Pure and Applied Mathematics Quarterly
Volume 16, Number 4, 1053–1066, 2020

Parametrization simple irreducible plane curve singularities in arbitrary characteristic

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Abstract: We study the classification of plane curve singularities in arbitrary characteristic. We first give a bound for the determinacy of a plane curve singularity with respect to pararametrization equivalence in terms of its conductor. We then define the notion of parametrization modality for plane curve singularities. Finally, we revisit Mehmood-Pfister's classification of parametrization simple plane curve singularities which are irreducible by giving a concrete list of normal forms of equations and parametrizations. In characteristic zero, the classification of parametrization simple irreducible plane curve singularities was achieved by Bruce and Gaffney.

1. Introduction

We classify irreducible plane curve singularities $f \in K[[x,y]]$ which are simple with respect to parametrization equivalence, where K is an algebraically closed field of arbitrary characteristic. That is, the irreducible plane singularities whose parametrizations have modality 0 up to the change of coordinates in the source and target spaces (or, left-right equivalence, see Section 2.1). The notion of modality was introduced by Arnold in the seventies into the singularity theory for real and complex singularities. He classified simple, unimodal and bimodal hypersurface singularities with respect to right equivalence, i.e. the hypersurface singularities of right modality 0,1,2 respectively [1], [2], [3]. The classifications of contact simple and unimodal complete intersection singularities were done by Giusti [12] and Wall [22]. Classification of contact simple space curve singularities was obtained by Giusti [12] and Frühbis-Krüger [9]. In positive characteristic, the right simple, unimodal and bimodal hypersurface singularities were recently classified by Greuel and the

Received January 2, 2019.
*The author's research is supported by Juan de la Cierva Incorporación IJCI-2016-29891, the ERCEA Consolidator Grant 615655 NMST and the National Foundation for Science and Technology Development (NAFOSTED), Grant number 101.04-2019.316, Vietnam.

author in [14] and [21]. The classification of contact simple hypersurface singularities were achieved by Greuel-Kröning [11], while classifications of contact unimodal and bimodal singularities are still unknown.

Curve singularities can be also described by parametrizations. Two plane curve singularities are parametrization equivalent if their parametrizations are left-right equivalent. The first results on classification of simple curve singularities with respect to parametrization equivalence were obtained by Bruce and Gaffney, for complex irreducible plane curve singularities in $\mathbb{C}\{x, y\}$ [6]. The classifications were extended to irreducible space curves by Gibson and Hobbs [10], irreducible curves of any embedding dimension by Arnold [4], and reducible curves by Kolgushkin and Sadykov [17]. In this paper, after introducing the notion of parametrization modality, we revisit Mehmood-Pfister's classification of parametrization simple irreducible plane curve singularities. This generalizes the result of Bruce and Gaffney to the singularities in arbitrary characteristic (Theorem 3.3). We give lists of normal forms of equations and parametrizations of parametrization simple plane curve singularities which are irreducible (Tables 1, 2, 3 in Section 3). We first study in Section 2 the problem of determinacy with respect to parametrization equivalence. The theory of determinacy was systematically studied by Mather in [18], where he defined the equivalence relations $\mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{L}$ and $\mathcal{A}$ and obtained necessary and sufficient conditions for finite determinacy with respect to them. He also gave estimates for the corresponding determinacy. Lower estimates were provided later by Gaffney, Bruce, du Plessis and Wall. The problem of determinacy in positive characteristic with respect to $\mathcal{R}, \mathcal{K}$ was treated by Boubakri, Greuel and Markwig in [5] and recently by Greuel and Pham [15, 16]. We show that reduced plane curve singularities are finitely determined with respect to parametrization equivalence. Moreover, we give a lower bound for parametrization determinacy of a plane curve singularity in terms of its conductor (Theorem 2.1).

2. Parametrization determinacy

2.1. Parametrization equivalence

For a *plane curve singularity* f, i.e. an element in the maximal ideal $\mathfrak{m}$ in $K[[x, y]]$, there is a unique (up to multiplication with units) decomposition $f = f_1^{\rho_1} \cdot \ldots \cdot f_r^{\rho_r}$, with $f_i \in \mathfrak{m}$ irreducible in $K[[x, y]]$. We assume, in this note, that f is *reduced*, i.e. $\rho_i = 1$ for all $i = 1, \ldots, r$. The integral closure of $R := R_f := K[[x, y]]/\langle f\rangle$ (in the total quotient ring $\mathrm{Quot}(R)$) is isomorphic to $\bar{R} := \bigoplus_{i=1}^r K[[t]]$ (see [7], [13]). A composition $K[[x, y]] \twoheadrightarrow R \hookrightarrow \bar{R} = \bigoplus_{i=1}^r K[[t]]$

of the natural projection $K[[x,y]] \twoheadrightarrow R$ and a normalization $R \hookrightarrow \bar{R}$, is called a *parametrization* of f. It is an element in the space $J := \mathrm{Hom}_K(K[[x,y]], \bar{R})$ of morphisms of local K-algebras. Any element $\psi \in J$ can be identified with the image $\psi(x), \psi(y)$ of x and y in $\bar{R}$. Hence, it is often written as a tuple of r pairs $(x_i(t), y_i(t))$.

