

# On the BV structure on the cohomology of moduli space

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**Abstract:** The question of vanishing of the BV operator on the cohomology of the moduli space of Riemann surfaces is investigated. The BV structure, which comprises a BV operator and an antibracket, is identified, vanishing theorems are proven, and a counterexample is provided.

**Keywords:** Moduli space of Riemann surfaces, Deligne-Mumford compactification, spectral sequence, mixed Hodge structure, stable cohomology, tautological class, BV-algebra, Maurer-Cartan equation, quantum master equation.

## Introduction

Since the early nineties, there has been considerable interest in nonperturbative methods in string field theory, in particular, setting up a *quantum master equation* (QME)

$$(1) \quad dS + \hbar\Delta S + \frac{1}{2}\{S, S\} = 0$$

in the dg-BV-algebra  $C_c^\bullet(\mathcal{M})$  of compactly supported cochains with rational coefficients on the moduli space  $\mathcal{M}$  of Riemann surfaces and describing a solution  $S$  of the QME, see, for example, [44, 45, 39, 24, 10, 30, 6, 7, 16, 22, 20, 32, 23, 9, 5]. One work that caught the eye was that of Kevin Costello [7], in which he used homotopical-algebraic methods and certain elementary facts about the cohomology of moduli spaces to prove the existence and uniqueness, up to homotopy, of a solution  $S$  of the QME. The proof consisted in a clever reduction of the QME to a linear equation

$$(2) \quad \hat{d}S' = 0$$

on a related  $S'$  in a homotopy abelian subquotient of  $C_c^\bullet(\mathcal{M})$  for the differential  $\hat{d} = d + \hbar\Delta$  and transferring the problem to cohomology.

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Note that the QME (1) is actually a *Maurer-Cartan equation*

$$\hat{d}S + \frac{1}{2}\{S, S\} = 0$$

in the dg-Lie algebra of cochains on the moduli space and if this dg-Lie algebra is homotopy abelian, then the Maurer-Cartan equation is equivalent to the cocycle equation (2). It is known, see [40, 21], that in a dg-BV-algebra, the homotopy abelianness of the underlying dg-Lie algebra may be derived from the degeneration at  $E_1$  of a spectral sequence associated to the double complex  $(C_c^\bullet(\mathcal{M}), d, \Delta)$ , which entails the vanishing of the differential  $d_1 = \Delta$  on the cohomology  $H_c^\bullet(\mathcal{M})$ .

Given Costello's homotopy abelianness result, one might hope that homotopy abelianness takes place for the whole dg-Lie algebra  $C_c^\bullet(\mathcal{M})$  and that the general argument of [40, 21] would imply Costello's existence and uniqueness result. On the other hand, any general statement, such as the vanishing of a cohomology operation,  $\Delta$  in particular, on the cohomology of the moduli space is extremely interesting, given how little is still known about it.

This paper grew out of our investigation of the vanishing of  $\Delta$  on the cohomology  $H_c^\bullet(\mathcal{M})$  of the moduli space. It has turned out that  $\Delta$  vanishes on vast ranges of these cohomology groups, appearing to provide convincing evidence for vanishing everywhere, see Section 2 below. On the other hand, we have found out that total vanishing does not actually take place. We provide a counterexample in Corollary 2.7. In the process, we obtain useful description of the BV operator  $\Delta$  and the antibracket  $\{-, -\}$ , relating them to the differential  $d_1$  in the spectral sequence associated to the topological filtration on the compactified moduli space  $\overline{\mathcal{M}}$ . This differential may be expressed as the Gysin homomorphism given by Poincaré residue, see Section 1.3. It is somewhat surprising that the differentials on the first terms of the spectral sequences associated to the double complex  $(C_c^\bullet(\mathcal{M}), d, \Delta)$  and the topological filtration on  $\overline{\mathcal{M}}$  are closely related, see Theorem 1.2.

### Conventions

In this paper,  $\mathcal{M}_{g,n}$  denotes the (usual) moduli space of smooth, connected, compact, genus  $g$  Riemann surfaces with  $n$  labeled punctures. Here and throughout the paper, we assume that our Riemann surfaces are *stable*, i.e.,  $n \geq 3$  if  $g = 0$  and  $n \geq 1$  if  $g = 1$ ; otherwise  $g, n \geq 0$ . Let  $S_n$  be the group of permutations of  $n$  objects. Then  $\mathcal{M}_g(n) := \mathcal{M}_{g,n}/S_n$  is the moduli space of compact Riemann surfaces with  $n$  unlabeled punctures.

We work with homology  $H_\bullet(X) := H_\bullet(X, \mathbb{Q})$  and cohomology  $H^\bullet(X) := H^\bullet(X, \mathbb{Q})$  with rational coefficients. Vector spaces are also assumed to be over  $\mathbb{Q}$ .

### 1. The BV structure on the (co)homology of moduli spaces

#### 1.1. Operations on the homology $H_\bullet(\mathcal{M}_g(n), \mathbb{Q})$

The following operations have been originally defined by Zwiebach [45] and modified by Costello [7].

