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Critical points and mKdV hierarchy of type $C_n^{(1)}$

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Abstract: We consider the population of critical points, generated from the critical point of the master function with no variables, which is associated with the trivial representation of the twisted affine Lie algebra $C_n^{(1)}$. The population is naturally partitioned into an infinite collection of complex cells $\mathbb{C}^m$, where m are positive integers. For each cell we define an injective rational map $\mathbb{C}^m \to \mathcal{M}(C_n^{(1)})$ of the cell to the space $\mathcal{M}(C_n^{(1)})$ of Miura opers of type $C_n^{(1)}$. We show that the image of the map is invariant with respect to all mKdV flows on $\mathcal{M}(C_n^{(1)})$ and the image is point-wise fixed by all mKdV flows $\frac{\partial}{\partial t_r}$ with index r greater than $2m$.

Keywords: Critical points, master functions, mKdV hierarchies, Miura opers, affine Lie algebras.

1. Introduction

Let $\mathfrak{g}$ be a Kac-Moody algebra with invariant scalar product $(\ ,\ )$, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra, $\alpha_0, \dots, \alpha_n$ simple roots. Let $\Lambda_1, \dots, \Lambda_N$ be dominant integral weights, $k_0, \dots, k_n$ nonnegative integers, $k = k_0 + \dots + k_n$.

Consider $\mathbb{C}^N$ with coordinates $z = (z_1, \dots, z_N)$. Consider $\mathbb{C}^k$ with coordinates u collected into $n+1$ groups, the j-th group consisting of k_j variables,

$$u = (u^{(0)}, \dots, u^{(n)}), \qquad u^{(j)} = (u_1^{(j)}, \dots, u_{k_j}^{(j)}).$$

The *master function* is the multivalued function on $\mathbb{C}^k \times \mathbb{C}^N$ defined by the formula

$$(1.1) \qquad \Phi(u,z) = \sum_{a<b} (\Lambda_a, \Lambda_b) \ln(z_a - z_b) - \sum_{a,i,j} (\alpha_j, \Lambda_a) \ln(u_i^{(j)} - z_a) +$$

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$$+\sum_{j<j'}\sum_{i,i'}(\alpha_j,\alpha_{j'})\ln(u_i^{(j)}-u_{i'}^{(j')})+\sum_{j}\sum_{i<i'}(\alpha_j,\alpha_j)\ln(u_i^{(j)}-u_{i'}^{(j)}),$$

with singularities at the places where the arguments of the logarithms are equal to zero.

A point in $\mathbb{C}^k\times\mathbb{C}^N$ can be interpreted as a collection of particles in $\mathbb{C}$: $z_a, u_i^{(j)}$. A particle z_a has weight Λ_a, a particle $u_i^{(j)}$ has weight $-\alpha_j$. The particles interact pairwise. The interaction of two particles is determined by the scalar product of their weights. The master function is the "total energy" of the collection of particles.

Notice that all scalar products are integers. So the master function is the logarithm of a rational function. From a "physical" point of view, all interactions are integer multiples of a certain unit of measurement. This is important for what will follow.

The variables u are the *true* variables, variables z are *parameters*. We may think that the positions of z-particles are fixed and the u-particles can move.

There are "global" characteristics of this situation,

$$I(z,\kappa)=\int e^{\Phi(u,z)/\kappa}A(u,z)du,$$

where $A(u,z)$ is a suitable density function, κ a parameter, and there are "local" characteristics – critical points of the master function with respect to the u-variables,

$$d_u\Phi(u,z)=0.$$

A critical point is an equilibrium position of the u-particles for fixed positions of the z-particles. In this paper we are interested in the equilibrium positions of the u-particles.

Examples of master functions associated with $\mathfrak{g}=\mathfrak{sl}_2$ were considered by Stieltjes and Heine in 19th century, see for example [12]. The master functions were introduced in [10] to construct integral representations for solutions of the KZ equations, see also [13, 14].

The critical points of master functions with respect to the u-variables were used to find eigenvectors in the associated Gaudin models by the Bethe ansatz method, see [2, 9, 15]. In important cases the algebra of functions on the critical set of a master function is closely related to Schubert calculus, see [5].

In [11, 6] it was observed that the critical points of master functions with respect to the u-variables can be deformed and form families. Having one critical point, one can construct a family of new critical points. The family is called a population of critical points. A point of the population is a critical point of the same master function or of another master function associated with the same $\mathfrak{g}, \Lambda_1, \dots, \Lambda_N$ but with different integer parameters $k_0, \dots, k_n$. The population is a variety isomorphic to the flag variety of the Kac-Moody algebra $\mathfrak{g}^t$ Langlands dual to $\mathfrak{g}$, see [6, 7, 4].

In [17], it was discovered that the population, originated from the critical point of the master function associated with the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$ and the parameters $N = 0, k_0 = \cdots = k_n = 0$, is connected with the mKdV integrable hierarchy associated with $\widehat{\mathfrak{sl}}_{n+1}$. Namely, that population can be naturally embedded into the space of $\widehat{\mathfrak{sl}}_{n+1}$ Miura opers so that the image of the embedding is invariant with respect to all mKdV flows on the space of Miura opers. For $n = 1$, that result follows from the classical paper by M. Adler and J. Moser [1], which served as a motivation for [17].

The case of the twisted affine Lie algebra $A_{2n}^{(2)}$ was considered in [16, 18]. In this paper we prove analogous statements for the twisted affine Lie algebra $C_n^{(1)}$.

In Sections 2–4 we follow the paper [3] by V. Drinfled and V. Sokolov. We review the affine Lie algebras $A_{2n-1}^{(1)}$ and $C_n^{(1)}$, the associated mKdV and KdV hierarchies, Miura maps. In particular, we describe the $C_n^{(1)}$ mKdV hierarchy as a sequence of commuting flows on the infinite-dimensional space $\mathcal{M}(C_n^{(1)})$ of the $C_n^{(1)}$ Miura opers.

In Section 5 we study the tangent maps to Miura maps. In Section 6, we introduce our master functions,

$$
\begin{aligned}
(1.2) \qquad \Phi(u,k) \;&=\; 2\sum_{i<i'} \ln(u_i^{(0)} - u_{i'}^{(0)}) + 4\sum_{j=1}^{n-1}\sum_{i<i'} \ln(u_i^{(j)} - u_{i'}^{(j)}) \\
&+\; 2\sum_{i<i'} \ln(u_i^{(n)} - u_{i'}^{(n)}) - 2\sum_{j=0}^{n-1}\sum_{i,i'} \ln(u_i^{(j)} - u_{i'}^{(j+1)}).
\end{aligned}
$$

This master function is the special case of the master function in (1.1). The master function in (1.2) is defined by formula (1.1) for $\mathfrak{g}$ being the Langlands dual to $C_n^{(1)}$ and $N = 0$, see a remark in Section 6.1.

Following [6, 7, 17], we describe the generation procedure of new critical points starting from a given critical point of $\Phi(u, k)$. We define the population of critical points generated from the critical point of the function with no variables, namely, the function corresponding to the parameters $k_0 = \cdots = k_n = 0$. That population is partitioned into complex cells $\mathbb{C}^m$ labeled by degree increasing sequences $J = (j_1, \dots, j_m)$, see the definition in Section 6.5.

In Theorem 6.4 we deduce from [8] that every critical point of the master function in (1.2) with arbitrary parameters $k_0, \dots, k_n$ belongs a cell of our population. Moreover, a function in (1.2) with some parameters $k_0, \dots, k_n$ either does not have critical points at all or its critical points form a cell $\mathbb{C}^m$ corresponding to a degree increasing sequence.

In Section 7, to every degree increasing sequence J we assign a rational injective map $\mu^J : \mathbb{C}^m \to \mathcal{M}(C_n^{(1)})$ of the cell corresponding to J to the space $\mathcal{M}(C_n^{(1)})$ of Miura opers of type $C_n^{(1)}$. We describe properties of that map.

In Section 8, we formulate and prove our main result. Theorem 8.1 says that for any degree increasing sequence, the variety $\mu^J(\mathbb{C}^m)$ is invariant with respect to all mKdV flows on $\mathcal{M}(C_n^{(1)})$ and that variety is point-wise fixed by all flows $\frac{\partial}{\partial t_r}$ with index r greater than $2m$, see Theorem 8.1.

This theorem shows that there is a deep interrelation between the critical set of the master functions of the form (1.2) and rational finite-dimensional submanifolds of the space $\mathcal{M}(C_n^{(1)})$, invariant with respect to all flows of the $C_n^{(1)}$ mKdV hierarchy.

Initially the critical points of the master functions were related to quantum integrable systems of the Gaudin type through the Bethe ansatz, [10, 2, 9, 15]. Our result shows that the critical points are also related to the classical integrable systems, namely, the mKdV hierarchies.

In the next papers we plan to extend this result to other affine Lie algebras.

2. Kac-Moody algebra of type $A_{2n-1}^{(1)}$

In this section we follow [3, Section 5].

2.1. Definition

For $n \geqslant 2$, consider the $2n \times 2n$ Cartan matrix of type $A_{2n-1}^{(1)}$,

$$A_{2n-1}^{(1)} = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,2n-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{2n-1,0} & a_{2n-1,1} & \dots & a_{2n-1,2n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

For example, for $n = 2$, we have

$$A_3^{(1)} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

The Kac-Moody algebra $\mathfrak{g}(A_{2n-1}^{(1)})$ *of type* $A_{2n-1}^{(1)}$ is the Lie algebra with *canonical generators* $E_i, H_i, F_i \in \mathfrak{g}(A_{2n-1}^{(1)})$, $i = 0, \dots, 2n-1$, subject to the relations:

$$[E_i, F_j] = \delta_{i,j} H_i,$$
$$[H_i, E_j] = a_{i,j} E_j, \qquad [H_i, F_j] = -a_{i,j} F_j, \qquad (\operatorname{ad} E_i)^{1-a_{i,j}} E_j = 0,$$
$$(\operatorname{ad} F_i)^{1-a_{i,j}} F_j = 0, \qquad [H_i, H_j] = 0, \qquad \sum_{i=0}^{2n-1} H_i = 0,$$

see [3, Section 5]. The Lie algebra $\mathfrak{g}(A_{2n-1}^{(1)})$ is graded with respect to the *standard grading*, $\deg E_i = 1, \deg F_i = -1$, $i = 0, \dots, 2n-1$. Let $\mathfrak{g}(A_{2n-1}^{(1)})^j = \{x \in \mathfrak{g}(A_{2n-1}^{(1)}) \mid \deg x = j\}$, then $\mathfrak{g}(A_{2n-1}^{(1)}) = \oplus_{j\in\mathbb{Z}} \mathfrak{g}(A_{2n-1}^{(1)})^j$.

Notice that $\mathfrak{g}(A_{2n-1}^{(1)})^0$ is the $2n-1$-dimensional space generated by the H_i. Denote $\mathfrak{h} = \mathfrak{g}(A_{2n-1}^{(1)})^0$. Introduce elements α_j of the dual space $\mathfrak{h}^*$ by the conditions $\langle \alpha_j, H_i \rangle = a_{i,j}$ for $i, j = 0, \dots, 2n-1$. For $j = 0, 1, \dots, 2n-1$, we denote by $\mathfrak{n}_j^- \subset \mathfrak{g}(A_{2n-1}^{(1)})$ the Lie subalgebra generated by F_i, $i \in \{0, 1, \dots, 2n-1\}$, $i \neq j$. For example, $\mathfrak{n}_0^-$ is generated by $F_1, F_2, \dots, F_{2n-1}$.

2.2. Realizations of $\mathfrak{g}(A_{2n-1}^{(1)})$

Consider the complex Lie algebra $\mathfrak{sl}_{2n}$ with standard basis $e_{i,j}$, $i, j = 1, \dots, 2n$. Let $w = e^{2\pi i/2n}$. Define the *Coxeter automorphism* $C : \mathfrak{sl}_{2n} \to \mathfrak{sl}_{2n}$ of order $2n$ by the formula

$$C(X) = SXS^{-1}, \quad S = \operatorname{diag}(1, w, \dots, w^{2n-1}).$$

Denote $(\mathfrak{sl}_{2n})_j = \{x \in \mathfrak{sl}_{2n} \mid Cx = w^j x\}$. The twisted Lie subalgebra $L(\mathfrak{sl}_{2n}, C) \subset \mathfrak{sl}_{2n}[\xi, \xi^{-1}]$ is the subalgebra

$$L(\mathfrak{sl}_{2n}, C) = \oplus_{j\in\mathbb{Z}} \xi^j \otimes (\mathfrak{sl}_{2n})_{j \bmod 2n}.$$

The isomorphism $\tau_C : \mathfrak{g}(A^{(1)}_{2n-1}) \to L(\mathfrak{sl}_{2n}, C)$ is defined by the formula,

$$\begin{aligned} E_0 &\mapsto \xi \otimes e_{1,2n}, & E_i &\mapsto \xi \otimes e_{i+1,i},\\ F_0 &\mapsto \xi^{-1} \otimes e_{2n,1}, & F_i &\mapsto \xi^{-1} \otimes e_{i,i+1},\\ H_0 &\mapsto 1 \otimes (e_{1,1} - e_{2n,2n}), & H_i &\mapsto 1 \otimes (-e_{i,i} + e_{i+1,i+1}), \end{aligned}$$

for $i = 1, \dots, 2n-1$. Under this isomorphism we have $\mathfrak{g}(A^{(1)}_{2n-1})^j = \xi^j \otimes (\mathfrak{sl}_{2n})_j$.