Two morphisms of K-algebras $\psi, \psi' \colon K[[x,y]] \to \bar{R} = \bigoplus_{i=1}^r K[[t]]$ are called *left-right equivalent* (or, $\mathcal{A}$-equivalent), $\psi \sim_{\mathcal{A}} \psi'$, if there exist an automorphism ϕ of $\bar{R}$ and an automorphism $\Phi \in Aut_K(K[[x,y]])$ such that $\psi \circ \Phi = \phi \circ \psi'$. By an automorphism of $\bar{R}$ we mean a tuple of automorphisms of $K[[t]]$. Two plane curves $f, g \in K[[x,y]]$ are called *parametrization equivalent*, denoted by $f \sim_p g$, if there exist a parametrization ψ of f and a parametrization ψ' of g such that $\psi \sim_{\mathcal{A}} \psi'$. It was known that two plane curve singularities are parametrization equivalent if and only if they are contact equivalent ([20, Prop. 1.2.10], see also [6, Lemma 2.2] for f irreducible).

2.2. Parametrization determinacy

For each $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geq 0}$, the $\mathbf{k}$-jet of ψ is defined to be the composition $j^{\mathbf{k}}\psi \colon K[[x,y]] \xrightarrow{\psi} \bigoplus_{i=1}^r K[[t]] \to \bigoplus_{i=1}^r K[[t]]/(t^{k_i+1})$. We call ψ *parametrization* $\mathbf{k}$*-determined* if it is parametrization equivalent to every ψ' whose $\mathbf{k}$-jet coincides with that of ψ. We say that f is *parametrization finitely determined* if one (and therefore all) of its parametrizations is parametrization $\mathbf{k}$-determined for some $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geq 0}$. A minimum $\mathbf{k}$ with this property is called a parametrization determinacy of f (or ψ). We show, in the present note, that f is $\mathbf{d}$-parametrization determined, where $\mathbf{d}$ is concretely given by the conductor of f.

Let $\mathcal{C} := (R : \bar{R}) := \{u \in R \mid u\bar{R} \subset R\}$ be the *conductor ideal* of $\bar{R}$ in R (cf. [23]). Then $\mathcal{C}$ is an ideal of both R and $\bar{R}$. So one has $\mathcal{C} = (t^{c_1}) \times \cdots \times (t^{c_r})$ for some $c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0}$. We call $\mathbf{c} := \mathbf{c}(f) := (c_1, \ldots, c_r) \in \mathbb{Z}^r_{\geq 0}$ the *conductor (exponent)* of f. The conductor $\mathbf{c} = (c_1, \ldots, c_r)$ of f is related to the ones of its branches and other invariants by the following beautiful formulas

$$c_i = c(f_i) + \sum_{j \neq i} i(f_i, f_j) \tag{2.1}$$

and

$$|\mathbf{c}| := c_1 + \ldots + c_r = 2\delta, \tag{2.2}$$

where δ is the delta invariant of f, defined as $\delta := \dim_K \bar{R}/R$.

Here for $g, h \in K[[x, y]]$, $i(g, h)$ denotes the *intersection multiplicity* of g, h defined by $i(g, h) := \dim_K K[[x, y]]/(g, h)$. Note that, if h is irreducible and ψ is a parametrization of h, then $i(g, h) = \operatorname{ord} \psi(g)$. Furthermore, the intersection multiplicity is additive, i.e. if $h = h_1 \cdot \ldots \cdot h_r$, then $i(g, h) = i(g, h_1) + \ldots + i(g, h_r)$. The following theorem generalizes a result of Zariski.

Theorem 2.1. *Let $f \in \mathfrak{m} \subset K[[x, y]]$ be reduced, r the number of the irreducible components, $\mathbf{c} \in \mathbb{Z}^r_{\geq 0}$ its conductor, and let*

$$\mathbb{Z}^r_{\geq 0} \ni \mathbf{d} := \begin{cases} 1 & \textit{if } \operatorname{mt}(f) = 1 \\ \mathbf{c} + 1 & \textit{if } \operatorname{mt}(f) = 2 \textit{ and } r = 1 \\ \mathbf{c} & \textit{if } \operatorname{mt}(f) = 2 \textit{ and } r = 2 \\ \mathbf{c} - \mathbf{1} & \textit{if } \operatorname{mt}(f) > 2. \end{cases}$$

Then f is parametrization $\mathbf{d}$-determined. In particular, f is always parametrization $(\mathbf{c} + \mathbf{1})$-determined.

