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# Generalization of the Weierstrass $\wp$ function and Maass lifts of weak Jacobi forms

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*Dedicated to Professor Kyoji Saito on the occasion of his 75th birthday*

**Abstract:** Typically, a Maass lift is a map from (holomorphic) Jacobi forms of index 1 to Siegel modular forms of degree 2 or other kinds of modular forms. In this paper, we construct Maass lifts from weak Jacobi forms to (non-holomorphic) Siegel modular forms of degree 2 with or without levels and characters, as formal series. By the Koecher principle, the images of our lifts are not holomorphic at cusps, even if the formal series converge. When the level is equal or less than 3 and the character is trivial, the image of our Maass lift is in the space of meromorphic Siegel modular forms.

**Keywords:** $\wp$ function, Maass lifts, weak Jacobi forms.

## 1. Introduction

Three important keywords in this paper are '$\wp$ function', 'Maass lifts' and 'weak Jacobi forms'. All of these three keywords are closely related to the theory of Jacobi forms. In 1985, Eichler and Zagier introduced the concept of Jacobi forms in their book [EZ]. In their book, they gave the relation between our keywords to Jacobi forms:

(i) The Weierstrass $\wp$ function can be recognized as a meromorphic Jacobi form of weight 2 and index 0 with respect to $\mathrm{SL}(2,\mathbb{Z})$.
(ii) They constructed a Maass lift. Actually, they gave a map from Jacobi forms of index 1 to Siegel modular forms of degree 2.
(iii) They introduced the concept of weak Jacobi forms, which is just like Jacobi forms but do not satisfy the condition on their Fourier coefficients corresponding to the Koecher principle.

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Nevertheless, in their book, these three topics were treated independently. In fact, all Siegel modular forms satisfy the Koecher principle, hence we can not apply this Maass lift to weak Jacobi forms directly in the range of holomorphic functions.

Ten years later, these three were connected by Borcherds. In 1995, Borcherds [Bo] constructed modular forms on some orthogonal groups by using infinite products (so called Borcherds products). In his paper, he mentioned not only Borcherds products but also Maass lifts (additive lifts) of weak Jacobi forms. Actually, he gave maps from weak Jacobi forms of index 1 to modular forms on some orthogonal groups, by using the Weierstrass $\wp$ function. His idea seems to be able to apply to a map to Siegel modular forms of degree 2, however, he did not give a complete proof in this case.

In this paper, first, we construct a Maass lift from weak Jacobi forms to Siegel modular forms of degree 2, according to the idea of Eichler, Zagier and Borcherds. After a slight review of basic properties of modular forms and Jacobi forms in section 2, we construct a Maass lift without levels in section 3. This part does not contain any new idea. The only thing the author do is to give a complete proof of their idea, however, to prepare suitable notation is useful to study the case with levels.

In 2012, Ibukiyama [Ib] constructed Maass lifts from Jacobi forms to Siegel modular forms of degree 2 with levels and characters. In this paper, second, we apply his idea to weak Jacobi forms. There are two difficulties in this process. The first one is to find a meromorphic Jacobi form with levels and characters corresponding to the $\wp$ function in the case of without levels. We construct it in section 4. Another one is on the convergence of our Maass lifts. We discuss it in section 5. Our discussion in section 5 includes some complicated calculation. We show the precise process of it in section 6.

## 2. Preliminaries

In this section we review elliptic modular forms, Siegel modular forms and Jacobi forms. To prepare suitable notation is useful to study Maass lifts.

### 2.1. Elliptic modular forms

First, we review elliptic modular forms. We denote the complex upper half plane by

$$\mathbb{H} := \{\, \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \,\}.$$

The special linear group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$\mathbb{H} \ni \tau \mapsto g\langle\tau\rangle := \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \in \mathbb{H} \qquad \left(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\right).$$

For any fixed $k \in \mathbb{Z}$, this action induces the $\mathrm{SL}(2,\mathbb{R})$-action on the set of all holomorphic functions on $\mathbb{H}$ by

$$\mathrm{Hol}(\mathbb{H}) \ni f \mapsto (f|_k g)(\tau) := (\gamma\tau+\delta)^{-k} f(g\langle\tau\rangle) \in \mathrm{Hol}(\mathbb{H}),$$

where we denote the set of all holomorphic functions on $X$ by $\mathrm{Hol}(X)$.

Let

$$\mathrm{SL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{R}) \cap \mathrm{Mat}(2\times 2,\mathbb{Z})$$

and $\Gamma^{(1)}$ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Let $\psi^{(1)}$ be a character of $\Gamma^{(1)}$, namely, a homomorphism $\psi^{(1)} : \Gamma^{(1)} \to S^1 := \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$. Roughly, an elliptic modular form is a $\Gamma^{(1)}$-invariant (with $\psi^{(1)}$) holomorphic function on $\mathbb{H}$. However, strictly, we should discuss with their behavior at cusps. Let $f$ be a $\Gamma^{(1)}$-invariant holomorphic function on $\mathbb{H}$, namely, we assume that $f$ satisfies $\psi^{(1)}(g_1)\, f = f|_k g_1$ for any $g_1 \in \Gamma^{(1)}$. Then for any $g \in \mathrm{SL}(2,\mathbb{Z})$, $f|_k g$ is $\left(g^{-1}\Gamma^{(1)}g\right)$-invariant (with a character). Since $\Gamma^{(1)}$ is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$, there exists $h_g > 0$ such that

$$\left\{ t \in \mathbb{R} \;\middle|\; \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in g^{-1}\Gamma^{(1)}g, \quad \psi^{(1)}\left(g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g^{-1}\right) = 1 \right\} = h_g\mathbb{Z}.$$

Therefore, $f|_k g$ has a Fourier expansion

$$(f|_k g)(Z) = \sum_{n \in h_g^{-1}\mathbb{Z}} c_g(n)\mathbf{e}(n\tau), \tag{2.1}$$

where $\mathbf{e}(*) := \exp(2\pi i *)$.

**Definition 1.** *For $f \in \mathrm{Hol}(\mathbb{H})$ and $k \in \mathbb{Z}$, we say $f$ is an elliptic modular form of weight $k$ with character $\psi^{(1)}$ with respect to $\Gamma^{(1)}$ if $f$ satisfies the following two conditions:*
*(1) $\psi^{(1)}(g_1)\, f = f|_k g_1$ for any $g_1 \in \Gamma^{(1)}$.*
*(2) On (2.1), $c_g(n) = 0$ for any $g \in \mathrm{SL}(2,\mathbb{Z})$ and $n < 0$.*

We denote by $\mathbb{M}_k^{(1)}(\Gamma^{(1)};\psi^{(1)})$ the space of all elliptic modular forms of weight $k$ with character $\psi^{(1)}$ with respect to $\Gamma^{(1)}$. It is well known that

$\mathbb{M}_k^{(1)}(\Gamma^{(1)};\psi^{(1)})$ is finite-dimensional and especially

$$\mathbb{M}_k^{(1)}(\Gamma^{(1)};\psi^{(1)}) = \{0\} \qquad (k \leqq 0)$$

except when $k = 0$ and $\psi^{(1)}$ is the trivial character:

$$\mathbb{M}_0^{(1)}(\Gamma^{(1)};\mathbf{1}) = \mathbb{C},$$

where we denote the trivial character (map to 1) by $\mathbf{1}$.

### 2.2. Siegel modular forms of degree 2

Second, we review Siegel modular forms of degree 2. We denote the Siegel upper half space of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{C}) \;\middle|\; \mathrm{Im}\, Z > 0 \right\}.$$

The symplectic group

$$\mathrm{Sp}(2,\mathbb{R}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}(4\times 4, \mathbb{R}) \middle| {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

For any fixed $k \in \mathbb{Z}$, this action induces the $\mathrm{Sp}(2,\mathbb{R})$-action on the set of all holomorphic functions on $\mathbb{H}_2$ by

$$\mathrm{Hol}(\mathbb{H}_2) \ni F \mapsto (F|_kM)(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle) \in \mathrm{Hol}(\mathbb{H}_2).$$

Let

$$\mathrm{Sp}(2,\mathbb{Z}) := \mathrm{Sp}(2,\mathbb{R}) \cap \mathrm{Mat}(4\times 4, \mathbb{Z})$$

and $\Gamma^{(2)}$ be a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})$. Let $\psi^{(2)}$ be a character of $\Gamma^{(2)}$. Roughly, a Siegel modular form is a $\Gamma^{(2)}$-invariant (with $\psi^{(2)}$) holomorphic function on $\mathbb{H}_2$. Actually, the definition of Siegel modular forms is as follows:

**Definition 2.** *For $F \in \mathrm{Hol}(\mathbb{H}_2)$ and $k \in \mathbb{Z}$, we say $F$ is a Siegel modular form of weight $k$ with character $\psi^{(2)}$ with respect to $\Gamma^{(2)}$ if $F$ satisfies the conditions $\psi^{(2)}(M_1)\, F = F|_kM_1$ for any $M_1 \in \Gamma^{(2)}$.*

We denote by $\mathbb{M}_k^{(2)}(\Gamma^{(2)};\psi^{(2)})$ the space of all Siegel modular forms of weight $k$ with character $\psi^{(2)}$ with respect to $\Gamma^{(2)}$. It is well known that $\mathbb{M}_k^{(2)}(\Gamma^{(2)};\psi^{(2)})$ is finite-dimensional and especially

$$\mathbb{M}_k^{(2)}(\Gamma^{(2)};\psi^{(2)}) = \{0\} \qquad (k \leqq 0)$$

except when $k=0$ and $\psi^{(2)}$ is the trivial character:

$$\mathbb{M}_0^{(2)}(\Gamma^{(2)};\mathbf{1}) = \mathbb{C}.$$

Although this definition of Siegel modular forms does not contain any condition at their cusps, it is better to discuss with their behavior at cusps here, because we need to see it carefully later. For $(s,t,u) \in \mathbb{R}^3$, put

$$T\,(s,t,u) := \begin{pmatrix} 1 & 0 & s & t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R}).$$

Let $F \in \mathbb{M}_k^{(2)}(\Gamma^{(2)};\psi^{(2)})$. Then for any $M \in \mathrm{Sp}(2,\mathbb{Z})$, $F|_kM$ is $\left(M^{-1}\Gamma^{(2)}M\right)$-invariant (with a character). Since $\Gamma^{(2)}$ is a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})$,

$$\Lambda_M := \left\{ (s,t,u) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} T\,(s,t,u) \in M^{-1}\Gamma^{(2)}M, \\ \psi^{(2)}\left(MT\,(s,t,u)\,M^{-1}\right) = 1 \end{array} \right\}$$

is a rank 3 lattice in $\mathbb{R}^3$. Therefore, $F|_kM$ has a Fourier expansion

$$(F|_kM)(Z) = \sum_{(n,l,m)\in\Lambda_M^\sharp} c_M(n,l,m)\mathbf{e}(n\tau + lz + m\omega), \tag{2.2}$$

where

$$\Lambda_M^\sharp := \left\{ (n,l,m) \in \mathbb{R}^3 \mid \forall (s,t,u) \in \Lambda_M,\ ns + lt + mu \in \mathbb{Z} \right\}$$

is the dual lattice of $\Lambda_M$. On (2.2), sometimes we write $c_M\begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix}$ instead of $c_M(n,l,m)$ and then we have

$$(F|_kM)(Z) = \sum_{(n,l,m)\in\Lambda_M^\sharp} c_M\begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix} \mathbf{e}\left(\mathrm{tr}\left(\begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix}\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right)\right).$$

For $g \in \mathrm{GL}(2,\mathbb{R})$, put

$$R\,(g) := \begin{pmatrix} {}^t g & O_2 \\ O_2 & g^{-1} \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R})$$

and

$$R\left(\Gamma^{(2)}; M\right) := \left\{ \, g \in \mathrm{GL}(2,\mathbb{R}) \;\; \middle| \;\; R\,(g) \in M^{-1}\Gamma^{(2)} M \, \right\}.$$

For any $g \in R\left(\Gamma^{(2)}; M\right)$, we have

$$c_M \begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix} = \psi^{(2)}\left(MR\,(g)\,M^{-1}\right)(\det g)^{-k}\, c_M\left(g \begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix} {}^t g\right). \tag{2.3}$$

Hence, since the series (2.2) converges absolutely on $\mathbb{H}_2$, if $c_M(n,l,m) \neq 0$,

$$\sum_{g \in R\left(\Gamma^{(2)};M\right)} \left| \mathbf{e}\left(\mathrm{tr}\left({}^t g \begin{pmatrix} n & \frac{l}{2} \\ \frac{l}{2} & m \end{pmatrix} g \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right)\right)\right|,$$

where $g$ runs over $R\left(\Gamma^{(2)}; M\right)$, which is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$, should converge on $\mathbb{H}_2$. Consequently we have the following proposition.

**Proposition 3** (Koecher principle)**.** *On (2.2), if $n < 0$ or if $4nm - l^2 < 0$, then $c_M(n,l,m) = 0$.*

To study more details of Siegel modular forms, we can refer a lot of good textbooks.[1]

At the end of this subsection, we give the definition of meromorphic Siegel modular forms. We denote the set of all meromorphic functions on $X$ by $\mathrm{Mer}(X)$.

**Definition 4.** *For $F \in \mathrm{Mer}(\mathbb{H}_2)$ and $k \in \mathbb{Z}$, we say $F$ is a meromorphic Siegel modular form of weight $k$ with character $\psi^{(2)}$ with respect to $\Gamma^{(2)}$ if there exists $F_1 \in \mathbb{M}^{(2)}_{k_1}(\Gamma^{(2)}; \mathbf{1})$ such that $FF_1 \in \mathbb{M}^{(2)}_{k+k_1}(\Gamma^{(2)}; \psi^{(2)})$.*

### 2.3. Jacobi forms

Jacobi forms were first studied by Eichler and Zagier in their book [EZ]. In this book, they studied Jacobi forms with respect to the full modular group,

[1] When the author was a graduate student, my supervisor Professor Kyoji Saito recommends two textbooks [Fr] and Klingen [Kl].

mainly. Based on their book, we study Jacobi forms for arbitrary finite index subgroup in this subsection. For details, see [AI] or [Ao4].

Let $T := \{\pm T(0,0,u) \mid u \in \mathbb{R}\}$. In the sense of the action of $\mathrm{Sp}(2,\mathbb{R})$ on $\mathbb{H}_2$, $T$ is the set of all parallel transformation with respect to the variable $\omega$. Let $\mathrm{Sp}(2,\mathbb{R})^{\mathrm{J}}$ and $\mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}}$ be the centralizer and normalizer of $T$ in $\mathrm{Sp}(2,\mathbb{R})$, respectively. Namely,

$$\mathrm{Sp}(2,\mathbb{R})^{\mathrm{J}} := \{\ M \in \mathrm{Sp}(2,\mathbb{R}) \mid \forall M_1 \in T,\ M^{-1}M_1M = M_1\ \}$$

and

$$\mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}} := \{\ M \in \mathrm{Sp}(2,\mathbb{R}) \mid \forall M_1 \in T,\ M^{-1}M_1M \in T\ \}.$$

To study them, we define elements of $\mathrm{Sp}(2,\mathbb{R})$ by

$$S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad U(x,y) := \begin{pmatrix} 1 & 0 & 0 & y \\ x & 1 & y & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (x,y \in \mathbb{R})$$

and

$$C(g_1) := \begin{pmatrix} \frac{\alpha}{\sqrt{\lambda_1}} & 0 & \frac{\beta}{\sqrt{\lambda_1}} & 0 \\ 0 & \sqrt{\lambda_1} & 0 & 0 \\ \frac{\gamma}{\sqrt{\lambda_1}} & 0 & \frac{\delta}{\sqrt{\lambda_1}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{\lambda_1}} \end{pmatrix} \quad \left( \begin{matrix} g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}) \\ \lambda_1 = \det g_1 \end{matrix} \right),$$

where $\mathrm{GL}^+(2,\mathbb{R}) := \{\ g \in \mathrm{GL}(2,\mathbb{R}) \mid \det g > 0\ \}$. The following lemma is easy to show by direct calculation.

**Lemma 5.** *Any* $M \in \mathrm{Sp}(2,\mathbb{R})^{\mathrm{J}}$ *(resp.* $\mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}}$*) can be written as*

$$M = \pm C(g_1)\, U(x,y)\, T(0,0,u) \tag{2.4}$$

*uniquely, where* $g_1 \in \mathrm{SL}(2,\mathbb{R})$ *(resp.* $\mathrm{GL}^+(2,\mathbb{R})$*) and* $x,y,u \in \mathbb{R}$.