1. For  $n \geq 0$ , define the *BV operator*

$$(3) \quad \Delta : H_k(\mathcal{M}_g(n+2)) \rightarrow H_{k+1}(\mathcal{M}_{g+1}(n))$$

as follows. If  $Z \in H_k(\mathcal{M}_g(n+2))$  is a homology class represented by a singular cycle  $Z = \sum_{i=1}^m c_i \sigma_i$  for  $\sigma_i : \Delta^k \rightarrow \mathcal{M}_g(n+2)$ , then set

$$\Delta(Z) := \sum_i c_i \sum_{\{\beta, \gamma\}} (\text{twist-gluing of } \sigma_i \text{ at punctures } \{\beta, \gamma\}),$$

where the summation runs over unordered pairs  $\{\beta, \gamma\}$  of punctures and *twist-gluing* is cutting out small holomorphic disks around the punctures  $\{\beta, \gamma\}$ , and gluing their complement(s) along the boundaries. More precisely, we consider holomorphic disks at each puncture (i.e., holomorphic embeddings of the standard disk  $|z| < 1$  centered at the puncture and not containing other punctures). Then we cut out the disks  $|z| \leq r$  and  $|w| \leq r$  for some  $r = 1 - \epsilon$  at sewn punctures and identify the annuli  $r < |z| < 1/r$  and  $r < |w| < 1/r$  via  $w = e^{it}/z$ , with  $t$  running over the interval  $[0, 2\pi]$ , and thereby increasing the degree of the chain  $\sigma_i$ . This gives a new chain  $S^1 \times \Delta^k \rightarrow \mathcal{M}_{g+1}(n)$ , resulting in a new cycle  $\Delta(Z)$ . Moreover, the map  $\Delta$  is well-defined on  $H_\bullet(\mathcal{M}_g(n+2))$ , i.e., it is independent of the choice of holomorphic disks involved in twist-gluing (see [45, 7]).

2. Similarly, for  $n_1, n_2 \geq 0$ , define the *antibracket*

$$(4) \quad \{-, -\} : H_{k_1}(\mathcal{M}_{g_1}(n_1+1)) \otimes H_{k_2}(\mathcal{M}_{g_2}(n_2+1)) \\ \rightarrow H_{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1+n_2))$$

as follows. If  $Z_1 \in H_{k_1}(\mathcal{M}_{g_1}(n_1+1))$  and  $Z_2 \in H_{k_2}(\mathcal{M}_{g_2}(n_2+1))$  are homology classes represented by singular cycles  $Z_1 = \sum_{i=1}^m c_i \sigma_i$  for  $\sigma_i : \Delta^{k_1} \rightarrow \mathcal{M}_{g_1}(n_1+1)$

1) and  $Z_2 = \sum_{j=1}^m d_j \tau_j$  for  $\tau_j : \Delta^{k_2} \rightarrow \mathcal{M}_{g_2}(n_2 + 1)$ , then set

$$\{Z_1, Z_2\} := \sum_{i,j} c_i d_j \sum_{\beta,\gamma} (\text{twist-gluing of } \sigma_i \text{ with } \tau_j \text{ at punctures } \beta, \gamma),$$

where  $\beta$  runs over the punctures of the surface in  $\mathcal{M}_{g_1}(n_1 + 1)$  and  $\gamma$  runs over the punctures of the surface in  $\mathcal{M}_{g_2}(n_2 + 1)$ . This gives a new chain  $S^1 \times \Delta^{k_1} \times \Delta^{k_2} \rightarrow \mathcal{M}_{g_1+g_2}(n_1 + n_2)$ , resulting in a new cycle  $\{Z_1, Z_2\} \in H_{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2))$ , independent of the choices made along the way (see *ibid.*).

The reason why these homology operations are introduced is to set up a quantum master equation, actually, at the chain, rather than homology level:

$$dS + \hbar \Delta S + \frac{1}{2} \{S, S\} = 0$$

on a formal power series  $S = \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} S_{g,n} \lambda^{2g-2+n} \hbar^g$  with coefficients  $S_{g,n}$  being chains in  $C_{6g-6+2n}(\mathcal{M}_g(n))$  and  $d$  denoting chain boundary. However, for the above equation to be sensible, degree considerations suggest to change grading on chains and homology to grading by codimension, as well as assume that the formal variables  $\lambda$  and  $\hbar$  have degree zero. This change of grading may be regarded as the application of canonical *Poincaré-Lefschetz duality*

$$H_k(\mathcal{M}_g(n)) = H_c^{6g-6+2n-k}(\mathcal{M}_g(n)),$$

where  $H_c^\bullet$  denotes cohomology with compact support. Thus, under Poincaré-Lefschetz duality, the BV operator (3) and antibracket (4) turn into

$$(5) \quad \Delta : H_c^k(\mathcal{M}_g(n+2)) \rightarrow H_c^{k+1}(\mathcal{M}_{g+1}(n))$$

and

$$(6) \quad \{-, -\} : H_c^{k_1}(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_c^{k_2}(\mathcal{M}_{g_2}(n_2 + 1)) \rightarrow H_c^{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2)),$$

respectively. We hope that, given that Poincaré-Lefschetz duality identifies these operations (5) and (6) with (3) and (4), respectively, our use of the same notation,  $\Delta$  and  $\{-, -\}$ , will not create serious confusion.

From [2, 45] we know that these two operations define the structure of a dg-Lie algebra with a differential  $\Delta$  of degree 1 on

$$(7) \quad \mathfrak{g} := \bigoplus_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} H_c^\bullet(\mathcal{M}_g(n))[1],$$

where  $V[1]$  denotes the desuspension of a graded vector space  $V$ :  $V[1]^k := V^{k+1}$ .