The multiplicity of f, $\operatorname{mt}(f)$, is defined to be the maximum of integers k for which $\langle f \rangle \subset \mathfrak{m}^k$. For the proof of the theorem we need the two following lemmas, which give several relations between the conductor ($\mathbf{c}$) and the maximal contact multiplicity ($\bar{\beta_1}$) of a reduced power series f in some concrete cases. Recall that the *maximal contact multiplicity* of f is defined by

$$\bar{\beta_1}(f) := \sup\{\min_{i=1,\ldots,r} i(f_i, \gamma) | \gamma \text{ regular}\},$$

where $f_1, \ldots, f_r$ are the irreducible components of f.

Lemma 2.2. *Let $f = f_1 \cdot f_2 \in K[[x, y]]$ be reduced such that f_1, f_2 are regular. Then*

$$\bar{\beta_1}(f) = i(f_1, f_2).$$

Proof. By definition, one has

$$\bar{\beta_1}(f) \geq \min\{i(f_1, f_1), i(f_2, f_1)\} = i(f_1, f_2).$$

It remains to prove $\min\{i(f_1, \gamma), i(f_2, \gamma)\} \leq i(f_1, f_2)$, for every non-singular series γ. Since γ is non-singular, there exists a coordinate change $\Phi \in Aut_K K[[x, y]]$ such that $\Phi(\gamma) = y$. Since the intersection multiplicity is invariant under automorphisms of $K[[x, y]]$, it suffices to show $\min\{i(F_1, y), i(F_2, y)\} \leq i(F_1, F_2)$, where $F_1 := \Phi(f_1)$ and $F_2 := \Phi(f_2)$. Since $i(F_1, F_2) \geq 1$, we may assume that $i(F_1, y) > 1$ and $i(F_2, y) > 1$, because, if otherwise, the desired inequality would be trivial. Then

$$F_1(x,y) = a_{01}y + a_{k0}x^k + \sum_{i+kj>k} a_{ij}x^iy^j;\ a_{01}, a_{k0} \neq 0$$

and

$$F_2(x,y) = c_{01}y + c_{l0}x^l + \sum_{i+lj>l} c_{ij}x^iy^j;\ c_{01}, c_{l0} \neq 0,$$

where $k := i(F_1, y) > 1$ and $l := i(F_2, y) > 1$. Here a_{01} and c_{01} are different from zero, since F_1 and F_2 are regular. Thus F_1 has a parametrization

$$x(t) = t;\ y(t) = at^k + \text{ terms of higher order.}$$

Therefore

$$\begin{aligned} F_2(x(t), y(t)) &= ac_{01}t^k + \text{ terms of higher order} \\ &+ c_{l0}t^l + \text{ terms of higher order.} \end{aligned}$$

Hence

$$i(F_1, F_2) = \text{ord } F_2(x(t), y(t)) \geq \min\{k, l\} = \min\{i(F_1, y), i(F_2, y)\}. \qquad \square$$

Lemma 2.3. *Let $f \in K[[x,y]]$ be irreducible.*

(i) If $\mathrm{mt}(f) = 2$, then $c(f) = \bar{\beta}_1(f) - 1$.
(ii) If $\mathrm{mt}(f) > 2$, then $c(f) > \bar{\beta}_1(f)$.

Proof. The lemma follows from the conductor formula (cf. [7, Proposition 4.4.5]). See also [20, Lemma 2.5.5] for an elementary proof. $\square$

Proof of Theorem 2.1. Note that $\mathbf{d} + \mathbf{1} \geq \mathbf{c}$, i.e. $d_i + 1 \geq c_i$ for all $i = 1, \ldots, r$. Let $\psi = (\psi_1, \ldots, \psi_r)\colon K[[x,y]] \to \bar{R}$ be a parametrization of f and let $\psi'\colon K[[x,y]] \to \bar{R}$ such that $j^{\mathbf{d}}(\psi) = j^{\mathbf{d}}(\psi')$. It suffices to show that $\psi \sim_{\mathcal{A}} \psi'$.