The relations of these matrices in the above lemma are as follows:

$$\begin{aligned} C(g_1)\, C(g_2) &= C(g_1g_2), \\ U(x,y)\, C(g_1) &= C(g_1)\, U\left(\frac{1}{\det g_1}(x,y)g_1\right), \end{aligned}$$

$$
\begin{aligned}
T\left(0,0,u\right)C\left(g_1\right) &= C\left(g_1\right)T\left(0,0,\frac{u}{\det g_1}\right),\\
U\left(x_1,y_1\right)U\left(x_2,y_2\right) &= U\left(x_1+x_2,y_1+y_2\right)T\left(0,0,x_1y_2-x_2y_1\right),\\
T\left(0,0,u\right)U\left(x,y\right) &= U\left(x,y\right)T\left(0,0,u\right)
\end{aligned}
$$

and

$$
T\left(0,0,u_1\right)T\left(0,0,u_2\right)=T\left(0,0,u_1+u_2\right).
$$

By using the decomposition (2.4), we define a group homomorphism $\lambda : \mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}} \to \mathbb{R}^+ := \{\ \lambda_1 \in \mathbb{R} \mid \lambda_1 > 0\ \}$ by $\lambda(M) := \lambda_1$.

Let $k \in \mathbb{Z}$. The group action of $\mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}}$ on $\mathrm{Hol}(\mathbb{H}\times\mathbb{C})\times\mathbb{R}$ is defined by

$$
(\phi, m)|_k M := (\phi|_{k,m}M, \lambda(M)m) \quad \left(M \in \mathrm{Sp}(2,\mathbb{R})^{\mathrm{N}}\right),
$$

where $\phi|_{k,m}M$ is given by

$$
\left(\phi|_{k,m}M\right)(\tau,z)\mathbf{e}(m\lambda(M)\omega) = \left(\phi(\tau,z)\mathbf{e}(m\omega)\right)|_k M.
$$

We can see $\phi|_{k,m}M \in \mathrm{Hol}(\mathbb{H}\times\mathbb{C})$ by using Lemma 5. Actually, we have

$$
\begin{aligned}
\left(\phi|_{k,m}C\left(g_1\right)\right)(\tau,z) &:= \lambda_1^k\left(\gamma\tau+\delta\right)^{-k}\mathbf{e}\left(\frac{-m\lambda_1\gamma z^2}{\gamma\tau+\delta}\right)\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta},\frac{\lambda_1 z}{\gamma\tau+\delta}\right),\\
\left(\phi|_{k,m}U\left(x,y\right)\right)(\tau,z) &:= \mathbf{e}\left(m(x^2\tau+2xz+xy)\right)\phi(\tau,z+x\tau+y)
\end{aligned}
$$

and

$$
\left(\phi|_{k,m}T\left(s,t,u\right)\right)(\tau,z) := \mathbf{e}(mu)\phi(\tau+s,z+t).
$$

Let

$$
\mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}} := \mathrm{Sp}(2,\mathbb{R})^{\mathrm{J}} \cap \mathrm{Mat}(4\times 4,\mathbb{Z})
$$

and $\Gamma^{(\mathrm{J})}$ be a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}}$. Let $m \in \mathbb{R}$ and $\psi^{(\mathrm{J})}$ be a character of $\Gamma^{(\mathrm{J})}$. Roughly, a Jacobi form is a $\Gamma^{(\mathrm{J})}$-invariant (with $\psi^{(\mathrm{J})}$) holomorphic function on $\mathbb{H}\times\mathbb{C}$. However, as the Koecher principle does not hold on Jacobi forms, we should discuss with their behavior at cusps. Let $\phi$ be a $\Gamma^{(\mathrm{J})}$-invariant holomorphic function on $\mathbb{H}\times\mathbb{C}$, namely, we assume that $\phi$ satisfies $\psi^{(\mathrm{J})}\left(M_1\right)\phi = \phi|_{k,m}M_1$ for any $M_1 \in \Gamma^{(\mathrm{J})}$. The following proposition holds (cf. [Ao4, Proposition 6]).

**Proposition 6.** *Let $k \in \mathbb{Z}$, $m \in \mathbb{R}$ and $\psi^{(\mathrm{J})}$ be a character of $\Gamma^{(\mathrm{J})}$. We assume that $\phi \in \mathrm{Hol}(\mathbb{H} \times \mathbb{C})$ satisfies $\psi^{(\mathrm{J})}(M_1)\,\phi = \phi|_{k,m}M_1$ for any $M_1 \in \Gamma^{(\mathrm{J})}$. If $m < 0$, then $\phi = 0$. If $m = 0$, then $\phi$ does not depend on $\tau$.*

Hence we may assume $m \geqq 0$. For any $g \in \mathrm{SL}(2,\mathbb{Z})$, $\phi|_{k,m}C(g)$ is $\left(C(g)^{-1}\Gamma^{(\mathrm{J})}C(g)\right)$-invariant (with a character). Since $\Gamma^{(\mathrm{J})}$ is a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}}$,

$$\Lambda_g^{\mathrm{J}} := \left\{ (s,t) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} T(s,t,0) \in C(g)^{-1}\Gamma^{(\mathrm{J})}C(g), \\ \psi^{(\mathrm{J})}\left(C(g)T(s,t,0)C(g)^{-1}\right) = 1 \end{array} \right\}$$

is a rank 2 lattice in $\mathbb{R}^2$. Therefore, $\phi|_kC(g)$ has a Fourier expansion

$$(\phi|_kC(g))(Z) = \sum_{(n,l)\in\left(\Lambda_g^{\mathrm{J}}\right)^\sharp} c_g(n,l)\mathbf{e}(n\tau + lz), \tag{2.5}$$

where

$$\left(\Lambda_g^{\mathrm{J}}\right)^\sharp := \left\{ (n,l) \in \mathbb{R}^2 \mid \forall(s,t) \in \Lambda_g^{\mathrm{J}},\ ns + lt \in \mathbb{Z} \right\}$$

is the dual lattice of $\Lambda_g^{\mathrm{J}}$.

**Definition 7.** *For $\phi \in \mathrm{Hol}(\mathbb{H}\times\mathbb{C})$, $k \in \mathbb{Z}$, $m \in \mathbb{R}^+ \cup \{0\}$, we say $\phi$ is a weak Jacobi form of weight $k$ and index $m$ with character $\psi^{(\mathrm{J})}$ with respect to $\Gamma^{(\mathrm{J})}$ if $\phi$ satisfies the following two conditions:*
*(1) $\psi^{(\mathrm{J})}(M_1)\,\phi = \phi|_{k,m}M_1$ for any $M_1 \in \Gamma^{(\mathrm{J})}$.*
*(2) On (2.5), $c_g(n,l) = 0$ for any $g \in \mathrm{SL}(2,\mathbb{Z})$ and $n < 0$.*
*We say a weak Jacobi form $\phi$ is a Jacobi form if $\phi$ satisfies one more condition:*
*(3) On (2.5), $c_g(n,l) = 0$ for any $g \in \mathrm{SL}(2,\mathbb{Z})$ and $4nm - l^2 < 0$.*

We denote by $\mathbb{J}_{k,m}(\Gamma^{(\mathrm{J})};\psi^{(\mathrm{J})})$ (resp. $\mathbb{J}_{k,m}^{\mathrm{weak}}(\Gamma^{(\mathrm{J})};\psi^{(\mathrm{J})})$) the space of all Jacobi forms (resp. all weak Jacobi forms) of weight $k$ and index $m$ with character $\psi^{(\mathrm{J})}$ with respect to $\Gamma^{(\mathrm{J})}$. Clearly, $\mathbb{J}_{k,m}(\Gamma^{(\mathrm{J})};\psi^{(\mathrm{J})}) \subset \mathbb{J}_{k,m}^{\mathrm{weak}}(\Gamma^{(\mathrm{J})};\psi^{(\mathrm{J})})$ holds.

At the end of this subsection, we give the definition of meromorphic Jacobi forms.

**Definition 8.** *For $\phi \in \mathrm{Mer}(\mathbb{H}\times\mathbb{C})$, $k \in \mathbb{Z}$ and $m \in \mathbb{R}$, we say $\phi$ is a meromorphic Jacobi form of weight $k$ and index $m$ with character $\psi^{(\mathrm{J})}$ with respect to $\Gamma^{(\mathrm{J})}$ if there exists $\phi_1 \in \mathbb{J}_{k_1,m_1}(\Gamma^{(\mathrm{J})};\mathbf{1})$ such that $\phi\phi_1 \in \mathbb{J}_{k+k_1,m+m_1}(\Gamma^{(\mathrm{J})};\psi^{(\mathrm{J})})$.*

### 2.4. Congruent subgroups

For any natural number $N \in \mathbb{N} := \{1, 2, 3, \dots\}$, let

$$\Gamma_0^{(2)}(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z}) \;\middle|\; C \equiv O_2 \pmod{N} \right\}$$

and

$$\Gamma_0^{(1)}(N) := \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; \gamma \equiv 0 \pmod{N} \right\}$$

be congruent subgroups of level $N$. Let

$$\Gamma_0^{(\mathrm{J})}(N) := \Gamma_0^{(2)}(N) \cap \mathrm{Sp}(2, \mathbb{Z})^{\mathrm{J}}.$$

The Maass lifts (of usual Jacobi forms) with respect to $\Gamma_0^{(2)}(N)$ were precisely studied in the paper by Ibukiyama [Ib] and the aim of this paper is to extend them to weak Jacobi forms. According to the paper by Ibukiyama [Ib], here we introduce a character induced from a Dirichlet character of modulo $N$. Let $\chi$ be a Dirichlet character modulo $N$, namely, $\chi : \mathbb{Z} \to \mathbb{C}$ is a function satisfying the following three conditions:
(1) $\chi(\delta) = 0$ if $(\delta, N) \neq 1$ and $\chi(\delta) \in S^1$ if $(\delta, N) = 1$.
(2) $\chi(\delta_1) = \chi(\delta_2)$ if $\delta_1 \equiv \delta_2 \pmod{N}$.
(3) $\chi(\delta_1\delta_2) = \chi(\delta_1)\chi(\delta_2)$ for any $\delta_1, \delta_2 \in \mathbb{Z}$.
Then

$$\psi_\chi^{(1)}(g) := \chi(\delta) \qquad \left( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0^{(1)}(N) \right)$$

$$\psi_\chi^{(2)}(M) := \chi(\det D) \qquad \left( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N) \right)$$

and

$$\psi_\chi^{(\mathrm{J})}(M) := \chi(\det D) \qquad \left( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(\mathrm{J})}(N) \right)$$

are characters of $\Gamma_0^{(1)}(N)$, $\Gamma_0^{(2)}(N)$ and $\Gamma_0^{(\mathrm{J})}(N)$, respectively. For simplicity, we write

$$\mathbb{M}_k^{(1)}(N; \chi) := \mathbb{M}_k^{(1)}(\Gamma_0^{(1)}(N); \psi_\chi^{(1)}), \qquad \mathbb{M}_k^{(2)}(N; \chi) := \mathbb{M}_k^{(2)}(\Gamma_0^{(2)}(N); \psi_\chi^{(2)}),$$

$$\mathbb{J}_{k,m}(N;\chi) := \mathbb{J}_{k,m}(\Gamma_0^{(\mathrm{J})}(N);\psi_\chi^{(\mathrm{J})}) \text{ and } \mathbb{J}_{k,m}^{\mathrm{weak}}(N;\chi) := \mathbb{J}_{k,m}^{\mathrm{weak}}(\Gamma_0^{(\mathrm{J})}(N);\psi_\chi^{(\mathrm{J})}).$$

The following two propositions are important to construct Maass lifts (cf. [AI]).

**Lemma 9.** *Any* $M \in \Gamma_0^{(\mathrm{J})}(N)$ *can be written as*

$$M = \pm C\,(g_1)\,U\,(x,y)\,T\,(0,0,u) \tag{2.6}$$

*uniquely, where* $g_1 \in \Gamma_0^{(1)}(N)$ *and* $x, y, u \in \mathbb{Z}$.

**Lemma 10.** *The group* $\Gamma_0^{(2)}(N)$ *is generated by* $\Gamma_0^{(\mathrm{J})}(N)$ *and* $S$.

## 2.5. Fourier-Jacobi expansion

Let $k \in \mathbb{Z}$, $N \in \mathbb{N}$ and $\chi$ be a Dirichlet character of modulo $N$. Let $F \in \mathbb{M}_k^{(2)}(N;\chi)$. Since $T\,(0,0,1) \in \Gamma_0^{(2)}(N)$ and $\psi_\chi^{(2)}\,(T\,(0,0,1)) = 1$, $F$ has a Fourier expansion with respect to $\omega$. By Koecher principle (Proposition 3), it is

$$F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau,z)\mathbf{e}(m\omega) \qquad (\phi_m \in \mathbb{J}_{k,m}(N;\chi))\,, \tag{2.7}$$

which we usually call Fourier-Jacobi expansion of $F$. On the Fourier expansion

$$\phi_m(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{l\in\mathbb{Z} \\ (4nm-l^2\geqq 0)}} c(n,l,m)\mathbf{e}\,(n\tau + lz)\,,$$

since $S \in \Gamma_0^{(2)}(N)$ and $\psi_\chi^{(2)}\,(S) = \chi(-1)$, we have

$$c(n,l,m) = (-1)^k\chi(-1)c(m,l,n).$$

Now we introduce the formal series of Jacobi forms. For $k \in \mathbb{Z}$, we define

$$\mathbb{FM}_k(N;\chi) := \left\{ (\phi_m)_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}(N;\chi) \;\middle|\; (\mathrm{Sym}) \right\}$$

and

$$\mathbb{FM}_k^{\mathrm{weak}}(N;\chi) := \left\{ (\phi_m)_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}^{\mathrm{weak}}(N;\chi) \;\middle|\; (\mathrm{Sym}) \right\},$$

where (Sym) means the following condition:

$$\text{(Sym)}: \quad \begin{array}{c}\text{On the Fourier expansion } \phi_m(\tau,z)=\sum_{n,l} c(n,l,m)\mathbf{e}\left(n\tau+lz\right),\\ c(n,l,m)=(-1)^k\chi(-1)c(m,l,n) \text{ holds for any } n,l,m\in\mathbb{Z}.\end{array}$$

By using this notation, we can regard the Fourier-Jacobi expansion (2.7) as a map

$$\text{FJ}: \mathbb{M}_k^{(2)}(N;\chi) \hookrightarrow \mathbb{FM}_k(N;\chi) \tag{2.8}$$

by identifying $(\phi_m)_{m=0}^{\infty}$ with their formal sum $\sum_{m=0}^{\infty}\phi_m(\tau,z)\mathbf{e}(m\omega)$. The map FJ is injective but may not be surjective. Nevertheless, by Lemma 10, if an element of $\mathbb{FM}_k(N;\chi)$ or $\mathbb{FM}_k^{\text{weak}}(N;\chi)$ converges locally uniformly on $\mathbb{H}_2$, then it is a Siegel modular form, namely in the image of FJ. In [Ao1], the author proved the following theorem:

**Theorem 11.** *When $N=1$, the map* FJ *is surjective. Actually we have*

$$\mathbb{M}_k^{(2)}(1;\mathbf{1}) = \mathbb{FM}_k(1;\mathbf{1}) = \mathbb{FM}_k^{\text{weak}}(1;\mathbf{1}).$$

The author conjecture that the map (2.8) is surjective even when $N\geq 2$, but we could not give a proof yet.

**Conjecture 12.** *Let $N\in\mathbb{N}$ and $\chi$ be a Dirichlet character of modulo $N$. Then the map* FJ *is surjective, namely we have*

$$\mathbb{M}_k^{(2)}(N;\chi) = \mathbb{FM}_k(N;\chi) \quad (?).$$

We remark that $\mathbb{FM}_k(N;\chi)=\mathbb{FM}_k^{\text{weak}}(N;\chi)$ is not always true when $N \geqq 2$. If there exists $\phi\in\mathbb{J}_{k,1}^{\text{weak}}(N;\chi)$ whose Fourier expansion looks like Jacobi form at $i\infty$ but not a Jacobi form by its behavior at another cusps, then its image by our Maass lift $\text{ML}(\phi)$, defined in section 5, is in $\mathbb{FM}_k^{\text{weak}}(N;\chi)$ but not in $\mathbb{FM}_k(N;\chi)$. Nevertheless, for $N\leqq 4$ we can show a bit weak theorem. We define

$$\mathbb{FM}_k^{+}(N;\chi) := \left\{ (\phi_m)_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}(N;\chi) \;\middle|\; \text{(Sym+)} \right\}$$

and

$$\mathbb{FM}_k^{\text{weak}+}(N;\chi) := \left\{ (\phi_m)_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{J}_{k,m}^{\text{weak}}(N;\chi) \;\middle|\; \text{(Sym+)} \right\},$$

where (Sym+) means the following condition:

$$\text{(Sym+)}: \qquad (2.3) \text{ holds for any } M \in \mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}},$$

while the condition (Sym) is equivalent to

$$\text{(Sym)}: \qquad (2.3) \text{ holds for } M = E_4 \in \mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}}.$$

Then the following theorem holds. (cf. [Ao2])

**Theorem 13.** *When $N \leq 4$, we have*

$$\mathbb{M}_k^{(2)}(N;\chi) = \mathbb{FM}_k^{+}(N;\chi).$$

Let

$$\mathrm{FJ}_1 \ : \ \mathbb{M}_k^{(2)}(N;\chi) \ni F \mapsto \phi_1 \in \mathbb{J}_{k,1}(N;\chi).$$

The Maass lift (of Jacobi forms) of level $N$ and character $\chi$ is a map

$$\mathrm{ML} \ : \ \mathbb{J}_{k,1}(N;\chi) \to \mathbb{M}_k^{(2)}(N;\chi)$$

such that $\mathrm{FJ}_1 \circ \mathrm{ML} = \mathrm{Id}$. From the next section, we see these Maass lifts precisely.