**Remark 1.1.** *The graded symmetric algebra  $S(\mathfrak{g}[-1])$  on the suspension of a dg-Lie algebra  $\mathfrak{g}$  is known to carry the structure of a BV-algebra, see [42, Example 2.5], which uses a slightly different grading convention:  $\deg \Delta = -1$  and  $\deg \hbar = 2$ . In the case of our dg-Lie algebra  $\mathfrak{g}$  as in (7), the graded symmetric algebra  $S(\mathfrak{g}[-1])$  is isomorphic to the (co)homology  $H_c^\bullet(\mathcal{M})$  of Zwiebach’s dg-BV-algebra  $C_c^\bullet(\mathcal{M})$  of compactly supported chains in the moduli space  $\mathcal{M}$  of not necessarily connected, closed Riemann surfaces with unlabeled punctures.*

### 1.2. Construction via the real Deligne-Mumford compactification

Let  $\underline{\mathcal{M}}_g(n)$  be the real Deligne-Mumford compactification of  $\mathcal{M}_g(n)$  to an orbifold with corners (see [24]). It is known that  $\underline{\mathcal{M}}_g(n)$  is homotopy equivalent to  $\mathcal{M}_g(n)$  and  $H_\bullet(\underline{\mathcal{M}}_g(n)) \cong H_\bullet(\mathcal{M}_g(n))$ . In this construction,  $\Delta$  uses twist-attaching (in which there are  $S^1$  ways of attaching a pair of punctures on a Riemann surface  $\Sigma \in \underline{\mathcal{M}}_g(n)$ , or  $S^1$  worth of choices of real rays in the tensor product (over  $\mathbb{C}$ ) of the tangent spaces to  $\Sigma$  at these two punctures). The antibracket  $\{-, -\}$  uses twist-attaching in a similar way.

### 1.3. Construction via the Deligne-Mumford compactification

**1.3.1. The “topological” spectral sequence** Let  $\overline{\mathcal{M}}_g(n)$  denote the Deligne-Mumford (DM) compactification of  $\mathcal{M}_g(n)$  and  $\overline{\mathcal{M}}_g(n)^{(p)}$  be the moduli space of Riemann surfaces with at least  $p$  double points (also known as *nodes*). Then we have a decreasing filtration

$$\dots \subset \overline{\mathcal{M}}_g(n)^{(p+1)} \subset \overline{\mathcal{M}}_g(n)^{(p)} \dots \subset \overline{\mathcal{M}}_g(n)^{(0)} = \overline{\mathcal{M}}_g(n).$$

It generates a spectral sequence satisfying

$$(8) \quad E_{p,q}^1 = H_{p+q}(\overline{\mathcal{M}}_g(n)^{(-p)}, \overline{\mathcal{M}}_g(n)^{(-p+1)}) \Rightarrow H_\bullet(\overline{\mathcal{M}}_g(n))$$

with  $E_{p,q}^1 = 0$  unless  $q \geq -p \geq 0$  and a differential

$$d_{p,q}^1 : H_{p+q}(\overline{\mathcal{M}}_g(n)^{(-p)}, \overline{\mathcal{M}}_g(n)^{(-p+1)}) \rightarrow H_{p+q-1}(\overline{\mathcal{M}}_g(n)^{(-p+1)}, \overline{\mathcal{M}}_g(n)^{(-p+2)}),$$

given by the boundary map in the long exact sequence of the triple  $(\overline{\mathcal{M}}_g(n)^{(-p)}, \overline{\mathcal{M}}_g(n)^{(-p+1)}, \overline{\mathcal{M}}_g(n)^{(-p+2)})$ . We will also utilize a *coboundary map*, the linear dual

$$(9) \quad \begin{aligned} (d_{p,q}^1)^* : H_c^{p+q-1}(\overline{\mathcal{M}}_g(n)^{(-p+1)} \setminus \overline{\mathcal{M}}_g(n)^{(-p+2)}) \\ \rightarrow H_c^{p+q}(\overline{\mathcal{M}}_g(n)^{(-p)} \setminus \overline{\mathcal{M}}_g(n)^{(-p+1)}) \end{aligned}$$

of  $d_{p,q}^1$  later on.

After applying Poincaré-Lefschetz duality and a linear change of variables  $p, q$  to the spectral sequence (8), we get a cohomological spectral sequence with

$$(10) \quad E_1^{p,q} = H^{q-p}(\overline{\mathcal{M}}_g(n)^{(p)} \setminus \overline{\mathcal{M}}_g(n)^{(p+1)}) \Rightarrow H^\bullet(\overline{\mathcal{M}}_g(n))$$

with  $q \geq p \geq 0$  and a differential

$$d_1^{p,q} : H^{q-p}(\overline{\mathcal{M}}_g(n)^{(p)} \setminus \overline{\mathcal{M}}_g(n)^{(p+1)}) \rightarrow H^{q-p-1}(\overline{\mathcal{M}}_g(n)^{(p+1)} \setminus \overline{\mathcal{M}}_g(n)^{(p+2)}).$$

This is the *Poincaré residue map* (see [8, 11, 25]), which might be thought of as a Gysin homomorphism, integration along the boundary of the tubular neighborhood of  $\overline{\mathcal{M}}_g(n)^{(p+1)}$  inside  $\overline{\mathcal{M}}_g(n)^{(p)}$ . The linear dual is

$$(11) \quad (d_1^{p,q})^* : H_{q-p-1}(\overline{\mathcal{M}}_g(n)^{(p+1)} \setminus \overline{\mathcal{M}}_g(n)^{(p+2)}) \rightarrow H_{q-p}(\overline{\mathcal{M}}_g(n)^{(p)} \setminus \overline{\mathcal{M}}_g(n)^{(p+1)})$$

may also be thought of as a Gysin homomorphism in homology. We will now demonstrate that the BV operator (3) and antibracket (4) are essentially these Gysin homomorphisms  $(d_1^{p,q})^*$ , see (11), for  $p = 0$ . This will imply that the corresponding operators (5) and (6) on the compactly supported cohomology are essentially the coboundary homomorphisms  $(d_{p,q}^1)^*$ , see (9), for  $p = 0$ .