Indeed, we have

$$\psi(x) - \psi'(x) \in t^{\mathbf{d}+\mathbf{1}}\bar{R} \subset R \text{ and } \psi(y) - \psi'(y) \in t^{\mathbf{d}+\mathbf{1}}\bar{R} \subset R.$$

Thus there exist $g_1, g_2 \in K[[x,y]]$ such that

$$\psi(g_1) = \psi(x) - \psi'(x) \in t^{\mathbf{d}+\mathbf{1}}\bar{R} \text{ and } \psi(g_2) = \psi(y) - \psi'(y) \in t^{\mathbf{d}+\mathbf{1}}\bar{R}.$$

The following claim shows that, the map $\Phi\colon K[[x,y]] \longrightarrow K[[x,y]]$ sending x, y to $x - g_1(x,y), y - g_2(x,y)$ respectively, is an automorphism of $K[[x,y]]$ and hence $\psi \sim_{\mathcal{A}} \psi'$ as required, since $\psi \circ \Phi = \psi'$.

Claim 2.4. $\mathrm{mt}(g_1) > 1$ *(similarly,* $\mathrm{mt}(g_2) > 1$*).*

Proof of the claim: Since the case $\mathrm{mt}(f) = 1$ is evident, we assume that $\mathrm{mt}(f) \geq 2$. We argue by contradiction. Suppose that it is not true, i.e. $\mathrm{mt}(g_1) = 1$. Then by the definition of the maximal contact multiplicity $\bar{\beta}_1(f)$,

$$\min\{i(f_i, g_1) | i = 1, \dots, r\} \leq \bar{\beta}_1(f). \tag{2.3}$$

The following three steps comprise the proof:

Step 1: $\mathrm{mt}(f) = 2$ and $r = 1$. Then $d = c + 1$ and $\psi(g_1) \in t^{d+1}K[[t]]$. This implies

$$i(f, g_1) = \mathrm{ord}\ \psi(g_1) \geq d + 1 = c + 2 = \bar{\beta}_1(f) + 1,$$

where the last equality is due to Lemma 2.3. This contradicts to (2.2).

Step 2: $\mathrm{mt}(f) = 2$ and $r = 2$. Then $f = f_1 \cdot f_2$ with $\mathrm{mt}(f_1) = \mathrm{mt}(f_2) = 1$ and $\mathbf{d} = \mathbf{c}$. It follows from (2.1) that $c_1 = c_2 = i(f_1, f_2)$. Since $\psi_1(g_1) \in t^{d_1+1}K[[t]]$,

$$i(g_1, f_1) = \mathrm{ord}\ \psi_1(g_1) \geq d_1 + 1 = i(f_1, f_2) + 1.$$

Similarly, $i(g_1, f_2) \geq i(f_1, f_2) + 1$. Combining Lemma 2.2 and (2.2) we get

$$i(f_1, f_2) + 1 \leq \min\{i(f_1, g_1); i(f_2, g_1)\} \leq \bar{\beta}_1(f) = i(f_1, f_2),$$

a contradiction.

Step 3: $\mathrm{mt}(f) > 2$. Then $\mathbf{d} = \mathbf{c} - \mathbf{1}$. Let $f = f_1 \cdot \ldots \cdot f_r$ be an irreducible decomposition of f such that $\mathrm{mt}(f_1) \leq \ldots \leq \mathrm{mt}(f_r)$. We consider the three following cases:

• If $\mathrm{mt}(f_r) > 2$, then $i(f_r, g_1) = \mathrm{ord}\ \psi_r(g_1) \geq d_r + 1 = c_r$. By Lemma 2.3 and by the definition of the maximal contact multiplicity of f_r, one deduce that

$$c(f_r) > \bar{\beta}_1(f_r) \geq i(f_r, g_1) \geq c_r > c(f_r),$$

a contradiction.

• If $\mathrm{mt}(f_r) = 2$, then $r > 1$ and $i(f_r, g_1) = \mathrm{ord}\ \psi_r(g_1) \geq d_r + 1 = c_r$. This implies that $\bar{\beta}_1(f_r) \geq c_r$. By (2.1) and the inequality $i(f_1, f_r) \geq \mathrm{mt}(f_r) = 2$,

$$c_r \geq c(f_r) + i(f_1, f_r) > c(f_r) + 1.$$

It follows from Lemma 2.3 that $c(f_r) = \bar{\beta}_1(f_r) - 1 \geq c_r - 1 > c(f_r)$, which is a contradiction.