The standard isomorphism $\tau_0 : \mathfrak{g}(A^{(1)}_{2n-1}) \to \mathfrak{sl}_{2n}[\lambda, \lambda^{-1}]$ is defined by the formula,

$$\begin{aligned} E_0 &\mapsto \lambda \otimes e_{1,2n}, & E_i &\mapsto 1 \otimes e_{i+1,i},\\ F_0 &\mapsto \lambda^{-1} \otimes e_{2n,1}, & F_i &\mapsto 1 \otimes e_{i,i+1},\\ H_0 &\mapsto 1 \otimes (e_{1,1} - e_{2n,2n}), & H_i &\mapsto 1 \otimes (-e_{i,i} + e_{i+1,i+1}), \end{aligned}$$

for $i = 1, \dots, 2n-1$.

2.3. Element $\Lambda^{(1)}$

Denote by $\Lambda^{(1)}$ the element $\sum_{j=0}^{2n-1} E_j \in \mathfrak{g}(A^{(1)}_{2n-1})$. Then $\mathfrak{z}(A^{(1)}_{2n-1}) = \{x \in \mathfrak{g}(A^{(1)}_{2n-1}) \mid [\Lambda^{(1)}, x] = 0\}$ is an abelian Lie subalgebra of $\mathfrak{g}(A^{(1)}_{2n-1})$. Denote $\mathfrak{z}(A^{(1)}_{2n-1})^j = \mathfrak{z}(A^{(1)}_{2n-1}) \cap \mathfrak{g}(A^{(1)}_{2n-1})^j$, then $\mathfrak{z}(A^{(1)}_{2n-1}) = \oplus_{j\in\mathbb{Z}} \mathfrak{z}(A^{(1)}_{2n-1})^j$. We have $\dim \mathfrak{z}(A^{(1)}_{2n-1})^j = 1$ if $j \neq 0 \bmod 2n$ and $\dim \mathfrak{z}(A^{(1)}_{2n-1})^j = 0$ otherwise.

Let $\mathfrak{g}(A^{(1)}_{2n-1})$ be realized as $L(\mathfrak{sl}_{2n}, C)$ and written out as $2n\times 2n$-matrices. For $m \in \mathbb{Z}$ and $1 \leqslant j < 2n$, we introduce the element

$$A_{(2n)m+j} = \xi^{(2n)m+j} \otimes \begin{pmatrix} 0 & I_j \\ I_{2n-j} & 0 \end{pmatrix} \quad \in \quad L(\mathfrak{sl}_{2n}, C),$$

where I_j is the $j \times j$ identity matrix. We have $A_{(2n)m+j} = (A_1)^{(2n)m+j}$.

If $\mathfrak{g}(A^{(1)}_{2n-1})$ is realized as $L(\mathfrak{sl}_{2n}, \sigma_0)$, we introduce the element

$$B_{(2n)m+j} = \begin{pmatrix} 0 & \lambda^{m+1} \otimes I_j \\ \lambda^m \otimes I_{2n-j} & 0 \end{pmatrix} \quad \in \quad L(\mathfrak{sl}_{2n}, \sigma_0).$$

We have $B_{(2n)m+j} = (B_1)^{(2n)m+j}$.

Lemma 2.1. *For any $m \in \mathbb{Z}$, $1 \leqslant j < 2n$, the elements $(\tau_C)^{-1}(A_{(2n)m+j})$, $(\tau_0)^{-1}(B_{(2n)m+j})$ of $\mathfrak{z}(A^{(1)}_{2n-1})^{(2n)m+j}$ are equal.* □

Denote by $\Lambda^{(1)}_{(2n)m+j}$ the elements $(\tau_C)^{-1}(A_{(2n)m+j})$ and $(\tau_0)^{-1}(B_{(2n)m+j})$ of $\mathfrak{g}(A^{(1)}_{2n-1})$. Notice that $\Lambda^{(1)}_1 = \sum_{i=0}^{2n-1} E_i = \Lambda^{(1)}$. For any $m \in \mathbb{Z}, 1 \leqslant j < 2n$, the element $\Lambda^{(1)}_{(2n)m+j}$ generates $\mathfrak{z}(A^{(1)}_{2n-1})^{(2n)m+j}$.

Let $T = \sum_{j=-\infty}^{m} T_j$ be a formal series with $T_j \in \mathfrak{g}(A^{(1)}_{2n-1})^j$. Denote $T^+ = \sum_{j=0}^{m} T_j,\ \ T^- = \sum_{j<0} T_j$. Let $\mathfrak{g}(A^{(1)}_{2n-1})$ be realized as $\mathfrak{sl}_{2n}[\lambda, \lambda^{-1}]$. Consider $\Lambda^{(1)} = B_1$ as a $2n \times 2n$ matrix depending on the parameter λ. By [3, Lemma 3.4], we may represent T uniquely in the form $T = \sum_{j=-\infty}^{k} b_j\, (\Lambda^{(1)})^j$, $b_j \in$ Diag, where Diag $\subset \mathfrak{gl}_{2n}$ is the space of diagonal $2n \times 2n$ matrices. Denote $(T)^+_{\Lambda^{(1)}} = \sum_{j=0}^{k} b_j\, (\Lambda^{(1)})^j,\ \ (T)^-_{\Lambda^{(1)}} = \sum_{j<0} b_j\, (\Lambda^{(1)})^j$.

Lemma 2.2. *We have* $(T)^+_{\Lambda^{(1)}} = T^+$, $(T)^-_{\Lambda^{(1)}} = T^-$, $b_0 = T^0$.

Proof. The isomorphism $\iota : \mathfrak{sl}_{2n}[\lambda, \lambda^{-1}] \to L(\mathfrak{sl}_{2n}, C)$ is given by the formula $\lambda^m \otimes e_{k,l} \mapsto \xi^{(2n)m+k-l} \otimes e_{k,l}$. We have $\iota(b_0) = \iota(1 \otimes (b_0^1 e_{1,1} + \cdots + b_0^{2n} e_{2n,2n})) = 1 \otimes (b_0^1 e_{1,1} + \cdots + b_0^{2n} e_{2n,2n}) \in \mathfrak{g}(A^{(1)}_{2n-1})^0$, $\iota(b_1 \Lambda^{(1)}) = \iota((b_1^1 e_{1,1} + b_1^2 e_{2,2} + \cdots + b_1^{2n} e_{2n,2n})(e_{2,1} + \cdots + e_{2n-1,2n-2} + e_{2n,2n-1} + \lambda e_{1,2n})) = \iota(b_1^1 \lambda e_{1,2n} + b_1^2 e_{2,1} + \cdots + b_1^{2n} e_{2n,2n-1}) = \xi \otimes (b_1^1 e_{1,2n} + b_1^2 e_{2,1} + \cdots + b_1^{2n} e_{2n,2n-1}) \in \mathfrak{g}(A^{(1)}_{2n-1})^1$, $\iota(b_{-1} (\Lambda^{(1)})^{-1}) = \iota((b_{-1}^1 e_{1,1} + b_{-1}^2 e_{2,2} + \cdots + b_{-1}^{2n} e_{2n,2n})(e_{1,2} + \cdots + e_{2n-1,2n} + \lambda^{-1} e_{2n,1})) = \iota(b_{-1}^1 e_{1,2} + \cdots + b_{-1}^{2n-1} e_{2n-1,2n} + b_{-1}^{2n} \lambda^{-1} e_{2n,1}) = \xi^{-1} \otimes b_{-1}^1 e_{1,2} + \cdots + b_{-1}^{2n-1} e_{2n-1,2n} + b_{-1}^{2n} e_{2n,1}) \in \mathfrak{g}(A^{(1)}_{2n-1})^{-1}$. Similarly one checks that $\iota(b_j\, (\Lambda^{(1)})^j) \in \mathfrak{g}(A^{(1)}_{2n-1})^j$ for any j. □

We have $(\Lambda^{(1)})^{-1} = \sum_{i=1}^{2n-1} e_{i,i+1} + \lambda^{-1} e_{2n,1}$, and

$$E_0 = \Lambda^{(1)} e_{2n,2n},\ \ E_i = \Lambda^{(1)} e_{i,i},\ \ F_0 = e_{2n,2n} (\Lambda^{(1)})^{-1},\ \ F_i = e_{i,i} (\Lambda^{(1)})^{-1},$$

for $i = 1, \ldots, 2n-1$.

Lemma 2.3. *Consider the elements* $F_0, F_i + F_{2n-i}$ *for* $i = 1, \ldots, n-1$ *as* $2n \times 2n$ *matrices. Let* $g \in \mathbb{C}$. *Then*

$$\begin{aligned} e^{gF_0} &= 1 + g\, e_{2n,2n} (\Lambda^{(1)})^{-1}, \\ e^{g(F_i + F_{2n-i})} &= 1 + g\, (e_{i,i} + e_{2n-i,2n-i}) (\Lambda^{(1)})^{-1}, \\ e^{gF_n} &= 1 + g\, e_{n,n} (\Lambda^{(1)})^{-1}. \end{aligned} \tag{2.1}$$

□

Lemma 2.4. *We have*

$$e_{i+1,i+1} \Lambda^{(1)} = \Lambda^{(1)} e_{i,i}, \qquad e_{i,i} (\Lambda^{(1)})^{-1} = (\Lambda^{(1)})^{-1} e_{i+1,i+1}, \tag{2.2}$$

for all i, *where we set* $e_{2n+1,2n+1} = e_{1,1}$. □

3. Kac-Moody algebra of type $C_n^{(1)}$

In this section we follow [3, Section 5].

3.1. Definition

For $n \geqslant 2$, consider the $(n+1) \times (n+1)$ Cartan matrix of type $C_n^{(1)}$,

$$\begin{aligned} C_n^{(1)} &= \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n,0} & a_{n,1} & \dots & a_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -2 & 2 & -1 & 0 & \dots & \dots & \dots \\ 0 & -1 & 2 & -1 & \dots & \dots & \dots \\ \dots & 0 & -1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2 & -1 & 0 \\ \dots & \dots & \dots & \dots & -1 & 2 & -2 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}. \end{aligned}$$

For example, for $n = 2$, we have

$$C_n^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

The Kac-Moody algebra $\mathfrak{g}(C_n^{(1)})$ *of type* $C_n^{(1)}$ is the Lie algebra with *canonical generators* $e_i, h_i, f_i \in \mathfrak{g}(C_n^{(1)})$, $i = 0, \dots, n$, subject to the relations

$$\begin{aligned} [e_i, f_j] &= \delta_{i,j} h_i, \qquad [h_i, e_j] = a_{i,j} e_j, \qquad [h_i, f_j] = -a_{i,j} f_j, \\ (\operatorname{ad} e_i)^{1-a_{i,j}} e_j &= 0, \quad (\operatorname{ad} f_i)^{1-a_{i,j}} f_j = 0, \qquad [h_i, h_j] = 0, \\ & h_0 + \dots + h_n = 0, \end{aligned}$$

see [3, Section 5].

The Lie algebra $\mathfrak{g}(C_n^{(1)})$ is graded with respect to the *standard grading*, $\deg e_i = 1$, $\deg f_i = -1$, $i = 0, \dots, n$. Let $\mathfrak{g}(C_n^{(1)})^j = \{x \in \mathfrak{g}(C_n^{(1)}) \mid \deg x = j\}$, then $\mathfrak{g}(C_n^{(1)}) = \oplus_{j \in \mathbb{Z}} \mathfrak{g}(C_n^{(1)})^j$.

Notice that $\mathfrak{g}(C_n^{(1)})^0$ is the n-dimensional space generated by the h_i. Denote $\mathfrak{h} = \mathfrak{g}(C_n^{(1)})^0$. Introduce elements α_j of the dual space $\mathfrak{h}^*$ by the conditions $\langle \alpha_j, h_i \rangle = a_{i,j}$ for $i, j = 0, \dots, n$.

3.2. Realizations of $\mathfrak{g}(C_n^{(1)})$

Recall the twisted Lie subalgebra $L(\mathfrak{sl}_{2n}, C)$. We have an embedding $\tilde{\tau}_C : \mathfrak{g}(C_n^{(1)}) \hookrightarrow L(\mathfrak{sl}_{2n}, C)$ defined by the formula

$$\begin{aligned} &e_0 \mapsto \xi \otimes e_{1,2n}, && e_n \mapsto \xi \otimes e_{n+1,n} \\ &f_0 \mapsto \xi^{-1} \otimes e_{2n,1}, && f_n \mapsto \xi^{-1} \otimes e_{n,n+1} \\ &e_i \mapsto \xi \otimes (e_{i+1,i} + e_{2n+1-i,2n-i}), && f_i \mapsto \xi^{-1} \otimes (e_{i,i+1} + e_{2n-i,2n+1-i}), \\ &h_0 \mapsto 1 \otimes (e_{1,1} - e_{2n,2n}), && h_n \mapsto 1 \otimes (-e_{n,n} + e_{n+1,n+1}), \\ &h_i \mapsto 1 \otimes (-e_{i,i} + e_{i+1,i+1} - e_{2n-i,2n-i} + e_{2n+1-i,2n+1-i}), \end{aligned}$$

for $i = 1, \ldots, n-1$. Under this embedding we have $\mathfrak{g}(C_n^{(1)})^j \subset \xi^j \otimes (\mathfrak{sl}_{2n})_j$.