**1.3.2. Identification of the BV operator** Let

$$\begin{aligned} \alpha_p : H_\bullet(\overline{\mathcal{M}}_g(n+2)^{(p)} \setminus \overline{\mathcal{M}}_g(n+2)^{(p+1)}) \\ \rightarrow H_\bullet(\overline{\mathcal{M}}_{g+1}(n)^{(p+1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(p+2)}) \end{aligned}$$

be the composition  $\alpha_p := \pi_*\rho^!$ , where  $\pi_*$  is the pushforward of the map  $\pi$  attaching the last two punctures and  $\rho^!$  is the transfer map for  $\rho$  from the following diagram of étale morphisms:

$$\begin{array}{ccc} (\overline{\mathcal{M}}_{g,n+2}^{(p)} \setminus \overline{\mathcal{M}}_{g,n+2}^{(p+1)})/S_n \times S_2 & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g+1}(n)^{(p+1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(p+2)} \\ \downarrow \rho & & \\ \overline{\mathcal{M}}_g(n+2)^{(p)} \setminus \overline{\mathcal{M}}_g(n+2)^{(p+1)}, & & \end{array}$$

in which  $\overline{\mathcal{M}}_{g,n+2}^{(p)}$  is the moduli space of Riemann surfaces with  $n+2$  labeled punctures and at least  $p$  nodes,  $S_n$  permutes the first  $n$  punctures in  $\mathcal{M}_{g,n+2}$ , whereas  $S_2$  permutes the last two punctures, and  $\rho$  forgets the division of the punctures into two groups: the first  $n$  ones and the last two.

Let us consider the particular case  $p = 0$ , where  $\overline{\mathcal{M}}_g(n+2)^{(0)} \setminus \overline{\mathcal{M}}_g(n+2)^{(1)} = \mathcal{M}_g(n+2)$  and  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$  is the moduli space of Riemann surfaces with exactly one node. Then we have

$$\alpha_0 : H_\bullet(\mathcal{M}_g(n+2)) \rightarrow H_\bullet(\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}),$$

where  $\alpha_0 = \pi_*\rho^!$  for the diagram

$$(12) \quad \begin{array}{ccc} \mathcal{M}_{g,n+2}/S_n \times S_2 & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)} \\ \downarrow \rho & & \\ \mathcal{M}_g(n+2) & & \end{array}$$

of étale morphisms.

**Theorem 1.2.** *1. The BV operator  $\Delta$  as in Eq. (3) is equal to the composition  $(d_1^{0,\bullet})^*\alpha_0$  below:*

$$\begin{aligned} H_\bullet(\mathcal{M}_g(n+2)) &\xrightarrow{\alpha_0} H_\bullet(\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}) \\ &\xrightarrow{(d_1^{0,\bullet})^*} H_{\bullet+1}(\mathcal{M}_{g+1}(n)). \end{aligned}$$

*2. The BV operator  $\Delta$  as in Eq. (5) is equal to the composition  $(d_0^1,\bullet)^*\alpha_0$  below:*

$$\begin{aligned} H_c^\bullet(\mathcal{M}_g(n+2)) &\xrightarrow{\alpha_0} H_c^\bullet(\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}) \\ &\xrightarrow{(d_0^1,\bullet)^*} H_c^{\bullet+1}(\mathcal{M}_{g+1}(n)), \end{aligned}$$

where  $\alpha_0 = \pi_* \rho^!$  is the morphism induced by the étale morphisms  $\pi$  and  $\rho$  from (12) on cohomology with compact support.

*Proof.* Claims (1) and (2) are equivalent by Poincaré-Lefschetz duality. To prove claim (1), let us trace what happens under the maps  $\rho^!$  and  $\pi_*$  mapping up the map  $\alpha_0$ , as well as under the map  $(d_1^{0,\bullet})^*$ . Given a singular chain of smooth Riemann surfaces of genus  $g$  with  $n + 2$  unlabeled punctures, the transfer map  $\rho^!$  sums up all possible ways of picking a pair of punctures, whereas the map  $\pi_*$  attaches these two punctures to form a node and thereby places this chain within the part  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$  of the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{g+1}(n)$  which corresponds to stable curves with exactly one node. This is exactly what the BV operator  $\Delta$  does, except that twist-gluing at the chosen pair of punctures is replaced so far with attaching. Now, the map  $(d_1^{0,\bullet})^*$  in homology linear dual to the Poincaré residue map  $d_1^{0,\bullet}$  in cohomology is the umkehr map, which associates to a cycle in  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$  its pre-image in the boundary of the tubular neighborhood of  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$  inside  $\overline{\mathcal{M}}_{g+1}(n)$ . The tubular neighborhood forms an  $S^1$ -bundle over  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$  and may locally be built out of Riemann surfaces obtained by twist-gluing at the node of a surface in  $\overline{\mathcal{M}}_{g+1}(n)^{(1)} \setminus \overline{\mathcal{M}}_{g+1}(n)^{(2)}$ . Thus, composing the map  $(d_1^{0,\bullet})^*$  sends the homology class of stable curves with one node to the homology class obtained by twist-gluing of stable curves at the node. This reconciles the composition  $(d_1^{0,\bullet})^* \alpha_0$  with  $\Delta$ . □

**1.3.3. Identification of the antibracket** Given that the antibracket has the same nature as the BV operator and is, namely, the *derived bracket* for the BV operator  $\Delta$  on  $S(\mathfrak{g}[-1]) \cong H_\bullet(\mathcal{M})$ , see Remark 1.1, it is not surprising that there is a similar identification of the antibracket via the Gysin homomorphisms, coming from the topological spectral sequences.

