Pure and Applied Mathematics Quarterly
Volume 16, Number 3, 557–572, 2020

On the area formulas of inscribed polygons in classical geometry

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Dedicated to Prof. Kyoji Saito on the occasion of his 75th birthday

Abstract: We show that there is no area formula of the general inscribed n-gon for $n \geq 5$ only by using arithmetic operations and k-th roots of its side lengths in classical geometry.

Keywords: Euclidean geometry, hyperbolic geometry, spherical geometry.

1. Introduction

For any triangle on the Euclidean plane, its congruence class is uniquely determined by its side lengths. Therefore the area, which is an invariant of the congruence class, can be written in terms of the side lengths, known as Heron's formula (see Step 1 in Section 2). On the other hand, like rhombi, the congruence class of a quadrilateral is not uniquely determined by its side lengths. When we confine ourselves to a quadrilateral inscribed in a round circle, its congruence class is uniquely determined by its side lengths, and there is a area formula in terms of its side lengths knows as Brahmagupta's formula (see Step 1 in Section 2). In general, the side lengths of a convex n-gon inscribed in a circle, which we will call *a cyclic n-gon*, determine its congruence class uniquely so that there should be an area formula in terms of its side lengths. For this problem, Matsumoto et al. [2] proved the following result:

Theorem 1 ([2, Theorem 1]). *In Euclidean geometry, there is no area formula of the general cyclic n-gon for $n \geq 5$ in terms of its side lengths by using only four arithmetic operations of addition, subtraction, multiplication and division, and k-th roots.*

Received June 20, 2019.
2010 Mathematics Subject Classification: Primary 51Kxx, 51M10; secondary 51N20.

This result recalls the Abel-Ruffini theorem that there is no formula of a solution of the general polynomial of degree n for $n \geq 5$ in terms of its coefficients, using only arithmetic operations and k-th roots [5]. In this paper we show that the similar result also holds for other classical geometry, namely hyperbolic and spherical geometry:

Theorem 2. *Let S be the area of a cyclic n-gon for $n \geq 5$ in hyperbolic and spherical geometry whose side lengths are $a_1, a_2, \cdots .a_n$. Then there is no formula of $\cos \frac{S}{2}$ in terms of $s(a_1), s(a_2), \cdots, s(a_n)$, only by using arithmetic operations and k-th roots, where $s(x)$ is equal to $\sinh \frac{x}{2}$ for hyperbolic geometry while it is equal to $\sin \frac{x}{2}$ for spherical geometry.*

The content of this paper is as follows; in Section 2 we will recall the idea of Matsumoto et al. for Euclidean geometry step by step. After considering the comparison between Euclidean and hyperbolic distances, we will check that the argument of Matsumoto et al. also works for hyperbolic geometry in Section 3. In Section 4 we will treat the case of spherical geometry.

2. On the area of a Euclidean cyclic polygon

In this Section, we review the proof due to Matsumoto et al. [2] that there is no area formula for a Euclidean cyclic n-gon for $n \geq 5$ in terms of the four arithmetic operations and k-th roots of its side lengths step by step.

Step 1

There are well-known area formulas for a triangle and a cyclic quadrilateral; Heron's formula: The area S of a triangle with its side lengths a, b and c can be written as

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$.
Brahmagupta's formula: The area S of a cyclic quadrilateral with its side lengths a, b, c and d can be written as

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{a+b+c+d}{2}$.

For a pentagon $ABCDE$ in Figure 1, let us denote the length of the diagonal AC by x. Since the area of the pentagon is the sum of those of the

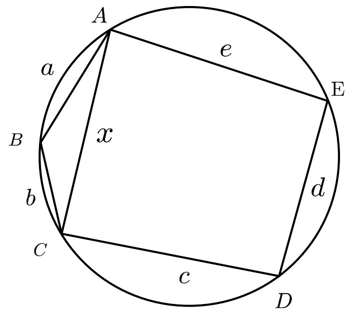

Figure 1: The inscribed pentagon ABCDE.

triangle ABC and the quadrilateral $ACDE$, we have the following equation by means of Heron's formula and Brahmagupta's formula:

$$
\begin{aligned}
&2(a^2+b^2-c^2-d^2-e^2)x^2-8cdex+16S^2-a^4-b^4+c^4+d^4+e^4\\
&-2(-a^2b^2+c^2d^2+d^2e^2+e^2c^2)=8S\sqrt{-x^4+2(a^2+b^2)x^2-(a^2-b^2)^2},
\end{aligned}
$$

which implies that the diagonal length x satisfies a quartic equation whose coefficients belong to $\mathbb{Q}(a,b,c,d,e,S)$ [2, lemma1]. From Galois theory [5], we see that x can be expressed by the four arithmetic operations and k-th roots of a,b,c,d,e,S.

