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Modular forms from the Weierstrass functions

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Abstract: We construct holomorphic elliptic modular forms of weight 2 and weight 1, by special values of Weierstrass $\wp$-functions, and by differences of special values of Weierstrass ζ-functions, respectively. Also we calculated the values of these forms at some cusps.

Keywords: Weierstrass $\wp$-function, Weierstrass ζ-function, elliptic modular forms, period integral.

Introduction

In the study of Jacobi inversion problem for the period maps associated with primitive forms of types $\mathrm{A}_2, \mathrm{B}_2$ and G_2, the second named author has introduced a concept of Eisenstein series of types $\mathrm{A}_2, \mathrm{B}_2$ and G_2 (cf. [Sa, §8]).

First, Eisenstein series of type A_2 are nothing but the classical Eisenstein series. In this case their weights are always equal or greater than 3.

Second, Eisenstein series of types B_2 and G_2, for the case when their weights are equal or greater than 3, are described by shifted classical Eisenstein series [Sa, §8]. Their holomorphicity at cusps and the values at cusps can be shown and calculated similar to the classical Eisenstein series by helps of Riemann's zeta-function or Dirichlet's L-functions.

Third, Eisenstein series of types B_2 and G_2, for the case when their weights are equal or less than 2, have completely different expressions. The weight 2 Eisenstein series of types B_2 and G_2 have the expressions as special values of Weierstrass $\wp$-functions. The weight 1 Eisenstein series of type G_2 has the expression as a difference of special values of Weierstrass ζ-functions.

All of these Eisenstein series of types A_2, B_2 and G_2, should be elliptic modular forms, due to the theory of period maps introduced by the second author. Actually, in the first and second cases, it is also clear by their expressions. However, in the third case, it is not so obvious. In the present paper, we give a short way to construct modular forms of weight 2 and 1, including the above third case.

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1. From the Weierstrass $\wp$-function

In this section we construct elliptic modular forms of weight 2 from the Weierstrass $\wp$-function. Since the Weierstrass $\wp$-function can be recognized as a meromorphic Jacobi form of weight 2 and index 0, its special value at $z = s\tau + t$ ($(s,t) \in \mathbb{Q}^2 - \mathbb{Z}^2$) has a modularity. Therefore, it turns out to be an elliptic modular form, by showing the holomorphicity at each cusps. It has been already done by direct calculation (cf. [DS, §4.6]), however, here we give a proof by using Jacobi forms.

1.1. Definition and notation

Let $\Omega \subset \mathbb{C}$ be a $\mathbb{Z}$-module generated by two $\mathbb{R}$-linearly independent elements. The Weierstrass $\wp$-function is defined by

$$\begin{aligned}\wp(\Omega, z) &:= \frac{1}{z^2} + \sum_{\omega\in\Omega-\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \\ &= \frac{1}{z^2} + \sum_{\omega\in\Omega-\{0\}} \frac{(2\omega - z)z}{(\omega - z)^2\omega^2} \\ &= \frac{1}{z^2} + \sum_{\omega\in\Omega-\{0\}} \frac{\left(2-\frac{z}{\omega}\right)z}{\left(1-\frac{z}{\omega}\right)^2} \cdot \frac{1}{\omega^3}.\end{aligned}$$

For a while we fix Ω. Since the sum in the third line of the above definition converges absolutely and locally uniformly with respect to z, $\wp(\Omega, z)$ is a meromorphic function on z in $\mathbb{C}$. The set of all poles of $\wp(\Omega, z)$ is Ω and the order at each pole is 2. Also, it is doubly periodic, namely,

$$\wp(\Omega, z) = \wp(\Omega, z+\omega) \quad (\omega \in \Omega).$$

In this paper we move Ω as well as z. It is easy to see that

$$\wp(\Omega, z) = j^2 \wp(j\Omega, jz) \quad (j \in \mathbb{C} - \{0\}).$$

Here we put

$$\wp(\tau, z) := \wp(\tau\mathbb{Z} + \mathbb{Z}, z).$$

By a similar argument as above, we know that $\wp$ is a meromorphic function on $\mathbb{H} \times \mathbb{C}$, where we denote the complex upper half plane by

$$\mathbb{H} := \{\, \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \,\}.$$

The set of all poles of $\wp(\tau, z)$ is $\{\, (\tau, z) \mid \exists s, t \in \mathbb{Z} \text{ s.t. } z = s\tau + t \,\}$.