A diagram analogous to diagram (12), which defines the homomorphism  $\alpha_0$ , is the following pair of étale morphisms:

$$(13) \quad \begin{array}{ccc} \mathcal{M}_{g_1, n_1+1}/S_{n_1} \times \mathcal{M}_{g_2, n_2+1}/S_{n_2} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(1)} \setminus \overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(2)} \\ & & \downarrow \rho \\ & & \mathcal{M}_{g_1}(n_1 + 1) \times \mathcal{M}_{g_2}(n_2 + 1). \end{array}$$

Here  $S_{n_i}$  permutes the first  $n_i$  punctures in  $\mathcal{M}_{g_i, n_i+1}$ ,  $i = 1, 2$ , and  $\pi$  attaches the last,  $n_1 + 1$ st puncture on the Riemann surface representing a point in  $\mathcal{M}_{g_1, n_1+1}/S_{n_1}$  to the last,  $n_2 + 1$ st puncture on the Riemann surface representing a point in  $\mathcal{M}_{g_2, n_2+1}/S_{n_2}$ . A diagram analogous to the one defining  $\alpha_p$  may



be written down similarly, but we are skipping it for the sake of simplicity. Define a map

$$\beta_0 : H_\bullet(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_\bullet(\mathcal{M}_{g_2}(n_2 + 1)) \rightarrow H_\bullet(\overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(1)} \setminus \overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(2)})$$

as  $\beta_0 = \pi_*\rho^!$  for  $\pi$  and  $\rho$  from the previous diagram. The following theorem is proven exactly in the same way as Theorem 1.2.

**Theorem 1.3.** *1. The antibracket  $\{-, -\}$  as in Eq. (4) is equal to the composition  $(d_1^{0,\bullet})^*\beta_0$  below:*

$$\begin{aligned} & H_\bullet(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_\bullet(\mathcal{M}_{g_2}(n_2 + 1)) \\ & \xrightarrow{\beta_0} H_\bullet(\overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(1)} \setminus \overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(2)}) \\ & \xrightarrow{(d_1^{0,\bullet})^*} H_{\bullet+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2)). \end{aligned}$$

*2. The antibracket  $\{-, -\}$  as in Eq. (6) is equal to the composition  $(d_{0,\bullet}^1)^*\beta_0$  below:*

$$\begin{aligned} & H_c^\bullet(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_c^\bullet(\mathcal{M}_{g_2}(n_2 + 1)) \\ & \xrightarrow{\beta_0} H_c^\bullet(\overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(1)} \setminus \overline{\mathcal{M}}_{g_1+g_2}(n_1 + n_2)^{(2)}) \\ & \xrightarrow{(d_{0,\bullet}^1)^*} H_c^{\bullet+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2)), \end{aligned}$$

where  $\beta_0 = \pi_*\rho^!$  is the morphism induced by the étale morphisms  $\pi$  and  $\rho$  from (13) on cohomology with compact support.

## 2. Vanishing results for the BV operator and antibracket

### 2.1. The genus $g = 0$ case

Let us consider the case  $g = 0$  first, as the BV operator and antibracket vanish on the moduli space of genus-zero Riemann surfaces for the trivial reason of it being rationally acyclic.

**Theorem 2.1.**

$$H^k(\mathcal{M}_0(n)) = \begin{cases} \mathbb{Q}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

This theorem follows from Arnold’s computation [1, 4] of the rational cohomology of the braid group, as observed by Looijenga [27]. See also a different argument of Westerland [43]. We also need the following classical result.

**Theorem 2.2** (Mumford [31]).  $H^1(\mathcal{M}_{1,n}) = 0$ .

These results imply the desired vanishing.

**Corollary 2.3.** 1. *The BV operator*

$$H_k(\mathcal{M}_0(n + 2)) \xrightarrow{\Delta} H_{k+1}(\mathcal{M}_1(n))$$

vanishes for all  $k \geq 0, n \geq 1$ . In cohomology with compact support,

$$H_c^l(\mathcal{M}_0(n + 2)) \xrightarrow{\Delta} H_c^{l+1}(\mathcal{M}_1(n))$$

vanishes for all  $l \geq 0, n \geq 1$ . 2. *The antibracket*

$$H_{k_1}(\mathcal{M}_0(n_1 + 1)) \otimes H_{k_2}(\mathcal{M}_0(n_2 + 1)) \xrightarrow{\{-,-\}} H_{k_1+k_2+1}(\mathcal{M}_0(n_1 + n_2))$$

vanishes for  $k_1, k_2 \geq 0, n_1, n_2 \geq 1$ . In cohomology with compact support, the antibracket

$$H_c^{l_1}(\mathcal{M}_0(n_1 + 1)) \otimes H_c^{l_2}(\mathcal{M}_0(n_2 + 1)) \xrightarrow{\{-,-\}} H_c^{l_1+l_2+1}(\mathcal{M}_0(n_1 + n_2))$$

vanishes when  $l_1, l_2 \geq 0, n_1, n_2 \geq 1$ .