## 3. Maass lifts without levels

The Maass lifts with levels include much more tedious calculations than without levels. Hence, before discussing the lifts with levels precisely, we view the story of the Maass lift without levels in this section.

Throughout this section, we fix $N = 1$ and therefore its Dirichlet character $\chi$ should be trivial. Hence, for simplicity, we write $\mathbb{M}_k^{(1)}$, $\mathbb{M}_k^{(2)}$, $\mathbb{J}_{k,m}, \ldots$ instead of $\mathbb{M}_k^{(1)}(1;\mathbf{1})$, $\mathbb{M}_k^{(2)}(1;\mathbf{1})$, $\mathbb{J}_{k,m}(1;\mathbf{1}), \ldots$.

### 3.1. Elliptic modular forms without levels

First of all, we remark that $\mathbb{M}_k^{(1)} = \{0\}$ when $k$ is odd, because $-E_2 \in \Gamma_0^{(2)}(1) = \mathrm{SL}(2,\mathbb{Z})$. Hence we may assume $k$ is even. To construct the Maass lift, first we review some special elliptic modular forms.

The Dedekind eta function is defined by

$$\eta(\tau) := \mathbf{e}\left(\frac{1}{24}\tau\right) \prod_{n=1}^{\infty} \left(1 - \mathbf{e}(n\tau)\right).$$

This function $\eta$ is holomorphic on $\mathbb{H}$, does not have any zero on $\mathbb{H}$ and satisfies two functional equations

$$\eta(\tau+1) = \mathbf{e}\left(\frac{1}{24}\right)\eta(\tau) \quad \text{and} \quad \frac{1}{\sqrt{\tau}}\,\eta\left(-\frac{1}{\tau}\right) = \mathbf{e}\left(-\frac{1}{8}\right)\,\eta(\tau),$$

where we choose $0 < \arg\left(\sqrt{\tau}\right) < \frac{\pi}{2}$. In this paper we do not give the definition of elliptic modular forms of fractional weights, however, since $\mathrm{SL}(2,\mathbb{Z})$ is generated by two matrices $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, roughly, this $\eta$ is an elliptic modular form of weight $\frac{1}{2}$. To have an elliptic modular form of integral weight from it, we raise it to 24th power. The Ramanujan Delta function is defined by

$$\Delta(\tau) := \eta(\tau)^{24} = \mathbf{e}(\tau)\prod_{n=1}^{\infty}\left(1-\mathbf{e}(n\tau)\right)^{24} \in \mathbb{M}_{12}^{(1)}.$$

This $\Delta$ is an elliptic modular form of weight 12 and does not have any zero in $\mathbb{H}$.

For even $k \geq 4$, the Eisenstein series of weight $k$ (with level 1) is a holomorphic function on $\mathbb{H}$ defined by

$$G_k(\tau) := \sum_{(\gamma,\delta)\in\mathbb{Z}^2}{}' \frac{1}{(\gamma\tau+\delta)^k} \in \mathbb{M}_k^{(1)}, \tag{3.1}$$

where $\sum'$ is the summation over all pairs of integers $(\gamma,\delta)$ except $(0,0)$. More generally, in this paper we denote by $\sum'$ the summation except the zero vector. The Fourier expansion of $G_k$ is given by

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\mathbf{e}(n\tau), \tag{3.2}$$

where $\sigma_{k-1}$ is a divisor function defined by $\sigma_{k-1}(n) := \sum_{\alpha|n}\alpha^{k-1}$ and $\zeta$ is the Riemann zeta function.

When $k = 2$, the sum in the equation (3.1) does not converge, however, we can define $G_2$ as a holomorphic function on $\mathbb{H}_2$ by the equation (3.2):

$$\begin{aligned} G_2(\tau) :=& 2\zeta(2) + 2(2\pi i)^2\sum_{n=1}^{\infty}\sigma_1(n)\mathbf{e}(n\tau) \\ =& -\frac{(2\pi i)^2}{12}\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\mathbf{e}(n\tau)\right). \end{aligned} \tag{3.3}$$

It is well known that $G_2$ is not an elliptic modular form but a quasi-modular form of weight 2 with respect to $\mathrm{SL}(2,\mathbb{Z})$. Namely, $G_2$ satisfies the functional equation

$$G_2(\tau) = -2\pi i \frac{\gamma}{\gamma\tau+\delta} + (\gamma\tau+\delta)^{-2} G_2\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

for any $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$.

We remark that there are several ways to define Eisenstein series. Sometimes we use $\widetilde{G}_k$ or $e_k$ defined by

$$\widetilde{G}_k(\tau) := \frac{(k-1)!}{2(2\pi i)^k} G_k(\tau) = \frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\mathbf{e}(n\tau) \tag{3.4}$$

or

$$e_k(\tau) := \frac{1}{2\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\mathbf{e}(n\tau) \tag{3.5}$$

instead of $G_k$, where we use the functional equation of the Riemann zeta function and the Bernoulli numbers $B_{2k}$, defined by

$$\frac{z}{\exp(z)-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

### 3.2. Jacobi forms and the Weierstrass $\wp$ function

Second we review some special (weak, meromorphic) Jacobi forms.

The structure of the bigraded ring of all weak Jacobi forms are determined in the book of Eichler and Zagier [EZ]. This is given by

$$\bigoplus_{k,m\in\mathbb{Z}} \mathbb{J}^{\mathrm{weak}}_{k,m} = \left(\bigoplus_{k\in\mathbb{Z}} \mathbb{M}^{(1)}_k\right)[\phi_{-2,1},\phi_{0,1}] \oplus \phi_{-1,2}\left(\bigoplus_{k\in\mathbb{Z}} \mathbb{M}^{(1)}_k\right)[\phi_{-2,1},\phi_{0,1}],$$

where

$$\begin{aligned}\phi_{-2,1}(\tau,z) = {} & (\mathbf{e}(z) - 2 + \mathbf{e}(-z)) \\ & + (-2\mathbf{e}(2z) + 8\mathbf{e}(z) - 12 + 8\mathbf{e}(-z) - 2\mathbf{e}(-2z))\,\mathbf{e}(\tau) \\ & + \cdots \in \mathbb{J}^{\mathrm{weak}}_{-2,1}, \\ \phi_{0,1}(\tau,z) = {} & (\mathbf{e}(z) + 10 + \mathbf{e}(-z))\end{aligned}$$

$$+ (10\mathbf{e}(2z) - 64\mathbf{e}(z) + 108 - 64\mathbf{e}(-z) + 10\mathbf{e}(-2z))\,\mathbf{e}(\tau) + \cdots \in \mathbb{J}_{0,1}^{\mathrm{weak}}$$

and

$$\begin{aligned}\phi_{-1,2}(\tau,z) = {} & (\mathbf{e}(z) - \mathbf{e}(-z)) \\ & + (\mathbf{e}(3z) + 3\mathbf{e}(z) - 3\mathbf{e}(-z) - 3\mathbf{e}(-3z))\,\mathbf{e}(\tau) \\ & + \cdots \in \mathbb{J}_{-1,2}^{\mathrm{weak}}.\end{aligned}$$

Especially, we have

$$\bigoplus_{k\in 2\mathbb{Z}} \bigoplus_{m\in\mathbb{Z}} \mathbb{J}_{k,m}^{\mathrm{weak}} = \left(\bigoplus_{k\in\mathbb{Z}} \mathbb{M}_k^{(1)}\right)[\phi_{-2,1}, \phi_{0,1}] \tag{3.6}$$

and

$$\bigoplus_{k\in\mathbb{Z}} \mathbb{J}_{k,1}^{\mathrm{weak}} = \left(\bigoplus_{k\in\mathbb{Z}} \mathbb{M}_k^{(1)}\right)\phi_{-2,1} \oplus \left(\bigoplus_{k\in\mathbb{Z}} \mathbb{M}_k^{(1)}\right)\phi_{0,1}. \tag{3.7}$$

We remark that $\phi_{-2,1}(\tau,0) = 0$ and $\phi_{0,1}(\tau,0) = 12$. Therefore, from (3.6), if $\phi \in \mathbb{J}_{k,m}^{\mathrm{weak}}$ satisfies the conditions $\phi(\tau,0) = 0$ and $k \in 2\mathbb{Z}$, then there exists $\tilde{\phi} \in \mathbb{J}_{k+2,m-1}^{\mathrm{weak}}$ such that $\phi = \tilde{\phi}\phi_{-2,1}$. More precisely, we have

$$\phi_{-2,1}(\tau,z) = \frac{(1-\mathbf{e}(z))^2}{\mathbf{e}(z)} \prod_{n=1}^{\infty} \frac{(1-\mathbf{e}(n\tau+z))^2\,(1-\mathbf{e}(n\tau-z))^2}{(1-\mathbf{e}(n\tau))^4} \quad \text{(cf. [Bo])}. \tag{3.8}$$

If we fix $\tau \in \mathbb{H}$ and consider $\phi_{-2,1}(\tau,z)$ as a holomorphic function with respect to $z$ on $\mathbb{C}$, the equation (3.8) means that the set of all zeros of $\phi_{-2,1}$ is $\mathbb{Z}+\tau\mathbb{Z}$ and at each zero its order is 2.

The Weierstrass $\wp$ function is a meromorphic function on $\mathbb{H}\times\mathbb{C}$ defined by

$$\wp(\tau,z) := \frac{1}{z^2} + \sum_{(x,y)\in\mathbb{Z}^2}{}' \left(\frac{1}{(z-x\tau-y)^2} - \frac{1}{(x\tau+y)^2}\right).$$

If we fix $\tau \in \mathbb{H}$ and consider $\wp(\tau,z)$ as a meromorphic function with respect to $z$ on $\mathbb{C}$, the set of all poles of $\wp$ is $\mathbb{Z}+\tau\mathbb{Z}$ and at each pole its order is 2. It is well known that the Weierstrass $\wp$ function satisfies the functional equations

$$\wp(\tau, z) = \wp(\tau, z + x\tau + y) = (\gamma\tau + \delta)^{-2}\wp\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \frac{z}{\gamma\tau + \delta}\right)$$

for any $x, y \in \mathbb{Z}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Actually, $\wp$ is a meromorphic Jacobi form of weight 2 and index 0 (with level 1):

$$\wp(\tau, z) = \frac{(2\pi i)^2}{12}\frac{\phi_{0,1}(\tau, z)}{\phi_{-2,1}(\tau, z)}. \qquad \text{(cf. [EZ])} \tag{3.9}$$

The Laurent expansion of $\wp$ at $z = 0$ is

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)G_{2k}(\tau)z^{2k-2}.$$

Let

$$\mathbb{D} := \frac{1}{2\pi i}\frac{\partial}{\partial z} \qquad \text{and} \qquad p_1(z) := \frac{1}{1 - \mathbf{e}(z)}.$$

Borcherds gave an interesting expression of $\wp$ in his paper [Bo]:

$$\begin{aligned} \frac{1}{(2\pi i)^2}\wp(\tau, z) = (\mathbb{D}\, p_1)(z) - \frac{1}{(2\pi i)^2}G_2(\tau) \\ + \sum_{n=1}^{\infty}\sum_{\alpha \mid n}\alpha\left(\mathbf{e}(n\tau + \alpha z) + \mathbf{e}(n\tau - \alpha z)\right). \end{aligned} \tag{3.10}$$

The right hand side converges in $\{(\tau, z) \in \mathbb{H} \times \mathbb{C} \mid |\operatorname{Im} z| < \operatorname{Im}\tau\}$. From this expression, immediately we have

$$\begin{aligned} \left(\frac{1}{2\pi i}\right)^2\left(\mathbb{D}^{k-2}\wp\right)(\tau, z) = \left(\mathbb{D}^{k-1}p_1\right)(z) \\ + \sum_{n=1}^{\infty}\sum_{\alpha \mid n}\alpha^{k-1}\Big(\mathbf{e}(n\tau + \alpha z) + (-1)^k\mathbf{e}(n\tau - \alpha z)\Big) \quad (k \geq 3). \end{aligned} \tag{3.11}$$

### 3.3. Maass lift of Jacobi forms

In the case of level 1, the Maass lift was essentially given by Maass to prove the Saito-Kurokawa conjecture (cf. [EZ, Section 6]).

The Maass lift is constructed by using Hecke operators $V_m : \mathbb{J}_{k,1} \ni \phi \mapsto \phi|V_m \in \mathbb{J}_{k,m}$, defined below. Let

$$\phi(\tau, z) = \sum_{(n,l)\in\mathbb{Z}^2} c(n,l)\mathbf{e}(n\tau + lz) \in \mathbb{J}_{k,1}$$

be a Jacobi form of weight $k$ and index 1. We remark that $c(n,l) = 0$ when $4n - l^2 < 0$. By (3.7), we may assume $k$ is even. Moreover we may assume $k \geqq 4$, because $\mathbb{J}_{k,1} \neq \{0\}$ if and only if $k$ is an even integer equal or greater than 4 (cf. [EZ, Theorem 3.5]).

(i) First, for $m \in \mathbb{N}$, let

$$\Delta(m) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \alpha\delta - \beta\gamma = m \right\}$$

and define

$$(\phi|V_m)(\tau, z) := \frac{1}{m} \sum_{[g]\in \mathrm{SL}(2,\mathbb{Z})\backslash\Delta(m)} (\phi|_{k,1}C(g))(\tau, z).$$

Since

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{array}{ll} \alpha, \delta \in \mathbb{N}, & \alpha\delta = m, \\ \beta \in \mathbb{Z}, & 0 \leqq \beta < \delta \end{array} \right\} \tag{3.12}$$

is a complete set of representatives of a coset $\mathrm{SL}(2,\mathbb{Z})\backslash\Delta(m)$, the Fourier expansion of $\phi|V_m$ is given by

$$(\phi|V_m)(\tau, z) = \sum_{(n,l)\in\mathbb{Z}^2} \left( \sum_{\alpha|(n,l,m)} \alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz)$$

and therefore $\phi|V_m \in \mathbb{J}_{k,m}$.

(ii) Second, for $m = 0$, let

$$\begin{aligned}(\phi|V_0)(\tau) :=& c(0,0)\widetilde{G}_k(\tau) \\ =& c(0,0)\left(\frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\mathbf{e}(n\tau)\right).\end{aligned}$$

Since $k \geqq 4$, it is clear that $\phi|V_0 \in \mathbb{J}_{k,0}$.

Thus the Maass lift of $\phi$, defined by

$$(\mathrm{ML}(\phi))(Z) := \sum_{m=0}^{\infty} (\phi|V_m)(\tau, z)\mathbf{e}(m\omega),$$

has the Fourier expansion

$$\begin{aligned}(\mathrm{ML}(\phi))(Z) &= \frac{c(0,0)}{2}\zeta(1-k) \\ &\quad + \sum_{(n,m)\in\mathbb{N}_0^2}{}' \sum_{l\in\mathbb{Z}} \left( \sum_{\alpha|(n,l,m)} \alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz + m\omega),\end{aligned}$$

where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Therefore $\mathrm{ML}(\phi) \in \mathbb{FM}_k$. Hence, by Theorem 11, we have $\mathrm{ML}(\phi) \in \mathbb{M}_k^{(2)}$.

On of the important example of this lift is for

$$\phi_{10,1}(\tau, z) := \Delta(\tau)\phi_{-2,1}(\tau, z) \in \mathbb{J}_{10,1}.$$

The image of the Maass lift of this $\phi_{10,1}$ is Igusa's modular form of weight 10:

$$\Delta_{10}(Z) := \mathrm{ML}(\phi_{10,1}) = \sum_{m=1}^{\infty} (\phi_{10,1}|V_m)(\tau, z)\mathbf{e}(m\omega).$$

As $\phi_{10,1}(\tau, 0) = 0$, we have $(\phi_{10,1}|V_m)(\tau, 0) = 0$. Hence we have

$$\frac{(\phi_{10,1}|V_m)}{\phi_{-2,1}} \in \mathbb{J}^{\mathrm{weak}}_{12,m-1} \tag{3.13}$$

and therefore $\Delta_{10}(Z)$ has zeros of at least order 2 at the image of $\{ Z \in \mathbb{H} \mid z = 0 \}$ by each element of $\mathrm{Sp}(2, \mathbb{Z})$.