1.2. Construction of elliptic modular forms

For $(s,t) \in \mathbb{Q}^2 - \mathbb{Z}^2$, we define a holomorphic function on $\mathbb{H}$ by

$$f_{(s,t)}(\tau) := \wp(\tau, s\tau + t).$$

Then the following lemma holds.

Lemma 1. *We have*

$$\left(f_{(s,t)}|_2 A\right)(\tau) := (c\tau + d)^{-2} f_{(s,t)}\left(\frac{a\tau + b}{c\tau + d}\right) = f_{(s,t)A}(\tau)$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Proof.

$$\begin{aligned}
\left(f_{(s,t)}|_2 A\right)(\tau) &:= (c\tau + d)^{-2} f_{(s,t)}\left(\frac{a\tau + b}{c\tau + d}\right) \\
&= (c\tau + d)^{-2} \wp\left(\frac{a\tau + b}{c\tau + d}, s\frac{a\tau + b}{c\tau + d} + t\right) \\
&= (c\tau + d)^{-2} \wp\left(\frac{a\tau + b}{c\tau + d}\mathbb{Z} + \mathbb{Z}, s\frac{a\tau + b}{c\tau + d} + t\right) \\
&= \wp\big((a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}, s(a\tau + b) + t(c\tau + d)\big) \\
&= \wp\big(\tau\mathbb{Z} + \mathbb{Z}, (sa + tc)\tau + (sb + td)\big) \\
&= \wp\big(\tau, (sa + tc)\tau + (sb + td)\big) \\
&= f_{(s,t)A}(\tau).
\end{aligned}$$

□

Hence, especially, we have $f_{(s,t)}|_2 A = f_{(s,t)}$ for any $A \in \Gamma_{(s,t)}$, where we denote by

$$\Gamma_{(s,t)} := \left\{ A \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; (s,t)A - (s,t) \in \mathbb{Z}^2 \right\}.$$

We remark that $\Gamma_{(s,t)}$ is an elliptic modular group, since $\Gamma_{(s,t)}$ contains the principal congruence subgroup of level L

$$\Gamma(L) := \left\{ A \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{L} \right\},$$

where L is the common denominator of s and t.

In 1985, Eichler and Zagier mentioned that the function $\wp(\tau, z)$ is a meromorphic Jacobi form of weight 2 and index 1. Namely, in their book [EZ], they gave the explicit formula of $\wp(\tau, z)$ as a quotient of holomorphic Jacobi forms [EZ, Theorem. 3.6. (p. 39)]:

$$-\frac{3}{\pi^2}\wp(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\phi_{10,1}(\tau, z)}$$
$$= \frac{\mathbf{e}(z) + 10 + \mathbf{e}(-z)}{\mathbf{e}(z) - 2 + \mathbf{e}(-z)} + 12(\mathbf{e}(z) - 2 + \mathbf{e}(-z))\mathbf{e}(\tau) + \cdots ,$$

where $\mathbf{e}(*) := \exp(2\pi i *)$. This series converges absolutely and locally uniformly in $\{(\tau, z) \in \mathbb{H} \times \mathbb{C} \mid |\operatorname{Im} z| < \operatorname{Im} \tau\}$ (cf. [Ao, §3.2.]). Hence we have

$$\lim_{\tau \to i\infty} f_{(s,t)}(\tau) = \begin{cases} -\dfrac{\pi^2}{3} \cdot \dfrac{\mathbf{e}(t) + 10 + \mathbf{e}(-t)}{\mathbf{e}(t) - 2 + \mathbf{e}(-t)} & (s = 0) \\ -\dfrac{\pi^2}{3} & (0 < |s| < 1) \end{cases}.$$

Since $f_{(s+m,t)} = f_{(s,t)}$ for any $m \in \mathbb{Z}$, we know that $f_{(s,t)}$ is holomorphic at $i\infty$ for any $(s, t) \in \mathbb{Q}^2 - \mathbb{Z}^2$, namely, we have

$$\lim_{\tau \to i\infty} f_{(s,t)}(\tau) = \begin{cases} -\dfrac{\pi^2}{3} \cdot \dfrac{\mathbf{e}(t) + 10 + \mathbf{e}(-t)}{\mathbf{e}(t) - 2 + \mathbf{e}(-t)} & (s \in \mathbb{Z}) \\ -\dfrac{\pi^2}{3} & (s \notin \mathbb{Z}) \end{cases}. \tag{1.1}$$

Therefore, we have the following theorem.