Step 2

We denote the lengths of the diagonals AC and AD of a pentagon in Figure 2 by x and y respectively.

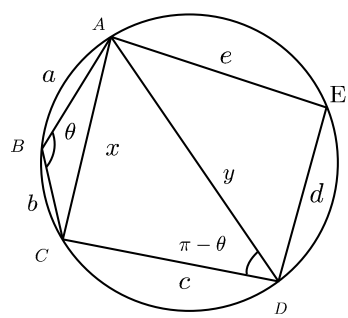

Figure 2: The inscribed pentagon ABCDE.

The inscribed quadrilateral $ABCD$ is decomposed into two triangles ABC and ACD. If we denote the angle $\angle ABC$ by θ, then $\angle ADC = \pi - \theta$, and by using the cosine formula for triangles ABC and ACD, we have

$$x^2 = a^2 + b^2 - 2ab\cos\theta = c^2 + y^2 - 2cy\cos(\pi - \theta) = c^2 + y^2 + 2cy\cos\theta.$$

Eliminating $cos\theta$, we get [2, lemma2]

$$x^2 = \frac{(a^2+b^2)cy + (c^2+y^2)ab}{ab+cy}. \tag{1}$$

Similarly considering the quadrilateral $ACDE$, we have

$$y^2 = \frac{(x^2+c^2)de + (d^2+e^2)cx}{cx+de}. \tag{2}$$

Eliminating y in (1) and (2), the diagonal length x satisfies the following polynomial equation of degree 7 whose coefficients belong to $\mathbb{Q}(a, b, c, d, e)$:

(3)
$$\begin{aligned}
&cdex^7 + (c^2d^2 + d^2e^2 + e^2c^2 - a^2b^2)x^6 \\
&+ cde\{(c^2 + d^2 + e^2) - 2(a^2 + b^2)\}x^5 \\
&+ \{c^2d^2e^2 + 2a^2b^2(c^2 + d^2 + e^2) - 2(a^2 + b^2)(c^2d^2 + d^2e^2 + e^2c^2)\}x^4 \\
&+ cde\{(a^2 + b^2)^2 + 4a^2b^2 - 2(a^2 + b^2)(c^2 + d^2 + e^2)\}x^3 \\
&+ \{(a^2 + b^2)^2(c^2d^2 + d^2e^2 + e^2c^2) - 2c^2d^2e^2(a^2 + b^2) \\
&- a^2b^2(c^2 + d^2 + e^2)^2\}x^2 + cde(c^2 + d^2 + e^2)(a^2 - b^2)^2x + c^2d^2e^2(a^2 - b^2)^2 \\
&= 0.
\end{aligned}$$

Assuming $a = b$, the above equation (3) reduces to the following quintic equation [2, lemma2]:

$$\begin{aligned}
&cdex^5 + (c^2d^2 + d^2e^2 + e^2c^2 - a^4)x^4 \\
&+ cde\{(c^2 + d^2 + e^2) - 4a^2\}x^3 \\
&+ \{c^2d^2e^2 + 2a^4(c^2 + d^2 + e^2) - 4a^2(c^2d^2 + d^2e^2 + e^2c^2)\}x^2 \\
&+ 4a^2cde\{(2a^2 - (c^2 + d^2 + e^2)\}x \\
&+ a^2\{4a^2(c^2d^2 + d^2e^2 + e^2c^2) - 4c^2d^2e^2 - a^2(c^2 + d^2 + e^2)^2\} = 0.
\end{aligned} \tag{4}$$

Step 3

For a cyclic pentagon with side lengths $(a, b, c, d, e) = (1, 1, 2, 3, 4)$ (for the existence of such a pentagon, see [2, Proposition 4]), the diagonal length x becomes a solution of the following quintic equation reduced from (4):

$$f(x) = 8x^5 + 81x^4 + 200x^3 - 114x^2 - 864x - 723 = 0. \tag{5}$$

Then $f(x)$ is irreducible over $\mathbb{Q}$ whose Galois group is isomorphic to the symmetric group S_5 of five letters which is not solvable [2, lemma3].

Now we can prove Theorem 1 for $n = 5$ as follows. Suppose that the area S of a cyclic pentagon can be written in terms of the four arithmetic operations and k-th roots of its side lengths. Then x could also be calculated in the same way from the side lengths because of Step 1. Applying Galois theory [5], we see that Galois group of the minimal polynomial of x over $Q(a, b, c, d, e)$ is solvable, which contradicts that the Galois group of the equation (5) is isomorphic to S_5.