### 2.2. The genus $g > 0$ case

From [14] we have the following bounds on homology:

$$H_k(\mathcal{M}_{g,n}) = 0 \quad \text{for} \quad \begin{cases} g = 0, k > n - 3, \\ n = 0, k > 4g - 5, \\ n > 0, g > 0, k > 4g - 4 + n. \end{cases}$$

Equivalently,

$$H_c^l(\mathcal{M}_{g,n}) = 0 \quad \text{for} \quad \begin{cases} g = 0, l < n - 3, \\ n = 0, l < 2g - 1, \\ n > 0, g > 0, l < 2g - 2 + n. \end{cases}$$

These above bounds imply the following. For the BV operator

$$H_k(\mathcal{M}_g(n+2)) \xrightarrow{\Delta} H_{k+1}(\mathcal{M}_{g+1}(n))$$

as in Equation (3), the left-hand side vanishes for  $k > 4g - 2 + n$ ,  $n \geq 0$ , while the right-hand side is zero for  $k > 4g - 1 + n$ , if  $n > 0$ , and  $k > 4g - 2$ , if  $n = 0$ . On the other hand, for the antibracket

$$H_{k_1}(\mathcal{M}_{g_1}(n_1+1)) \otimes H_{k_2}(\mathcal{M}_{g_2}(n_2+1)) \xrightarrow{\{-,-\}} H_{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1+n_2))$$

as in Equation (4), the left-hand side is zero for  $k_1 > 4g_1 - 3 + n_1$  or  $k_2 > 4g_2 - 3 + n_2$ , but the right-hand side vanishes for  $k_1+k_2 > 4g_1+4g_2-5+n_1+n_2$ , if  $n_1+n_2 > 0$ , and  $k_1+k_2 > 4g_1+4g_2-6$ , if  $n_1+n_2 = 0$ . Therefore, we have

**Theorem 2.4.** 1. *The BV operator*

$$H_k(\mathcal{M}_g(n+2)) \xrightarrow{\Delta} H_{k+1}(\mathcal{M}_{g+1}(n))$$

vanishes for  $k > 4g - 2 + n$ ,  $g > 0$ ,  $n \geq 0$ . In cohomology with compact support,

$$H_c^l(\mathcal{M}_g(n+2)) \xrightarrow{\Delta} H_c^{l+1}(\mathcal{M}_{g+1}(n))$$

vanishes for  $l < 2g + n$ ,  $g > 0$ ,  $n \geq 0$ . In particular,  $\Delta = 0$  on compactly supported cohomology as  $g \rightarrow \infty$  or  $n \rightarrow \infty$  and  $l$  being fixed, i.e., stably

2. *The antibracket*

$$H_{k_1}(\mathcal{M}_{g_1}(n_1+1)) \otimes H_{k_2}(\mathcal{M}_{g_2}(n_2+1)) \xrightarrow{\{-,-\}} H_{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1+n_2))$$

vanishes for  $k_1 > 4g_1 - 3 + n_1$  or  $k_2 > 4g_2 - 3 + n_2$ ,  $g_1, g_2 > 0$  and  $n_1, n_2 \geq 0$ . In cohomology with compact support, the antibracket

$$H_c^{l_1}(\mathcal{M}_{g_1}(n_1+1)) \otimes H_c^{l_2}(\mathcal{M}_{g_2}(n_2+1)) \xrightarrow{\{-,-\}} H_c^{l_1+l_2+1}(\mathcal{M}_{g_1+g_2}(n_1+n_2))$$

vanishes when  $l_1 < 2g_1 - 1 + n_1$  or  $l_2 < 2g_2 - 1 + n_2$ ,  $g_1, g_2 > 0$  and  $n_1, n_2 \geq 0$ .

Likewise, homological stability implies the vanishing of the BV operator and antibracket within the *stable range*  $k < \frac{2}{3}(g-1)$ . Indeed, Harer’s stability theorem [13], as improved by Ivanov [17, 18, 19], Harer himself [15], Boldsen [3], and Randal-Williams [34]:

$$H^k(\mathcal{M}_g) \cong H^k(\mathcal{M}_{g+1}) \quad \text{for } k < \frac{2}{3}(g-1),$$

combined with Madsen and Weiss’s proof [29] of Mumford’s conjecture, stating that

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \rightarrow H^\bullet(\mathcal{M}_g),$$

where  $\kappa_i \in H^{2i}(\mathcal{M}_g)$  is the  $i$ th “tautological”  $\kappa$  class,  $i = 1, 2, \dots$ , is an isomorphism in degree  $\leq \frac{2}{3}(g - 1)$ , and Looijenga’s relation [28] with the case of pointed Riemann surfaces, which asserts that

$$H^\bullet(\mathcal{M}_g)[\psi_1, \psi_2, \dots, \psi_n] \rightarrow H^\bullet(\mathcal{M}_{g,n}),$$

where  $\psi_i \in H^2(\mathcal{M}_g)$  is the  $i$ th “tautological”  $\psi$  class,  $i = 1, \dots, n$ , is an isomorphism in degree  $\leq \frac{2}{3}(g - 1)$ , implies that

$$\mathbb{Q}[\psi_1, \dots, \psi_n, \kappa_1, \kappa_2, \dots] \rightarrow H^\bullet(\mathcal{M}_{g,n})$$

is an isomorphism in degree  $\leq \frac{2}{3}(g - 1)$ . This, in particular, forces the rational cohomology  $H^\bullet(\mathcal{M}_{g,n})$  in the stable range to be concentrated in even degrees. Taking invariants of the  $S_n$ -action on cohomology does not affect these statements. Now, given that the BV operator and antibracket have degree 1, we obtain the following vanishing result.