• If $\mathrm{mt}(f_r) = 1$ then $\mathrm{mt}(f_1) = \mathrm{mt}(f_2) = \ldots = \mathrm{mt}(f_r) = 1$ and $r = \mathrm{mt}(f) > 2$. Due to (2.1) one has $c_1 \geq i(f_1, f_2) + i(f_1, f_r) \geq i(f_1, f_2) + 1$. Hence

$$i(f_1, g_1) = \text{ord } \psi_1(g_1) \geq d_1 + 1 = c_1 \geq i(f_1, f_2) + 1.$$

Similarly $i(f_2, g_1) \geq i(f_1, f_2) + 1$ and then $i(f_1, f_2) + 1 \leq \min\{i(f_1, g_1); i(f_2, g_1)\}$. It hence follows from Lemma 2.2 that

$$i(f_1, f_2) + 1 \leq \min\{i(f_1, g_1); i(f_2, g_1)\} \leq \bar{\beta}_1(f_1 \cdot f_2) = i(f_1, f_2),$$

a contradiction. This completes the theorem. □

Example 2.5. 1. Let $f = x^2 - y^5$. Then $r(f) = 1$ and $\mathbf{c}(f) = 4$. It is easy to see that f is not parametrization 4-determined.

2. Let $f = (x - y^3)(x - y^5)$. Then $r(f) = 2$, $\mathbf{c}(f) = (3, 3)$ and

$$\psi \colon K[[x, y]] \to K[[t]] \oplus K[[t]], g \mapsto g(t^3, t) \oplus g(t^5, t)$$

is a parametrization of f. It can be easily verified that f is parametrization (3, 2)- but not (2, 2)- determined.

3. Parametrization simple singularities

In this section, we first define the notion of parametrization modality. Then we classifiy parametrization irreducible plane curve singularities whose parametrization modality is zero. This should be considered as a revisited version of [19].

3.1. Parametrization modality

Consider an action of algebraic group G on a variety X (over a given algebraically closed field K) and a *Rosenlicht stratification* $\{(X_i, p_i), i = 1, \ldots, s\}$ of X w.r.t. G. That is, a stratification $X = \cup_{i=1}^{s} X_i$, where the stratum X_i is a locally closed G-invariant subvariety of X such that the projection $p_i : X_i \to X_i/G$ is a geometric quotient. For each open subset $U \subset X$ the modality of U, G-mod(U), is the maximal dimension of the images of $U \cap X_i$ in X_i/G. The modality G-mod(x) of a point $x \in X$ is the minimum of G-mod(U) over all open neighbourhoods U of x.

Let $\mathcal{L} := Aut(K[[x, y]])$ resp. $\mathcal{R} := Aut(\bar{R})$ the left group resp. the right group. The left-right group $\mathcal{A} := \mathcal{R} \times \mathcal{L}$ acts on $J = \mathrm{Hom}_K(K[[x, y]], \bar{R})$

by $((\phi,\Phi),\psi) \mapsto \Phi^{-1}\circ\psi\circ\phi$. Then, two elements $\psi,\psi' \in J$ are left-right equivalent, if and only if they belong to the same $\mathcal{A}$-orbit.

For each $k \in \mathbb{Z}$, denoted by J_k the k-jet space of J, that is, the space of morphisms

$$K[[x,y]] \to \bar{R}_k := \bigoplus_{i=1}^{r} K[[t]]/(t^{k+1}).$$

We may identify an element ψ in J_k with the pair $(\psi(x),\psi(y))$ in $K[[t]]/(t^{k+1})\times K[[t]]/(t^{k+1})$, and therefore J_k can be identified with the variety $\bar{R}_k^2 \cong \mathbb{A}_K^{2(k+1)}$. For each element $\psi \in J$, denoted the $j^k\psi$ the image of ψ by the map induced by the projection $\bar{R} \to \bar{R}_k$. We call ψ to be *left-right k-determined* if it is left-right equivalent to any element in J whose k-jet coincides with $j^k\psi$. A number k is called *left-right sufficiently large* for ψ, if there exists a neighbourhood U of the $j^k\psi$ in J_k such that every $\psi' \in J$ with $j^k\psi' \in U$ is left-right k-determined. We also consider the k-jet of the left-right group $\mathcal{A}$ defined by $\mathcal{A}_k := \mathcal{R}_k \times \mathcal{L}_k$. This group acts naturally on the k-jet space J_k. The *left-right modality* of ψ, $\mathcal{A}$-mod(ψ), is defined to be the $\mathcal{A}_k$-modality of $j^k\psi$ in J_k with k right sufficiently large for ψ.