We also have the standard embedding $\tilde{\tau}_0 : \mathfrak{g}(C_n^{(1)}) \hookrightarrow \mathfrak{sl}_{2n}[\lambda, \lambda^{-1}]$ defined by the formula

$$\begin{aligned} &e_0 \mapsto \lambda \otimes e_{1,2n}, && e_i \mapsto 1 \otimes (e_{i+1,i} + e_{2n+1-i,2n-i}), \\ &f_0 \mapsto \lambda^{-1} \otimes e_{2n,1}, && f_i \mapsto 1 \otimes (e_{i,i+1} + e_{2n-i,2n+1-i}), \\ &e_n \mapsto 1 \otimes e_{n+1,n}, && f_n \mapsto 1 \otimes e_{n,n+1}, \\ &h_0 \mapsto 1 \otimes (e_{1,1} - e_{2n,2n}), && h_n \mapsto 1 \otimes (-e_{n,n} + e_{n+1,n+1}), \\ &h_i \mapsto 1 \otimes (-e_{i,i} + e_{i+1,i+1} - e_{2n-i,2n-i} + e_{2n+1-i,2n+1-i}), \end{aligned}$$

for $i = 1, \ldots, n-1$.

3.3. Element $\Lambda^{(2)}$

Denote by $\Lambda^{(2)}$ the element $\sum_{i=0}^{n} e_i \in \mathfrak{g}(C_n^{(1)})$. Then $\mathfrak{z}(C_n^{(1)}) = \{x \in \mathfrak{g}(C_n^{(1)}) \mid [\Lambda^{(2)}, x] = 0\}$ is an abelian Lie subalgebra of $\mathfrak{g}(C_n^{(1)})$. Denote $\mathfrak{z}^j(C_n^{(1)}) = \mathfrak{z}(C_n^{(1)}) \cap \mathfrak{g}(C_n^{(1)})^j$, then $\mathfrak{z}(C_n^{(1)}) = \oplus_{j \in \mathbb{Z}} \mathfrak{z}(C_n^{(1)})^j$. We have $\dim \mathfrak{z}(C_n^{(1)})^j = 0$ if j is even, and $\dim \mathfrak{z}(C_n^{(1)})^j = 1$ otherwise.

If $\mathfrak{g}(C_n^{(1)})$ is realized as a subalgebra of $L(\mathfrak{sl}_{2n}, C)$ and written out as $2n \times 2n$ matrices, then for odd j, $1 \leqslant j < 2n$, we introduce the element

$$A_{(2n)m+j} = \xi^{(2n)m+j} \otimes \begin{pmatrix} 0 & I_j \\ I_{2n-j} & 0 \end{pmatrix},$$

where I_j is the $j \times j$ identity matrix. We have $A_{(2n)m+j} = (A_1)^{(2n)m+j}$.

If $\mathfrak{g}(C_n^{(1)})$ is realized as a subalgebra of $\mathfrak{sl}_{2n}[\lambda,\lambda^{-1}]$ and written out as $2n\times 2n$ matrices, then for odd j, $1\leqslant j<2n$, we introduce the element

$$B_{(2n)m+j}=\begin{pmatrix} 0 & \lambda^{m+1}\otimes I_j \\ \lambda^m\otimes I_{2n-j} & 0\end{pmatrix}.$$

We have $B_{(2n)m+j}=(B_1)^{(2n)m+j}$.

Lemma 3.1. *For any $m\in\mathbb{Z}$, odd j, $1\leqslant j<2n$, the elements*

$$(\tilde\tau_C)^{-1}(A_{(2n)m+j}),\qquad (\tilde\tau_0)^{-1}(B_{(2n)m+j}),$$

of $\mathfrak{z}(C_n^{(1)})^{(2n)m+j}$ are equal. □

Denote the elements $(\tilde\tau_C)^{-1}(A_{(2n)m+j})$ of $\mathfrak{g}(C_n^{(1)})$ by $\Lambda^{(2)}_{(2n)m+j}$. Notice that $\Lambda^{(2)}_1=\sum_{i=0}^n e_i=\Lambda^{(2)}$. We set $\Lambda^{(2)}_j=0$ if j is even. The element $\Lambda^{(2)}_{(2n)m+j}$ generates $\mathfrak{z}(C_n^{(1)})^{(2n)m+j}$.

3.4. Lie algebra $\mathfrak{g}(C_n^{(1)})$ as a subalgebra of $\mathfrak{g}(A_{2n-1}^{(1)})$

The map $\rho:\mathfrak{g}(C_n^{(1)})\to\mathfrak{g}(A_{2n-1}^{(1)})$,

$$\begin{aligned}
&e_0\mapsto E_0, && e_i\mapsto E_i+E_{2n-i}, && e_n\mapsto E_n,\\
&f_0\mapsto F_0, && f_i\mapsto F_i+F_{2n-i}, && f_n\mapsto F_n,\\
&h_0\mapsto H_0, && h_i\mapsto H_i+H_{2n-i}, && h_n\mapsto H_n,
\end{aligned}$$

where $i=1,\dots,n-1$, realizes the Lie algebra $\mathfrak{g}(C_n^{(1)})$ as a subalgebra of $\mathfrak{g}(A_{2n-1}^{(1)})$. This embedding preserves the standard grading and $\rho(\Lambda^{(2)})=\Lambda^{(1)}$. We have $\rho(\mathfrak{z}(C_n^{(1)})^j)\subset\mathfrak{z}(A_{2n-1}^{(1)})^j$.

4. mKdV equations

In this section we follow [3].

4.1. The mKdV equations of type $A_{2n-1}^{(1)}$

Denote by $\mathcal{B}$ the space of complex-valued functions of one variable x. Given a finite dimensional vector space W, denote by $\mathcal{B}(W)$ the space of W-valued functions of x. Denote by ∂ the differential operator $\frac{d}{dx}$.

Consider the Lie algebra $\tilde{\mathfrak{g}}(A_{2n-1}^{(1)})$ of the formal differential operators of the form $c\partial + \sum_{i=-\infty}^{k} p_i$, $c \in \mathbb{C}$, $p_i \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^i)$. Let $U = \sum_{i<0} U_i$, $U_i \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^i)$. If $\mathcal{L} \in \tilde{\mathfrak{g}}(A_{2n-1}^{(1)})$, define

$$e^{\operatorname{ad} U}(\mathcal{L}) = \mathcal{L} + [U, \mathcal{L}] + \frac{1}{2!}[U, [U, \mathcal{L}]] + \dots .$$

The operator $e^{\operatorname{ad} U}(\mathcal{L})$ belongs to $\tilde{\mathfrak{g}}(A_{2n-1}^{(1)})$. The map $e^{\operatorname{ad} U}$ is an automorphism of the Lie algebra $\tilde{\mathfrak{g}}(A_{2n-1}^{(1)})$. The automorphisms of this type form a group. If elements of $\mathfrak{g}(A_{2n-1}^{(1)})$ are realized as matrices depending on a parameter as in Section 2.2, then $e^{\operatorname{ad} U}(\mathcal{L}) = e^U \mathcal{L} e^{-U}$.

A *Miura oper* of type $A_{2n-1}^{(1)}$ is a differential operator of the form

$$\mathcal{L} = \partial + \Lambda^{(1)} + V, \tag{4.1}$$

where $\Lambda^{(1)} = \sum_{i=0}^{2n-1} E_i \in \mathfrak{g}(A_{2n-1}^{(1)})$ and $V \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^0)$. Any Miura oper of type $A_{2n-1}^{(1)}$ is an element of $\tilde{\mathfrak{g}}(A_{2n-1}^{(1)})$. Denote by $\mathcal{M}(A_{2n-1}^{(1)})$ the space of all Miura opers of type $A_{2n-1}^{(1)}$.

Proposition 4.1 ([3, Proposition 6.2])**.** *For any Miura oper $\mathcal{L}$ of type $A_{2n-1}^{(1)}$ there exists an element $U = \sum_{i<0} U_i$, $U_i \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^i)$, such that the operator $\mathcal{L}_0 = e^{\operatorname{ad}U}(\mathcal{L})$ has the form*

$$\mathcal{L}_0 = \partial + \Lambda^{(1)} + H,$$

where $H = \sum_{j<0} H_j, H_j \in \mathcal{B}(\mathfrak{z}(A_{2n-1}^{(1)})^j)$. If $U, \tilde{U}$ are two such elements, then $e^{\operatorname{ad}U} e^{-\operatorname{ad}\tilde{U}} = e^{\operatorname{ad}T}$, where $T = \sum_{j<0} T_j$, $T_j \in \mathfrak{z}(A_{2n-1}^{(1)})^j$. □

Let $\mathcal{L}, U$ be as in Proposition 4.1. Let $r \neq 0 \bmod 2n$. The element $\phi(\Lambda_r^{(1)}) = e^{-\operatorname{ad}U}(\Lambda_r^{(1)})$ does not depend on the choice of U in Proposition 4.1.

The element $\phi(\Lambda_r^{(1)})$ is of the form $\sum_{i=-\infty}^{k} \phi(\Lambda_r^{(1)})^i$, $\phi(\Lambda_r^{(1)})^i \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^i)$. We set $\phi(\Lambda_r^{(1)})^+ = \sum_{i=0}^{k} \phi(\Lambda_r^{(1)})^i$, $\phi(\Lambda_r^{(1)})^- = \sum_{i<0} \phi(\Lambda_r^{(1)})^i$.

Let $r \in \mathbb{Z}_{>0}$ and $r \neq 0 \bmod 2n$. The differential equation

$$\frac{\partial \mathcal{L}}{\partial t_r} = [\phi(\Lambda_r^{(1)})^+, \mathcal{L}] \tag{4.2}$$

is called the *r-th mKdV equation* of type $A_{2n-1}^{(1)}$.

Equation (4.2) defines vector fields $\frac{\partial}{\partial t_r}$ on the space $\mathcal{M}(A^{(1)}_{2n-1})$ of Miura opers of type $A^{(1)}_{2n-1}$. For all r, s, the vector fields $\frac{\partial}{\partial t_r}$, $\frac{\partial}{\partial t_s}$ commute, see [3, Section 6].

Lemma 4.2 ([3])**.** *We have*

$$\frac{\partial \mathcal{L}}{\partial t_r} = -\frac{d}{dx}\phi(\Lambda^{(1)}_r)^0. \tag{4.3}$$

□

4.2. mKdV equations of type $C^{(1)}_n$

A *Miura oper* of type $C^{(1)}_n$ is a differential operator of the form

$$\mathcal{L} = \partial + \Lambda^{(2)} + V, \tag{4.4}$$

where $\Lambda^{(2)} = \sum_{i=0}^{n} e_i \in \mathfrak{g}(C^{(1)}_n)$ and $V \in \mathcal{B}(\mathfrak{g}(C^{(1)}_n)^0)$. Denote by $\mathcal{M}(C^{(1)}_n)$ the space of all Miura opers of type $C^{(1)}_n$.

Proposition 4.3 ([3, Proposition 6.2])**.** *For any Miura oper $\mathcal{L}$ of type $C^{(1)}_n$ there exists an element $U = \sum_{i<0} U_i$, $U_i \in \mathcal{B}(\mathfrak{g}(C^{(1)}_n)^i)$, such that the operator $\mathcal{L}_0 = e^{\mathrm{ad} U}(\mathcal{L})$ has the form*

$$\mathcal{L}_0 = \partial + \Lambda^{(2)} + H,$$

where $H = \sum_{j<0} H_j, H_j \in \mathcal{B}(\mathfrak{z}(C^{(1)}_n)^j)$. If $U, \tilde{U}$ are two such elements, then $e^{\mathrm{ad} U} e^{-\mathrm{ad}\tilde{U}} = e^{\mathrm{ad} T}$, where $T = \sum_{j<0} T_j$, $T_j \in \mathfrak{z}(C^{(1)}_n)^j$. □

Let $\mathcal{L}, U$ be as in Proposition 4.3. Let r be odd. The element $\phi(\Lambda^{(2)}_r) = e^{-\mathrm{ad} U}(\Lambda^{(2)}_r)$ does not depend on the choice of U in Proposition 4.3.

The element $\phi(\Lambda^{(2)}_r)$ is of the form $\sum_{i=-\infty}^{k} \phi(\Lambda^{(2)}_r)^i$, $\phi(\Lambda^{(2)}_r)^i \in \mathcal{B}(\mathfrak{g}(C^{(1)}_n)^i)$. We set $\phi(\Lambda^{(2)}_r)^+ = \sum_{i=0}^{k} \phi(\Lambda^{(2)}_r)^i$, $\phi(\mathcal{L}^{(2)} a_r)^- = \sum_{i<0} \phi(\Lambda^{(2)}_r)^i$.

Let $r \in \mathbb{Z}_{>0}$, r odd. The differential equation

$$\frac{\partial \mathcal{L}}{\partial t_r} = [\phi(\Lambda^{(2)}_r)^+, \mathcal{L}] \tag{4.5}$$

is called the r*-th mKdV equation* of type $C^{(1)}_n$.