Theorem 2. *$f_{(s,t)}$ is an elliptic modular form of weight 2 with respect to $\Gamma_{(s,t)}$.*

Example 3. *The following two forms appear as Eisenstein series of types* B_2 *and* G_2 *in Saito [Sa, §8], respectively. (Eisenstein series appears in Saito [Sa] should be distinguished from the usual Eisenstein series appears in the theory of modular forms.)*

- $f_{(0,\frac{1}{2})}$, *which appears as* $\omega_0^2\wp\left(\frac{1}{2}\omega_0\right)$ *in Saito [Sa], is an elliptic modular form of weight 2 with respect to* $\Gamma_0(2)$. *We have*

$$\lim_{\tau \to \infty} f_{(0,\frac{1}{2})}(\tau) = \frac{2}{3}\pi^2, \qquad \lim_{\tau \to \infty} \left(f_{(0,\frac{1}{2})}|_2 S\right)(\tau) = -\frac{1}{3}\pi^2.$$

Hence we have $f_{(0,\frac{1}{2})} = \frac{2}{3}\pi^2\alpha_2$*, which is an unique modular forms of weight* 2 *with respect to* $\Gamma_0(2)$ *up to constant multiplier.*

- $f_{(0,\frac{1}{3})}$*, which appears as* $\omega_0^2\wp\left(\frac{1}{3}\omega_0\right)$ *in Saito [Sa], is an elliptic modular form of weight* 2 *with respect to* $\Gamma_0(3)$*. We have*

$$\lim_{\tau\to\infty} f_{(0,\frac{1}{3})}(\tau) = \pi^2, \qquad \lim_{\tau\to\infty}\left(f_{(0,\frac{1}{3})}|_2 S\right)(\tau) = -\frac{1}{3}\pi^2.$$

Hence $f_{(0,\frac{1}{3})} = \pi^2\alpha_1^2$*, which is an unique modular forms of weight* 2 *with respect to* $\Gamma_0(3)$ *up to constant multiplier.*

Here we put $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and denote the Hecke congruence subgroup of level* L *by*

$$\Gamma_0(L) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; b \equiv 0 \pmod{L} \right\}.$$

Notation α_2 *and* α_1 *correspond to* α *which appears in Aoki-Ibukiyama [AI, §6] as elliptic modular forms of weight* 2 *and level* 2 *and weight* 1 *and level* 3*, respectively.*

1.3. Values at cusps

Although we have already known the values of $f_{(s,t)}$ at all cusps in previous subsection, here we calculate them directly from the definition without using the expression of $\wp$ by Jacobi forms. It is much easier than the calculation of the Fourier expansion of $f_{(s,t)}$, essentially given in [DS, §4.6].

By Lemma 1, it is enough to calculate

$$\lim_{\tau\to i\infty} f_{(s,t)}(\tau)$$

for any fixed $0 \leqq s < 1$ and $0 \leqq t < 1$. Let L be a common denominator of s and t. Since $f_{(s,t)}(\tau + L) = f_{(s,t)}(\tau)$, we may assume $\mathrm{Re}(\tau) < L$. Also we assume $\mathrm{Im}(\tau) > L$. Recall that

$$f_{(s,t)}(\tau) = \frac{1}{z(\tau)^2} + \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{\left(2 - \frac{z(\tau)}{\omega(\tau)}\right) z(\tau)}{\left(1 - \frac{z(\tau)}{\omega(\tau)}\right)^2} \cdot \frac{1}{\omega(\tau)^3},$$

where $\omega(\tau) := c\tau + d$ and $z(\tau) := s\tau + t$, converges absolutely. We decompose it as