### 3.4. Maass lift of weak Jacobi forms

Now we apply the idea of the Maass lift to weak Jacobi forms. The Maass lifts of weak Jacobi forms was first studied in 1990's by Borcherds. In his paper [Bo], he constructed maps from weak Jacobi forms to modular forms on orthogonal groups $\mathrm{SO}(2, n+2)$ associated with unimodular lattices (hence $n \in 8\mathbb{N}$). Here we consider a lift to Siegel modular forms of degree 2, that corresponds to the case $n = 1$ in Borcherds' paper. He did not give a complete proof when $n = 1$, however, our construction in this subsection is totally based on his idea, except for the part using Theorem 11.

Let

$$\phi(\tau, z) = \sum_{(n,l)\in\mathbb{Z}^2} c(n,l)\mathbf{e}(n\tau + lz) \in \mathbb{J}_{k,1}^{\text{weak}}$$

be a weak Jacobi form of weight $k$ and index 1. We remark that $c(n,l) = 0$ when $n < 0$. By (3.7), we may assume $k$ is even. Here we add the assumption $k > 0$. For any $x \in \mathbb{Z}$, since $\phi|_{k,1}U(x,0) = \phi$, we have $c(n,l) = c(n + xl + x^2, l + 2x)$. Therefore, from the definition of weak Jacobi forms, we have $c(0,l) = 0$ except for $l = -1, 0, 1$. As $k$ is even and $\phi|_{k,1}C(-E_2) = \phi$, we have $c(0,-1) = c(0,1)$. For $m \in \mathbb{N}$, we don't change the definition of $V_m$. Namely,

$$\begin{aligned}(\phi|V_m)(\tau, z) &:= \frac{1}{m}\sum_{[g]\in \mathrm{SL}(2,\mathbb{Z})\backslash\Delta(m)} (\phi|_{k,1}C(g))(\tau,z) \\ &= \sum_{(n,l)\in\mathbb{Z}^2}\left(\sum_{\alpha|(n,l,m)} \alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right)\mathbf{e}(n\tau + lz).\end{aligned}$$

Then we have $\phi|V_m \in \mathbb{J}_{k,m}^{\text{weak}}$. For $m = 0$, we need to modify the definition of $\phi|V_0$. When $k > 2$, we define

$$(\phi|V_0)(\tau,z) := c(0,0)\widetilde{G}_k(\tau) + c(0,1)\frac{1}{(2\pi i)^2}\left(\mathbb{D}^{k-2}\wp\right)(\tau,z).$$

By using (3.4) and (3.11), we have

$$\begin{aligned}(\phi|V_0)(\tau,z) =& \frac{c(0,0)}{2}\zeta(1-k) + c(0,1)\left(\mathbb{D}^{k-1}p_1\right)(z) \\ &+ \sum_{(n,l)\in\mathbb{N}\times\mathbb{Z}}\left(\sum_{\alpha|(n,l)}\alpha^{k-1}c\left(0,\frac{l}{\alpha}\right)\right)\mathbf{e}(n\tau + lz).\end{aligned} \tag{3.14}$$

When $k = 2$, we define

$$(\phi|V_0)(\tau,z) := c(0,1)\frac{1}{(2\pi i)^2}\wp(\tau,z).$$

Here we remark that $\phi(\tau, 0)$ is an elliptic modular form of weight 2 with respect to $\mathrm{SL}(2,\mathbb{Z})$. Hence $\phi(\tau,0) = 0$ and therefore $2c(0,1) + c(0,0) = 0$. By using (3.3) and (3.10), we have

$$\begin{aligned}(\phi|V_0)(\tau,z) =& c(0,1)(\mathbb{D}\,p_1)(z) + c(0,1)\frac{1}{12}\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)\mathbf{e}(n\tau)\right) \\ &+ c(0,1)\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha\left(\mathbf{e}(n\tau+\alpha z)+\mathbf{e}(n\tau-\alpha z)\right) \\ =& \frac{c(0,0)}{2}\zeta(-1) + c(0,1)(\mathbb{D}\,p_1)(z) \\ &+ \sum_{(n,l)\in\mathbb{N}\times\mathbb{Z}}\left(\sum_{\alpha|(n,l)}\alpha\, c\left(0,\frac{l}{\alpha}\right)\right)\mathbf{e}(n\tau+lz).\end{aligned}$$

Hence the equation (3.14) holds even for $k=2$. We remark that $\phi|V_0$ has poles of order $k$ at each $z\in\mathbb{Z}+\tau\mathbb{Z}$.

The Maass lift of a weak Jacobi form $\phi$ is defined by

$$(\mathrm{ML}(\phi))(Z) := \sum_{m=0}^{\infty}(\phi|V_m)(\tau,z).$$

It has the Fourier expansion

$$\begin{aligned}(\mathrm{ML}(\phi))(Z) =& \frac{c(0,0)}{2}\zeta(1-k) + c(0,1)\left(\mathbb{D}^{k-1}p_1\right)(z) \\ &+ \sideset{}{'}\sum_{(n,m)\in\mathbb{N}_0^2}\sum_{l\in\mathbb{Z}}\left(\sum_{\alpha|(n,l,m)}\alpha^{k-1}c\left(\frac{nm}{\alpha^2},\frac{l}{\alpha}\right)\right)\mathbf{e}(n\tau+lz+m\omega)\end{aligned}$$

and therefore the Fourier coefficients of $\mathrm{ML}(\phi)$ has the symmetry under the exchange of the variables $\tau$ and $\omega$. Nevertheless, if $\phi\notin\mathbb{J}_{k,1}$, we have $\mathrm{ML}(\phi)\notin\mathbb{FM}_k^{\mathrm{weak}}$, because $\phi|V_0$ is not a holomorphic but a meromorphic Jacobi form. To identify this $\mathrm{ML}(\phi)$, we consider the product of two formal series of Jacobi forms $\Delta_{10}(Z)^{\frac{k}{2}}$ and $(\mathrm{ML}(\phi))(Z)$. By the equations (3.9) and (3.13), we have

$$\left(\Delta_{10}(Z)\right)^{\frac{k}{2}}\left((\mathrm{ML}(\phi))(Z)\right)\in\mathbb{FM}_{6k}^{\mathrm{weak}}.$$

Thus, by Theorem 11, $\left(\Delta_{10}(Z)\right)^{\frac{k}{2}}\left((\mathrm{ML}(\phi))(Z)\right)\in\mathbb{M}_{6k}^{(2)}$. Hence we have the following theorem.

**Theorem 14.** *For any $\phi\in\mathbb{J}_{k,1}^{\mathrm{weak}}$, $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form of weight $k$.*

## 4. Generalized $\wp$ function

Hereafter we assume $N \geq 2$ and $\chi$ is a Dirichlet character modulo $N$. As $N \neq 1$, we have $\chi(0) = 0$.

### 4.1. Eisenstein series with levels

First of all, we remark that $\mathbb{M}_k^{(1)}(N;\chi) = \{0\}$ when $\chi(-1) \neq (-1)^k$, because $-E_2 \in \Gamma_0^{(2)}(N)$. Hence, here we assume $\chi(-1) = (-1)^k$. There are some kinds of definitions of Eisenstein series with levels, however, in this paper, we introduce one appeared in the paper by Ibukiyama [Ib]. The Eisenstein series of weight $k$ and level $N$ with character $\chi$ is defined by

$$\widetilde{G}_{k,\chi}(\tau) := \frac{1}{2}L(1-k,\chi) + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n)\mathbf{e}(n\tau) \in \mathbb{M}_k^{(1)}(N;\chi), \tag{4.1}$$

where $L(1-k,\chi)$ is the special value of (analytic continued) Dirichlet L function and $\sigma_{k-1,\chi}$ is the divisor function with character $\chi$ defined by

$$\sigma_{k-1,\chi}(n) := \sum_{\delta|n} \chi(\delta)\delta^{k-1}.$$

The generalized Bernoulli numbers $B_{n,\chi}$ are defined by the generating function

$$\frac{\sum_{b=1}^{N} \chi(b) z \exp(bz)}{\exp(Nz) - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{z^n}{n!} \qquad \text{(cf. [AIK, §4])}. \tag{4.2}$$

These Bernoulli numbers are closely related to special values of the Dirichlet $L$ function:

$$L(1-k,\chi) = -\frac{1}{k}B_{k,\chi} \qquad (k \in \mathbb{N}) \qquad \text{(cf. [AIK, §9])}.$$

Hence we have

$$\widetilde{G}_{k,\chi}(\tau) := -\frac{1}{2k}B_{k,\chi} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n)\mathbf{e}(n\tau).$$

In (4.2), substituting $-z$ for $z$, we have

$$\frac{\sum_{b=1}^{N} \chi(b) z \exp((N-b)z)}{\exp(Nz) - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{(-z)^n}{n!}$$

and therefore

$$\frac{\chi(-1)\sum_{b=1}^{N}\chi(b)z\exp(bz)}{\exp(Nz)-1}=\sum_{n=0}^{\infty}(-1)^n B_{n,\chi}\frac{z^n}{n!}.$$

Hence, if $\chi(-1)\neq(-1)^n$, we have $B_{n,\chi}=0$.

Let $\chi_0$ be the principal character module $N$, namely

$$\chi_0(\delta):=\begin{cases}1 & (\ (\delta,N)=1\ )\\ 0 & (\ (\delta,N)\neq 1\ )\end{cases}.$$

In this case, we have

$$\widetilde{G}_{k,\chi_0}(\tau)=\sum_{\delta|N}\mu(\delta)\delta^{k-1}\widetilde{G}_k(\delta\tau) \tag{4.3}$$

for any positive even integer $k$, where $\mu$ is the Möbius function.

## 4.2. Jacobi forms with levels

As in the case of level 1, the structure of all weak Jacobi forms of level $N$ are determined by Aoki-Ibukiyama [AI] or more precisely Aoki [Ao4]. As with (3.7), we have

$$\bigoplus_{k\in\mathbb{Z}}\mathbb{J}_{k,1}^{\mathrm{weak}}(N;\chi)=\left(\bigoplus_{k\in\mathbb{Z}}\mathbb{M}_k^{(1)}(N;\chi)\right)\phi_{-2,1}\oplus\left(\bigoplus_{k\in\mathbb{Z}}\mathbb{M}_k^{(1)}(N;\chi)\right)\phi_{0,1}. \tag{4.4}$$

Hence $\mathbb{J}_{k,1}^{\mathrm{weak}}(N;\chi)=\{0\}$ when $\chi(-1)\neq(-1)^k$.

## 4.3. Generalization of the $\wp$ function

We define a meromorphic function $R_{N,\chi}$ on $\mathbb{C}$ by

$$R_{N,\chi}(z):=\frac{\sum_{b=1}^{N}\chi(b)\mathbf{e}(bz)}{1-\mathbf{e}(Nz)}=\sum_{n=0}^{\infty}\frac{-B_{n,\chi}}{n!}(2\pi iz)^{n-1}.$$

From this definition, we have $R_{N,\chi}(z)=-\chi(-1)R_{N,\chi}(-z)$ and

$$R_{N,\chi}(z)=\sum_{j=0}^{\infty}\frac{-B_{2j+\epsilon,\chi}}{(2j+\epsilon)!}(2\pi iz)^{2j+\epsilon-1}, \tag{4.5}$$

where we put

$$\epsilon := \begin{cases} 0 & (\ \chi(-1) = 1\ ) \\ 1 & (\ \chi(-1) = -1\ ) \end{cases} .$$

This meromorphic function $R_{N,\chi}$ has at most a pole of order 1 at each $z \in \frac{1}{N}\mathbb{Z}$. The residue of $R_{N,\chi}$ at $z = \frac{s}{N}$ is given by

$$\text{(4.6)} \qquad \operatorname{Res}\left(R_{N,\chi}, \frac{s}{N}\right) = \lim_{z \to \frac{s}{N}} \left(z - \frac{s}{N}\right) R_{N,\chi}(z) = -\frac{\sum_{b=1}^{N} \chi(b) \mathbf{e}\left(\frac{bs}{N}\right)}{2\pi i N}.$$

For example, we have

$$\text{(4.7)} \qquad \operatorname{Res}\left(R_{N,\chi}, 0\right) = \frac{-B_{0,\chi}}{2\pi i} = \begin{cases} \frac{-\varphi(N)}{2\pi i N} & (\ \chi = \chi_0\ ) \\ 0 & (\ \chi \neq \chi_0\ ) \end{cases} ,$$

where $\varphi$ is the Euler's totient function. We remark that if $\operatorname{Res}\left(R_{N,\chi}, \frac{s}{N}\right) = 0$ then $z = \frac{s}{N}$ is not a pole of $R_{N,\chi}$. If we need the explicit residue at each pole, we can calculate the sum in (4.6) by using the Gauss sum of Dirichlet characters, however, we do not use it in this paper. From the definition, we can see that the equation $R_{N,\chi}(z) = R_{N,\chi}(z+1)$ holds.

We define a generalized $\wp$ function by

$$P_{N,\chi}(\tau, z) := R_{N,\chi}(z) + Q_{N,\chi}(\tau, z),$$

where

$$\begin{aligned} Q_{N,\chi}(\tau, z) := &\sum_{n=1}^{\infty} \sum_{\alpha=1}^{\infty} \chi(\alpha) \mathbf{e}(n\alpha\tau + \alpha z) \\ &- \sum_{n=1}^{\infty} \sum_{\alpha=1}^{\infty} \chi(-\alpha) \mathbf{e}(n\alpha\tau - \alpha z). \end{aligned}$$

In the definition of $Q_{N,\chi}$, the sum in the first line and the second line converge when $\max\{0, -\operatorname{Im} z\} < \operatorname{Im} \tau$ and $\max\{0, \operatorname{Im} z\} < \operatorname{Im} \tau$, respectively. Hence this $P_{N,\chi}$ is a meromorphic function on $\{\ (\tau, z) \in \mathbb{C}^2 \mid |\operatorname{Im} z| < \operatorname{Im} \tau\ )\}$.

For a while we assume $0 < \operatorname{Im} z < \operatorname{Im} \tau$. Under this assumption, we have

$$\begin{aligned} &P_{N,\chi}(\tau, z) - P_{N,\chi}(\tau, z - \tau) \\ = &R_{N,\chi}(z) - R_{N,\chi}(z - \tau) - \sum_{\alpha=1}^{\infty} \chi(\alpha) \mathbf{e}(\alpha z) - \sum_{\alpha=1}^{\infty} \chi(-\alpha) \mathbf{e}(\alpha\tau - \alpha z) \end{aligned}$$

$$= \left( R_{N,\chi}(z) - \sum_{\alpha=1}^{\infty} \chi(\alpha)\mathbf{e}(\alpha z) \right) + \chi(-1) \left( R_{N,\chi}(\tau - z) - \sum_{\alpha=1}^{\infty} \chi(\alpha)\mathbf{e}(\alpha\tau - \alpha z) \right)$$
$$= 0.$$

By using this relation for analytic continuation, we can regard $P_{N,\chi}$ as a meromorphic function on $\mathbb{H} \times \mathbb{C}$, satisfying the functional equation

$$P_{N,\chi}(\tau, z) = P_{N,\chi}(\tau, z+1) = P_{N,\chi}(\tau, z+\tau). \tag{4.8}$$

If we fix $\tau \in \mathbb{H}$ and consider $P_{N,\chi}(\tau, z)$ as a meromorphic function with respect to $z$ on $\mathbb{C}$, it is an elliptic function (doubly periodic meromorphic function) with two periods 1 and $\tau$. It has at most a pole of order 1 at each $z \in \frac{1}{N}\mathbb{Z} + \tau\mathbb{Z}$. Moreover, we can show

$$P_{N,\chi}(\tau, z) = -\chi(-1) P_{N,\chi}(\tau, -z).$$

Next we will show that $P_{N,\chi}$ has an automorphic property with respect to $\Gamma_0^{(1)}(N)$, the congruent subgroups of level $N$. For $g_1 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0^{(1)}(N)$, let

$$\left(P_{N,\chi}|g_1\right)(\tau, z) := \chi(\alpha)(\gamma\tau+\delta)^{-1} P_{N,\chi}\left( \frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{z}{\gamma\tau+\delta} \right).$$

From (4.8), $P_{N,\chi}|M$ satisfies functional equation

$$\left(P_{N,\chi}|g_1\right)(\tau, z) = \left(P_{N,\chi}|g_1\right)(\tau, z+1) = \left(P_{N,\chi}|g_1\right)(\tau, z+\tau). \tag{4.9}$$

Therefore, from (4.8) and (4.9), both $P_{N,\chi}$ and $P_{N,\chi}|g_1$ are meromorphic elliptic functions.