Equation (4.5) defines vector fields $\frac{\partial}{\partial t_r}$ on the space $\mathcal{M}(C^{(1)}_n)$ of Miura opers. For all r, s, the vector fields $\frac{\partial}{\partial t_r}$, $\frac{\partial}{\partial t_s}$ commute, see [3, Section 6].

Lemma 4.4 ([3]). *We have*

$$\frac{\partial \mathcal{L}}{\partial t_r} = -\frac{d}{dx}\phi(\Lambda_r^{(2)})^0. \tag{4.6}$$ □

4.3. Comparison of mKdV equations of types $C_n^{(1)}$ and $A_{2n-1}^{(1)}$

Consider $\mathfrak{g}(C_n^{(1)})$ as a Lie subalgebra of $\mathfrak{g}(A_{2n-1}^{(1)})$, see Section 3.4. If $\mathcal{L}$ is a Miura oper of type $C_n^{(1)}$, then it is also a Miura oper of type $A_{2n-1}^{(1)}$. We have $\mathcal{M}(C_n^{(1)}) \subset \mathcal{M}(A_{2n-1}^{(1)})$,

$$\begin{aligned} \mathcal{M}(A_{2n-1}^{(1)}) &= \{\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{i=1}^{2n} v_i e_{i,i} \mid \sum_{i=1}^{2n} v_i = 0\}, \\ \mathcal{M}(C_n^{(1)}) &= \{\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{i=1}^{2n} v_i e_{i,i} \mid \\ &\qquad \sum_{i=1}^{2n} v_i = 0,\ v_j + v_{2n+1-j} = 0, j = 1, \dots, n\}. \end{aligned} \tag{4.7}$$

Lemma 4.5. *Let r be odd, $r > 0$. Let $\mathcal{L}^{C_n^{(1)}}(t_r)$ be the solution of the r-th mKdV equation of type $C_n^{(1)}$ with initial condition $\mathcal{L}^{C_n^{(1)}}(0) = \mathcal{L}$. Let $\mathcal{L}^{A_{2n-1}^{(1)}}(t_r)$ be the solution of the r-th mKdV equation of type $A_{2n-1}^{(1)}$ with initial condition $\mathcal{L}^{A_{2n-1}^{(1)}}(0) = \mathcal{L}$. Then $\mathcal{L}^{C_n^{(1)}}(t_r) = \mathcal{L}^{A_{2n-1}^{(1)}}(t_r)$ for all values of t_r.* □

Proof. The element U in Proposition 4.3 which is used to construct the mKdV equation of type $C_n^{(1)}$ can be used also to construct the mKdV equation of type $A_{2n-1}^{(1)}$. □

4.4. KdV equations of type $A_{2n-1}^{(1)}$

Let $\mathcal{B}((\partial^{-1}))$ be the algebra of formal pseudodifferential operators of the form $a = \sum_{i\in\mathbb{Z}} a_i\partial^i$, with $a_i \in \mathcal{B}$ and finitely many terms only with $i > 0$. The relations in this algebra are

$$\partial^k u - u\partial^k = \sum_{i=1}^{\infty} k(k-1)\dots(k-i+1)\frac{d^i u}{dx^i}\partial^{k-i}$$

for any $k \in \mathbb{Z}$ and $u \in \mathcal{B}$. For $a = \sum_{i\in\mathbb{Z}} a_i\partial^i \in \mathcal{B}((\partial^{-1}))$, define $a^+ = \sum_{i\geqslant 0} a_i\partial^i$.

Denote $\mathcal{B}[\partial] \subset \mathcal{B}((\partial^{-1}))$ the subalgebra of differential operators $a = \sum_{i=0}^{m} a_i\partial^i$ with $m \in \mathbb{Z}_{\geqslant 0}$. Denote $\mathcal{D} \subset \mathcal{B}[\partial]$ the affine subspace of differential operators of the form $L = \partial^{2n} + \sum\limits_{i=0}^{2n-2} u_i\partial^i$.

For $L \in \mathcal{D}$, there exists a unique $L^{\frac{1}{2n}} = \partial + \sum_{i\leqslant 0} a_i\partial^i \in \mathcal{B}((\partial^{-1}))$ such that $(L^{\frac{1}{2n}})^{2n} = L$. For $r \in \mathbb{N}$, we have $L^{\frac{r}{2n}} = \partial^r + \sum_{i=-\infty}^{r-1} b_i\partial^i$, $b_i \in \mathcal{B}$. We set $(L^{\frac{r}{2n}})^+ = \partial^r + \sum_{i=0}^{r-1} b_i\partial^i$.

For $r \in \mathbb{N}$, the differential equation

$$\frac{\partial L}{\partial t_r} = [L, (L^{\frac{r}{2n}})^+] \tag{4.8}$$

is called the *r-th KdV equation* of type $A_{2n-1}^{(1)}$.

Equation (4.8) defines flows $\frac{\partial}{\partial t_r}$ on the space $\mathcal{D}$. For all $r, s \in \mathbb{N}$ the flows $\frac{\partial}{\partial t_r}$ and $\frac{\partial}{\partial t_s}$ commute, see [3].

4.5. Miura maps

Let $\mathcal{L} = \partial + \Lambda^{(1)} + V$ be a Miura oper of type $A_{2n-1}^{(1)}$ with $V = \sum_{k=1}^{2n} v_k e_{k,k}$, $\sum_{k=1}^{2n} v_k = 0$. For $i = 0, \dots, 2n$, define the scalar differential operator $L_i = \partial^{2n} + \sum_{j=0}^{2n-2} u_{j,i}\partial^j \in \mathcal{D}$ by the formula:

$$\begin{aligned} L_0 = L_{2n} &= (\partial - v_{2n})(\partial - v_{2n-1})\dots(\partial - v_2)(\partial - v_1), \\ L_i &= (\partial - v_i)(\partial - v_{i-1})\dots(\partial - v_1)(\partial - v_{2n})\dots(\partial - v_{i+2})(\partial - v_{i+1}), \end{aligned} \tag{4.9}$$

for $i = 1, \dots, 2n-1$.

Theorem 4.6 ([3, Proposition 3.18])**.** *Let a Miura oper $\mathcal{L}$ satisfy the mKdV equation (4.2) for some r. Then for every $i = 0, \dots, 2n-1$ the differential operator L_i satisfies the KdV equation (4.8).*

For $i = 0, \dots, 2n$, we define the *i-th Miura map* by the formula

$$\mathfrak{m}_i \; : \; \mathcal{M}(A_{2n-1}^{(1)}) \;\to\; \mathcal{D}, \quad \mathcal{L} \;\mapsto\; L_i,$$

see (4.9).

For $i = 0, 1, \dots, 2n-1$, an *i-oper* is a differential operator of the form

$$\mathcal{L} = \partial + \Lambda^{(1)} + V + W,$$

with $V \in \mathcal{B}(\mathfrak{g}(A_{2n-1}^{(1)})^0)$ and $W \in \mathcal{B}(\mathfrak{n}_i^-)$. For $w \in \mathcal{B}(\mathfrak{n}_i^-)$ and an i-oper $\mathcal{L}$, the differential operator $e^{\operatorname{ad} w}(\mathcal{L})$ is an i-oper. The i-opers $\mathcal{L}$ and $e^{\operatorname{ad} w}(\mathcal{L})$ are called *i-gauge equivalent.* A Miura oper is an i-oper for any i.

Theorem 4.7 ([3, Proposition 3.10])**.** *If Miura opers $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are i-gauge equivalent, then $\mathfrak{m}_i(\mathcal{L}) = \mathfrak{m}_i(\tilde{\mathcal{L}})$.* □

5. Tangent maps to Miura maps

5.1. Tangent spaces

Consider the spaces of Miura opers $\mathcal{M}(C_n^{(1)}) \subset \mathcal{M}(A_{2n}^{(1)})$. The tangent space to $\mathcal{M}(C_n^{(1)})$ at a point $\mathcal{L}$ is

$$T_{\mathcal{L}}\mathcal{M}(C_n^{(1)}) = \{X = \sum_{i=1}^{2n} X_i e_{i,i} \mid \sum_{i=1}^{2n} X_i = 0\,, \quad X_j + X_{2n+1-j} = 0,\ j = 1, \dots, n\}, \tag{5.1}$$

where X_i are functions of variable x. Recall $\mathcal{D} = \{L = \partial^{2n} + \sum_{i=0}^{2n-2} u_i\partial^i\}$. The tangent space to $\mathcal{D}$ at a point L is $T_L\mathcal{D} = \{Z = \sum_{i=0}^{2n-2} Z_i\partial^i\}$, where Z_i are functions of x.

Consider the restrictions of Miura maps to $\mathcal{M}(C_n^{(1)})$ and the corresponding tangent maps

$$d\mathfrak{m}_i : T_{\mathcal{L}}\mathcal{M}(C_n^{(1)}) \to T_{\mathfrak{m}_i(\mathcal{L})}\mathcal{D}, \quad i = 1, \dots, 2n. \tag{5.2}$$

By definition, if $\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{i=1}^{2n} v_i e_{i,i} \in \mathcal{M}(C_n^{(1)})$, $X = \sum_{i=1}^{2n} X_i e_{i,i} \in T_{\mathcal{L}}\mathcal{M}(C_n^{(1)})$, $d\mathfrak{m}_i(X) = Z^i = \sum_{j=0}^{2n-2} Z^i_j\partial^j$, then

$$\begin{aligned} Z^i = {} & (-X_i)(\partial - v_{i-1})\dots(\partial - v_1)(\partial - v_{2n})\dots(\partial - v_{i+1}) \\ & + (\partial - v_i)(-X_{i-1})\dots(\partial - v_1)(\partial - v_{2n})\dots(\partial - v_{i+1}) + \dots \\ & + (\partial - v_i)(\partial - v_{i-1})\dots(-X_1)(\partial - v_{2n})\dots(\partial - v_{i+1}) \\ & + (\partial - v_i)(\partial - v_{i-1})\dots(\partial - v_1)(-X_{2n})\dots(\partial - v_{i+1}) + \dots \\ & + (\partial - v_i)(\partial - v_{i-1})\dots(\partial - v_1)(\partial - v_{2n})\dots(-X_{i+1}). \end{aligned} \tag{5.3}$$

In what follows we study the intersection of kernels of these tangent maps when i runs through certain subsets of $\{1, \dots, 2n\}$.

5.2. Formula for the first coefficient

Proposition 5.1. *Let* $\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{i=1}^{2n} v_i e_{i,i} \in \mathcal{M}(A_{2n}^{(1)})$, $X = \sum_{i=1}^{2n} X_i e_{i,i} \in T_{\mathcal{L}}\mathcal{M}(C_n^{(1)})$, $d\mathfrak{m}_i(X) = Z^i = \sum_{j=0}^{2n-2} Z_j^i \partial^j$. *Then*

$$Z_{2n-2}^i = -\left(\sum_{k=1}^{2n} v_k X_k + \sum_{k=1}^{i}(i-k)X_k' + \sum_{k=i+1}^{2n}(i+2n-k)X_k'\right). \tag{5.4}$$

Proof. The proof uses only the identity $\sum_{j=1}^{2n+1} v_j = 0$ and is straightforward.

$$\begin{aligned}
&Z_{2n-2}^i = (-X_i)\,[-v_{i-1} - v_{i-2} - \dots - v_1 - v_{2n} - \dots - v_{i+1}]\\
&+ (-X_{i-1})' + (-X_{i-1})[-v_i - v_{i-2} - \dots - v_1 - v_{2n} - \dots - v_{i+1}]\\
&+ (i-1)(-X_1)' + (-X_1)[-v_i - v_{i-1} - \dots - v_2 - v_{2n} - \dots - v_{i+1}]\\
&+ (i)(-X_{2n})' + (-X_{2n})[-v_i - v_{i-1} - \dots - v_2 - v_1 - v_{2n-1} - \dots\\
&- v_{i+1}] + \ (2n-1)(-X_{i+1})' + (-X_{i+1})[-v_i - v_{i-1} - \dots - v_1 - v_{2n}\\
&- \ \dots - v_{i+2}] = -\left(\sum_{k=1}^{2n} v_k X_k + \sum_{k=1}^{i}(i-k)X_k' + \sum_{k=i+1}^{2n}(i+2n-k)X_k'\right).
\end{aligned}$$

□

Notice that

$$\sum_{k=1}^{2n} v_k X_k = 2\sum_{k=1}^{n} v_k X_k.$$

5.3. Intersection of kernels of $d\mathfrak{m}_i$

Lemma 5.2. *Let* $\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{k=1}^{2n} v_k e_{k,k} \in \mathcal{M}(C_n^{(1)})$, $X = \sum_{k=1}^{2n} X_k e_{k,k} \in T_{\mathcal{L}}\mathcal{M}(C_n^{(1)})$, $d\mathfrak{m}_i(X) = Z^i = \sum_{j=0}^{2n-2} Z_j^i \partial^j$. *Assume that* $Z_{2n-2}^i = 0$ *for* $i = 1, \dots, 2n-1$, *then*

$$X_1' - 2v_1 X_1 = 2\sum_{k=2}^{n} v_k X_k, \qquad X_i' = 0, \quad i = 2, \dots, 2n-1. \tag{5.5}$$