$$f_{(s,t)}(\tau) = \frac{1}{z(\tau)^2} + \sum_{d\in\mathbb{Z}-\{0\}} \frac{\left(2 - \frac{z(\tau)}{d}\right) z(\tau)}{\left(1 - \frac{z(\tau)}{d}\right)^2} \cdot \frac{1}{d^3} + f^*_{(s,t)}(\tau),$$

where we put

$$f^*_{(s,t)}(\tau) := \sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \frac{\left(2 - \frac{z(\tau)}{\omega(\tau)}\right) z(\tau)}{\left(1 - \frac{z(\tau)}{\omega(\tau)}\right)^2} \cdot \frac{1}{\omega(\tau)^3}.$$

First we show that $\lim_{\tau\to i\infty} f^*_{(s,t)}(\tau) = 0$. Since

$$\left|\frac{z(\tau)}{\omega(\tau)}\right| < \frac{1}{2} \quad \text{for any } \tau$$

except for finitely many (c, d), we have

$$\left| f^*_{(s,t)}(\tau) - \sum_{\text{finite}} \frac{\left(2 - \frac{z(\tau)}{\omega(\tau)}\right) z(\tau)}{\left(1 - \frac{z(\tau)}{\omega(\tau)}\right)^2} \cdot \frac{1}{\omega(\tau)^3} \right| < 10 \sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \left|\frac{z(\tau)}{\omega(\tau)^3}\right|.$$

Here we use the following lemma.

Lemma 4. *Let $k \geqq 3$ be an integer. Then we have*

$$\sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \frac{1}{|\omega(\tau)|^k} < \frac{4}{(\operatorname{Im}\tau)^k}\,\zeta_{\mathrm{R}}(k) + \frac{2\pi}{(\operatorname{Im}\tau)^{k-1}}\,\zeta_{\mathrm{R}}(k-1),$$

where we denote the Riemann's zeta-function by ζ_{R}.

Proof.

$$\sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \frac{1}{|\omega(\tau)|^k} = 2\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} \frac{1}{|c\tau + d|^k}$$

$$< 4\sum_{c=1}^{\infty}\left(\frac{1}{(c\operatorname{Im}\tau)^k} + \int_0^{\infty} \frac{dx}{\left(\sqrt{(c\operatorname{Im}\tau)^2 + x^2}\right)^k}\right)$$

$$
\begin{aligned}
&= 4\sum_{c=1}^{\infty}\left(\frac{1}{(c\,\mathrm{Im}\,\tau)^k} + \int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{k-2}}{(c\,\mathrm{Im}\,\tau)^{k-1}} d\theta\right) \\
&< 4\sum_{c=1}^{\infty}\left(\frac{1}{(c\,\mathrm{Im}\,\tau)^k} + \frac{\pi}{2}\frac{1}{(c\,\mathrm{Im}\,\tau)^{k-1}}\right) \\
&= \frac{4}{(\mathrm{Im}\,\tau)^k}\,\zeta_{\mathrm{R}}(k) + \frac{2\pi}{(\mathrm{Im}\,\tau)^{k-1}}\,\zeta_{\mathrm{R}}(k-1) \qquad \square
\end{aligned}
$$

By using this lemma, we have

$$
\lim_{\tau\to i\infty} f^*_{(s,t)}(\tau) = \sum_{\text{finite}} \lim_{\tau\to i\infty}\left(\frac{\left(2-\frac{z(\tau)}{\omega(\tau)}\right) z(\tau)}{\left(1-\frac{z(\tau)}{\omega(\tau)}\right)^2} \cdot \frac{1}{\omega(\tau)^3}\right) = 0.
$$

Therefore, we have

$$
\lim_{\tau\to i\infty} f_{(s,t)}(\tau) = \lim_{\tau\to i\infty} \frac{1}{z(\tau)^2} + \lim_{\tau\to i\infty} \sum_{d\in\mathbb{Z}-\{0\}} \frac{\left(2-\frac{z(\tau)}{d}\right) z(\tau)}{\left(1-\frac{z(\tau)}{d}\right)^2} \cdot \frac{1}{d^3}. \tag{1.2}
$$