Step 4

For $n \geq 6$, suppose that there is an area formula $F(a_1, a_2, \cdots, a_n)$ for a generic cyclic n-gon in terms of the four arithmetic operations and k-th roots of its side lengths $a_1, a_2, \cdots, a_n$. Then there exists $\varepsilon > 0$ such that for any t satisfying $0 < t < \varepsilon$, there is a cyclic n-gon with side lengths $a_1 = a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 4, a_6 = t, \cdots, a_n = t$ [2, Proposition 4]. If we denote the area formula $F(1, 1, 2, 3, 4, t, \cdots, t)$ by $F(t)$ for simplicity, then $\lim_{t \to +0} F(t) = S_0$ holds where S_0 is the area of the cyclic pentagon with side lengths $1, 1, 2, 3, 4$ [2, Proposition 2]. If we consider t as a complex variable, $F(t)$ can be considered as a single-valued algebraic function on a domain in $\mathbb{C}$ containing the open interval $(0, \varepsilon)$. Then $F(t)$ can be written as the fraction of Puiseux series like $F(t) = \dfrac{c_0 + c_1 t^{\frac{1}{p}} + c_2 t^{\frac{2}{p}} + \cdots}{d_0 + d_1 t^{\frac{1}{q}} + c_2 t^{\frac{2}{q}} + \cdots}$, which implies $\lim_{t \to +0} F(t) = \frac{c_0}{d_0} = S_0$ [2, Proposition 2]. From the assumption, the area $F(t)$ can be written by the four arithmetic operations and k-th roots of its side lengths, hence the coefficients c_i, d_j of the numerator and the denominator of $F(t)$ are elements of a radical extension of $\mathbb{Q}$. Therefore $S_0 = \frac{c_0}{d_0}$ belongs to a radical extension of $\mathbb{Q}$ which contradicts to the conclusion for a pentagon in Step 3. To summarize, for $n \geq 6$, there is no area formula for a generic Euclidean cyclic n-gon in terms of the four arithmetic operations and k-th roots of its side lengths.

3. On the area of a hyperbolic cyclic polygon

Relations between Euclidean and hyperbolic distances

We take the unit disk $\mathbb{D}$ in $\mathbb{R}^2$ as a model of hyperbolic geometry. Let C be the Euclidean circle of radius $r_e < 1$ with its center at the origin O in $\mathbb{D}$. C can be also considered as the hyperbolic circle whose radius is denoted by r_h. Then we have the following relation between r_e and r_h [1]:

$$r_e = \tanh \frac{r_h}{2}. \tag{6}$$

Proposition 1. *For $P, Q \in C$, the Euclidean distance $d_e(P,Q)$ and the hyperbolic distance $d_h(P,Q)$ between P and Q satisfy*

$$s(d_h(P,Q)) = \sinh \frac{d_h(P,Q)}{2} = \frac{1}{1-r_e^2} d_e(P,Q). \tag{7}$$

By means of (6), it can be rewritten as

$$d_e(P,Q) = (1 - \tanh^2 \frac{r_h}{2}) s(d_h(P,Q)). \tag{8}$$

Proof. First we consider the Euclidean triangle OPQ and the hyperbolic triangle OPQ.

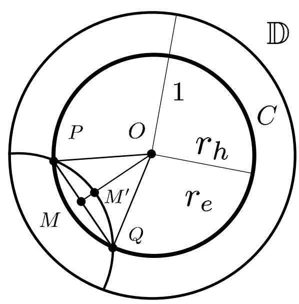

Figure 3: The Euclidean resp. hyperbolic triangle OPQ.

Let M and M' be the points of the Euclidean segment PQ and the hyperbolic segment PQ such that OM and OM' are perpendicular to these segments respectively. For the hyperbolic triangle $OM'Q$, the hyperbolic sine formula implies

$$\frac{\sinh r_h}{\sin \frac{\pi}{2}} = \frac{\sinh \frac{d_h(P,Q)}{2}}{\sin \frac{\angle POQ}{2}},$$

while for the Euclidean triangle OMQ the Euclidean sine formula implies

$$\frac{r_e}{\sin\frac{\pi}{2}} = \frac{\frac{d_e(P,Q)}{2}}{\sin\frac{\angle POQ}{2}}.$$

Eliminating $\sin\frac{\angle POQ}{2}$ in the above equations, we have

$$\sinh\frac{d_h(P,Q)}{2} = \frac{\sinh r_h}{2r_e} \times d_e(P,Q).$$

Also from (6) we have

$$\frac{\sinh r_h}{2r_e} = \frac{2\sinh\frac{r_h}{2}\cosh\frac{r_h}{2}}{2\tanh\frac{r_h}{2}} = \cosh^2\frac{r_h}{2} = \frac{1}{1-\tanh^2\frac{r_h}{2}} = \frac{1}{1-r_e^2}.$$