**Theorem 2.5.** 1. *The BV operator*

$$H_k(\mathcal{M}_g(n + 2)) \xrightarrow{\Delta} H_{k+1}(\mathcal{M}_{g+1}(n))$$

vanishes for  $k \leq \frac{2}{3}(g - 1)$ ,  $g > 0$ ,  $n \geq 0$ . In particular,  $\Delta = 0$  on homology as  $g \rightarrow \infty$  or  $n \rightarrow \infty$  and  $k$  being fixed, i.e., stably. In cohomology with compact support,

$$H_c^l(\mathcal{M}_g(n + 2)) \xrightarrow{\Delta} H_c^{l+1}(\mathcal{M}_{g+1}(n))$$

vanishes for  $l \geq \frac{16}{3}g - \frac{4}{3} + 2n$ ,  $g > 0$ ,  $n \geq 0$ .

2. *The antibracket*

$$H_{k_1}(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_{k_2}(\mathcal{M}_{g_2}(n_2 + 1)) \xrightarrow{\{-, -\}} H_{k_1+k_2+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2))$$

vanishes for  $k_1 \leq \frac{2}{3}(g_1 - 1)$ ,  $k_2 \leq \frac{2}{3}(g_2 - 1)$ ,  $k_1 + k_2 \leq \frac{2}{3}(g_1 + g_2) - \frac{5}{3}$ ,  $g_1, g_2 > 0$ , and  $n_1, n_2 \geq 0$ . In cohomology with compact support, the antibracket

$$H_c^{l_1}(\mathcal{M}_{g_1}(n_1 + 1)) \otimes H_c^{l_2}(\mathcal{M}_{g_2}(n_2 + 1)) \xrightarrow{\{-, -\}} H_c^{l_1+l_2+1}(\mathcal{M}_{g_1+g_2}(n_1 + n_2))$$

vanishes when  $l_1 \geq \frac{16}{3}g_1 - \frac{10}{3} + 2n_1$ ,  $l_2 \geq \frac{16}{3}g_2 - \frac{10}{3} + 2n_2$ ,  $l_1 + l_2 \geq \frac{16}{3}(g_1 + g_2) - \frac{19}{3} + 2(n_1 + n_2)$ ,  $g_1, g_2 > 0$ , and  $n_1, n_2 \geq 0$ .

### 2.3. A nonvanishing example

Given a vanishing range of the BV operator and antibracket on rational homology, one may wonder if they vanish identically. Concrete computations with moduli spaces are quite hard in general, and we do not have an example of nonvanishing of the antibracket. Here we present an example of nonvanishing of the BV operator. First off, let us analyze the differential in the spectral sequence (10).

**Proposition 2.6.** *The differential  $d_1^{0,6} : H^6(\mathcal{M}_3) \rightarrow H^5(\overline{\mathcal{M}}_3^{(1)} \setminus \overline{\mathcal{M}}_3^{(2)})$  is nontrivial. Moreover, so is its projection to  $H^5(\mathcal{M}_2(2))$ .*

*Proof.* It suffices to prove the second statement. By Looijenga’s computation [27] of the rational cohomology of  $\mathcal{M}_3$ , its *Poincaré-Serre polynomial*, in which the coefficient by  $t^k u^l$  is the dimension of the subquotient  $H^k(\mathcal{M}_3)$  of weight  $l$ , is equal to  $1 + t^2 u^2 + t^6 u^{12}$ . This implies that  $H^6(\mathcal{M}_3) \cong \mathbb{Q}$  of weight 12. Since the weight is not equal to the cohomology degree, the corresponding element in  $E_1^{0,6}$  will not survive to  $H^6(\overline{\mathcal{M}}_3)$  in the limit  $E_\infty^{0,6}$ . If all the differentials  $d_r^{0,6}$ ,  $r \geq 1$ , were zero on  $H^6(\mathcal{M}_3)$ , then  $H^6(\mathcal{M}_3)$  would contribute nontrivially to  $E_\infty^{0,6} \subset H^6(\overline{\mathcal{M}}_3)$  and thereby have weight 6, which would be contradiction. The plan is to show that all the higher differentials  $d_r^{0,6}$ ,  $r \geq 2$ , vanish. This would force  $d_1^{0,6}$  to be nontrivial.