(Case: $s \neq 0$**)**
From (1.2), we have

$$
\lim_{\tau\to i\infty} f_{(s,t)}(\tau) = \sum_{d\in\mathbb{Z}-\{0\}} \frac{-1}{d^2} = -2\,\zeta_{\mathrm{R}}(2).
$$

Comparing with (1.1), we have the famous formula

$$
\zeta_{\mathrm{R}}(2) = \frac{\pi^2}{6}. \tag{1.3}
$$

Since Jacobi forms $\phi_{12,1}$ and $\phi_{10,1}$ are constructed from (Jacobi or lattice) theta-functions, without using the Riemann's zeta-function, this is a new proof of (1.3).

(Case: $s = 0$**)**
From (1.2), we have

$$
\lim_{\tau\to i\infty} f_{(0,t)}(\tau) = \frac{1}{t^2} + \sum_{d\in\mathbb{Z}-\{0\}} \frac{(2-\frac{t}{d})\,t}{(1-\frac{t}{d})^2} \cdot \frac{1}{d^3}
$$

$$= \frac{1}{t^2} + \sum_{d \in \mathbb{Z}-\{0\}} \frac{t}{d^3} \sum_{n=0}^{\infty} (n+2) \left(\frac{t}{d}\right)^n$$
$$= \frac{1}{t^2} + 2 \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} (2n+1) \frac{t^{2n}}{d^{2n+2}}$$
$$= \frac{1}{t^2} + 2 \sum_{n=1}^{\infty} (2n+1)\, \zeta_{\mathrm{R}}(2n+2) t^{2n}.$$

Hence we have

$$\frac{1}{t^2} + 2 \sum_{n=1}^{\infty} (2n+1)\, \zeta_{\mathrm{R}}(2n+2) t^{2n} = -\frac{\pi^2}{3} \cdot \frac{\mathbf{e}(t) + 10 + \mathbf{e}(-t)}{\mathbf{e}(t) - 2 + \mathbf{e}(-t)}. \tag{1.4}$$

This is a relation between the special values of the Riemann's zeta-function. We remark that the equation (1.4) can be shown by using the Bernoulli numbers B_{2n}:

$$\frac{1}{2} z + \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}, \qquad \zeta_{\mathrm{R}}(2n+2) = (-1)^n \frac{B_{2n+2} (2\pi)^{2n+2}}{2(2n+2)!}.$$

2. From the Weierstrass ζ-function

In this section we construct elliptic modular forms of weight 1 from the Weierstrass ζ-function. Since the Weierstrass ζ-function is quasi-periodic and is not doubly periodic, its special value itself does not have a modularity. Classically, to gain a modularity, we modify it to the Hecke form by decreasing the Weierstrass η-function (cf. [La, Chapter 15] or [DS, §4.8]). However, here we can construct an elliptic modular form taking the difference of two special values properly. Although this is a corollary of the classical argument, here we calculate the values at some cusps by much easier calculation.

2.1. Definition and notation

Again let $\Omega \subset \mathbb{C}$ be a $\mathbb{Z}$-module generated by two $\mathbb{R}$-linearly independent elements. The Weierstrass ζ-function is defined by

$$\zeta(\Omega, z) := \frac{1}{z} + \sum_{\omega \in \Omega - \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$
$$:= \frac{1}{z} - \sum_{\omega \in \Omega - \{0\}} \frac{z^2}{(\omega - z)\omega^2}$$

$$:= \frac{1}{z} - \sum_{\omega \in \Omega - \{0\}} \frac{z^2}{(1-\frac{z}{\omega})} \cdot \frac{1}{\omega^3}.$$

For a while we fix Ω. Since the sum in the third line of the above definition converges absolutely and locally uniformly with respect to z, $\zeta(\Omega, z)$ is a meromorphic function on z in $\mathbb{C}$. The set of all poles of $\zeta(\Omega, z)$ is Ω and the order at each pole is 1. It is not doubly periodic, however, $\zeta(\Omega, z+\omega) - \zeta(\Omega, z)$ does not depend on z, but only on ω.