Hence we get

$$\sinh\frac{d_h(P,Q)}{2} = \frac{1}{1-r_e^2}d_e(P,Q). \qquad \square$$

For n points $A_1, A_2, \cdots, A_n$ of C in cyclic order, let a_k^e and a_k^h be the Euclidean and the hyperbolic distances between A_k and A_{k+1} $(k = 1, 2, \cdots, n)$ where we assume that $A_{n+1} = A_1$. Proposition 1 implies the following identities:

$$s(a_k^h) = \sinh\frac{a_k^h}{2} = \frac{1}{1-r_e^2}a_k^e,\ \ a_k^e = (1-\tanh^2\frac{r_h}{2})s(a_k^h)\ \ (k = 1, 2, \cdots, n).$$

Next result is due to Matsumoto et al. [2].

Proposition 2 ([2, Proposition 4])**.** *For positive numbers $a_1, a_2, \cdots, a_n$ satisfying that a_n is the largest, there is a Euclidean cyclic n-gon whose side lengths are $a_1, a_2, \cdots, a_n$ if and only if the following inequality holds:*

$$\sum_{k=1}^{n-1} a_k/a_n > 1.$$

The similar statement also holds for hyperbolic geometry.

Proposition 3. *For positive numbers $a_1, a_2, \cdots, a_n$ satisfying that a_n is the largest, there is a hyperbolic cyclic n-gon whose side lengths are $a_1, a_2, \cdots, a_n$ if and only if the following inequality holds:*

$$\sum_{k=1}^{n-1} s(a_k)/s(a_n) > 1. \tag{9}$$

Proof. We assume the existence of a hyperbolic n-gon inscribed in a circle of hyperbolic radius r_h whose hyperbolic side lengths are $a_1^h, a_2^h, \cdots, a_n^h$. Applying a hyperbolic isometry we may assume that the center of the circumscribed circle is the origin O in $\mathbb{D}$. Let $a_1^e, a_2^e, \cdots, a_n^e$ be the Euclidean side lengths of the Euclidean cyclic n-gon sharing vertices of this hyperbolic n-gon. Then from (8) and Proposition 2, we have

$$\sum_{k=1}^{n-1} s(a_k^h)/s(a_n^h) = \sum_{k=1}^{n-1} a_k^e/a_n^e > 1.$$

On the other hand, suppose that positive real numbers $a_1^h, a_2^h, \cdots, a_n^h$ satisfy the inequality

$$\sum_{k=1}^{n-1} s(a_k^h)/s(a_n^h) > 1$$

under the assumption that a_n^h is the largest among $a_1^h, a_2^h, \cdots, a_n^h$. Proposition 2 implies that there is a Euclidean cyclic n-gon whose side lengths are $s(a_1^h), s(a_2^h), \cdots, s(a_n^h)$. Denote the Euclidean radius of the circumscribed circle of this Euclidean n-gon by r_e. For any $\lambda > 0$, there is a Euclidean cyclic n-gon with the circumscribed circle of radius λr_e, whose side lengths are $\lambda s(a_1^h), \lambda s(a_2^h), \cdots, \lambda s(a_n^h)$ by similarity. Now for any λ satisfying $0 < \lambda < 1/r_e$, this Euclidean cyclic n-gon inscribes the circle of Euclidean radius $\lambda r_e < 1$. In particular for $\lambda_0 = \frac{\sqrt{1+4r_e^2}-1}{2r_e^2}$ we have

$$\frac{1}{1-(\lambda_0 r_e)^2}\lambda_0 s(a_k^h) = s(a_k^h).$$

Therefore by (7), there exists the hyperbolic cyclic n-gon having sides lengths $a_1^h, a_2^h, \cdots, a_n^h$ which shares vertices of this Euclidean cyclic n-gon. □

Corollary 1.

1. *There is a hyperbolic cyclic pentagon with side lengths* $a_1, a_2, \cdots, a_5$ *satisfying* $s(a_1) = s(a_2) = 1, s(a_3) = 2, s(a_4) = 3$ *and* $s(a_5) = 4$.
2. *For any* t *satisfying* $0 < t < 4$, *there is a hyperbolic cyclic* n*-gon with side lengths* $a_1, a_2, \cdots, a_n$ *satisfying* $s(a_1) = s(a_2) = 1, s(a_3) = 2, s(a_4) = 3, s(a_5) = 4, s(a_6) = t, \cdots, s(a_n) = t$.