Now, looking at different components of the boundary  $\partial\mathcal{M}_3$ , we find that they all have a trivial cohomology group  $H^5$ , except possibly  $\mathcal{M}_2(2)$ . Indeed,  $\overline{\mathcal{M}}_3^{(1)} \setminus \overline{\mathcal{M}}_3^{(2)} = \mathcal{M}_2(2) \amalg \mathcal{M}_{2,1} \times \mathcal{M}_{1,1}$ . The space  $\mathcal{M}_{1,1}$  of elliptic curves is known to be isomorphic to the affine line  $\mathbb{C}$ , whereas the following argument, borrowed from Dan Petersen [33], shows that  $\mathcal{M}_{2,1}$  has the rational homology of  $\mathbb{C}P^1$ . Indeed, consider the Leray-Serre spectral sequence for the forgetful map  $\pi : \mathcal{M}_{2,1} \rightarrow \mathcal{M}_2$ . The base  $\mathcal{M}_2$  is isomorphic to  $\mathcal{M}_0(6)$ , because every curve of genus 2 is hyperelliptic. We know from Theorem 2.1 that it has the cohomology of a singleton. On the other hand, the local systems  $R^0\pi_*\mathbb{Q}$  and  $R^2\pi_*\mathbb{Q}$  are trivial, while  $R^1\pi_*\mathbb{Q}$  has no rational cohomology, as the curve representing a point of  $\mathcal{M}_2$  has the hyperelliptic involution, which acts on the fiber of  $R^1\pi_*\mathbb{Q}$  by  $-1$ .

We conclude that the projection of  $d_1^{0,6} : H^6(\mathcal{M}_3) \rightarrow H^5(\overline{\mathcal{M}}_3^{(1)} \setminus \overline{\mathcal{M}}_3^{(2)})$  onto  $H^5(\mathcal{M}_{2,1} \times \mathcal{M}_{1,1}) = 0$  must be zero. If we show that the higher differentials  $d_r^{0,6}$ ,  $r \geq 2$ , vanish, it will imply that the projection of  $d_1^{0,6} : H^6(\mathcal{M}_3) \rightarrow H^5(\overline{\mathcal{M}}_3^{(1)} \setminus \overline{\mathcal{M}}_3^{(2)})$  to  $H^5(\mathcal{M}_2(2))$  is nontrivial.

For each  $r \geq 2$ , the higher differential  $d_r^{0,6}$  maps  $E_r^{0,6}$ , which is a subspace of  $E_1^{0,6}$ , to  $E_r^{r,5+r}$ , which is a subquotient of  $E_1^{r,5+r}$ . We claim that all these

terms  $E_1^{r,5+r}$  are zero. For  $r = 2$ , the term  $E_1^{r,5+r} = E_1^{2,7} = H^5(\overline{\mathcal{M}}_3^{(2)} \setminus \overline{\mathcal{M}}_3^{(3)})$  is the direct sum of the cohomology groups of connected components of  $\overline{\mathcal{M}}_3^{(2)} \setminus \overline{\mathcal{M}}_3^{(3)}$ . These components are quotients under finite group actions of the following spaces:

$$\mathcal{M}_{1,4}, \quad \mathcal{M}_{2,1} \times \mathcal{M}_{0,3}, \quad \mathcal{M}_{1,3} \times \mathcal{M}_{1,1}, \quad \mathcal{M}_{1,2} \times \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}.$$

Looking at the forgetful map  $\mathcal{M}_{1,4} \rightarrow \mathcal{M}_{1,1}$ , which is topologically a fiber bundle with fiber of the homotopy type of a three-dimensional CW complex and base having the homotopy type of a point, we see that  $H^5(\mathcal{M}_{1,4}) = 0$ . We have already seen that  $\mathcal{M}_{2,1}$  has the rational homology of  $\mathbb{C}P^1$ , and so does  $\mathcal{M}_{2,1} \times \mathcal{M}_{0,3}$ . Similar to  $\mathcal{M}_{1,4}$ , the space  $\mathcal{M}_{1,3}$  has the homotopy type of a CW complex of dimension two, and so does  $\mathcal{M}_{1,3} \times \mathcal{M}_{1,1}$ . A similar argument works for  $\mathcal{M}_{1,2} \times \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$ .

Analyzing similarly the groups  $E_1^{r,5+r} = H^5(\overline{\mathcal{M}}_3^{(r)} \setminus \overline{\mathcal{M}}_3^{(r+1)})$  for  $r \geq 3$ , we quickly see that all of them vanish for dimensional reasons. □

**Corollary 2.7.** *The BV operator*

$$\Delta : H_5(\mathcal{M}_2(2)) \rightarrow H_6(\mathcal{M}_3)$$

*does not vanish and is moreover an isomorphism between these one-dimensional vector spaces over  $\mathbb{Q}$ .*

*Proof.* Working at the dual, cohomology level in the proof of Proposition 2.6, we have seen that  $H^6(\mathcal{M}_3)$  is one-dimensional. We also have  $H^5(\mathcal{M}_2(2)) = \mathbb{Q}$ , given Tommasi’s computation [41, Corollary III.2.2] of the Poincaré-Serre polynomial of  $\mathcal{M}_2(2)$  as  $1 + t^2u^2 + t^5u^{10}$ .

Thus, to prove our claim, we only need to see that  $\Delta \neq 0$ . By Theorem 1.2, we know that  $\Delta = (d_1^{0,6})^* \alpha_0$ . Recall that  $\alpha_0 = \pi_* \rho^1$ , where  $\pi$  and  $\rho$  were given in diagram (12). Note that in this particular case,  $\rho = \text{id}$  and  $\pi$  is the inclusion of  $\mathcal{M}_2(2)$  as a connected component of  $\overline{\mathcal{M}}_3^{(1)} \setminus \overline{\mathcal{M}}_3^{(2)}$ . Thus,  $\Delta = (d_1^{0,6})^* \pi_*$ , which is exactly the linear dual of the projection of  $d_1^{0,6}$  to  $H^5(\mathcal{M}_2(2))$ , whose nonvanishing is proven in Proposition 2.6. □

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