In this paper we move Ω as well as z. It is easy to see that

$$\zeta(\Omega, z) = j\zeta(j\Omega, jz) \quad (j \in \mathbb{C} - \{0\}).$$

Here we put

$$\zeta(\tau, z) := \zeta(\tau\mathbb{Z} + \mathbb{Z}, z).$$

By a similar argument as above, we know that ζ is a meromorphic function on $\mathbb{H} \times \mathbb{C}$. The set of all poles of $\zeta(\tau, z)$ is $\{ (\tau, z) \mid \exists s, t \in \mathbb{Z} \text{ s.t. } z = s\tau + t \}$. We define $\eta_1(\tau)$ and $\eta_2(\tau)$ by

$$\zeta(\tau, z+\tau) - \zeta(\tau, z) = \eta_1(\tau) \quad \text{and} \quad \zeta(\tau, z+1) - \zeta(\tau, z) = \eta_2(\tau).$$

2.2. Construction of an elliptic modular form

For $(s, t) \in \mathbb{Q}^2 - \mathbb{Z}^2$, we define a holomorphic function on $\mathbb{H}$ by

$$g_{(s,t)}(\tau) := \zeta(\tau, s\tau + t).$$

Then the following lemma holds.

Lemma 5. *We have*

$$\left(g_{(s,t)}|_1 A\right)(\tau) := (c\tau + d)^{-1} g_{(s,t)}\left(\frac{a\tau + b}{c\tau + d}\right) = g_{(s,t)A}(\tau)$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Proof.

$$\begin{aligned}\left(g_{(s,t)}|_1 A\right)(\tau) &:= (c\tau + d)^{-1} g_{(s,t)}\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= (c\tau + d)^{-1} \zeta\left(\frac{a\tau + b}{c\tau + d}, s\frac{a\tau + b}{c\tau + d} + t\right)\end{aligned}$$

$$
\begin{aligned}
&= (c\tau+d)^{-1}\zeta\left(\frac{a\tau+b}{c\tau+d}\mathbb{Z}+\mathbb{Z}, s\frac{a\tau+b}{c\tau+d}+t\right)\\
&= \zeta\big((a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}, s(a\tau+b)+t(c\tau+d)\big)\\
&= \zeta\big(\tau\mathbb{Z}+\mathbb{Z}, (sa+tc)\tau+(sb+td)\big)\\
&= \zeta\big(\tau, (sa+tc)\tau+(sb+td)\big)\\
&= g_{(s,t)A}(\tau). \qquad \square
\end{aligned}
$$

However, $g_{(s,t)}$ itself is not modular, since we have

$$
\left(g_{(s,t)}|_1 A\right)(\tau) - g_{(s,t)}(\tau) = g_{(s,t)A}(\tau) - g_{(s,t)}(\tau) = u\eta_1(\tau) + v\eta_2(\tau) \tag{2.1}
$$

for any $A \in \Gamma_{(s,t)}$, where $u := s(a-1)+tc \in \mathbb{Z}$ and $v := sb+t(d-1) \in \mathbb{Z}$.

Now we take $r \in \mathbb{Z}-\{0\}$ such that $(rs, rt) \notin \mathbb{Z}^2$. We remark that $\Gamma_{(s,t)} \subset \Gamma_{(rs,rt)}$. Let

$$
h_{r,(s,t)}(\tau) := rg_{(s,t)}(\tau) - g_{(rs,rt)}(\tau).
$$

Then, from (2.1), $h_{r,(s,t)}$ has the automorphic property of weight 1 with respect to $\Gamma_{(s,t)}$, namely, we have $h_{r,(s,t)}|_1 A = h_{r,(s,t)}$ for any $A \in \Gamma_{(s,t)}$. The following lemma holds.

Lemma 6. *$h_{r,(s,t)}$ is bounded at each cusp.*

We give a proof of this lemma in the next subsection. Consequently, we have the following theorem.

Theorem 7. *$h_{r,(s,t)}$ is an elliptic modular form of weight 1 with respect to $\Gamma_{(s,t)}$.*

More generally, the following theorem holds.