In the following we will see the arguments in Section 2 also hold for hyperbolic geometry.

Step 1

The next formula for the area S of a hyperbolic triangle with side lengths a, b, c is a hyperbolic analog of Heron's formula [4]:

$$\cos\frac{S}{2} = \frac{1+\frac{1}{2}(s(a)^2+s(b)^2+s(c)^2)}{\sqrt{1+s(a)^2}\sqrt{1+s(b)^2}\sqrt{1+s(c)^2}} \tag{10}$$

where $s(x) = \sinh\frac{x}{2}$. Similarly the next formula for the area S of a hyperbolic cyclic quadrilateral with side lengths a, b, c, d is a hyperbolic analog of Brahmagupta's formula [4, Theorem 3.4.]:

$$\cos\frac{S}{2} = \frac{1+\frac{1}{2}(s(a)^2+s(b)^2+s(c)^2+s(d)^2)-s(a)s(b)s(c)s(d)}{\sqrt{1+s(a)^2}\sqrt{1+s(b)^2}\sqrt{1+s(c)^2}\sqrt{1+s(d)^2}}. \tag{11}$$

Let S be the area of a hyperbolic cyclic pentagon $ABCDE$ with side lengths a, b, c, d, e, and x be the hyperbolic length of the diagonal AC. We denote the areas of the hyperbolic triangle ABC and the hyperbolic cyclic quadrilateral $ACDE$ by S_1 and S_2 respectively. Then the additivity of areas $S = S_1 + S_2$ and the addition formula of cosine show

$$\cos\frac{S}{2} - \cos\frac{S_1}{2}\cos\frac{S_2}{2} = \sin\frac{S_1}{2}\sin\frac{S_2}{2}.$$

By squaring both sides, we have

$$\cos^2\frac{S_1}{2} + \cos^2\frac{S_2}{2} = 1 - \cos^2\frac{S}{2} + 2\cos\frac{S}{2}\cos\frac{S_1}{2}\cos\frac{S_2}{2}.$$

Applying (10) and (11) to $\cos\frac{S_1}{2}$ and $\cos\frac{S_2}{2}$, the following equation holds:

$$\begin{aligned}
& (s(a)^2+s(b)^2+s(x)^2+2)^2(1+s(c)^2)(1+s(d)^2)(1+s(e)^2) \\
+ \ & (s(x)^2+s(c)^2+s(d)^2+s(e)^2+2-2s(x)s(c)s(d)s(e))^2 \\
& (1+s(a)^2)(1+s(b)^2) \\
= \ & 4(1-\cos^2\frac{S}{2})(1+s(x)^2)(1+s(a)^2)(1+s(b)^2)(1+s(c)^2) \\
& (1+s(d)^2)(1+s(e)^2) \\
+ \ & 2\cos\frac{S}{2}(s(a)^2+s(b)^2+s(x)^2+2) \\
& (s(x)^2+s(c)^2+s(d)^2+s(e)^2+2-2s(x)s(c)s(d)s(e)) \\
& \sqrt{1+s(a)^2}\sqrt{1+s(b)^2}\sqrt{1+s(c)^2}\sqrt{1+s(d)^2}\sqrt{1+s(e)^2}.
\end{aligned}$$

Therefore $s(x)$ is a solution of the quartic equation whose coefficients belong to $\mathbb{Q}(s(a), s(b), s(c), s(d), s(e), \sqrt{1+s(a)^2}, \sqrt{1+s(b)^2}, \sqrt{1+s(c)^2}, \sqrt{1+s(d)^2}, \sqrt{1+s(e)^2}, \cos\frac{S}{2})$. From Galois theory $s(x)$ can be written in terms of arithmetic operations and k-th roots of $s(a), s(b), s(c), s(d), s(e)$ and $\cos\frac{S}{2}$.

Step 2

After replacing a, b, c, d, e, x in (3) with $a_e, b_e, c_e, d_e, e_e, x_e$, and multiplying $(\frac{1}{1-r_e^2})^{10}$ to (3), from (7) we have