Now we fix $\tau \in \mathbb{H}$ and consider $P_{N,\chi}(\tau, z)$ as a meromorphic function with respect to $z$ on $\mathbb{C}$. As already mentioned, $P_{N,\chi}$ has at most a pole of order 1 at each $z \in \frac{1}{N}\mathbb{Z} + \tau\mathbb{Z}$. By (4.6), the residue of $R_{N,\chi}$ at $z = \frac{s}{N} + \tau t$ is given by

$$\operatorname{Res}\left( R_{N,\chi}, \frac{s}{N} + \tau t \right) = -\frac{\sum_{b=1}^{N} \chi(b)\mathbf{e}\left(\frac{bs}{N}\right)}{2\pi i N}. \tag{4.10}$$

We compare poles of $P_{N,\chi}|g_1$ with poles of $P_{N,\chi}$. From the definition of $P_{N,\chi,k}|g_1$, it has at most a pole at each $z \in \frac{\gamma\tau+\delta}{N}\mathbb{Z} + (\alpha\tau+\beta)\mathbb{Z} = \frac{1}{N}\mathbb{Z} + \tau\mathbb{Z}$. The residue of $P_{N,\chi,k}|g_1$ at $z = \frac{(\gamma\tau+\delta)s'}{N} + (\alpha\tau+\beta)t'$ is given by

$$\operatorname{Res}\left( R_{N,\chi}|g_1, \frac{(\gamma\tau+\delta)s'}{N} + (\alpha\tau+\beta)t' \right) = -\frac{\chi(\alpha)\sum_{b=1}^{N} \chi(b)\mathbf{e}\left(\frac{bs'}{N}\right)}{2\pi i N}$$

$$= -\frac{\sum_{b=1}^{N}\chi(b)\mathbf{e}\left(\frac{b\delta s'}{N}\right)}{2\pi i N}. \tag{4.11}$$

From (4.10), the residue of $P_{N,\chi,k}$ at the same point is given by

$$\operatorname{Res}\left(R_{N,\chi}, \frac{(\gamma\tau+\delta)s'}{N} + (\alpha\tau+\beta)t'\right) = -\frac{\sum_{b=1}^{N}\chi(b)\mathbf{e}\left(\frac{b\delta s'}{N}\right)}{2\pi i N}. \tag{4.12}$$

Comparing (4.11) with (4.12), two meromorphic elliptic functions $R_{N,\chi}$ and $R_{N,\chi}|g_1$ have completely same poles. Therefore, the difference $P_{N,\chi}(\tau,z) - (P_{N,\chi}|g_1)(\tau,z)$ is a holomorphic elliptic function, that is a constant function (with respect to $z$). We will show that it vanishes below. More precisely, we will show

$$\lim_{z\to 0}\left\{P_{N,\chi}(\tau,z) - (P_{N,\chi}|g_1)(\tau,z)\right\} = 0. \tag{4.13}$$

(i) First, we consider the case $\chi = \chi_0$. In this case, as for $R_{N,\chi}$ part, from the equations (4.5) and (4.7), we have

$$R_{N,\chi}(z) = \sum_{j=0}^{\infty}\frac{-B_{2j,\chi}}{(2j)!}(2\pi i z)^{2j-1} = \frac{-\varphi(N)}{2\pi i N}\frac{1}{z} + O(z),$$

and therefore

$$\lim_{z\to 0}\left\{R_{N,\chi}(z) - (\gamma\tau+\delta)^{-1}R_{N,\chi}\left(\frac{z}{\gamma\tau+\delta}\right)\right\} = 0.$$

As for $Q_{N,\chi}$ part, because $\chi(-1) = 1$, we have

$$Q_{N,\chi}(\tau,0) = 0.$$

Hence we have (4.13).

(ii) Second, we consider the case $\epsilon = 0,\ \chi \neq \chi_0$. In this case, from the equations (4.5) and (4.7), we have

$$R_{N,\chi}(z) = \sum_{j=0}^{\infty}\frac{-B_{2j,\chi}}{(2j)!}(2\pi i z)^{2j-1} = O(z),$$

hence $R_{N,\chi}(0) = 0$ and therefore we have (4.13).

(iii) Third, we consider the case $\epsilon = 1$. In this case, from the equations (4.5), we have

$$R_{N,\chi}(z) = \sum_{j=0}^{\infty} \frac{-B_{2j+1,\chi}}{(2j+1)!}(2\pi i z)^{2j} = -B_{1,\chi} + O(z),$$

hence $R_{N,\chi}(0) = -B_{1,\chi} = L(0,\chi)$. Therefore, we have

$$\begin{aligned} P_{N,\chi}(\tau,0) =& R_{N,\chi}(0) + Q_{N,\chi}(\tau,0) \\ =& L(0,\chi) + 2\sum_{n=1}^{\infty}\sum_{\alpha=1}^{\infty} \chi(\alpha)\mathbf{e}(n\alpha\tau) \\ =& 2\widetilde{G}_{1,\chi}(\tau). \end{aligned}$$

Thus, as $\widetilde{G}_{1,\chi}(\tau) = \chi(\alpha)(\gamma\tau+\delta)^{-1}\widetilde{G}_{1,\chi}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$, we have (4.13).

From the above, now we have $P_{N,\chi}(\tau,z) = (P_{N,\chi}|g_1)(\tau,z)$ for any $g_1 \in \Gamma_0^{(1)}(N)$.

At the end of this subsection, we give the Laurent expansion of $P_{N,\chi}$ at $z=0$. As for $R_{N,\chi}$ part, by (4.7), from (4.5), we have

$$\begin{aligned} R_{N,\chi}(z) =& \sum_{j=0}^{\infty} \frac{-B_{2j+\epsilon,\chi}}{(2j+\epsilon)!}(2\pi i z)^{2j+\epsilon-1} \\ =& -\frac{B_{0,\chi}}{2\pi i}\frac{1}{z} + \sum_{j=0}^{\infty}\frac{-B_{2j+2-\epsilon,\chi}}{(2j+2-\epsilon)!}(2\pi i z)^{2j+1-\epsilon} \\ =& -\frac{B_{0,\chi}}{2\pi i}\frac{1}{z} + \sum_{j=0}^{\infty}\frac{L(-1-2j+\epsilon,\chi)}{(2j+1-\epsilon)!}(2\pi i z)^{2j+1-\epsilon}. \end{aligned}$$

As for $Q_{N,\chi}$ part, we have

$$\begin{aligned} Q_{N,\chi}(\tau,z) =& \sum_{n=1}^{\infty}\sum_{\alpha=1}^{\infty}\chi(\alpha)\mathbf{e}(n\alpha\tau+\alpha z) - \sum_{n=1}^{\infty}\sum_{\alpha=1}^{\infty}\chi(-\alpha)\mathbf{e}(n\alpha\tau-\alpha z) \\ =& \sum_{n=1}^{\infty}\sum_{\alpha|n}\chi(\alpha)\mathbf{e}(n\tau+\alpha z) - \sum_{n=1}^{\infty}\sum_{\alpha|n}\chi(-\alpha)\mathbf{e}(n\tau-\alpha z) \\ =& \sum_{n=1}^{\infty}\sum_{\alpha|n}\chi(\alpha)\mathbf{e}(n\tau)\left(\mathbf{e}(\alpha z) - \chi(-1)\mathbf{e}(-\alpha z)\right) \\ =& 2\sum_{n=1}^{\infty}\sum_{\alpha|n}\chi(\alpha)\mathbf{e}(n\tau)\sum_{j=0}^{\infty}\frac{(2\pi i\alpha z)^{2j+1-\epsilon}}{(2j+1-\epsilon)!} \end{aligned}$$

$$=\sum_{j=0}^{\infty}\frac{2(2\pi i)^{2j+1-\epsilon}}{(2j+1-\epsilon)!}\sum_{n=1}^{\infty}\sigma_{2j+1-\epsilon,\chi}(n)\mathbf{e}(n\tau)z^{2j+1-\epsilon}.$$

Hence we have

$$P_{N,\chi}(\tau,z)=-\frac{B_{0,\chi}}{2\pi i}\frac{1}{z}+\sum_{j=0}^{\infty}\frac{2(2\pi i)^{2j+1-\epsilon}}{(2j+1-\epsilon)!}\widetilde{G}_{2j+2-\epsilon,\chi}(\tau)z^{2j+1-\epsilon}.$$

If we use

$$\widetilde{G}_{k,\chi}(\tau):=\frac{(k-1)!}{2(2\pi i)^k}G_{k,\chi}(\tau)$$

instead of $G_{k,\chi}$, it is

$$(2\pi i)P_{N,\chi}(\tau,z)=\begin{cases}-\dfrac{\varphi(N)}{N}\dfrac{1}{z}+\displaystyle\sum_{k=1}^{\infty}G_{2k,\chi}(\tau)z^{2k-1} & (\ \chi=\chi_0\ )\\ \displaystyle\sum_{k=1}^{\infty}G_{2k,\chi}(\tau)z^{2k-1} & (\ \epsilon=0,\ \chi\neq\chi_0\ )\\ \displaystyle\sum_{k=0}^{\infty}G_{2k+1,\chi}(\tau)z^{2k} & (\ \epsilon=1\ )\end{cases}.$$

Especially, we have

$$(2\pi i)^2\left(\mathbb{D}\,P_{N,\chi_0}\right)(\tau,z)=\frac{\varphi(N)}{N}\frac{1}{z^2}+\sum_{k=1}^{\infty}(2k-1)G_{2k,\chi_0}(\tau)z^{2k-2}.$$

This is just a generalization of the Weierstrass $\wp$ function:

$$\wp(\tau,z)+G_2(\tau)=\frac{1}{z^2}+\sum_{k=1}^{\infty}(2k-1)G_{2k}(\tau)z^{2k-2}.$$

Since $\varphi(N)=\sum_{\delta|N}\mu(\delta)\frac{N}{\delta}$, by (4.3), we have

$$(2\pi i)^2\left(\mathbb{D}\,P_{N,\chi_0}\right)(\tau,z)=\sum_{\delta|N}\mu(\delta)\delta\left(\wp(\delta\tau,\delta z)+G_2(\delta\tau)\right). \tag{4.14}$$

## 5. Maass lifts with levels

### 5.1. Maass lifts of Jacobi forms

The Maass lifts of Jacobi forms with levels were studied by several researchers, and finally Ibukiyama [Ib] gave the explicit formula for arbitrary levels with characters. In this subsection, we review his work.

The Maass lifts are constructed by using Hecke operators

$$V_m : \mathbb{J}_{k,1}(N;\chi) \ni \phi \mapsto \phi|V_m \in \mathbb{J}_{k,m}(N;\chi),$$

defined below. Let

$$\phi(\tau,z) = \sum_{(n,l)\in\mathbb{Z}^2} c(n,l)\mathbf{e}(n\tau + lz) \in \mathbb{J}_{k,1}(N;\chi)$$

be a Jacobi form of weight $k$ and index 1. We remark that $c(n,l) = 0$ when $4n - l^2 < 0$. By (4.4), we may assume $\chi(-1) = (-1)^k$. Moreover we may assume $k > 0$, because $\mathbb{J}_{k,1}(N;\chi) = \{0\}$ if $k \leqq 0$.

(i) First, for $m \in \mathbb{N}$, let

$$\Delta(N;m) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha\delta - \beta\gamma = m, \quad (\alpha, N) = 1 \\ \gamma \equiv 0 \pmod N \end{matrix} \right\}$$

and define

$$(\phi|V_m)(\tau,z) := \frac{1}{m} \sum_{[g]\in\Gamma_0^{(1)}(N)\backslash\Delta(N;m)} \chi((g)_{11})\,(\phi|_{k,1}C(g))(\tau,z),$$

where $(g)_{11}$ means the (1,1)-component of the matrix $g$. Since

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \quad (\alpha, N) = 1, \\ \beta \in \mathbb{Z}, \quad 0 \leqq \beta < \delta \end{matrix} \right\}$$

is a complete set of representatives of a coset $\mathrm{SL}(2,\mathbb{Z})\backslash\Delta(N;m)$, the Fourier expansion of $\phi|V_m$ is given by

$$(\phi|V_m)(\tau,z) = \sum_{(n,l)\in\mathbb{Z}^2} \left( \sum_{\alpha|(n,l,m)} \chi(\alpha)\alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz)$$

and therefore $\phi|V_m \in \mathbb{J}_{k,m}(N;\chi)$.

(ii) Second, for $m = 0$, let

$$(\phi|V_0)(\tau) := c(0,0)\widetilde{G}_{k,\chi}(\tau)$$

$$=c(0,0)\left(\frac{1}{2}L(1-k,\chi)+\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)\mathbf{e}(n\tau)\right).$$

It is clear that $\phi|V_0 \in \mathbb{J}_{k,0}(N;\chi)$.

Thus the Maass lift of $\phi$ defined by

$$\left(\mathrm{ML}(\phi)\right)(Z) := \sum_{m=0}^{\infty}\left(\phi|V_m\right)(\tau,z)$$

has the Fourier expansion

$$\begin{aligned}\left(\mathrm{ML}(\phi)\right)(Z) = &\frac{c(0,0)}{2}L(1-k,\chi)\\ &+\sideset{}{'}\sum_{(n,m)\in\mathbb{N}_0^2}\sum_{l\in\mathbb{Z}}\left(\sum_{\alpha|(n,l,m)}\chi(\alpha)\alpha^{k-1}c\left(\frac{nm}{\alpha^2},\frac{l}{\alpha}\right)\right)\mathbf{e}(n\tau+lz+m\omega).\end{aligned}$$

Therefore $\mathrm{ML}(\phi) \in \mathbb{FM}_k(N;\chi)$ as a formal series of Jacobi forms. As Conjecture 12 has not been proven, we could not show the convergence of $\mathrm{ML}(\phi)$ just like as section 3. Nevertheless, we can show that $\mathrm{ML}(\phi)$ converges on $\mathbb{H}_2$ by direct calculation. Hence we have $\mathrm{ML}(\phi) \in \mathbb{M}_k^{(2)}(N;\chi)$.

### 5.2. Maass lifts of weak Jacobi forms

Now we apply the idea of Ibukiyama's Maass lifts to weak Jacobi forms. Our procedure is just like the case of level 1. In the case of level 1 we use the Weierstrass $\wp$ function, which is a meromorphic Jacobi form of index 0 with level 1. In the case of level $N$, we use the function $P_{N,\chi}$, which is a meromorphic Jacobi form of index 0 with level $N$ and character $\chi$.

Let

$$\phi(\tau,z) = \sum_{(n,l)\in\mathbb{Z}^2} c(n,l)\mathbf{e}(n\tau+lz) \in \mathbb{J}_{k,1}^{\mathrm{weak}}(N;\chi)$$

be a weak Jacobi form of weight $k$ and index 1 with level $N$ and character $\chi$. We remark that $c(n,l)=0$ when $n<0$. By (4.4), we may assume $\chi(-1)=(-1)^k$. Here we add the assumption $k>0$. As in the case without levels, we have $c(0,l)=0$ except for $l=-1,0,1$ and we have $c(0,-1)=c(0,1)$. For $m\in\mathbb{N}$, we don't change the definition of $V_m$. Namely,

$$\left(\phi|V_m\right)(\tau,z) := \frac{1}{m}\sum_{[g]\in\Gamma_0^{(1)}(N)\backslash\Delta(N;m)}\chi(\alpha)\left(\phi|_{k,1}C(g)\right)(\tau,z)$$

$$= \sum_{(n,l)\in\mathbb{Z}^2} \left( \sum_{\alpha|(n,l,m)} \chi(\alpha)\alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz).$$

Then we have $\phi|V_m \in \mathbb{J}^{\mathrm{weak}}_{k,m}(N;\chi)$. For $m = 0$, we need to modify the definition of $\phi|V_0$. We define

$$(\phi|V_0)(\tau, z) := c(0,0)\widetilde{G}_{k,\chi}(\tau) + c(0,1)\left(\mathbb{D}^{k-1} P_{N,\chi}\right)(\tau, z).$$

Then we have

$$\begin{aligned}(\phi|V_0)(\tau, z) =& \frac{c(0,0)}{2} L(1-k,\chi) + c(0,1)\left(\mathbb{D}^{k-1} R_{N,\chi}\right)(z) \\ &+ \sum_{(n,l)\in\mathbb{N}\times\mathbb{Z}} \left( \sum_{\alpha|(n,l)} \chi(\alpha)\alpha^{k-1} c\left(0, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz).\end{aligned}$$

The Maass lift of a weak Jacobi form $\phi$ is defined by

$$(\mathrm{ML}(\phi))(Z) := \sum_{m=0}^{\infty} (\phi|V_m)(\tau, z).$$

It has the Fourier expansion

$$\begin{aligned}(\mathrm{ML}(\phi))(Z) =& \frac{c(0,0)}{2} L(1-k,\chi) + c(0,1)\left(\mathbb{D}^{k-1} R_{N,\chi}\right)(z) \\ &+ \sideset{}{'}\sum_{(n,m)\in\mathbb{N}_0^2} \sum_{l\in\mathbb{Z}} \left( \sum_{\alpha|(n,l,m)} \chi(\alpha)\alpha^{k-1} c\left(\frac{nm}{\alpha^2}, \frac{l}{\alpha}\right)\right) \mathbf{e}(n\tau + lz + m\omega)\end{aligned} \tag{5.1}$$

and therefore the Fourier coefficients of $\mathrm{ML}(\phi)$ has the symmetry under the exchange of the variables $\tau$ and $\omega$. Nevertheless, if $\phi \notin \mathbb{J}_{k,1}(N;\chi)$, we have $\mathrm{ML}(\phi) \notin \mathbb{FM}^{\mathrm{weak}}_k(N;\chi)$, because $\phi|V_0$ is not a holomorphic but a meromorphic Jacobi form. To identify this $\mathrm{ML}(\phi)$, we consider the product of two formal series of Jacobi forms $\Delta_{10}(NZ)^{\lceil \frac{k}{2} \rceil}$ and $(\mathrm{ML}(\phi))(Z)$, where $\lceil \frac{k}{2} \rceil$ means the least integer greater than or equal to $\frac{k}{2}$. Since $\Delta_{10}(NZ) \in \mathbb{M}^{(2)}_{10}(N;\chi_0)$ and $P_{N,\chi}$ has at most a pole of order 1 at each $z \in \frac{1}{N}\mathbb{Z} + \tau\mathbb{Z}$, we have

$$\left(\Delta_{10}(NZ)\right)^{\lceil \frac{k}{2} \rceil} \left((\mathrm{ML}(\phi))(Z)\right) \in \mathbb{FM}^{\mathrm{weak}}_{10\lceil \frac{k}{2} \rceil + k}(N;\chi). \tag{5.2}$$

Therefore, if we could show the convergence of the left hand side of (5.2), we know $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form.