Let $f \in \mathfrak{m} \subset K[[x,y]]$ be reduced plane curve singularity and let ψ be its parametrization. By Theorem 2.1, ψ is left-right $(|\mathbf{c}|+1)$-determined, where $|\mathbf{c}|$ denotes the sum $c_1+\ldots+c_r$ for $\mathbf{c}=(c_1,\ldots,c_r)$. Note that,

$$|\mathbf{c}| = \sum_{i=1}^{r}\left(c(f_i)+\sum_{j\neq i} i(f_i,f_j)\right) = 2\delta(f).$$

It yields that ψ is left-right $(2\delta(f)+1)$-determined. From the upper semi-continuity of the delta function δ (see [8]), we can show, by using the same argument as in [14], that $k = 2\delta(f)+1$ is left-right sufficiently large for ψ. The *parametrization modality* of f, denoted by $\mathcal{P}$-mod(f), is defined to be the left-right modality of ψ, i.e the number $\mathcal{A}_k$-mod$(j^k\psi)$.

A plane curve singularity $f \in K[[x,y]]$ is called *parametrization simple, uni-modal, bi-modal* or *r-modal* if its parametrization modality is equal to 0,1,2 or r respectively. These notions are independent of the choice of a parametrization, and its sufficiently large number k. This may be proved in much the same way as [14, Prop. 2.6, 2.12]. The simpleness can be also described by deformation theory. A deformation of ψ over $\mathbb{A}^l$ is a morphism $\psi_{\mathbf{s}}\colon K[[x,y]] \to \bar{R}[\mathbf{s}] = \bigoplus_{i=1}^{r} K[\mathbf{s}][[t]]$ such that $\psi_{\mathbf{0}} = \psi$, where $\mathbf{s} = (s_1,\ldots,s_l)$. A plane curve singularity $f \in K[[x,y]]$ is parametrization simple if its parametrization is of finite deformation type, i.e. its parametrization can be deformed only into finitely many left-right classes in J. The following properties are consequences of our definition.

Proposition 3.1. *(i) If f and g are parametrization equivalent, then they have the same parametrization modality.*

(ii) (Semicontinuity of modality) Let $\psi_{\mathbf{s}}$ be deformation of ψ over $\mathbb{A}^l$, then there exists an open neighbourhood U of 0 in $\mathbb{A}^l$ such that $\mathcal{A}\text{-mod}(\psi_s) \leq \mathcal{A}\text{-mod}(\psi)$ for all s in U.

Proof. (i) It follows immediately from our definition of modality.

(ii) By Theorem 2.1, ψ is k-determined with $k = 2\delta(f) + 1$. From the upper semi-continuity of the delta function δ (see [8]), as discussed above, we can show that $k = 2\delta(f) + 1$ is left-right sufficiently large for ψ. The claim is now done by the same argument as in [14, Proposition 2.7]. □

3.2. Parametrization simple irreducible plane curve singularities

Theorem 3.2 ([6, Theorem 3.8])**.** *Let* $\mathrm{char}(K) = 0$*. An irreducible plane curve singularity $f \in \mathfrak{m}^2 \subset K[[x, y]]$ is parametrization simple if and only if one of its parametrizations is $\mathcal{A}$-equivalent to one of the singularities in the Tables 1 (where $\varepsilon \in \{0, 1\}$ and $c_k(y) = a_0 + a_1 y + \ldots + a_k y^k \in K[y]$).*

Table 1: Irreducible simple plane curve singularities ($p = 0$)

Name	Equations	Parametrizations	Conditions
A_{2k}	$x^2 + y^{2k+1}$	(t^2, t^{2k+1})	$k \geq 1$
E_{6k}	$x^3 + y^{3k+1} + c_{k-2}(y)x^2y^{2k+1}$	$(t^3, t^{3k+1} + \varepsilon t^{3(k+q)+2})$	$0 \leq q \leq k-2$
E_{6k+2}	$x^3 + y^{3k+2} + c_{k-2}(y)x^2y^{2k+1}$	$(t^3, t^{3k+2} + \varepsilon t^{3(k+q)+4})$	$0 \leq q \leq k-2$
W_{12}	$x^4 + y^5 + ax^2y^3$	$(t^4, t^5 + \varepsilon t^7)$	
W_{18}	$x^4 + y^7 + c_1(y)x^2y^4$	$(t^4, t^7 + \varepsilon t^q)$	$q = 9, 13$
$\mathrm{W}^{\sharp}_{2q-1}$	$(x^2 + y^3)^2 + c_1(y)xy^{q+4}$	$(t^4, t^6 + t^{2q+5})$	$q \geq 1$

Theorem 3.3. *Let $p = \mathrm{char}(K)$. An irreducible plane curve singularity $f \in \mathfrak{m}^2 \subset K[[x, y]]$ is parametrization simple if and only if one of its parametrizations is $\mathcal{A}$-equivalent to one of the singularities in the Tables 2, 3, 4 (where $e \in \{1, 2\}$ and $\varepsilon \in \{0, 1\}$, $\varepsilon(y) \in \{0, 1, y\}$ and $c_k(y) = a_0 + a_1 y + \ldots + a_k y^k \in K[y]$).*

Proof. The theorem follows from Propositions 3.5 and 3.7 below. □

Remark 3.4. 1. The indices in Tables 2, 3, 4 are the conductors of the corresponding singularities, except for the singularities of type $\mathrm{W}^{\sharp}_{2q-1}$ whose the conductors are $2q + 14$.