$$\begin{aligned}
&s(c)s(d)s(e)s(x)^7 + (s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2 - s(a)^2s(b)^2)s(x)^6\\
&+ s(c)s(d)s(e)\{(s(c)^2 + s(d)^2 + s(e)^2) - 2(s(a)^2 + s(b)^2)\}s(x)^5\\
&+ \{s(c)^2s(d)^2s(e)^2 + 2s(a)^2s(b)^2(s(c)^2 + s(d)^2 + s(e)^2) - 2(s(a)^2 + s(b)^2)\\
&(s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2)\}s(x)^4 + 4s(a)^2s(b)^2\\
&+ s(c)s(d)s(e)\{(s(a)^2 + s(b)^2)^2x - 2(s(a)^2 + s(b)^2)(s(c)^2 + s(d)^2 + s(e)^2)\}\\
&s(x)^3 + \{(s(a)^2 + s(b)^2)^2(s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2) - 2s(c)^2\\
&s(d)^2s(e)^2(s(a)^2 + s(b)^2) - s(a)^2s(b)^2(s(c)^2 + s(d)^2 + s(e)^2)^2\}s(x)^2\\
&+ s(c)s(d)s(e)(s(c)^2 + s(d)^2 + s(e)^2)(s(a)^2 - s(b)^2)^2s(x) + s(c)^2s(d)^2s(e)^2\\
&(s(a)^2 - s(b)^2)^2 = 0,
\end{aligned}$$

where $s(a) = \sinh\frac{a}{2} = \frac{1}{1-r_e^2}a_e$, see Figure 4.

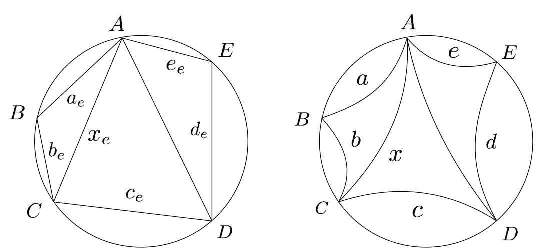

Figure 4: The Euclidean resp. hyperbolic inscribed pentagon $ABCDE$.

In particular assuming $s(a) = s(b)$, we have the following equation analogous to (4):

$$\begin{aligned}
&s(c)s(d)s(e)s(x)^5 + (s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2 - s(a)^4)s(x)^4\\
&+ s(c)s(d)s(e)\{(s(c)^2 + s(d)^2 + s(e)^2) - 4s(a)^2\}s(x)^3
\end{aligned}$$

$$\begin{aligned}
&+\{s(c)^2s(d)^2s(e)^2+2s(a)^4(s(c)^2+s(d)^2+s(e)^2)\\
&-4s(a)^2(s(c)^2s(d)^2+s(d)^2s(e)^2+s(e)^2s(c)^2)\}s(x)^2\\
&+4s(a)^2s(c)s(d)s(e)\{(2s(a)^2-(s(c)^2+s(d)^2+s(e)^2)\}s(x)\\
&+s(a)^2\{4s(a)^2(s(c)^2s(d)^2+s(d)^2s(e)^2+s(e)^2s(c)^2)\\
&-4s(c)^2s(d)^2s(e)^2-s(a)^2(s(c)^2+s(d)^2+s(e)^2)^2\}\\
&=0.
\end{aligned}$$

Step 3

By Corollary 1 (1) there is a hyperbolic cyclic pentagon with side lengths a, b, c, d, e satisfying $(s(a), s(b), s(c), s(d), s(e)) = (1, 1, 2, 3, 4)$. Hence for the hyperbolic length x of AC, $s(x)$ satisfies the following quintic equation which is analog of (5) in Step 3 of Section 2.

$$8s(x)^5+81s(x)^4+200s(x)^3-114s(x)^2-864s(x)-723=0.$$

The similar argument of Step 3 in Section 2 implies that for the area S of a hyperbolic cyclic pentagon with side lengths a, b, c, d, e, we cannot represent $\cos\frac{S}{2}$ in terms of the four arithmetic operations and k-th roots of $s(a), s(b), s(c), s(d), s(e)$.

Step 4

For $n \geq 6$, suppose that there is a formula $F(s(a_1), s(a_2), \cdots, s(a_n))$ of $\cos\frac{S}{2}$ for the area S of a generic hyperbolic cyclic n-gon with side lengths $a_1, a_2, \cdots, a_n$ in terms of the four arithmetic operations and k-th roots of $s(a_1), s(a_2), \cdots, s(a_n)$. By Corollary 1 (2), there exists $\varepsilon > 0$ such that for any t satisfying $0 < t < \varepsilon$, there is a hyperbolic cyclic n-gon satisfying $s(a_1) = s(a_2) = 1, s(a_3) = 2, s(a_4) = 3, s(a_5) = 4, s(a_6) = t, \cdots, s(a_n) = t$. Then the same argument of Step 4 in Section 2 implies that for $n \geq 5$, there is no formula of $\cos\frac{S}{2}$ in terms of the four arithmetic operations and k-th roots of $s(a_1), s(a_2), \cdots, s(a_n)$ for a generic hyperbolic cyclic n-gon.