**Theorem 15.** *If the left hand side of (5.2) converges absolutely and locally uniformly on $\mathbb{H}_2$, then $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form.*

However, we do not have any good general method to show the convergence as far as the author knows.

## 6. Convergence of our Mass lifts

In this section, we will show that $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form when $N = 2, 3$ and $\chi$ is the principle character.

### 6.1. Sketch of the proof

Since we assume $\chi(-1) = (-1)^k$, $k$ is even when $\chi$ is the principle character $\chi_0$. Let

$$(6.1) \qquad (\mathrm{ML}^*(\phi))(Z) := \Big(\Delta_{10}(NZ)\Big)^{\frac{k}{2}} \Big((\mathrm{ML}(\phi))(Z)\Big).$$

In this section, first we show the following proposition.

**Proposition 16.** *If $N = 2$ or $N = 3$, we have*

$$\mathrm{ML}^*(\phi) \in \mathbb{FM}^{+}_{6k}(N; \chi_0).$$

By using this proposition and Theorem 13, we know that $\mathrm{ML}^*(\phi)$ is a (holomorphic) Siegel modular form when $N = 2, 3$ and $\chi$ is the principle character. This means that $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form. In this section we also give a proof of Theorem 13.

### 6.2. Proof of Proposition 16

Let $p$ be a prime number. In this subsection we write $p$ instead of $N$, because our proof holds for any prime $N$ for the most part.

Since $\mathrm{ML}^*(\phi) \in \mathbb{FM}^{\mathrm{weak}}_{6k}(p; \chi_0)$, the only thing we need to show in this subsection is that the Fourier coefficients of $\mathrm{ML}^*(\phi)$ at each cusp satisfy the condition of Jacobi forms and (Sym+). From Lemma 9, a complete set of representatives of a coset $\Gamma_0^{(\mathrm{J})}(N)\backslash \mathrm{Sp}(2,\mathbb{Z})^{\mathrm{J}}$ is given by $\{C(g_0), C(g_1), \dots\}$, where $\{g_0, g_1, \dots\}$ is a complete set of representatives of a coset $\Gamma_0^{(1)}(N)\backslash \mathrm{SL}(2,\mathbb{Z})$.

When $N$ is a prime number $p$, a complete set of representatives of a coset $\Gamma_0^{(1)}(p)\backslash\mathrm{SL}(2,\mathbb{Z})$ is given by $\{g_0, g_1, \ldots, g_p\}$, where

$$g_0 := E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$g_t := \begin{pmatrix} 0 & -1 \\ 1 & t-1 \end{pmatrix} = g_1 \begin{pmatrix} 1 & t-1 \\ 0 & 1 \end{pmatrix} \quad (t = 2, \ldots, p).$$

Therefore, it is enough to calculate the Fourier coefficients of $\mathrm{ML}^*(\phi)$ at $C\,(g_0)$ and $C\,(g_1)$.

(A) As for $C\,(g_0)$, we have nearly finished the calculation. We have

$$R\left(\Gamma_0^{(1)}(p); C\,(g_0)\right) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \det g \in \{1, -1\} \right\}$$

easily. Let

$$(6.2) \qquad (\mathrm{ML}^*(\phi))\,(Z) = \sum_{m=0}^{\infty} \phi_m(\tau, z)\mathbf{e}(m\omega)$$

and

$$\phi_m(\tau, z) = \sum_{n=0}^{\infty} \sum_{l\in\mathbb{Z}} c_0(n, l, m)\mathbf{e}(n\tau + lz).$$

We put $c_0(n, l, m) = 0$ for $m < 0$ or for $n < 0$.

**Lemma 17.** *We have*

$$(6.3) \qquad c_0(n, l, m) = c_0(n, -l, m)$$

*and*

$$(6.4) \qquad c_0(n, l, m) = c_0(n + xl + x^2 m, l + 2xm, m)$$

*for any* $n, l, m, x \in \mathbb{Z}$. *Namely,* $\mathrm{ML}^*(\phi)$ *is* $C\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$*-invariant and* $U\,(x, 0)$*-invariant.*

*Proof.* Since $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^{\mathrm{weak}}(p;\chi_0)$, each $\phi_m \in \mathbb{J}_{6k,m}^{\mathrm{weak}}(p;\chi_0)$ and therefore $\phi_m$ is $C\left(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)\right)$-invariant and $U(x,0)$-invariant. □

**Lemma 18.** *We have*

$$c_0(n,l,m) = c_0(m,l,n) \tag{6.5}$$

*for any $n,l,m \in \mathbb{Z}$. Namely, $\mathrm{ML}^*(\phi)$ is $S$-invariant.*

*Proof.* This is a direct conclusion from $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^{\mathrm{weak}}(p;\chi_0)$. □

On the Lemmas 17 and 18, we remark that

$$C\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right) = R\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right), \quad U(x,0) = R\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

and

$$S = R\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right).$$

Since these $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ generate $R\left(\Gamma_0^{(1)}(p); C(g_0)\right)$, $\mathrm{ML}^*(\phi)$ is $R\left(\Gamma_0^{(1)}(p); C(g_0)\right)$-invariant.

(B) As for $C(g_1)$, we need a bit hard calculation. We have

$$R\left(\Gamma_0^{(1)}(p); C(g_1)\right) = \left\{ g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{Mat}(2\times 2,\mathbb{Z}) \;\middle|\; \begin{array}{c}\det g \in \{1,-1\}\\ \gamma \equiv 0 \pmod{p}\end{array} \right\}$$

easily. On

$$\left(\left(\mathrm{ML}^*(\phi)\right)|_{6k} C(g_1)\right)(Z) = \sum_{m=0}^{\infty} \left(\phi_m|_{6k,m} C(g_1)\right)(\tau,z)\mathbf{e}(m\omega),$$

let

$$\left(\phi_m|_{6k,m} C(g_1)\right)(\tau,z) = \sum_{n=0}^{\infty}\sum_{l\in\mathbb{Z}} c_1(n,l,m)\mathbf{e}\left(\frac{n\tau}{p} + lz\right).$$

We put $c_1(n,l,m) = 0$ for $m < 0$ or for $n < 0$.

**Lemma 19.** *We have*

$$c_1(n,l,m) = c_1(n,-l,m) \tag{6.6}$$

*and*

$$c_1(n,l,m) = c_1\left(n + xpl + x^2pm, l + 2xm, m\right) \tag{6.7}$$

*for any* $n,l,m,x \in \mathbb{Z}$. *Namely,* $\mathrm{ML}^*(\phi)|_{6k}C(g_1)$ *is* $C\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$*-invariant and* $U(x,0)$*-invariant.*

*Proof.* Since $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^{\mathrm{weak}+}(p;\chi_0)$, each $\phi_m \in \mathbb{J}_{6k,m}^{\mathrm{weak}}(p;\chi_0)$ and therefore $\phi_m|_{6k,m}C(g_1)$ is $C(g_1)^{-1}\Gamma_0^{(\mathrm{J})}(p)C(g_1)$-invariant. It is easy to see that $C\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) \in C(g_1)^{-1}\Gamma_0^{(\mathrm{J})}(p)C(g_1)$ and $U(x,0) \in C(g_1)^{-1}\Gamma_0^{(\mathrm{J})}(p)C(g_1)$ for any $x \in \mathbb{Z}$. □

Moreover, the following lemma holds.

**Lemma 20.** *We have*

$$c_1(n,l,m) = c_1\left(n, l + 2xn, m + xpl + x^2pn\right) \tag{6.8}$$

*for any* $n,l,m,x \in \mathbb{Z}$. *Namely,* $\mathrm{ML}^*(\phi)|_{6k}C(g_1)$ *is* $R\left(\left(\begin{smallmatrix} 1 & 0 \\ px & 1 \end{smallmatrix}\right)\right)$*-invariant.*

However, we need much calculation to show this lemma. Therefore, now we proceed our story without the proof of this lemma and we will show this lemma at the end of this subsection. On the Lemmas 19 and 20, since the group generated by these $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ px & 1 \end{smallmatrix}\right)$ coincides with $R\left(\Gamma_0^{(1)}(p); C(g_0)\right)$ when $p = 2, 3$, $(\mathrm{ML}^*(\phi))|_{6k}C(g_1)$ is $R\left(\Gamma_0^{(1)}(p); C(g_1)\right)$-invariant.

Up to this point we have shown $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^{\mathrm{weak}+}(N;\chi_0)$. Next we will show $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^{+}(N;\chi_0)$. We will continue to see the Fourier coefficients of $\mathrm{ML}^*(\phi)$ at $C(g_0)$ and $C(g_1)$.

(A) As for $C(g_0)$, from the equation (6.1), we have $c_0(n,l,m) = 0$ if $\min\{m,n\} < \frac{pk}{2}$. From Lemmas 17 and 18, we have the following lemma.

**Lemma 21.** *If* $(n,l,m) \in \mathbb{Z}^3$ *satisfies* $c_0(n,l,m) \neq 0$ *and* $|l| > \min\{m,n\}$, *then there exists* $(n',l',m') \in \mathbb{Z}^3$ *such that* $4nm - l^2 = 4n'm' - l'^2$, $c_0(n,l,m) = c_0(n',l',m')$ *and* $|l| > |l'|$.

This lemma induces the following proposition immediately.

**Proposition 22.** *If* $(n,l,m) \in \mathbb{Z}^3$ *satisfies* $c_0(n,l,m) \neq 0$, *then there exists* $(n',l',m') \in \mathbb{Z}^3$ *such that* $4nm - l^2 = 4n'm' - l'^2$, $c_0(n,l,m) = c_0(n',l',m')$ *and* $|l| \leqq \min\{m,n\}$. *Hence* $4mn - l^2 \geqq 0$.

(B) As for $C\left(g_1\right)$, since

$$\Delta_{10}^*(Z) := \left(\Delta_{10}(pZ)\right)|_k C\left(g_1\right) = p^{-10}\Delta_{10}\begin{pmatrix} \frac{\tau}{p} & z \\ z & p\omega \end{pmatrix},$$

we have $c_1(n,l,m) = 0$ if $\min\{m, pn\} < \frac{pk}{2}$. From Lemmas 19 and 20, we have the following lemma.

**Lemma 23.** *If $(n,l,m) \in \mathbb{Z}^3$ satisfies $c_1(n,l,m) \neq 0$ and $|l| > \min\{m,n\}$, then there exists $(n',l',m') \in \mathbb{Z}^3$ such that $4nm - pl^2 = 4n'm' - pl'^2$, $c_1(n,l,m) = c_1(n',l',m')$ and $|l| > |l'|$.*

This lemma induces the following proposition immediately.

**Proposition 24.** *If $(n,l,m) \in \mathbb{Z}^3$ satisfies $c_1(n,l,m) \neq 0$, then there exists $(n',l',m') \in \mathbb{Z}^3$ such that $4nm - pl^2 = 4n'm' - pl'^2$, $c_1(n,l,m) = c_1(n',l',m')$ and $|l| \leqq \min\{m,n\}$. Hence, if $p = 2$ or $p = 3$, we have $4mn - pl^2 \geqq 0$.*

Up to this point we have shown $\mathrm{ML}^*(\phi) \in \mathbb{FM}_{6k}^+(N;\chi_0)$ except the proof of Lemma 20. Now we give its proof.

First, we calculate $\left(\mathrm{ML}(\phi)\right)|_k C\left(g_1\right)$ explicitly. For $\phi \in \mathbb{J}_{k,1}^{\mathrm{weak}}(p;\chi_0)$, we put

$$\phi(\tau,z) = \left(\phi|_{k,1} C\left(g_0\right)\right)(\tau,z) = \sum_{(n,l)\in\mathbb{Z}^2} a_0(n,l)\mathbf{e}\left(n\tau + lz\right)$$

and

$$\left(\phi|_{k,1} C\left(g_1\right)\right)(\tau,z) = \sum_{(n,l)\in\mathbb{Z}^2} a_1(n,l)\mathbf{e}\left(\frac{n\tau}{p} + lz\right).$$

Just like Lemmas 17 and 19, the following lemma holds.

**Lemma 25.** *We have*

$$\begin{aligned} &a_0(n,l) = a_0(n,-l), && a_0(n,l) = a_0\left(n + xl + x^2, l + 2x\right), \\ &a_1(n,l) = a_1(n,-l) \quad \text{and} && a_1(n,l) = a_1\left(n + xpl + x^2p, l + 2x\right) \end{aligned}$$

*for any $n,l,m,x \in \mathbb{Z}$. Hence $a_0(n,l)$ only depends on $4n - l^2$ and $a_1(n,l)$ only depends on $4n - pl^2$.*

Hence, hereafter, we write $a_0\left(4n - l^2\right)$ and $a_1\left(4n - pl^2\right)$ instead of $a_0(n,l)$ and $a_1(n,l)$, respectively.

(i) First, we discuss the case $m \in \mathbb{N}$. By (3.12), a complete set of representatives of a coset $\Gamma_0^{(1)}(p)\backslash\Delta(m)$ is given by

$$\left\{ g_t \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} t \in \{0,1,\ldots,p\}, \\ \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \\ 0 \leqq \beta < \delta \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \\ 0 \leqq \beta < \delta \end{matrix} \right\}$$
$$\bigcup \left\{ g_1 \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \\ 0 \leqq \beta < p\delta \end{matrix} \right\}.$$

Hence a complete set of representatives of a coset $\Gamma_0^{(1)}(p)\backslash\Delta(p;m)g_1$ is given by

$$\Delta_0(p;m) \cup \Delta_1(p;m),$$

where

$$\Delta_0(p;m) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \quad 0 \leqq \beta < \delta, \\ \delta \equiv 0 \pmod{p}, \quad (\beta,p) = 1 \end{matrix} \right\}$$
$$\Delta_1(p;m) = \left\{ g_1 \begin{pmatrix} \alpha & p\beta \\ 0 & \delta \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{Z}) \;\middle|\; \begin{matrix} \alpha,\delta \in \mathbb{N}, \quad \alpha\delta = m, \\ (\delta,p) = 1, \quad 0 \leqq \beta < \delta \end{matrix} \right\}.$$

Therefore, we have

$$\begin{aligned} &\left(\left(\phi|V_m\right)|_{k,m} C\left(g_1\right)\right)(\tau,z) \\ =&\frac{1}{m} \sum_{g\in\Delta_0(p;m)} \chi_0\left(\left(g g_1^{-1}\right)_{11}\right)\left(\phi|_{k,1} C\left(g\right)\right)(\tau,z) \\ &+ \frac{1}{m} \sum_{g\in\Delta_1(p;m)} \chi_0\left(\left(g g_1^{-1}\right)_{11}\right)\left(\phi|_{k,1} C\left(g\right)\right)(\tau,z). \end{aligned}$$

Now we calculate this sum.