Theorem 8. *Let $U := \{u_1, u_2, \ldots, u_m\}$ be a set of m tuples, where $u_j := (s_j, t_j) \in \mathbb{Q}^2 - \mathbb{Z}^2$. We put*

$$
h_U(\tau) := \sum_{j=1}^{m} g_{u_j}(\tau)
$$

and

$$
\Gamma_U := \bigcap_{j=1}^{m} \Gamma_{u_j}.
$$

If $\sum_{j=1}^{m} u_j = (0,0)$, then h_U is an elliptic modular form of weight 1 with respect to Γ_U.

Here we remark that $g_{(-s,-t)}(\tau) = -g_{(s,t)}(\tau)$. Therefore, this theorem contains Theorem 7. This theorem can be shown by the same way as Theorem 7.

2.3. Values at cusps

In this subsection, we give a proof of Lemma 6, in a similar manner as subsection 1.3. Here we fix s and t. By the argument in the previous section, we may assume that $0 \leqq s < 1$ and $0 \leqq t < 1$. By Lemma 5, it is enough to show that $h_{r,(s,t)}(\tau)$ is bounded for sufficiently large $\operatorname{Im}\tau$. Let L be a common denominator of s and t. Since $h_{r,(s,t)}(\tau + L) = h_{r,(s,t)}(\tau)$, we may assume $\operatorname{Re}(\tau) < L$. Also we assume $\operatorname{Im}(\tau) > L$. From the definition, we have

$$h_{r,(s,t)}(\tau) = \frac{r^2-1}{rz(\tau)} + \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{r(r-1)z(\tau)^2}{\left(1-\frac{z(\tau)}{\omega(\tau)}\right)\left(1-\frac{rz(\tau)}{\omega(\tau)}\right)} \cdot \frac{1}{\omega(\tau)^3}, \tag{2.2}$$

where $\omega(\tau) := c\tau + d$ and $z(\tau) := s\tau + t$. The sum in (2.2) converges absolutely. We decompose it as

$$h_{r,(s,t)}(\tau) = \frac{r^2-1}{rz(\tau)} + \sum_{d\in\mathbb{Z}-\{0\}} \frac{r(r-1)z(\tau)^2}{\left(1-\frac{z(\tau)}{d}\right)\left(1-\frac{rz(\tau)}{d}\right)} \cdot \frac{1}{d^3} + h^*_{r,(s,t)}(\tau),$$

where we put

$$h^*_{r,(s,t)}(\tau) := \sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \frac{r(r-1)z(\tau)^2}{\left(1-\frac{z(\tau)}{\omega(\tau)}\right)\left(1-\frac{rz(\tau)}{\omega(\tau)}\right)} \cdot \frac{1}{\omega(\tau)^3}.$$

Here we show that $h^*_{r,(s,t)}(\tau)$ is bounded for large $\operatorname{Im}\tau$. Since

$$\left|\frac{z(\tau)}{\omega(\tau)}\right| < \frac{1}{2} \quad \text{for any } \tau$$

except for finitely many (c, d), we have

$$\left| h^*_{r,(s,t)}(\tau) - \sum_{\text{finite}} \frac{r(r-1)z(\tau)^2}{\left(1-\frac{z(\tau)}{\omega(\tau)}\right)\left(1-\frac{rz(\tau)}{\omega(\tau)}\right)} \cdot \frac{1}{\omega(\tau)^3} \right| < 4r(r-1) \sum_{\substack{c\in\mathbb{Z}-\{0\}\\ d\in\mathbb{Z}}} \left|\frac{z(\tau)^2}{\omega(\tau)^3}\right|.$$

Hence, by using this Lemma 4, we know that $h^*_{r,(s,t)}(\tau)$ is bounded for large $\operatorname{Im}\tau$. Therefore, $h_{r,(s,t)}(\tau)$ is bounded for large $\operatorname{Im}\tau$ also.

Although we know that $h_{r,(s,t)}(\tau)$ is bounded for large $\operatorname{Im}\tau$, to calculate the value at $i\infty$ is not so easy, because $h_{r,(s,t)}$ does not vanish at $i\infty$ except when $s = 0$. Hereafter, we calculate the value at $i\infty$ only in the case of $s = 0$.