2. Our lists of irreducible simple plane curve singularities coincide with the lists in [19] except for the case of characteristic 3. More precisely, the singularities of type W_{18} and $\mathrm{W}^\sharp_{2q-1}$ are excluded in our lists (i.e. they are not parametrization simple).
3. In contrast to the case of characteristic zero, in positive characteristic, there is only one infinite family of simple singualrities (A_{2k}). For example, there are only finitely many simple singualrities of multiplicity at least 3.
4. Since all deformations of parametrization induce a deformation of equation, it follows that, all contact simple singularities are parametrization simple. A direct comparison shows that the lists of Greuel-Kröning in [11] are contained in our lists. However, the singularities of type $E_{6k}, E_{6k+2}, k > 1$ and W in Tables 2, 3, 4 are not contact simple.

Table 2: Irreducible simple plane curve singularities ($p > 3$)

Name	Equations	Parametrizations	Conditions
A_{2k}	$x^2 + y^{2k+1}$	(t^2, t^{2k+1})	$k \geq 1$
E_6	$x^3 + y^4$	(t^3, t^4)	
E_8	$x^3 + y^5$	(t^3, t^5)	$p > 5$
	$x^3 + y^5 + \varepsilon xy^4$	$(t^3, t^5 + \varepsilon t^7)$	$p = 5$
E_{6k}	$x^3 + y^{3k+1} + c_{k-2}(y)x^2y^{2k+1}$	$(t^3, t^{3k+1} + \varepsilon t^{3(k+q)+2})$	$3k \leq ep + 5$
			$e = p \mod 3$
			$0 \leq q \leq k-2$
			$q < k$ if $3k+1 = dp$
E_{6k+2}	$x^3 + y^{3k+2} + c_{k-2}(y)x^2y^{2k+1}$	$(t^3, t^{3k+2} + \varepsilon t^{3(k+q)+4})$	$3k \leq ep + 4$
			$e = 2p \mod 3$
			$0 \leq q \leq k-2$
			$q < k$ if $3k+2 = ep$
W_{12}	$x^4 + y^5 + ax^2y^3$	$(t^4, t^5 + \varepsilon t^q)$	$q = 6, 7$
			$q = 7$ if $p > 5$
W_{18}	$x^4 + y^7 + c_1(y)x^2y^4$	$(t^4, t^7 + \varepsilon t^q)$	$p \neq 7, 13;\ q = 9, 13$
$\mathrm{W}^\sharp_{2q-1}$	$(x^2 + y^3)^2 + c_1(y)xy^{q+4}$	$(t^4, t^6 + t^{2q+5})$	$p \neq 13, q \geq 1$
			$q = 1$ if $p = 5$
			$q \leq 4$ if $p = 7$
			$q \leq 10$ if $p = 11$
			$q \leq \frac{p-13}{2}$ if $p \geq 17$

Proposition 3.5. *Let $f \in \mathfrak{m}^2 \subset K[[x, y]]$ be an irreducible plane curve singularity and let $(x(t), y(t))$ be its parametrization with $m = \mathrm{ord}x(t) = \mathrm{mt}(f) < \mathrm{ord}y(t) = n$. Then f is not parametrization simple if either*

(1) $m > 4$ *or*

Table 3: Irreducible simple plane curve singularities in characteristic 3

Name	Equations	Parametrizations	Conditions
A_{2k}	$x^2 + y^{2k+1}$	(t^2, t^{2k+1})	$1 \le k$
E_6	$x^3 + y^4 + \varepsilon x^2y^2$	$(t^3 + \varepsilon t^5, t^4)$	
E_8	$x^3 + y^5 + \varepsilon(y)x^2y^2$	$(t^3 + \varepsilon t^q, t^5)$	$q = 4, 7$
W_{12}	$x^4 + y^5 + ax^2y^3$	$(t^4, t^5 + \varepsilon t^q)$	$q = 7, 11$