(ia) As for the sum of $\Delta_0(p;m)$, it appears only when $p|m$. Hence we put $m = pm'$. Then we have

$$\frac{1}{m} \sum_{g\in\Delta_0(p;m)} \chi_0\left(\left(g g_1^{-1}\right)\right)\left(\phi|_{k,1} C\left(g\right)\right)(\tau,z)$$

$$
\begin{aligned}
=&m^{k-1}\sum_{\substack{\alpha\delta=m\\ \delta\equiv 0 \pmod p}}\sum_{\beta=0}^{\delta-1}\chi_0(-\beta)\delta^{-k}\phi\left(\frac{\alpha\tau+\beta}{\delta},\alpha z\right)\\
=&m^{k-1}\sum_{\alpha\delta=m'}\sum_{\beta=0}^{p\delta-1}\chi_0(-\beta)(p\delta)^{-k}\phi\left(\frac{\alpha\tau+\beta}{p\delta},\alpha z\right)\\
=&m^{k-1}\sum_{\alpha\delta=m'}\sum_{\beta=0}^{p-1}\sum_{j=0}^{\delta-1}\chi_0(-pj-\beta)(p\delta)^{-k}\phi\left(\frac{\alpha\tau+pj+\beta}{p\delta},\alpha z\right)\\
=&m^{k-1}\sum_{\alpha\delta=m'}\sum_{\beta=0}^{p-1}\sum_{j=0}^{\delta-1}\chi_0(-\beta)(p\delta)^{-k}\sum_{n,l\in\mathbb{Z}}a_0\left(4n-l^2\right)\mathbf{e}\left(n\frac{\alpha\tau+pj+\beta}{p\delta}+\alpha lz\right)\\
=&m^{k-1}\sum_{\alpha\delta=m'}\sum_{\beta=0}^{p-1}\chi_0(-\beta)p^{-k}\delta^{-k+1}\sum_{n,l\in\mathbb{Z}}a_0\left(4\delta n-l^2\right)\mathbf{e}\left(n\frac{\alpha\tau+\beta}{p}+\alpha lz\right)\\
=&\sum_{\alpha|m'}\sum_{\beta=0}^{p-1}\chi_0(-\beta)p^{-1}\alpha^{k-1}\sum_{n,l\in\mathbb{Z}}a_0\left(\frac{4nm'}{\alpha}-l^2\right)\mathbf{e}\left(n\frac{\alpha\tau+\beta}{p}+\alpha lz\right)\\
=&\sum_{\alpha|m'}\alpha^{k-1}\sum_{n,l\in\mathbb{Z}}a_0\left(\frac{4nm}{\alpha}-l^2\right)\mathbf{e}\left(n\alpha\tau+\alpha lz\right)\\
&-\frac{1}{p}\sum_{\alpha|m'}\alpha^{k-1}\sum_{n,l\in\mathbb{Z}}a_0\left(\frac{4nm'}{\alpha}-l^2\right)\mathbf{e}\left(\frac{\alpha n\tau}{p}+\alpha lz\right)\\
=&\sum_{n,l\in\mathbb{Z}}\sum_{\alpha|(n,l,m')}\alpha^{k-1}a_0\left(\frac{4nm-l^2}{\alpha^2}\right)\mathbf{e}\left(n\tau+lz\right)\\
&-\frac{1}{p}\sum_{n,l\in\mathbb{Z}}\sum_{\alpha|(n,l,m')}\alpha^{k-1}a_0\left(\frac{4nm'-l^2}{\alpha^2}\right)\mathbf{e}\left(\frac{n\tau}{p}+lz\right).
\end{aligned}
$$

(ib) As for the sum of $\Delta_1(p;m)$, we have

$$
\begin{aligned}
&\frac{1}{m}\sum_{g\in\Delta_1(p;m)}\chi_0\left(\left(gg_1^{-1}\right)\right)\left(\phi|_{k,1}C\left(g\right)\right)(\tau,z)\\
=&m^{k-1}\sum_{\substack{\alpha\delta=m\\(\delta,p)=1}}\sum_{\beta=0}^{\delta-1}\chi_0(\delta)\delta^{-k}\left(\phi|_{k,1}C\left(g_1\right)\right)\left(\frac{\alpha\tau+p\beta}{\delta},\alpha z\right)\\
=&m^{k-1}\sum_{\substack{\alpha\delta=m\\(\delta,p)=1}}\sum_{\beta=0}^{\delta-1}\chi_0(\delta)\delta^{-k}\sum_{n,l\in\mathbb{Z}}a_1\left(4n-pl^2\right)\mathbf{e}\left(\frac{n}{p}\frac{\alpha\tau+p\beta}{\delta}+\alpha lz\right)
\end{aligned}
$$

$$=m^{k-1}\sum_{\substack{\alpha\delta=m\\(\delta,p)=1}}\chi_0(\delta)\delta^{-k+1}\sum_{n,l\in\mathbb{Z}}a_1\left(4\delta n-pl^2\right)\mathbf{e}\left(\frac{n\alpha\tau}{p}+\alpha lz\right)$$

$$=\sum_{n,l\in\mathbb{Z}}\sum_{\alpha|(n,l,m)}\chi_0\left(\frac{m}{\alpha}\right)\alpha^{k-1}a_1\left(\frac{4nm-pl^2}{\alpha^2}\right)\mathbf{e}\left(\frac{n\tau}{p}+lz\right).$$

Hence, on the Fourier expansion

$$\left(\left(\phi|V_m\right)|_{k,m}C\left(g_1\right)\right)(\tau,z)=\sum_{n,l\in\mathbb{Z}}c_2(n,l,m)\mathbf{e}\left(\frac{n\tau}{p}+lz\right),$$

we have

$$c_2(n,l,m)=c_A(n,l,m)+c_B(n,l,m)+c_C(n,l,m)+c_D(n,l,m), \tag{6.9}$$

where

$$c_A(n,l,m)=\begin{cases}\displaystyle\sum_{\alpha|(n',l,m')}\alpha^{k-1}a_0\left(\frac{4nm-pl^2}{p\alpha^2}\right) & (n\equiv m\equiv 0 \pmod p)\\ 0 & (\text{otherwise})\end{cases},$$

$$c_B(n,l,m)=\begin{cases}\displaystyle -\frac{1}{p}\sum_{\alpha|(n,l,m')}\alpha^{k-1}a_0\left(\frac{4nm-pl^2}{p\alpha^2}\right) & (m\equiv 0 \pmod p)\\ 0 & (m\not\equiv 0 \pmod p)\end{cases},$$

$$c_C(n,l,m)=\sum_{\alpha|(n,l,m)}\alpha^{k-1}a_1\left(\frac{4nm-pl^2}{\alpha^2}\right)$$

and

$$c_D(n,l,m)=\begin{cases}\displaystyle -\sum_{\alpha|(n,l,m')}\alpha^{k-1}a_1\left(\frac{4nm-pl^2}{\alpha^2}\right) & (m\equiv 0 \pmod p)\\ 0 & (m\not\equiv 0 \pmod p)\end{cases}.$$

(ii) Second, we discuss the case $m=0$. By the equation (4.14), we have

$$(2\pi i)^2\left(\mathbb{D}\,P_{p,\chi_0}\right)(\tau,z)=\left(\left(\wp(\tau,z)+G_2(\tau)\right)-p\left(\wp(p\tau,pz)+G_2(p\tau)\right)\right).$$

Hence we have

$$\left(\left(\mathbb{D}\,P_{p,\chi_0}\right)|_{2,0}C\left(g_1\right)\right)(\tau,z)$$

$$
\begin{aligned}
=&\frac{1}{(2\pi i)^2}\left(\left(\wp(\tau,z)+G_2(\tau)+2\pi i\frac{1}{\tau}\right)\right.\\
&\qquad\left.-p\left(\frac{1}{p^2}\wp\left(\frac{\tau}{p},z\right)+\frac{1}{p^2}G_2\left(\frac{\tau}{p}\right)+2\pi i\frac{1}{p\tau}\right)\right)\\
=&\frac{1}{(2\pi i)^2}\left((\wp(\tau,z)+G_2(\tau))-\frac{1}{p}\left(\wp\left(\frac{\tau}{p},z\right)+G_2\left(\frac{\tau}{p}\right)\right)\right)\\
=&\left((\mathbb{D}\,p_1)(z)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha\left(\mathbf{e}(n\tau+\alpha z)+\mathbf{e}(n\tau-\alpha z)\right)\right)\\
&\quad-\frac{1}{p}\left((\mathbb{D}\,p_1)(z)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha\left(\mathbf{e}\left(\frac{n\tau}{p}+\alpha z\right)+\mathbf{e}\left(\frac{n\tau}{p}-\alpha z\right)\right)\right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\left(\left(\mathbb{D}^{k-1}P_{p,\chi_0}\right)|_{k,0}C(g_1)\right)(\tau,z)\\
=&\,\mathbb{D}^{k-2}\left((\mathbb{D}\,P_{p,\chi_0})|_{2,0}C(g_1)\right)(\tau,z)\\
=&\left(\left(\mathbb{D}^{k-1}p_1\right)(z)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha^{k-1}\left(\mathbf{e}(n\tau+\alpha z)+\mathbf{e}(n\tau-\alpha z)\right)\right)\\
&-\frac{1}{p}\left(\left(\mathbb{D}^{k-1}p_1\right)(z)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha^{k-1}\left(\mathbf{e}\left(\frac{n\tau}{p}+\alpha z\right)+\mathbf{e}\left(\frac{n\tau}{p}-\alpha z\right)\right)\right).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\left(\widetilde{G}_{k,\chi_0}|_k g_1\right)(\tau)\\
=&\widetilde{G}_k(\tau)-\frac{1}{p}\widetilde{G}_k\left(\frac{\tau}{p}\right)\\
=&\left(\frac{1}{2}\zeta(1-k)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha^{k-1}\mathbf{e}(n\tau)\right)-\frac{1}{p}\left(\frac{1}{2}\zeta(1-k)+\sum_{n=1}^{\infty}\sum_{\alpha|n}\alpha^{k-1}\mathbf{e}\left(\frac{n\tau}{p}\right)\right).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\left((\phi|V_0)|_{k,0}C(g_1)\right)(\tau,z)\\
=&a_0(0)\left(\widetilde{G}_{k,\chi_0}|_k g_1\right)(\tau)+a_0(-1)\left(\left(\mathbb{D}^{k-1}P_{p,\chi_0}\right)|_{k,0}C(g_1)\right)(\tau,z)
\end{aligned}
$$

$$= \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k)+a_0(-1)\left(\mathbb{D}^{k-1}\,p_1\right)(z)\right)$$
$$+\left(\sum_{n=1}^{\infty}\sum_{\alpha\mid n}\alpha^{k-1}\left(a_0(0)\mathbf{e}(n\tau)+a_0(-1)\left(\mathbf{e}(n\tau+\alpha z)+\mathbf{e}(n\tau-\alpha z)\right)\right)\right)$$
$$-\frac{1}{p}\left(\sum_{n=1}^{\infty}\sum_{\alpha\mid n}\alpha^{k-1}\right.$$
$$\left.\left(a_0(0)\mathbf{e}\left(\frac{n\tau}{p}\right)+a_0(-1)\left(\mathbf{e}\left(\frac{n\tau}{p}+\alpha z\right)+\mathbf{e}\left(\frac{n\tau}{p}-\alpha z\right)\right)\right)\right)$$
$$= \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k)+a_0(-1)\left(\mathbb{D}^{k-1}\,p_1\right)(z)\right)$$
$$+\left(\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{\alpha\mid(n,l)}\alpha^{k-1}a_0\left(\frac{-l^2}{\alpha^2}\right)\mathbf{e}(n\tau+lz)\right)$$
$$-\frac{1}{p}\left(\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{\alpha\mid n}\alpha^{k-1}a_0\left(\frac{-l^2}{\alpha^2}\right)\mathbf{e}\left(\frac{n\tau}{p}+lz\right)\right).$$

Therefore we have

$$\left(\left(\mathrm{ML}(\phi)\right)|_k C\left(g_1\right)\right)(Z)$$
$$= \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k)+a_0(-1)\left(\mathbb{D}^{k-1}\,p_1\right)(z)\right)$$
$$+\sum_{(n,m)\in\mathbb{N}_0^2}{}^{\prime}\sum_{l\in\mathbb{Z}}c_2(n,l,m)\mathbf{e}\left(\frac{n\tau}{p}+lz+m\omega\right),$$

where $c_2(n,l,m)$ is defined by above (6.9). We put

$$\left(\left(\mathrm{ML}(\phi)\right)|_k C\left(g_1\right)\right)(Z) = \widetilde{\phi}(\omega,z)+F_\phi(Z),$$

where

$$\widetilde{\phi}(\omega,z) := \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k)+a_0(-1)\left(\mathbb{D}^{k-1}\,p_1\right)(z)\right)$$
$$+\sum_{m=1}^{\infty}\sum_{l\in\mathbb{Z}}c_2(0,l,m)\mathbf{e}\left(m\omega+lz\right)$$

and

$$F_\phi(Z) := \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l\in\mathbb{Z}} c_2(n,l,m)\mathbf{e}\left(\frac{n\tau}{p} + lz + m\omega\right).$$

As for $F_\phi$, we have

$$c_2(n,l,m) = c_2\left(n, l+2xn, m+xpl+x^2pn\right)$$

directly form the equation (6.9). Namely, $F_\phi(Z)$ is $R\left(\left(\begin{smallmatrix}1 & 0\\ px & 1\end{smallmatrix}\right)\right)$-invariant as a formal power series.

As for $\widetilde{\phi}(\omega,z)$, we have

$$\begin{aligned}
\widetilde{\phi}(\omega,z) = {} & \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k) + a_0(-1)\left(\mathbb{D}^{k-1}p_1\right)(z)\right) \\
& + \left(1-\frac{1}{p}\right)\sum_{m=1}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{\alpha|(m,l)} \alpha^{k-1}a_0\left(\frac{-l^2}{\alpha^2}\right)\mathbf{e}\left(mp\omega+lz\right) \\
& + \sum_{m=1}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{\alpha|(m,l)} \alpha^{k-1}a_1\left(\frac{-pl^2}{\alpha^2}\right)\mathbf{e}\left(m\omega+lz\right) \\
& - \sum_{m=1}^{\infty}\sum_{l\in\mathbb{Z}}\sum_{\alpha|(m,l)} \alpha^{k-1}a_1\left(\frac{-pl^2}{\alpha^2}\right)\mathbf{e}\left(mp\omega+lz\right) \\
= {} & \left(1-\frac{1}{p}\right)\left(\frac{a_0(0)}{2}\zeta(1-k) + a_0(-1)\left(\mathbb{D}^{k-1}p_1\right)(z)\right) \\
& + \left(1-\frac{1}{p}\right)a_0(0)\sum_{m=1}^{\infty}\sigma_{k-1}(m)\mathbf{e}\left(mp\omega\right) \\
& + \left(1-\frac{1}{p}\right)a_0(-1)\sum_{m=1}^{\infty}\sum_{\alpha|m}\alpha^{k-1}\left(\mathbf{e}\left(mp\omega+\alpha z\right)+\mathbf{e}\left(mp\omega-\alpha z\right)\right) \\
& + a_1(0)\sum_{m=1}^{\infty}\sigma_{k-1}(m)\mathbf{e}\left(m\omega\right) \\
& + a_1(-p)\sum_{m=1}^{\infty}\sum_{\alpha|m}\alpha^{k-1}\left(\mathbf{e}\left(m\omega+\alpha z\right)+\mathbf{e}\left(m\omega-\alpha z\right)\right) \\
& - a_1(0)\sum_{m=1}^{\infty}\sigma_{k-1}(m)\mathbf{e}\left(mp\omega\right) \\
& - a_1(-p)\sum_{m=1}^{\infty}\sum_{\alpha|m}\alpha^{k-1}\left(\mathbf{e}\left(mp\omega+\alpha z\right)+\mathbf{e}\left(mp\omega-\alpha z\right)\right).
\end{aligned}$$

Hence, when $k > 2$, we have

$$\begin{aligned}\widetilde{\phi}(\omega, z) =& a_0(0)\left(1-\frac{1}{p}\right)\widetilde{G}_k(p\omega) \\ &+ a_0(-1)\left(1-\frac{1}{p}\right)\frac{1}{(2\pi i)^2}\left(\mathbb{D}^{k-2}\wp(p\omega, z)\right) \\ &+ a_1(0)\left(\widetilde{G}_k(\omega)-\widetilde{G}_k(p\omega)\right) \\ &+ a_1(-p)\frac{1}{(2\pi i)^2}\left(\mathbb{D}^{k-2}\wp(\omega, z)-\mathbb{D}^{k-2}\wp(p\omega, z)\right)\end{aligned} \tag{6.10}$$

and when $k = 2$, we have

$$\begin{aligned}\widetilde{\phi}(\omega, z) =& a_0(0)\left(1-\frac{1}{p}\right)\widetilde{G}_2(p\omega) \\ &+ a_0(-1)\left(1-\frac{1}{p}\right)\frac{1}{(2\pi i)^2}\left(\wp(p\omega, z)+G_2(p\omega)\right) \\ &+ a_1(0)\left(\widetilde{G}_2(\omega)-\widetilde{G}_2(p\omega)\right) \\ &+ a_1(-p)\frac{1}{(2\pi i)^2}\left(\wp(\omega, z)+G_2(\omega)-\wp(p\omega, z)-G_2(p\omega)\right).\end{aligned} \tag{6.11}$$

Therefore, in either case, we have

$$\widetilde{\phi}(\omega, z) = \widetilde{\phi}(\omega, z + px\omega),$$

namely, $\widetilde{\phi}(\omega, z)$ is $R\left(\left(\begin{smallmatrix}1 & 0\\ px & 1\end{smallmatrix}\right)\right)$-invariant as a meromorphic function.