(Case: $s = 0$**)**
When $s = 0$, we have

$$\lim_{\tau \to i\infty} h^*_{r,(0,t)}(\tau) = 0.$$

Therefore we have

$$\begin{aligned}
\lim_{\tau \to i\infty} h_{r,(0,t)}(\tau) &= \frac{r^2-1}{rt} + \sum_{d \in \mathbb{Z}-\{0\}} \frac{r(r-1)t^2}{\left(1-\frac{t}{d}\right)\left(1-\frac{rt}{d}\right)} \cdot \frac{1}{d^3} \\
&= \frac{r^2-1}{rt} + r \sum_{d \in \mathbb{Z}-\{0\}} \frac{t^2}{d^3} \sum_{n=0}^{\infty} \left(r^{n+1}-1\right)\left(\frac{t}{d}\right)^n \\
&= \frac{r^2-1}{rt} + 2r \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} \left(r^{2n}-1\right) \frac{t^{2n+1}}{d^{2n+2}} \\
&= \frac{r^2-1}{rt} + 2r \sum_{n=1}^{\infty} \left(r^{2n}-1\right) \zeta_{\mathrm{R}}(2n+2) t^{2n+1} \\
&= \frac{r^2-1}{rt} + r \sum_{n=1}^{\infty} \left(r^{2n}-1\right) \frac{B_{2n+2}}{(2n+2)!} (-1)^n (2\pi)^{2n+2} t^{2n+1} \\
&= \frac{r^2-1}{rt} + \left(-\frac{1}{rt} \sum_{n=1}^{\infty} \frac{B_{2n+2}}{(2n+2)!} (2\pi i r t)^{2n+2} + \frac{r}{t} \sum_{n=1}^{\infty} \frac{B_{2n+2}}{(2n+2)!} (2\pi i t)^{2n+2} \right) \\
&= \frac{r^2-1}{rt} - \frac{1}{rt} \left(-1 + (\pi i r t) - \frac{1}{12}(2\pi i r t)^2 + \frac{2\pi i r t}{\mathbf{e}(rt)-1} \right) \\
&\quad + \frac{r}{t} \left(-1 + (\pi i t) - \frac{1}{12}(2\pi i t)^2 + \frac{2\pi i t}{\mathbf{e}(t)-1} \right) \\
&= 2\pi i \left(\frac{r-1}{2} + \frac{r}{\mathbf{e}(t)-1} - \frac{1}{\mathbf{e}(rt)-1} \right).
\end{aligned}$$

Example 9. *$h_{2,(0,\frac{1}{3})}$ appears as Eisenstein series of type G_2 in Saito [Sa, §8], that is, $\omega_0\left(2\zeta\left(\frac{1}{3}\omega_0\right) - \zeta\left(\frac{2}{3}\omega_0\right)\right)$. This $h_{2,(0,\frac{1}{3})}$ is an elliptic modular form of weight 1 with respect to $\Gamma_1(3)$. We have*

$$\lim_{\tau \to \infty} h_{2,(0,\frac{1}{3})}(\tau) = -\sqrt{3}\pi i.$$

Hence $h_{2,(0,\frac{1}{3})} = -\sqrt{3}\pi i \alpha_1$, which is an unique modular forms of weight 1 with respect to $\Gamma_1(3)$ up to constant multiplier. Here we put

$$\Gamma_1(L) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; \begin{array}{c} a \equiv d \equiv 1 \pmod{L}, \\ b \equiv 0 \pmod{L} \end{array} \right\}.$$

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References

[Ao] H. Aoki, Generalization of the Weierstrass ℘ function and Maass lifts of weak Jacobi forms.

[AI] H. Aoki and T. Ibukiyama, Simple graded rings of Siegel modular forms of small levels, differential operators and Borcherds products, *Int. J. Math.* **16-3** (2005), 249–279. MR2130626

[DS] F. Diamon and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics **228**, Springer-Verlag, New York (2005). MR2112196

[EZ] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser (1985). MR0781735

[La] S. Lang, *Introduction to Modular Forms*, Grundlehren der mathematischen Wissenschaften **222**, Springer-Verlag, Berlin–New York (1976). MR0429740

[Sa] K. Saito, A view on elliptic integrals from primitive forms.

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