Proof. By assumption we have the system of equations

$$X_{2n-2}' + 2X_{2n-3}' + \dots + (2n-2)X_1' + (2n-1)X_{2n}' + \sum_{k=1}^{2n} v_k X_k = 0, \tag{5.6}$$

$$X'_{2n-3} + 2X'_{2n-4} + \cdots + (2n-2)X'_{2n} + (2n-1)X'_{2n-1} + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$X'_{2n-4} + 2X'_{2n-5} + \cdots + (2n-2)X'_{2n-1} + (2n-1)X'_{2n-2} + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$\ldots$$
$$X'_1 + 2X'_{2n} + \cdots + (2n-2)X'_4 + (2n-1)X'_3 + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$X'_{2n} + \cdots + (2n-2)X'_3 + (2n-1)X'_2 + \sum_{k=1}^{2n} v_k X_k = 0.$$

By subtracting the first equation from the second we get $(2n-1)X'_{2n-1} - X'_{2n-2} - X'_{2n-3} - \cdots - X'_1 - X'_{2n} = 0$, equivalently $2n\, X'_{2n-1} - \sum_{k=1}^{2n} X'_k = 0$. Since $\sum_{k=1}^{2n} X_k = 0$, we get $X'_{2n-1} = 0$. By subtracting the second from the third we get $X'_{2n-2} = 0$. Similarly we obtain

$$X'_i = 0, \quad i = 2, \ldots, 2n-1. \tag{5.7}$$

Applying (5.7) to the last equation in (5.6) yields

$$X'_{2n} + \sum_{k=1}^{2n} v_k X_k = X'_{2n} + 2\sum_{k=1}^{n} v_k X_k = 0.$$

By pulling out the term for $k = 1$ we obtain

$$X'_{2n} + 2v_1 X_1 + 2\sum_{k=2}^{n} v_k X_k = -X'_1 + 2v_1 X_1 + 2\sum_{k=2}^{n} v_k X_k = 0.$$

□

Lemma 5.3. *Let $j \in \{1, \ldots, n-1\}$. Let $\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{k=1}^{2n} v_k e_{k,k} \in \mathcal{M}(C_n^{(1)})$, $X = \sum_{k=1}^{2n} X_k e_{k,k} \in T_{\mathcal{L}}\mathcal{M}(C_n^{(1)})$, $d\mathfrak{m}_i(X) = Z^i = \sum_{j=0}^{2n-2} Z^i_j \partial^j$. Assume that $Z^i_{2n-2} = 0$ for all $i \notin \{j, 2n-j\}$, then*

$$X'_j + v_j X_j + v_{j+1} X_{j+1} = -\sum_{k=1, k\neq j, j+1}^{n} v_k X_k, \quad X'_j + X'_{j+1} = 0, \quad X'_i = 0$$

for $i \notin \{j, j+1, 2n-j, 2n+1-j\}$.

Proof. By assumption we have the system of equations

$$X'_{2n-1} + 2X'_{2n-2} + \dots + (2n-2)X'_2 + (2n-1)X'_1 + \sum_{k=1}^{2n} v_k X_k = 0,$$

$$X'_{2n-2} + 2X'_{2n-3} + \dots + (2n-2)X'_1 + (2n-1)X'_{2n} + \sum_{k=1}^{2n} v_k X_k = 0,$$

$$\dots$$

$$X'_{2n-j} + \dots + (2n-j)X'_1 + (2n+1-j)X'_{2n} + \dots$$
$$\dots + (2n-1)X'_{2n+2-j} + \sum_{k=1}^{2n} v_k X_k = 0,$$

$$X'_{2n-2-j} + \dots + (2n-2-j)X'_1 + (2n-1-j)X'_{2n} + \dots$$
$$+ (2n-1)X'_{2n-j} + \sum_{k=1}^{2n} v_k X_k = 0,$$

$$\dots$$

$$X'_j + \dots + jX'_1 + (j+1)X'_{2n} + \dots + (2n-1)X'_{j+2} + \sum_{k=1}^{2n} v_k X_k = 0,$$

$$\dots$$

Subtracting the second line from the first gives $X'_{2n} = 0$, cf. the proof of Lemma 5.2. Similarly, for $i \notin \{j, j+1, 2n-j, 2n+1-j\}$ considering the difference $Z^{i-1}_{2n-2} - Z^{i}_{2n-2} = 0$ we obtain $X'_i = 0$.

Considering the difference $Z^{2n+1-j}_{2n-2} - Z^{2n-1-j}_{2n-2} = 0$ we obtain

$$X'_{2n-j} + \dots + (2n-j)X'_1 + (2n+1-j)X'_{2n} + \dots$$
$$+ (2n-1)X'_{2n+2-j} + \sum_{k=1}^{2n} v_k X_k$$
$$-\Big(X'_{2n-2-j} + \dots + (2n-2-j)X'_1 + (2n-1-j)X'_{2n} + \dots$$
$$+ (2n-1)X'_{2n-j} + \sum_{k=1}^{2n} v_k X_k\Big)$$
$$= -(2n)(X'_{2n-j} + X'_{2n+1-j}) + 2\sum_{k=1}^{2n} X'_k = 0.$$

Hence $X'_{2n-j} + X'_{2n+1-j} = 0$ and $X'_j + X'_{j+1} = 0$. Now we can rewrite equation $Z^{2n}_{2n-2} = 0$ as

$$(j-1)X'_{2n+1-j} + (j)X'_{2n-j} + (2n-1-j)X'_{j+1}$$

$$+ (2n-j)X_j' + \sum_{k=1}^{2n} v_k X_k = 0.$$

Or equivalently

$$2X_j' + \sum_{k=1}^{2n} v_k X_k = 2X_j' + 2\sum_{k=1}^{n} v_k X_k$$
$$= 2(X_j' + v_j X_j + v_{j+1}X_{j+1} + \sum_{k=1, k\neq j,j+1}^{n} v_k X_k) = 0.$$

□

Lemma 5.4. *Let* $\mathcal{L} = \partial + \Lambda^{(1)} + \sum_{k=1}^{2n} v_k e_{k,k} \in \mathcal{M}(C_n^{(1)})$, $X = \sum_{k=1}^{2n} X_k e_{k,k} \in T_{\mathcal{L}}\mathcal{M}(C_n^{(1)})$, $d\mathfrak{m}_i(X) = Z^i = \sum_{j=0}^{2n-2} Z_j^i \partial^j$. *Assume that* $Z_{2n-2}^i = 0$ *for all* $i \neq n$, *then*

$$X_n' + 2v_n X_n = -2\sum_{k=1}^{n-1} v_k X_k, \qquad X_i' = 0, \quad i \notin \{n, n+1\}.$$

Proof. By assumption we have the system of equations

$$X_{2n-1}' + 2X_{2n-2}' + \cdots + (2n-2)X_2' + (2n-1)X_1' + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$X_{2n-2}' + 2X_{2n-3}' + \cdots + (2n-2)X_1' + (2n-1)X_{2n}' + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$\ldots$$
$$X_n' + \cdots + (n)X_1' + (n+1)X_{2n}' + \ldots$$
$$+ (2n-1)X_{n+2}' + \sum_{k=1}^{2n} v_k X_k = 0,$$
$$X_{n-2}' + \cdots + (n-2)X_1' + (n-1)X_{2n+1}' + \ldots$$
$$+ 2nX_n' + \sum_{k=1}^{2n+1} v_k X_k = 0,$$
$$\ldots$$
$$X_{2n}' + 2X_{2n-1}' + \cdots + (2n-1)X_2' + \sum_{k=1}^{2n} v_k X_k = 0.$$

Subtracting the second line from the first gives $X_{2n}' = 0$, cf. the proof of Lemma 5.2. Similarly, for $i \notin \{n, n+1\}$ considering the difference $Z_{2n-2}^{i-1} - Z_{2n-2}^i = 0$ we obtain $X_i' = 0$.

Now we can rewrite equation $Z^{2n}_{2n-2} = 0$ as

$$(n-1)X'_{n+1} + (n)X'_n + \sum_{k=1}^{2n} v_k X_k = X'_n + 2v_n X_n + 2\sum_{k=1}^{n-1} v_k X_k = 0.$$

□

6. Critical points of master functions and generation of tuples of polynomials

In this section we follow [6]. For functions $f(x), g(x)$, we denote

$$\mathrm{Wr}(f,g) = f(x)g'(x) - f'(x)g(x)$$

the Wronskian determinant, and $f'(x) := \frac{df}{dx}(x)$.

6.1. Master function

Choose nonnegative integers $k = (k_0, k_1, \dots, k_n)$. Consider variables $u = (u_i^{(j)})$, where $j = 0, 1, \dots, n$ and $i = 1, \dots, k_j$. The *master function* $\Phi(u;k)$ is defined by the formula:

$$\begin{aligned}(6.1)\qquad \Phi(u,k) &= 2\sum_{i<i'} \ln(u_i^{(0)} - u_{i'}^{(0)}) + 4\sum_{j=1}^{n-1}\sum_{i<i'} \ln(u_i^{(j)} - u_{i'}^{(j)}) \\ &+ 2\sum_{i<i'} \ln(u_i^{(n)} - u_{i'}^{(n)}) - 2\sum_{j=0}^{n-1}\sum_{i,i'} \ln(u_i^{(j)} - u_{i'}^{(j+1)}).\end{aligned}$$

The product of symmetric groups $\Sigma_{\boldsymbol{k}} = \Sigma_{k_0} \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n}$ acts on the set of variables by permuting the coordinates with the same upper index. The function Φ is symmetric with respect to the $\Sigma_{\boldsymbol{k}}$-action. A point u is a *critical point* if $d\Phi = 0$ at u. In other words, u is a critical point if and only if the following expressions equal zero:

$$\begin{aligned}(6.2)\qquad & \sum_{l=1}^{k_1} \frac{-2}{u_j^{(0)} - u_l^{(1)}} + \sum_{s\neq j} \frac{2}{u_j^{(0)} - u_s^{(0)}}, \qquad j = 1, \dots, k_0, \\ & \sum_{l=1}^{k_{i-1}} \frac{-2}{u_j^{(i)} - u_l^{(i-1)}} + \sum_{l=1}^{k_{i+1}} \frac{-2}{u_j^{(i)} - u_l^{(i+1)}} + \sum_{s\neq j} \frac{4}{u_j^{(i)} - u_s^{(i)}}, \\ & \qquad\qquad i = 1, \dots, n-1,\ j = 1, \dots, k_i,\end{aligned}$$

$$\sum_{l=1}^{k_{n-1}} \frac{-2}{u_j^{(n)} - u_l^{(n-1)}} + \sum_{s\neq j} \frac{2}{u_j^{(n)} - u_s^{(n)}}, \qquad j = 1, \dots, k_n.$$

All the orbits have the same cardinality $\prod_{i=0}^{n} k_i!$. We do not make distinction between critical points in the same orbit.

Remark. The definition of master functions can be found in [10], see also [6, 7]. The master functions $\Phi(u, k)$ in (6.1) are associated with the Kac-Moody algebra with Cartan matrix of type

$$A = (a_{i,j}) = \begin{pmatrix} 2 & -2 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & \dots \\ 0 & -1 & 2 & -1 & \dots & \dots & \dots \\ 0 & 0 & -1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2 & -1 & 0 \\ \dots & \dots & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -2 & 2 \end{pmatrix}, \tag{6.3}$$

which is dual to the Cartan matrix $C_n^{(1)}$, see this type of Langlands duality in [6, 7, 18].

6.2. Polynomials representing critical points

Let $u = (u_i^{(j)})$ be a critical point of the master function Φ. Introduce the $(n + 1)$-tuple of polynomials $y = (y_0(x), \dots, y_n(x))$,

$$y_j(x) \ = \ \prod_{i=1}^{k_j} (x - u_i^{(j)}). \tag{6.4}$$

This tuple of polynomials defines a point in the direct product $(\mathbb{C}[x])^{n+1}$. We say that the tuple *represents the critical point.*

Each polynomial of the tuple will be considered up to multiplication by a nonzero number.

It is convenient to think that the $(n+1)$-tuple $y^\emptyset = (1, \dots, 1)$ of constant polynomials represents in $(\mathbb{C}[x])^{n+1}$, the critical point of the master function with no variables. This corresponds to the case $k = (0, \dots, 0)$.

We say that a given tuple $y \in (\mathbb{C}[x])^{n+1}$ is *generic* if each polynomial $y_i(x)$ has no multiple roots and for $i = 0, \dots, n-1$ the polynomials $y_i(x)$ and $y_{i+1}(x)$ have no common roots. If a tuple represents a critical point, then it is generic, see (6.2). For example, the tuple $y^\emptyset$ is generic.

6.3. Elementary generation

An $(n+1)$-tuple is called *fertile* if there exist polynomials $\tilde{y}_0, \dots, \tilde{y}_n \in (\mathbb{C}[x])^{n+1}$ such that

$$\mathrm{Wr}(\tilde{y}_j, y_j) = \prod_{i\neq j} y_i^{-a_{i,j}}, \qquad j = 0, 1, \dots, n, \tag{6.5}$$

where $a_{i,j}$ are the entries of the Cartan matrix of type $C_n^{(1)}$, that is,

$$\begin{aligned} &\mathrm{Wr}(\tilde{y}_0, y_0) = y_1^2, \qquad \mathrm{Wr}(\tilde{y}_i, y_i) = y_{i-1}y_{i+1}, \qquad i = 1, \dots, n-1, \\ &\qquad\qquad\qquad\qquad\quad \mathrm{Wr}(\tilde{y}_n, y_n) = y_{n-1}^2. \end{aligned} \tag{6.6}$$

For example, $y^{\emptyset}$ is fertile and $\tilde{y}_j = x + c_j$, where the c_j are arbitrary numbers.