Now we recall that

$$\Delta_{10}^*(Z) := \left(\Delta_{10}(pZ)\right)|_k C\left(g_1\right) = p^{-10}\Delta_{10}\begin{pmatrix}\frac{\tau}{p} & z\\ z & p\omega\end{pmatrix}$$

is $R\left(\left(\begin{smallmatrix}1 & 0\\ px & 1\end{smallmatrix}\right)\right)$-invariant both as a formal power series and as a holomorphic function. Then $(\Delta_{10}^*(Z))^{\frac{k}{2}} F_\phi(Z)$ is $R\left(\left(\begin{smallmatrix}1 & 0\\ px & 1\end{smallmatrix}\right)\right)$-invariant as a formal power series and $(\Delta_{10}^*(Z))^{\frac{k}{2}} \widetilde{\phi}(\omega, z)$ is $R\left(\left(\begin{smallmatrix}1 & 0\\ p & 1\end{smallmatrix}\right)\right)$-invariant as a holomorphic function. Hence, by using the uniqueness of the Fourier expansion, we have

$$(\Delta_{10}^*(Z))^{\frac{k}{2}}\left(\left(\mathrm{ML}(\phi)\right)|_k C\left(g_1\right)\right)(Z) = (\Delta_{10}^*(Z))^{\frac{k}{2}} F_\phi(Z) + (\Delta_{10}^*(Z))^{\frac{k}{2}}\widetilde{\phi}(\omega, z)$$

is $R\left(\left(\begin{smallmatrix}1 & 0\\ px & 1\end{smallmatrix}\right)\right)$-invariant as a formal power series. Here we complete the proof.

**Remark.** When $p = 2, 3$, since $\mathrm{ML}(\phi)$ is a meromorphic Siegel modular form in actual fact, $\widetilde{\phi}(\omega, z)$ should be not only $R\left(\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)\right)$-invariant but also $SC\left(\Gamma_0^{(1)}(p)\right)$ $S$-invariant. For $k > 2$, it is almost trivial form the equation (6.10). However, for $k = 2$, it may not be trivial from the equation (6.11). Here we calculate the equation (6.11) more precisely. By the equation (4.4), we have

$$\mathbb{J}_{2,1}^{\mathrm{weak}}(p; \chi_0) = \mathbb{M}_4^{(1)}(p; \chi_0)\phi_{-2,1} \oplus \mathbb{M}_2^{(1)}(p; \chi_0)\phi_{0,1}.$$

Here $\phi_{-2,1}(\tau, 0) = 0$ and $\dim_{\mathbb{C}} \mathbb{M}_2^{(1)}(p; \chi_0) = 1$ when $p = 2, 3$, namely $\mathbb{M}_2^{(1)}(p; \chi_0) = \mathbb{C}\widetilde{G}_{2,\chi_0}(\tau)$. Since

$$\widetilde{G}_{2,\chi_0}(\tau) = \widetilde{G}_2(\tau) - p\widetilde{G}_2(p\tau)$$

and

$$\left(\widetilde{G}_{2,\chi_0}|_2 g_1\right)(\tau) = \widetilde{G}_2(\tau) - \frac{1}{p}\widetilde{G}_2\left(\frac{\tau}{p}\right),$$

we have

$$\lim_{\tau \to i\infty} \widetilde{G}_{2,\chi_0}(\tau) = -p \lim_{\tau \to i\infty} \left(\widetilde{G}_{2,\chi_0}|_2 g_1\right)(\tau)$$

and therefore

$$a_0(0) + 2a_0(-1) = -p\left(a_1(0) + 2a_1(-p)\right).$$

Hence we have

$$\begin{aligned}
\widetilde{\phi}(\omega, z) =& \left(a_0(0) + 2a_0(-1)\right)\left(1 - \frac{1}{p}\right)\widetilde{G}_2(p\omega) \\
&+ a_0(-1)\left(1 - \frac{1}{p}\right)\frac{1}{(2\pi i)^2}\wp(p\omega, z) \\
&+ \left(a_1(0) + 2a_1(-p)\right)\left(\widetilde{G}_2(\omega) - \widetilde{G}_2(p\omega)\right) \\
&+ a_1(-p)\frac{1}{(2\pi i)^2}\left(\wp(\omega, z) - \wp(p\omega, z)\right) \\
=& a_0(-1)\left(1 - \frac{1}{p}\right)\frac{1}{(2\pi i)^2}\wp(p\omega, z) \\
&+ \left(a_1(0) + 2a_1(-p)\right)\left(\widetilde{G}_2(\omega) - p\widetilde{G}_2(p\omega)\right) \\
&+ a_1(-p)\frac{1}{(2\pi i)^2}\left(\wp(\omega, z) - \wp(p\omega, z)\right),
\end{aligned}$$

which is $SC\left(\Gamma_0^{(1)}(p)\right)$ $S$-invariant.

### 6.3. Proof of Theorem 13

The idea of the proof of Theorem 13 is essentially given in [Ao2]. Here we will show Theorem 13 according to [Ao2]. For $k \in \mathbb{Z}$, we define

$$\mathbb{FW}_k(N;\chi) := \left\{ (f_m)_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathbb{M}_k^{(1)}(N;\chi) \;\middle|\; \text{(Sym)} \right\}$$

where (Sym) means the following condition:

$$\text{(Sym)} : \quad \begin{array}{c}\text{On the Fourier expansion } f_m(\tau) = \sum_{n=0}^{\infty} a_0(n,m)\mathbf{e}(n\tau)\,, \\ a_0(n,m) = (-1)^k \chi(-1) a_0(m,n) \text{ holds for any } n,m \in \mathbb{Z}.\end{array}$$

Since we have

$$\mathbb{FW}_k(N;\chi) = \begin{cases} S^2\left(\mathbb{M}_k^{(1)}(N;\chi)\right) \text{ (symmetric tensor)} & ((-1)^k\chi(-1) = 1) \\ A^2\left(\mathbb{M}_k^{(1)}(N;\chi)\right) \text{ (alternating tensor)} & ((-1)^k\chi(-1) = -1) \end{cases},$$

we can determine the structure of $\mathbb{FW}_k(N;\chi)$. For $r \in \mathbb{N} \cup \{0\}$, we define a map $\mathbb{D}_r$ by

$$\mathbb{D}_r \;:\; \mathbb{FM}_k^+(N;\chi) \ni (\phi_m(\tau,z))_{m=0}^{\infty} \longmapsto ((\mathbb{D}^r\,\phi_m)(\tau,0))_{m=0}^{\infty} \in \prod_{m=0}^{\infty} \mathrm{Hol}(\mathbb{H}),$$

and then define a space $\mathbb{FM}_k^+(N;\chi)[r]$ by

$$\mathbb{FM}_k^+(N;\chi)[r] := \left\{ F \in \mathbb{FM}_k^+(N;\chi) \;\middle|\; \begin{array}{c}\mathbb{D}_s F = 0 \\ \text{for any } 0 \leqq s < r\end{array} \right\}.$$

When $r = 0$, $\mathbb{D}_0$ is a restriction map

$$\mathbb{D}_0 : \mathbb{FM}_k^+(N;\chi) \xrightarrow{z=0} \mathbb{FW}_k(N;\chi)$$

and $\mathbb{FM}_k^+(N;\chi)[0] = \mathbb{FM}_k^+(N;\chi)$. Easily we have

$$\mathbb{D}_r \;:\; \mathbb{FM}_k^+(N;\chi)[r] \longrightarrow \mathbb{FW}_{k+r}(N;\chi)$$

and the kernel of this linear map $\mathbb{D}_r$ is

$$\ker \mathbb{D}_r = \mathbb{FM}_k^+(N;\chi)[r+1].$$

Hence we have

$$\dim_{\mathbb{C}} \mathbb{FM}_k^+(N;\chi) = \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \left(\mathbb{D}_r \left(\mathbb{FM}_k^+(N;\chi)[r]\right)\right). \tag{6.12}$$

The idea used in [Ao2] is to estimate the right hand side of (6.12) cleverly.

Since our proof is a slight modification of the method used in [Ao2], here we give a proof only for the case $N = p \in \{2,3\}, \chi = \chi_0$ and $k \in 2\mathbb{Z}$. Continuing from the previous subsection, we put

$$g_0 := E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $r \in \mathbb{N} \cup \{0\}$, we define

$$\mathbb{FW}_k(p;\chi)[r] := \left\{ (f_m)_{m=0}^{\infty} \in \mathbb{FW}_k(p;\chi) \;\middle|\; (\mathrm{FW})_r \right\},$$

where $(\mathrm{FW})_r$ means the following condition:

$(\mathrm{FW})_r$ : On the Fourier expansion $f_m(\tau) = \sum_{n=0}^{\infty} a_0(n,m)\mathbf{e}(n\tau)$ and $(f_m|_k C(g_1))(\tau) = \sum_{n=0}^{\infty} a_1(n,m)\mathbf{e}\left(\frac{n\tau}{p}\right)$, $a_0(n,m) = 0$ and $a_1(n,m) = 0$ holds if $\min\{n,m\} < r$.

For $r \in \mathbb{N} \cup \{0\}$ and

$$F := (\phi_m)_{m=0}^{\infty} \in \mathbb{FM}_k^+(p;\chi_0),$$

we put

$$(f_m^r)_{m=0}^{\infty} := \mathbb{D}^r(F).$$

We put their Fourier coefficients

$$\begin{aligned}
\phi_m(\tau,z) = (\phi_m|_{k,m} C(g_0))(\tau,z) &= \sum_{n,l \in \mathbb{Z}^2} c_0(n,l,m)\mathbf{e}(n\tau + lz), \\
(\phi_m|_{k,m} C(g_1))(\tau,z) &= \sum_{n,l \in \mathbb{Z}^2} c_1(n,l,m)\mathbf{e}\left(\frac{n\tau}{p} + lz\right), \\
f_m(\tau,z) = (f_m|_{k+r} C(g_0))(\tau,z) &= \sum_{n \in \mathbb{Z}} a_0^r(n,m)\mathbf{e}(n\tau), \\
(f_m|_{k+r} C(g_1))(\tau,z) &= \sum_{n \in \mathbb{Z}} a_1^r(n,m)\mathbf{e}\left(\frac{n\tau}{p}\right).
\end{aligned}$$

Then we have

$$a_0^r(n,m) = \sum_{l\in\mathbb{Z}} c_0(n,l,m)l^r \tag{6.13}$$

and

$$a_1^r(n,m) = \sum_{l\in\mathbb{Z}} c_1(n,l,m)l^r, \tag{6.14}$$

where we consider $l^r = 1$ when $l = r = 0$. The following lemma is immediately induced from $F \in \mathbb{FM}_k^+(p;\chi_0)$.

**Lemma 26.** *We have*

$$c_0(n,l,m) = c_0(n,-l,m), \tag{6.3}$$
$$c_0(n,l,m) = c_0(n+xl+x^2m, l+2xm, m), \tag{6.4}$$
$$c_0(n,l,m) = c_0(m,l,n), \tag{6.5}$$
$$c_1(n,l,m) = c_1(n,-l,m), \tag{6.6}$$
$$c_1(n,l,m) = c_1\left(n+xpl+x^2pm, l+2xm, m\right) \tag{6.7}$$
$$\text{and}\quad c_1(n,l,m) = c_1\left(n, l+2xn, m+xpl+x^2pn\right) \tag{6.8}$$

*for any $n,l,m,x \in \mathbb{Z}$.*

Hence the following proposition holds.

**Proposition 27.** *When $r$ is odd, we have $\mathbb{D}_r\left(\mathbb{FM}_k^+(p;\chi_0)[r]\right) = \{0\}$.*

*Proof.* By the equations (6.3) and (6.13), we have $a_0^r(n,m) = 0$ when $r$ is odd. By the equations (6.6) and (6.14), we have $a_1^r(n,m) = 0$ when $r$ is odd. □

The following lemma is induced from lemma 26 in the same way as we did in the previous section.

**Lemma 28.** *The following two properties hold.*

- *If $(n,l,m) \in \mathbb{Z}^3$ satisfies $4nm - l^2 \geqq 0$ and $|l| > \min\{m,n\}$, then there exists $(n',l',m') \in \mathbb{Z}^3$ such that $c_0(n,l,m) = c_0(n',l',m')$ and $\min\{m,n\} > \min\{m',n'\}$.*
- *If $(n,l,m) \in \mathbb{Z}^3$ satisfies $4nm - pl^2 \geqq 0$ and $|l| > \min\{m,n\}$, then there exists $(n',l',m') \in \mathbb{Z}^3$ such that $c_1(n,l,m) = c_1(n',l',m')$ and $\min\{m,n\} > \min\{m',n'\}$.*

Hence the following proposition holds.

**Proposition 29.** *Let $r \in \mathbb{N} \cup \{0\}$ and $F \in \mathbb{FM}_k^+(p;\chi_0)[2r]$. If $\min\{n,m\} < r$, then we have $c_0(n,l,m) = c_1(n,l,m) = 0$. Hence we have*

$$\mathbb{D}_{2r}\left(\mathbb{FM}_k^+(p;\chi_0)[2r]\right) \subset \mathbb{FW}_{k+2r}(p;\chi_0)[r]$$

*Proof.* We will give a proof by induction on $r$. When $r = 0$, the assertion is trivial. When $r = 1$, if $n = 0$ or if $m = 0$, first we have $c_0(n,l,m) = 0$ and $c_1(n,l,m) = 0$ unless $l = 0$, since $F \in \mathbb{FM}_k^+(p;\chi_0)$. Then, since the equations (6.13) and (6.14) holds for $r = 0$, we have $c_0(n,0,m) = 0$ and $c_1(n,0,m) = 0$. Therefore we assume that the assertion holds for $r$ and prove that it also holds for $r+1$. First, by the assumption, we have $c_0(n,l,m) = c_1(n,l,m) = 0$ when $\min\{n,m\} < r$. Then, by lemma 28, we have $c_0(n,l,m) = 0$ and $c_1(n,l,m) = 0$ unless $l \leqq r$, when $\min\{n,m\} = r$. Hence, by the equations (6.3), (6.4), (6.13) and (6.14),

$$\sum_{l=-r}^{r} c_0(n,l,m) l^s = 0 \qquad\qquad c_0(n,l,m) = c_0(n,-l,m)$$

and

$$\sum_{l=-r}^{r} c_1(n,l,m) l^s = 0 \qquad\qquad c_1(n,l,m) = c_1(n,-l,m)$$

holds for $s = 0, 2, 4, \ldots, 2r$. Therefore, by Vthe andermonde's determinant formula, we have $c_0(n,l,m) = 0$ and $c_1(n,l,m) = 0$ when $\min\{m,n\} = r$. ☐

Up to this point, we have the following estimation

**Proposition 30.** *We have the exact sequence*

$$0 \to \mathbb{FM}_k^+(p;\chi_0)[2(r+1)] \hookrightarrow \mathbb{FM}_k^+(p;\chi_0)[2r] \xrightarrow{\mathbb{D}_{2r}} \mathbb{FW}_{k+2r}(p;\chi_0)[r]$$

*for $r \in \{0\} \cup \mathbb{N}$. Hence we have*

$$\dim_{\mathbb{C}} \mathbb{FM}_k^+(p;\chi_0) \leqq \sum_{r=0}^{\infty} \dim_{\mathbb{C}}\left(\mathbb{FW}_{k+2r}(p;\chi_0)[r]\right). \tag{6.15}$$

We can calculate the right hand side of the inequality (6.15) explicitly. Actually, when $p \in \{2,3\}$, by using Maass lifts, we have

$$\sum_{r=0}^{\infty} \dim_{\mathbb{C}}\left(\mathbb{FW}_{k+2r}(p;\chi_0)[r]\right) = \dim_{\mathbb{C}} \mathbb{M}_k^{(2)}(p;\chi_0).$$

Since $\dim_{\mathbb{C}} \mathbb{M}_k^{(2)}(p;\chi_0) \leqq \dim_{\mathbb{C}} \mathbb{FM}_k^{+}(p;\chi_0)$, we have

$$\dim_{\mathbb{C}} \mathbb{FM}_k^{+}(p;\chi_0) = \dim_{\mathbb{C}} \mathbb{M}_k^{(2)}(p;\chi_0),$$

that is the goal of this subsection.

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