Table 4: Irreducible simple plane curve singularities in characteristic 2

Name	Equations	Parametrizations	Conditions
A_{2k}	$x^2 + y^{2k+1} + \varepsilon xy^{2k-q}$	$(t^{2k+1}, t^2 + \varepsilon t^{2q+1})$	$1 \le q < k$
E_6	$x^3 + y^4 + \varepsilon x^2y^2$	$(t^4 + \varepsilon t^5, t^3)$	
E_8	$x^3 + y^5$	(t^5, t^3)	
E_{12}	$x^3 + y^7 + \varepsilon x^2y^5$	$(t^3, t^7 + \varepsilon t^8)$	

(2) $m = 4$ *and* $p = 2$ *or*
(3) $m = 4$ *and* $n > 7$ *or*
(4) $m = 4$ *and* $n = 7$ *and* $p = 7$*, or*
(5) $m = 4$ *and* $n = 6$ *and* $p \mid (c - 3)$*.*
(6) $m \ge 3, n \ge 6$ *and* $p = 3$ *or*
(7) $m = 3$ *and* $n \ge 8$ *and* $p = 2$ *or*
(8) $m = 3 \nmid n$ *and* $n > ep + 6$ *with* $e = np \mod 3$ *and* $e \in \{1, 2\}$*.*

Proof. All the statements of the proposition was proved in [19] except (6), which will be proved here. Assume that $m \ge 3, n \ge 6$ and $p = 3$. Then ψ can be deformed into a parametrization of the form

$$\left(t^3 + t^4 + O(5), t^6 + t^7 + O(8)\right),$$

which is obviously $\mathcal{A}$-equivalent to

$$\psi' = (x'(t), y'(t)) = \left(t^3 + t^4 + O(5), t^7\right),$$

by the coordinate change $x \mapsto x, y \mapsto -y + x^2$ and a compatible automorphism of form $\phi(t) = t + a_2t^2 + \ldots$. Consider the deformation $\psi'_s := (x'_s(t), y'_s(t))$ of ψ' with $x'_s(t) = x(t) + st^5, y'_s(t) = y'(t)$. It is then easy to see that ψ'_s is $\mathcal{A}$-equivalent to $\psi'_{s'}$ if and only if $s = s'$. This implies that ψ' is not simple, and hence neither is ψ by Proposition 3.1. $\square$

Proposition 3.6 ([19])**.** *Let* $f \in \mathfrak{m}^2 \subset K[[x, y]]$ *be an irreducible plane curve singularity and let* c *be its conductor and let* $(x(t), y(t))$ *be its parametrization with* $m = \operatorname{ord} x(t) < \operatorname{ord} y(t) = n$*.*

(1) If $m = 2$ and $c = 2k$ then f is parametrization equivalent to a singularity of type A_k.

(2) If ψ is $\mathcal{A}$-equivalent to

$$\left(t^3, t^{3k+1} + t^{3(k+q)+2} + \textit{higher order terms}\right)$$

with $p \nmid 3q+1$ then ψ is $\mathcal{A}$-equivalent to

$$\left(t^3, t^{3k+1} + t^{3(k+q)+2}\right).$$

(3) If ψ is $\mathcal{A}$-equivalent to

$$\left(t^3, t^{3k+2} + t^{3(k+q)+4} + \textit{higher order terms}\right)$$

with $p \nmid 3q+2$ then ψ is $\mathcal{A}$-equivalent to

$$\left(t^3, t^{3k+2} + t^{3(k+q)+4}\right).$$

(4) If $m = 4, n = 5$ or $n = 7$, then f is parametrization equivalent to a singularity of type W_c.

(5) If $m = 4, n = 6$ and $p \neq (c-3)$ then ψ is $\mathcal{A}$-equivalent to

$$\left(t^4, t^6 + t^{2q+5}\right)$$

with $c = 2q + 14$.

Proof. cf. [19], Lemma 11, Lemma 4, Lemma 7, Lemma 9. □

Proposition 3.7. *The singularities in Tables 3 are parametrization simple.*

Proof. We first note that the multiplicity is upper semicontinuous by definition. Furthermore, the conductor of irreducible plane curve singularities is equal to 2 times the delta invariant (cf. (2.2)) and hence also upper semicontinuous by [8]. The proposition therefore follows directly from Proposition 3.6 and the upper semicontinuity of the multiplicity m and of the conductor c. For instance, a singularity of type $W^\sharp_{2q-1}$ $(p > 3)$ can be deformed into at most the classes $A_k, E_k, W_{12}, W_{18}, W^\sharp_{2q'-1}$ with $k \leq 2q + 14$ and $q' \leq q$. □

Acknowledgement

A part of this article was done in my thesis under the supervision of Professor Gert-Martin Greuel at the Technische Universität Kaiserslautern. I am grateful to him for many valuable suggestions.

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