Assume that an $(n+1)$-tuple of polynomials $y = (y_0, \dots, y_n)$ is fertile. Equations (6.5) are linear first order inhomogeneous differential equations with respect to $\tilde{y}_i$. The solutions are

$$\tilde{y}_0 = y_0 \int \frac{y_1^2}{y_0^2} dx + c_0 y_0, \tag{6.7}$$

$$\tilde{y}_i = y_i \int \frac{y_{i-1}y_{i+1}}{y_i^2} dx + c_i y_i, \qquad i = 1, \dots, n-1, \tag{6.8}$$

$$\tilde{y}_n = y_n \int \frac{y_{n-1}^2}{y_n^2} dx + c_n y_n, \tag{6.9}$$

where $c_0, \dots, c_n$ are arbitrary numbers. For each $i = 0, \dots, n$, the tuple

$$\begin{aligned} &y^{(i)}(x, c_i) \\ &\quad = (y_0(x), \dots, y_{i-1}(x), \tilde{y}_i(x, c_i), y_{i+1}(x), \dots, y_n(x)) \;\in\; (\mathbb{C}[x])^{n+1} \end{aligned} \tag{6.10}$$

forms a one-parameter family. This family is called the *generation of tuples from y in the i-th direction*. A tuple of this family is called an immediate descendant of y in the i-th direction.

Theorem 6.1 ([6]).

(i) A generic tuple $y = (y_0, \dots, y_n)$, $\deg y_i = k_i$, represents a critical point of the master function $\Phi(u; k)$ if and only if y is fertile.

(ii) If y represents a critical point, then for any $c \in \mathbb{C}$ the tuple $y^{(j)}(x, c)$, $j = 0, \dots, n$ is fertile.

(iii) If y is generic and fertile, then for almost all values of the parameter $c \in \mathbb{C}$ the tuple $y^{(j)}(x,c)$ is generic. The exceptions form a finite set in $\mathbb{C}$.

(iv) Assume that a sequence $y_{[\ell]}$, $\ell = 1, 2, \dots,$ of fertile tuples has a limit $y_{[\infty]}$ in $(\mathbb{C}[x])^{n+1}$ as ℓ tends to infinity.

(a) Then the limiting tuple $y_{[\infty]}$ is fertile.

(b) For $j = 0, \dots, n$, let $y^{(j)}_{[\infty]}$ be an immediate descendant of $y_{[\infty]}$. Then for all j there exist immediate descendants $y^{(j)}_{[\ell]}$ of $y_{[\ell]}$ such that $y^{(j)}_{[\infty]}$ is the limit of $y^{(j)}_{[\ell]}$ as ℓ tends to infinity. □

6.4. Degree increasing generation

Let $y = (y_0, \dots, y_n)$ be a generic fertile $(n+1)$-tuple of polynomials. Define $k_j = \deg y_j$ for $j = 0, \dots, n$.

The polynomial $\tilde{y}_0$ in (6.7) is of degree k_0 or $\tilde{k}_0 = 2k_1 + 1 - k_0$. We say that the generation $(y_0, \dots, y_n) \to (\tilde{y}_0, \dots, y_n)$ is *degree increasing* in the 0-th direction if $\tilde{k}_0 > k_0$. In that case $\deg \tilde{y}_0 = \tilde{k}_0$ for all c.

For $i = 1, \dots, n-1$, the polynomial $\tilde{y}_i$ in (6.8) is of degree k_i or $\tilde{k}_i = k_{i-1} + k_{i+1} + 1 - k_i$. We say that the generation $(y_0, \dots, y_i, \dots, y_n) \to (y_0, \dots, \tilde{y}_i, \dots, y_n)$ is *degree increasing* in the i-th direction if $\tilde{k}_i > k_i$. In that case $\deg \tilde{y}_i = \tilde{k}_i$ for all c.

The polynomial $\tilde{y}_n$ in (6.9) is of degree k_n or $\tilde{k}_n = 2k_{n-1} + 1 - k_n$. We say that the generation $(y_0, \dots, y_{n-1}, y_n) \to (y_0, \dots, y_{n-1}, \tilde{y}_n)$ is *degree increasing* in the n-th direction if $\tilde{k}_n > k_n$. In that case $\deg \tilde{y}_n = \tilde{k}_n$ for all c.

For $i = 0, \dots, n$, if the generation is degree increasing in the i-th direction we normalize family (6.10) and construct a map $Y_{y,i} : \mathbb{C} \to (\mathbb{C}[x])^{n+1}$ as follows. First we multiply the polynomials $y_0, \dots, y_n$ by numbers to make them monic. Then we choose a monic polynomial $y_{i,0}$ satisfying the equation $\mathrm{Wr}(y_{i,0}, y_i) = \epsilon \prod_{j \neq i} y_j^{-a_{j,i}}$, for some nonzero integer ϵ, and such that the coefficient of x^{k_i} in $y_{i,0}$ equals zero. Set

$$\tilde{y}_i(x,c) = y_{i,0}(x) + c y_i(x), \tag{6.11}$$

and define

$$\begin{aligned} Y_{y,i} : \mathbb{C} \quad &\to \quad (\mathbb{C}[x])^{n+1}, \\ c \quad &\mapsto \quad y^{(i)}(x,c) = (y_0(x), \dots, \tilde{y}_i(x,c), \dots, y_n(x)). \end{aligned} \tag{6.12}$$

The polynomials of this $(n+1)$-tuple are monic.

6.5. Degree-transformations and generation of vectors of integers

The degree-transformations

$$\begin{aligned}(6.13)\quad k &:= (k_0,\dots,k_n) \mapsto k^{(0)} = (2k_1+1-k_0,\dots,k_n),\\ k &:= (k_0,\dots,k_n) \mapsto k^{(i)} = (k_0,\dots,k_{i-1}+k_{i+1}+1-k_i,\dots,k_n),\\ &\qquad\qquad i=1,\dots,n-1,\\ k &:= (k_0,\dots,k_n) \mapsto k^{(n)} = (k_0,\dots,2k_{n-1}+1-k_n),\end{aligned}$$

correspond to the shifted action of reflections $w_0,\dots,w_n \in W$, where W is the Weyl group associated with the Cartan matrix A in (6.3) and $w_0,\dots,w_n$ are the standard generators, see [6, Lemma 3.11] for more detail.

We take formula 6.13 as the definition of *degree-transformations*:

$$\begin{aligned}(6.14)\quad w_0 &: k \mapsto k^{(0)} = (2k_1+1-k_0,\dots,k_n),\\ w_i &: k \mapsto k^{(i)} = (k_0,\dots,k_{i-1}+k_{i+1}+1-k_i,\dots,k_n),\\ &\qquad\qquad i=1,\dots,n-1,\\ w_n &: k \mapsto k^{(n)} = (k_0,\dots,2k_{n-1}+1-k_n),\end{aligned}$$

acting on arbitrary vectors $k=(k_0,\dots,k_n)$.

We start with the vector $k^{\emptyset}=(0,\dots,0)$ and a sequence $J=(j_1,j_2,\dots,j_m)$ of integers such that $j_i\in\{0,\dots,n\}$ for all i. We apply the corresponding degree transformations to $k^{\emptyset}$ and obtain a sequence of vectors $k^{\emptyset}$, $k^{(j_1)} = w_{j_1}k^{\emptyset}$, $k^{(j_1,j_2)} = w_{j_2}w_{j_1}k^{\emptyset}$,...,

$$k^J = w_{j_m}\dots w_{j_2}w_{j_1}k^{\emptyset}. \tag{6.15}$$

We say that the *vector* k^J *is generated from* $(0,\dots,0)$ *in the direction of* J.

We call a sequence J *degree increasing* if for every i the transformation w_{j_i} applied to $w_{j_{i-1}}\dots w_{j_1}k^{\emptyset}$ increases the j_i-th coordinate.

6.6. Multistep generation

Let $J=(j_1,\dots,j_m)$ be a degree increasing sequence. Starting from $y^{\emptyset}=(1,\dots,1)$ and J, we construct a map

$$Y^J : \mathbb{C}^m \to (\mathbb{C}[x])^{n+1}$$

by induction on m. If $J=\emptyset$, the map $Y^{\emptyset}$ is the map $\mathbb{C}^0=(pt)\mapsto y^{\emptyset}$. If $m=1$ and $J=(j_1)$, the map $Y^{(j_1)} : \mathbb{C}\to(\mathbb{C}[x])^{n+1}$ is given by formula

(6.12) for $y = y^\emptyset$ and $j = j_1$. More precisely, equation (6.5) takes the form $\mathrm{Wr}(\tilde{y}_{j_1}, 1) = 1$. Then $\tilde{y}_{j_1,0} = x$ and

$$Y^{(j_1)} \ : \ \mathbb{C} \mapsto (\mathbb{C}[x])^{n+1}, \qquad c \mapsto (1, \dots, x + c, \dots, 1).$$

By Theorem 6.1 all tuples in the image are fertile and almost all tuples are generic (in this example all tuples are generic). Assume that for $\tilde{J} = (j_1, \dots, j_{m-1})$, the map $Y^{\tilde{J}}$ is constructed. To obtain Y^J we apply the generation procedure in the j_m-th direction to every tuple of the image of $Y^{\tilde{J}}$. More precisely, if

$$Y^{\tilde{J}} \ : \ \tilde{c} = (c_1, \dots, c_{m-1}) \ \mapsto \ (y_0(x, \tilde{c}), \dots, y_n(x, \tilde{c})), \tag{6.16}$$

then

$$Y^J : (\tilde{c}, c_m) \mapsto (y_0(x, \tilde{c}), \dots, y_{j_m,0}(x, \tilde{c}) + c_m y_{j_m}(x, \tilde{c}), \dots, y_n(x, \tilde{c})). \tag{6.17}$$

The map Y^J is called the *generation of tuples from $y^\emptyset$ in the J-th direction.*

Lemma 6.2. *All tuples in the image of Y^J are fertile and almost all tuples are generic. For any $c \in \mathbb{C}^m$ the $(n+1)$-tuple $Y^J(c)$ consists of monic polynomials. The degree vector of this tuple equals k^J.* □

Lemma 6.3. *The map Y^J sends distinct points of $\mathbb{C}^m$ to distinct points of $(\mathbb{C}[x])^{n+1}$.*

Proof. The lemma is easily proved by induction on m. □

6.7. Critical points and the population generated from $y^\emptyset$

The set of all tuples $(y_0, \dots, y_n) \in (\mathbb{C}[x])^{n+1}$ obtained from $y^\emptyset = (1, \dots, 1)$ by generations in all directions $J = (j_1, \dots, j_m)$, $m \geqslant 0$, (not necessarily degree increasing) is called the *population of tuples* generated from $y^\emptyset$, see [6, 7].

Theorem 6.4 ([8])**.** *If a tuple of polynomials $(y_0, \dots, y_n)$ represents a critical point of the master function $\Phi(u, k)$ defined in (6.1) for some parameters $k = (k_0, \dots, k_n)$, then $(y_0, \dots, y_n)$ is a point of the population generated from $y^\emptyset$ by a degree increasing generation, that is, there exist a degree increasing sequence $J = (j_1, \dots, j_m)$ and a point $c \in \mathbb{C}^m$ such that $(y_0(x), \dots, y_n(x)) = Y^J(x, c)$. Moreover, for any other critical point of that function $\Phi(u, k)$ there is a point $c' \in \mathbb{C}^m$ such that the tuple $Y^J(x, c')$ represents that other critical point.*

By Theorem 6.4 a function $\Phi(u,k)$ either does not have critical points at all or all of its critical points form one cell $\mathbb{C}^m$.

Proof. Theorem 3.8 in [7] says that $(y_0,\dots,y_n)$ is a point of the population generated from $y^\emptyset$. The fact that $(y_0,\dots,y_n)$ can be generated from $y^\emptyset$ by a degree increasing generation is a corollary of Lemmas 3.5 and 3.7 in [7]. The same lemmas show that any other critical point of the master function $\Phi(u,k)$ is represented by the tuple $Y^J(x,c')$ for a suitable $c'\in\mathbb{C}^m$. □

7. Critical points of master functions and Miura opers

7.1. Miura oper associated with a tuple of polynomials, [7]

We say that a Miura oper of type $C_n^{(1)}$, $\mathcal{L}=\partial+\Lambda^{(2)}+V$, is *associated to an* $(n+1)$-*tuple of polynomials* y if $V=-\sum_{i=0}^n \ln'(y_i)\,h_i$, where $\ln'(f(x))=\frac{f'(x)}{f(x)}$. If $\mathcal{L}$ is associated to y and $V=\sum_{i=1}^{2n} v_i e_{i,i}$, then for $i=1,\dots,n$,

$$v_i=-v_{2n+1-i}=\ln'\Big(\frac{y_i}{y_{i-1}}\Big),\qquad i=1,\dots,n. \tag{7.1}$$

We also have

$$\langle\alpha_j,V\rangle=\ln'\Big(\prod_{i=0}^n y_i^{-a_{i,j}}\Big), \tag{7.2}$$

where $a_{i,j}$ are entries of the Cartan matrix of type $C_n^{(1)}$. More precisely,

$$\begin{aligned}\langle\alpha_0,V\rangle &= \ln'\Big(\frac{y_1^2}{y_0^2}\Big),\\ \langle\alpha_i,V\rangle &= \ln'\Big(\frac{y_{i-1}y_{i+1}}{y_i^2}\Big),\quad i=1,\dots,n-1,\\ \langle\alpha_n,V\rangle &= \ln'\Big(\frac{y_{n-1}^2}{y_n^2}\Big).\end{aligned} \tag{7.3}$$

For example,

$$\mathcal{L}^\emptyset:=\partial+\Lambda^{(2)} \tag{7.4}$$

is associated to the tuple $y^\emptyset=(1,\dots,1)$.