4. On the area of a spherical cyclic polygon

Relations between Euclidean and spherical distances

For a model of spherical geometry, we consider the unit sphere S^2 in $\mathbb{R}^3$ with chordal metric [1]. We denote the image of $A \in S^2$ under the stereographic projection from the north pole $N = (0,0,1)$ on the xy-plane by A'. Then the origin $O = (0,0)$ is the image of the south pole $N = (0,0,-1)$. Let us denote the Euclidean length of the segment OA' on the xy-plane by r_e and the spherical length of the arc SA on S^2 by r_s. Then we have the following relation between r_e and r_s:

$$r_e = \tan \frac{r_s}{2}. \tag{12}$$

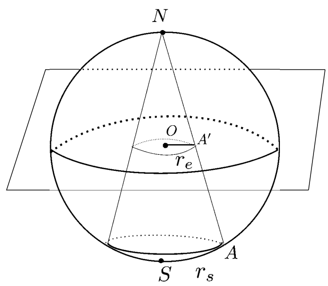

Figure 5: The stereographic projection.

Similar to Proposition 1 in Section 3, from the Euclidean and the spherical sine formulas we have

Proposition 4. *Let C_2 denote a spherical circle centered at S on S^2 and P, Q be points of C_2. Moreover denote the images of C_2, P, Q under the stereographic projection from N by C_1, P', Q' respectively. Then the Euclidean distance $d_e(P', Q')$ of P' and Q' and the spherical distance $d_s(P, Q)$ of P and Q satisfy the following relation:*

$$s(d_s(P,Q)) = \sin \frac{d_s(P,Q)}{2} = \frac{1}{1+r_e^2} d_e(P', Q'). \tag{13}$$

The next result is an analog of Proposition 3 in Section 3.

Proposition 5. *Let positive real numbers $a_1, a_2, \cdots, a_n$ be assumed that a_n is the largest. If $a_1, a_2, \cdots, a_n$ are sufficiently small and satisfy the following inequality*

$$\sum_{k=1}^{n-1} s(a_k)/s(a_n) > 1, \tag{14}$$

then there is a spherical cyclic n-gon with the spherical side lengths $a_1, a_2, \cdots, a_n$.

Proof. By Proposition 2, there is a cyclic n-gon with side lengths $s(a_1), s(a_2), \cdots, s(a_n)$ in Euclidean geometry. Denote the radius of the circumscribed circle by r_e. From now we assume that $r_e < \frac{1}{2}$ by taking $a_1, a_2, \cdots, a_n$ sufficiently small. Then $\lambda_0 = \frac{1-\sqrt{1-4r_e^2}}{2r_e^2}$ satisfies $0 < \lambda_0 < \frac{1}{r_e}$. In this case, there is a Euclidean n-gon in $\mathbb{D}$ inscribed in the circle centered at O with Euclidean radius $\lambda_0 r_e$ whose side lengths $\lambda_0 s(a_1), \lambda_0 s(a_2), \cdots, \lambda_0 s(a_n)$. Then considering the preimage of this Euclidean n-gon by the stereographic projection, we have a spherical cyclic n-gon on S^2 whose spherical side lengths are $a_1, a_2, \cdots, a_n$ because of $\frac{1}{1+(\lambda_0 r_e)^2}\lambda_0 s(a_k) = s(a_k)$. □

Corollary 2.

1. *For a sufficiently large k, there is a spherical cyclic pentagon satisfying $s(a_1) = s(a_2) = 1/k, s(a_3) = 2/k, s(a_4) = 3/k$ and $s(a_5) = 4/k$.*
2. *For k in (1), there exists $\varepsilon > 0$ such that for any t satisfying $0 < t < \varepsilon$, there is a spherical cyclic n-gon with side lengths $a_1, a_2, \cdots, a_n$ satisfying $s(a_1) = s(a_2) = 1/k, s(a_3) = 2/k, s(a_4) = 3/k, s(a_5) = 4/k, s(a_6) = t, \cdots, s(a_n) = t$.*

In the following, we will show that the arguments for Euclidean geometry in Section 2 also hold for spherical geometry step by step.

Step 1

The next formula for the area S of a spherical triangle with side lengths a, b, c is a spherical analog of Heron's formula [3]:

$$\cos\frac{S}{2} = \frac{1 - \frac{1}{2}(s(a)^2 + s(b)^2 + s(c)^2)}{\sqrt{1-s(a)^2}\sqrt{1-s(b)^2}\sqrt{1-s(c)^2}} \tag{15}$$

where $s(x) = \sin\frac{x}{2}$.

