suffices to show that a compact complex threefold V with $b_2 = 0$ is not bimeromorphic to a Kähler manifold. Assume the contrary, that V is

bimeromorphic to a Kähler manifold V' . In fact, we may assume that there is a surjective degree one morphism $f: V' \rightarrow V$. By [2, (5.3)] (cf. also [6, (5.22)]), the $\partial\bar{\partial}$ -lemma holds for V . As $b_2(V) = 0$, $h^2(V; \mathcal{O}_V) = h^0(V; \Omega_V^2) = 0$. Since $h^0(V; \Omega_V^2)$ is a birational invariant, $h^0(V'; \Omega_{V'}^2) = 0$ as well. Then $H^2(V'; \mathbb{C}) = H^{1,1}(V')$, so that there exists a Hodge metric on V' and thus V' is projective. Since $f: V' \rightarrow V$ is birational, there exists a hypersurface $E \subseteq V'$ such that $f(E)$ has codimension at least two and f induces an isomorphism $V' - E \cong V - f(E)$. Choose an irreducible curve C on V' not contained in E and an irreducible very ample divisor H on V' such that $H \cap C$ is finite and disjoint from E . Then $f(H)$ is a hypersurface in V and it meets $f(C)$ at a finite and nonempty set of points. Then $[f(H)] \smile [f(C)] > 0$, so that $[f(H)]$ is a nonzero element of $H^2(V; \mathbb{Q})$. This contradicts the assumption that $b_2(V) = 0$. \square

Remark 4.3. Let $\pi: \mathcal{X} \rightarrow \Delta$ be a degeneration of compact complex manifolds $X_t, t \neq 0$, to a singular X_0 . Under very general hypotheses, the arguments of Theorem 2.7 will show that, for $t \in \Delta^*$ small, the Hodge-de Rham spectral sequence for X_t degenerates at E_1 . For example, if all components of X_0 are bimeromorphic to Kähler manifolds, or if X_0 has normal crossings and all k -fold intersections $X_0^{[k]}$ satisfy the $\partial\bar{\partial}$ -lemma, then the Hodge-de Rham spectral sequence for X_t degenerates at E_1 for t small and $\neq 0$. On the other hand, it is easy to find examples for which the $\partial\bar{\partial}$ -lemma does not hold for $X_t, t \neq 0$. For example, let X_0 be the singular surface which is obtained by gluing the negative section σ_0 of the rational ruled surface \mathbb{F}_n to a disjoint section σ by some choice of isomorphism. Note that $\sigma_0^2 = -n$ and $\sigma^2 = n$, so that X_0 is d -semistable in the sense of [8]. Kodaira has shown [16] that, if $n \neq 0$, then there is a degeneration $\pi: \mathcal{X} \rightarrow \Delta$, such that $\pi^{-1}(0) \cong X_0$ and, for $t \neq 0$, $X_t = \pi^{-1}(t)$ is a Hopf surface. Then the Hodge-de Rham spectral sequence for X_t degenerates at E_1 , but the $\partial\bar{\partial}$ -lemma does not hold for any compact complex surface deformation equivalent to X_t .

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