Define the map

$$\mu:(\mathbb{C}[x]\setminus\{0\})^{n+1}\to\mathcal{M}(C_n^{(1)}),$$

which sends a tuple $y = (y_0, \dots, y_n)$ to the Miura oper $\mathcal{L} = \partial + \Lambda^{(2)} + V$ associated to y.

7.2. Deformations of Miura opers of type $C_n^{(1)}$, [7]

Lemma 7.1 ([7]). *Let $\mathcal{L} = \partial + \Lambda^{(2)} + V$ be a Miura oper of type $C_n^{(1)}$. Let α_j be the elements of the dual space defined in Section 3. Let $g \in \mathcal{B}$ and $j \in \{0, \dots, n\}$. Then*

$$e^{\operatorname{ad} gf_j}\mathcal{L} = \partial + \Lambda^{(2)} + V - gh_j - (g' - \langle \alpha_j, V\rangle g + g^2)f_j. \tag{7.5}$$

□

Corollary 7.2 ([7]). *Let $\mathcal{L} = \partial + \Lambda^{(2)} + V$ be a Miura oper of type $C_n^{(1)}$. Then $e^{\operatorname{ad} gf_j}\mathcal{L}$ is a Miura oper if and only if the scalar function g satisfies the Riccati equation*

$$g' - \langle \alpha_j, V\rangle g + g^2 = 0. \tag{7.6}$$

□

Let $\mathcal{L} = \partial + \Lambda^{(2)} + V$ be a Miura oper. For $j \in \{0, \dots, n\}$, we say that $\mathcal{L}$ is *deformable in the j-th direction* if equation (7.6) has a nonzero solution g, which is a rational function.

Theorem 7.3 ([7]). *Let $\mathcal{L} = \partial + \Lambda^{(2)} + V$ be the Miura oper associated to the tuple of polynomials $y = (y_0, \dots, y_n)$. Let $j \in \{0, \dots, n\}$. Then $\mathcal{L}$ is deformable in the j-th direction if and only if there exists a polynomial $\tilde{y}_j$ satisfying equation (6.5). Moreover, in that case any nonzero rational solution g of the Riccati equation (7.6) has the form $g = \ln'(\tilde{y}_j/y_j)$ where $\tilde{y}_j$ is a solution of equation (6.5). If $g = \ln'(\tilde{y}_j/y_j)$, then the Miura oper*

$$e^{\operatorname{ad} gf_j}\mathcal{L} = \partial + \Lambda^{(2)} + V - gh_j \tag{7.7}$$

is associated to the tuple $y^{(j)}$, which is obtained from the tuple y by replacing y_j with $\tilde{y}_j$.

7.3. Miura opers associated with the generation procedure

Let $J = (j_1, \dots, j_m)$ be a degree increasing sequence, see Section 6.5. Let $Y^J : \mathbb{C}^m \to (\mathbb{C}[x])^{n+1}$ be the generation of tuples from $y^\emptyset$ in the J-th direction. We define the associated family of Miura opers by the formula:

$$\mu^J \ : \ \mathbb{C}^m \ \to \ \mathcal{M}(C_n^{(1)}), \qquad c \ \mapsto \ \mu(Y^J(c)).$$

The map μ^J is called the *generation of Miura opers from $\mathcal{L}^\emptyset$ in the J-th direction*, see $\mathcal{L}^\emptyset$ in (7.4).

For $\ell = 1, \dots, m$, denote $J_\ell = (j_1, \dots, j_\ell)$ the beginning ℓ-interval of the sequence J. Consider the associated map $Y^{J_\ell} : \mathbb{C}^\ell \to (\mathbb{C}[x])^{n+1}$. Denote

$$Y^{J_\ell}(c_1, \dots, c_\ell) = (y_0(x, c_1, \dots, c_\ell; \ell), \dots, y_n(x, c_1, \dots, c_\ell; \ell)).$$

Introduce

(7.8)
$$g_1(x, c_1, \dots, c_m) = \ln'(y_{j_1}(x, c_1; 1)),$$
$$g_\ell(x, c_1, \dots, c_m) = \ln'(y_{j_\ell}(x, c_1 \dots, c_\ell; \ell)) - \ln'(y_{j_\ell}(x, c_1, \dots, c_{\ell-1}; \ell - 1)),$$

for $\ell = 2, \dots, m$. For $c \in \mathbb{C}^m$, define $U^J(c) = \sum_{i<0}(U^J(c))_i$, $(U^J(c))_i \in \mathcal{B}(\mathfrak{g}(A_{2n}^{(2)})^i)$, depending on $c \in \mathbb{C}^m$, by the formula

$$e^{-\operatorname{ad} U^J(c)} = e^{\operatorname{ad} g_m(x,c) f_{j_m}} \cdots e^{\operatorname{ad} g_1(x,c) f_{j_1}}. \tag{7.9}$$

Lemma 7.4. *For $c \in \mathbb{C}^m$, we have*

$$\mu^J(c) = e^{-\operatorname{ad} U^J(c)}(\mathcal{L}^\emptyset), \tag{7.10}$$

$$\mu^J(c) = \partial + \Lambda^{(2)} - \sum_{\ell=1}^m g_\ell(x, c) h_{j_\ell}. \tag{7.11}$$

Proof. The lemma follows from Theorem 7.3. □

Corollary 7.5. *Let $r > 0$, odd. Let $c \in \mathbb{C}^m$. Let $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)}$ be the value at $\mu^J(c)$ of the vector field of the r-th mKdV flow on the space $\mathcal{M}(C_n^{(1)})$, see (4.5). Then*

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} = -\frac{\partial}{\partial x}\Big(e^{-\operatorname{ad} U^J(c)}(\Lambda^{(2)})^r\Big)^0. \tag{7.12}$$

Proof. The corollary follows from (4.6) and (7.10). □

We have the natural embedding $\mathcal{M}(C_n^{(1)}) \hookrightarrow \mathcal{M}(A_{2n-1}^{(1)})$, see Section 3.4. Let $J = (j_1, j_2, \dots, j_m)$. Denote $\tilde{J} = (j_1, \dots, j_{m-1})$. Consider the associated family $\mu^{\tilde{J}} : \mathbb{C}^{m-1} \to \mathcal{M}(C_n^{(1)})$. Denote $\tilde{c} = (c_1, \dots, c_{m-1})$.

Proposition 7.6. *For any $r > 0$ the difference $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\big|_{\mu^{\tilde{J}}(\tilde{c})}$ has the following form for some scalar functions $u_1(x, c)$, $u_2(x, c)$:*

(i) if $j_m \in \{1, 2, \dots, n-1\}$, *then*

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)} = u_1(x,c)(e_{j_m+1,j_m+1} - e_{j_m,j_m}) + u_2(x,c)(e_{2n+1-j_m,2n+1-j_m} - e_{2n-j_m,2n-j_m}), \tag{7.13}$$

(ii) if $j_m = 0$, *then*

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)} = u_1(x,c)(e_{2n,2n} - e_{1,1}), \tag{7.14}$$

(iii) if $j_m = n$, *then*

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)} = u_1(x,c)(e_{n+1,n+1} - e_{n,n}). \tag{7.15}$$

Proof. We will write Λ for $\Lambda^{(2)} = \Lambda^{(1)}$. Denote

$$A_r = e^{g_{m-1}f_{j_{m-1}}} \dots e^{g_1 f_{j_1}} \Lambda^r e^{-g_1 f_{j_1}} \dots e^{-g_{m-1}f_{j_{m-1}}}.$$

Expand $A_r = \sum_i A_r^i \Lambda^i$ where $A_r^i = \sum_{l=1}^{2n} A_r^{i,l} e_{l,l}$ with scalar coefficients $A_r^{i,l}$. Then $\frac{\partial}{\partial t_r}\big|_{\mu^{\tilde J}(\tilde c)} = -\frac{\partial}{\partial x}A_r^0$. Assume that $j_m \in \{1, \dots, n-1\}$. Then

$$\begin{aligned}
\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} &= -\frac{\partial}{\partial x}[(1 + g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1})A_r \\
&\times (1 - g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1})]^0 = -\frac{\partial}{\partial x}A_r^0 \\
&-\frac{\partial}{\partial x}[g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r]^0 \\
&+\frac{\partial}{\partial x}[A_r g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}]^0 \\
&+\frac{\partial}{\partial x}[g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r \\
&\qquad \times\ g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}]^0.
\end{aligned}$$

The last term is zero since

$$\begin{aligned}
&[g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r g_m(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}]^0 \\
&= g_m^2[(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r\Lambda^{-1}(e_{j_m+1,j_m+1} + e_{2n+1-j_m,2n+1-j_m})]^0 \\
&= g_m^2(e_{j_m,j_m} + e_{2n-j_m,2n-j_m})[\Lambda^{-1}A_r\Lambda^{-1}]^0(e_{j_m+1,j_m+1} + e_{2n+1-j_m,2n+1-j_m}) \\
&= 0.
\end{aligned}$$

Consider now

$$\begin{aligned}
&\frac{\partial}{\partial x}\big[g_m(e_{j_m,j_m}+e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r\big]^0\\
&=\frac{\partial}{\partial x}\big[g_m(e_{j_m,j_m}+e_{2n-j_m,2n-j_m})\Lambda^{-1}A_r^1\Lambda^1\big]\\
&=\frac{\partial}{\partial x}\big[g_m(e_{j_m,j_m}+e_{2n-j_m,2n-j_m})\Lambda^{-1}\sum_{l=1}^{2n}A_r^{1,l}e_{l,l}\Lambda^1\big]\\
&=\frac{\partial}{\partial x}\big[g_m(e_{j_m,j_m}+e_{2n-j_m,2n-j_m})(A_r^{1,1}e_{2n,2n}+\sum_{l=2}^{2n}A_r^{1,l}e_{l-1,l-1})\big]\\
&=\frac{\partial}{\partial x}\big[g_m(A_r^{1,j_m+1}e_{j_m,j_m}+A_r^{1,2n+1-j_m}e_{2n-j_m,2n-j_m})\big].
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\frac{\partial}{\partial x}\big[A_r g_m(e_{j_m,j_m}+e_{2n-j_m,2n-j_m})\Lambda^{-1}\big]^0\\
&=\frac{\partial}{\partial x}\big[A_r\Lambda^{-1}g_m(e_{j_m+1,j_m+1}+e_{2n+1-j_m,2n+1-j_m})\big]^0\\
&=\frac{\partial}{\partial x}\big[A_r^1 g_m(e_{j_m+1,j_m+1}+e_{2n+1-j_m,2n+1-j_m})\big]\\
&=\frac{\partial}{\partial x}\big[A_r^{1,j_m+1}g_m e_{j_m+1,j_m+1}+A_r^{1,2n+1-j_m}g_m e_{2n+1-j_m,2n+1-j_m}\big].
\end{aligned}$$

So we get

$$\begin{aligned}
\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}-\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}&=(g_mA_r^{1,j_m+1})'(e_{j_m+1,j_m+1}-e_{j_m,j_m})\\
&\quad+\ (g_mA_r^{1,2n+1-j_m})'(e_{2n+1-j_m,2n+2-j_m}-e_{2n-j_m,2n-j_m}).
\end{aligned}$$

This proves the proposition for $j_m\in\{1,\dots,n-1\}$. The cases $j_m=0,n$ are proved similarly. If $j_m=0$, then

$$\begin{aligned}
&\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}=-\frac{\partial}{\partial x}\big[(1+g_m(e_{2n,2n})\Lambda^{-1})A_r(1-g_m(e_{2n,2n})\Lambda^{-1})\big]^0\\
&=-\frac{\partial}{\partial x}A_r^0-\frac{\partial}{\partial x}\big[g_m(e_{2n,2n})\Lambda^{-1}A_r\big]^0+\frac{\partial}{\partial x}\big[A_rg_m(e_{2n,2n})\Lambda^{-1}\big]^0\\
&\qquad+\frac{\partial}{\partial x}\big[g_m(e_{2n,2n})\Lambda^{-1}A_rg_m(e_{2n,2n})\Lambda^{-1}\big]^0.
\end{aligned}$$

The last term is zero since

$$[g_m(e_{2n,2n})\Lambda^{-1}A_r g_m(e_{2n,2n})\Lambda^{-1}]^0 = g_m^2(e_{2n,2n})[\Lambda^{-1}A_r\Lambda^{-1}]^0(e_{1,1}) = 0,$$

and we get

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)} \quad = \quad (g_m A_r^{1,1})'(e_{1,1} - e_{2n,2n}).$$

If $j_m = n$, then

$$\begin{aligned}\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} &= -\frac{\partial}{\partial x}\big[(1+g_m(e_{n,n})\Lambda^{-1})A_r(1-g_m(e_{n,n})\Lambda^{-1})\big]^0\\ &= -\frac{\partial}{\partial x}A_r^0 - \frac{\partial}{\partial x}\big[g_m(e_{n,n})\Lambda^{-1}A_r\big]^0 + \frac{\partial}{\partial x}\big[A_r g_m(e_{n,n})\Lambda^{-1}\big]^0\\ &\quad + \frac{\partial}{\partial x}\big[g_m(e_{n,n})\Lambda^{-1}A_r g_m(e_{n,n})\Lambda^{-1}\big]^0.\end{aligned}$$

The last term is zero since

$$[g_m(e_{n,n})\Lambda^{-1}A_r g_m(e_{n,n})\Lambda^{-1}]^0 = g_m^2(e_{n,n})[\Lambda^{-1}A_r\Lambda^{-1}]^0(e_{n+1,n+1}) = 0,$$

and we get

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} - \frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)} \quad = \quad (g_m A_r^{1,n+1})'(e_{n+1,n+1} - e_{n,n}).$$

□

Let $\mathfrak{m}_i : \mathcal{M}(A_{2n-1}^{(1)}) \to \mathcal{D},\ \mathcal{L} \mapsto L_i$, be the Miura maps defined in Section 4.5 for $i = 0, \dots, n$. Below we consider the composition of the embedding $\mathcal{M}(C_n^{(1)}) \hookrightarrow \mathcal{M}(A_{2n-1}^{(1)})$ and a Miura map.