Similarly the next formula for the area S of a spherical cyclic quadrilateral with side lengths a, b, c, d is a spherical analog of Brahmagupta's formula [3, Page 182, Proposition 5]:

$$\cos\frac{S}{2} = \frac{1 - \frac{1}{2}(s(a)^2 + s(b)^2 + s(c)^2 + s(d)^2) - s(a)s(b)s(c)s(d)}{\sqrt{1-s(a)^2}\sqrt{1-s(b)^2}\sqrt{1-s(c)^2}\sqrt{1-s(d)^2}}. \tag{16}$$

Let S be the area of a spherical cyclic pentagon $ABCDE$ with side lengths a, b, c, d, e, and x be the spherical length of the diagonal AC. We can calculate in the same way as a hyperbolic pentagon, so that $s(x)$ is a root of a quartic equation whose coefficients belong to $\mathbb{Q}(s(a),\ s(b),\ s(c),\ s(d),\ s(e),\ \sqrt{1-s(a)^2},\ \sqrt{1-s(b)^2},\ \sqrt{1-s(c)^2},\ \sqrt{1-s(d)^2},\ \sqrt{1-s(e)^2},\ \cos\frac{S}{2})$. From Galois theory $s(x)$ can be written in terms of arithmetic operations and k-th roots of $s(a), s(b), s(c), s(d), s(e), \cos\frac{S}{2}$.

Step 2

Replace a, b, c, d, e, x in (3) with $a_e, b_e, c_e, d_e, e_e, x_e$, multiply $(\frac{1}{1+r_e^2})^{10}$ to (3) and apply (13), we have the equation analogues to (3). In particular for the case that $s(a) = s(b)$, we have the following equation which is an analog of (4):

$$\begin{aligned}
&s(c)s(d)s(e)s(x)^5 + (s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2 - s(a)^4)s(x)^4\\
&+ s(c)s(d)s(e)\{(s(c)^2 + s(d)^2 + s(e)^2) - 4s(a)^2\}s(x)^3\\
&+ \{s(c)^2s(d)^2s(e)^2 + 2s(a)^4(s(c)^2 + s(d)^2 + s(e)^2)\\
&- 4s(a)^2(s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2)\}s(x)^2\\
&+ 4s(a)^2s(c)s(d)s(e)\{(2s(a)^2 - (s(c)^2 + s(d)^2 + s(e)^2)\}s(x)\\
&+ s(a)^2\{4s(a)^2(s(c)^2s(d)^2 + s(d)^2s(e)^2 + s(e)^2s(c)^2)-\\
&4s(c)^2s(d)^2s(e)^2 - s(a)^2(s(c)^2 + s(d)^2 + s(e)^2)^2\}\\
&= 0,
\end{aligned}$$

where $s(a) = \sin\frac{a}{2} = \frac{1}{1+r_e^2}a_e$.

Step 3

By Corollary 2 (1) there is a spherical cyclic pentagon with side lengths a, b, c, d, e satisfying $(s(a), s(b), s(c), s(d), s(e)) = (1/k, 1/k, 2/k, 3/k, 4/k)$ for a sufficiently large k. Hence for the spherical length x of AC, $s(x) = z/k$ satisfies the following quintic equation which is analog of (5) in Step 3 of Section 2:

$$8z^5 + 81z^4 + 200z^3 - 114z^2 - 864z - 723 = 0.$$

The same argument of Step 3 in Section 2 implies that for the area S of a spherical cyclic pentagon with side lengths a, b, c, d, e, $\cos\frac{S}{2}$ cannot be written in terms of the four arithmetic operations and k-th roots of $s(a), s(b), s(c), s(d), s(e)$.

Step 4

For $n \geq 6$, suppose that there is a formula $F(s(a_1), s(a_2), \cdots, s(a_n))$ of $\cos\frac{S}{2}$ for the area S of a generic spherical cyclic n-gon with side lengths $a_1, a_2, \cdots, a_n$ in terms of the four arithmetic operations and k-th roots of $s(a_1), s(a_2), \cdots, s(a_n)$. By Corollary 2 (2), there exists $\varepsilon > 0$ such that for any t satisfying $0 < t < \varepsilon$, there is a spherical cyclic n-gon satisfying $s(a_1) = s(a_2) = 1, s(a_3) = 2, s(a_4) = 3, s(a_5) = 4, s(a_6) = t, \cdots, s(a_n) = t$. Then the similar argument of Step 4 in Section 2 implies that for $n \geq 5$, there is no formula of $\cos\frac{S}{2}$ in terms of the four arithmetic operations and k-th roots of $s(a_1), s(a_2), \cdots, s(a_n)$ for a generic spherical cyclic n-gon.

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