Lemma 7.7. *If $j_m = 0$, we have $\mathfrak{m}_i \circ \mu^J(\tilde c, c_m) = \mathfrak{m}_i \circ \mu^{\tilde J}(\tilde c)$ for all $i \neq 0$. If $j_m = 1, \dots, n-1$, we have $\mathfrak{m}_i \circ \mu^J(\tilde c, c_m) = \mathfrak{m}_i \circ \mu^{\tilde J}(\tilde c)$ for all $i \neq j_m, 2n - j_m$. If $j_m = n$, we have $\mathfrak{m}_i \circ \mu^J(\tilde c, c_m) = \mathfrak{m}_i \circ \mu^{\tilde J}(\tilde c)$ for all $i \neq n$.*

Proof. The lemma follows from formula (7.10) and Theorem 4.7. □

Lemma 7.8. *If $j_m = 0$, then*

$$\frac{\partial \mu^J}{\partial c_m}(\tilde c, c_m) \;=\; -a\,\frac{y_1(x,\tilde c, m-1)^2}{y_0(x,\tilde c, c_m, m)^2}\,h_0 \qquad (7.16)$$

for some positive integer a. If $j_m = 1, \dots, n-1$, *then*

$$(7.17)\qquad \frac{\partial \mu^J}{\partial c_m}(\tilde{c}, c_m) \;=\; -a\; \frac{y_{j_m-1}(x,\tilde{c},m-1)y_{j_m+1}(x,\tilde{c},m-1)}{y_{j_m}(x,\tilde{c},c_m,m)^2}\, h_{j_m}$$

for some positive integer a. If $j_m = n$, *then*

$$(7.18)\qquad \frac{\partial \mu^J}{\partial c_m}(\tilde{c}, c_m) \;=\; -a\; \frac{y_{n-1}(x,\tilde{c},m-1)^2}{y_n(x,\tilde{c},c_m,m)^2}\, h_n$$

for some positive integer a.

Notice that the right-hand side of these formulas can be written as

$$(7.19)\qquad -a\prod_{i=0}^{n} y_i(x,c,m)^{-a_{i,j}} h_j.$$

Proof. Let $j_m = 0$. Then $y_0(x,\tilde{c},c_m,m) = y_{0,0}(x,\tilde{c}) + c_m y_0(x,\tilde{c},m-1)$, where $y_{0,0}(x,\tilde{c})$ is such that

$$\mathrm{Wr}(y_{0,0}(x,\tilde{c}), y_0(x,\tilde{c},m-1)) \;=\; a\; y_1(x,\tilde{c},m-1)^2,$$

for some positive integer a, see (6.11). We have $g_m = \ln'(y_0(x,\tilde{c},c_m,m)) - \ln'(y_0(x,\tilde{c},m-1))$.

By formula (7.11), we have

$$\begin{aligned}
\frac{\partial \mu^J}{\partial c_m}(\tilde{c}, c_m) &= -\frac{\partial g_m}{\partial c_m}(\tilde{c}, c_m) h_0 \\
&= -\frac{\partial}{\partial c_m}\left(\frac{y'_{0,0}(x,\tilde{c}) + c_m y'_0(x,\tilde{c},m-1)}{y_{0,0}(x,\tilde{c}) + c_m y_0(x,\tilde{c},m-1)}\right) h_0 \\
&= -\frac{\mathrm{Wr}(y_{0,0}(x,\tilde{c}), y_0(x,\tilde{c},m-1))}{(y_{0,0}(x,\tilde{c}) + c_m y_0(x,\tilde{c},m-1))^2} h_0 = -a\;\frac{y_1(x,\tilde{c},m-1)^2}{y_0(x,\tilde{c},c_m,m)^2} h_0\,.
\end{aligned}$$

This proves formula (7.16). The other formulas are proved similarly. □

7.4. Intersection of kernels of $d\mathfrak{m}_i$

Let $J = (j_1, \dots, j_m)$ be a degree increasing sequence and $\mu^J : \mathbb{C}^m \to \mathcal{M}(C_n^{(1)})$ the generation of Miura opers from $\mathcal{L}^\emptyset$ in the J-th direction. We have $\mu^J(c) = \partial + \Lambda^{(1)} + \sum_{k=1}^{2n} v_k(x,c)e_{k,k}$, where

$$\sum_{k=1}^{2n} v_k(x,c) = 0, \quad v_i(x,c) + v_{2n+1-i}(x,c) = 0, \qquad i = 1, \dots, n.$$

Let $X(c) = \sum_{k=1}^{2n} X_k(x,c)e_{k,k} \in T_{\mu^J(c)}\mathcal{M}(C_n^{(1)})$ be a field of tangent vectors to $\mathcal{M}(C_n^{(1)})$ at the points of the image of μ^J,

$$\sum_{k=1}^{2n} X_k(x,c) = 0, \quad X_i(x,c) + X_{2n+1-i}(x,c) = 0, \qquad i = 1,\dots,n.$$

Our goal is to show that under certain conditions we have

$$X(c) = A(c)\frac{\partial \mu^J}{\partial c_m}(c) \tag{7.20}$$

for some scalar function $A(c)$ on $\mathbb{C}^m$.

Proposition 7.9. *Let $j_m = 0$ and $X(c) \in T_{\mu^J(c)}\mathcal{M}(C_n^{(1)})$. Assume that $d\mathfrak{m}_i\big|_{\mu^J(c)}(X(c)) = 0$ for all $i = 1,\dots,2n-1$ and all $c \in \mathbb{C}^m$. Assume that $X(c)$ has the form indicated in the right-hand side of formula (7.14). Then equation (7.20) holds.*

Proof. Since $X_k(x,c) = 0$ for $k = 2,\dots,2n-1$, equation (5.5) takes the form $X_1' - 2v_1X_1 = 0$, or more precisely, $X_1' = 2\ln'\left(\frac{y_1(x,\tilde{c},m-1)}{y_0(x,\tilde{c},c_m,m)}\right)X_1$. Hence $X_1(x,c) = -X_{2n} = A(c)\frac{y_1(x,\tilde{c},m-1)^2}{y_0(x,\tilde{c},c_m,m)^2}$ for some scalar $A(c)$. Lemma 7.8 implies equation (7.20). □

Proposition 7.10. *Let $j_m \in \{1,\dots,n-1\}$ and $X(c) \in T_{\mu^J(c)}\mathcal{M}(C_n^{(1)})$. Assume that $d\mathfrak{m}_i\big|_{\mu^J(c)}(X(c)) = 0$ for all $i \notin \{j_m, 2n-j_m\}$ and all $c \in \mathbb{C}^m$. Assume that $X(c)$ has the form indicated in the right-hand side of formula (7.13). Then equation (7.20) holds.*

Proof. By Lemma 5.3 we have $X_{j_m}' + (v_{j_m} - v_{j_m+1})X_{j_m} = 0$. Then for $j_m = 1,\dots,n-1$, we have

$$\begin{aligned} X_{j_m} &= -X_{j_m+1} = X_{2n-j_m} = -X_{2n+1-j_m} \\ &= A(c)\,\frac{y_{j_m-1}(x,\tilde{c},m-1)y_{j_m+1}(x,\tilde{c},m-1)}{y_{j_m}(x,\tilde{c},c_m,m)^2}. \end{aligned}$$

Lemma 7.8 yields equation (7.20). □

Proposition 7.11. *Let $j_m = n$ and $X(c) \in T_{\mu^J(c)}\mathcal{M}(C_n^{(1)})$. Assume that $d\mathfrak{m}_i\big|_{\mu^J(c)}(X(c)) = 0$ for all $i \neq n$, and $c \in \mathbb{C}^m$. Assume that $X(c)$ has the form indicated in the right-hand side of formula (7.15). Then equation (7.20) holds.*

Proof. By assumptions we have $X_i = 0$ for $i \neq n, n+1$. By Lemma 5.4 we have $X_n' + 2v_n X_n = 0$, where $v_n = \ln' \frac{y_n}{y_{n-1}}$. Hence $X_n = -X_{n+1} = A(c)\frac{y_{n-1}(x,\tilde{c},m-1)^2}{y_n(x,\tilde{c},c_m,m)^2}$ for some scalar function $A(c)$. Lemma 7.8 yields equation (7.20). $\square$

8. Vector fields

8.1. Statement

Let $r > 0$ be odd. Recall that we denote by $\frac{\partial}{\partial t_r}$ the r-th mKdV vector field on the space $\mathcal{M}(C_n^{(1)})$ of Miura opers of type $C_n^{(1)}$. We also denote by $\frac{\partial}{\partial t_r}$ the r-th mKdV vector field of type $A_{2n-1}^{(1)}$ on the space $\mathcal{M}(A_{2n-1}^{(1)})$ of Miura opers of type $A_{2n-1}^{(1)}$. We have a natural embedding $\mathcal{M}(C_n^{(1)}) \hookrightarrow \mathcal{M}(A_{2n-1}^{(1)})$. Under this embedding the vector $\frac{\partial}{\partial t_r}$ on $\mathcal{M}(C_n^{(1)})$ equals the vector filed $\frac{\partial}{\partial t_r}$ on $\mathcal{M}(A_{2n-1}^{(1)})$ restricted to $\mathcal{M}(C_n^{(1)})$, see Section 4.3. We also denote by $\frac{\partial}{\partial t_r}$ the r-th KdV vector field on the space $\mathcal{D}$, see Section 4.4.

For a Miura map $\mathfrak{m}_i : \mathcal{M} \to \mathcal{D}$, $\mathcal{L} \mapsto L_i$, denote by $d\mathfrak{m}_i$ the associated derivative map $T\mathcal{M}(A_{2n-1}^{(1)}) \to T\mathcal{D}$ of tangent spaces. By Theorem 4.6 we have $d\mathfrak{m}_i : \frac{\partial}{\partial t_r}\big|_{\mathcal{L}} \mapsto \frac{\partial}{\partial t_r}\big|_{L_i}$.

Fix a degree increasing sequence $J = (j_1, \dots, j_m)$. Consider the associated family $\mu^J : \mathbb{C}^m \to \mathcal{M}(C_n^{(1)})$ of Miura opers. For a vector field Γ on $\mathbb{C}^m$, we denote by $\mathfrak{L}_\Gamma \mu^J$ the derivative of μ^J along the vector field. The derivative is well-defined since $\mathcal{M}(C_n^{(1)})$ is an affine space.

Theorem 8.1. *Let $r > 0$ be odd. Then there exists a polynomial vector field Γ_r on $\mathbb{C}^m$ such that*

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)} = \mathfrak{L}_{\Gamma_r}\mu^J(c) \tag{8.1}$$

for all $c \in \mathbb{C}^m$. If $r > 2m$, then $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = 0$ for all $c \in \mathbb{C}^m$.

Corollary 8.2. *The family μ^J of Miura opers is invariant with respect to all mKdV flows of type $C_n^{(1)}$ and the family is point-wise fixed by flows with $r > 2m$.*

In other words, every mKdV flow corresponds to a flow on the space of integration parameters $c \in \mathbb{C}^m$. Informally speaking, we may say, that the integration parameters $c = (c_1, \dots, c_m)$ are times of the mKdV flows.

8.2. Proof of Theorem 8.1 for $m = 1$

Let $J = (j_1)$. Then $\mu^J(c_1) = e^{g_1 f_{j_1}} \mathcal{L}^{\emptyset} e^{-g_1 f_{j_1}} = \partial + \Lambda - g_1 h_{j_1}$, where $g_1 = \frac{1}{x+c_1}$, see formula (7.9). We have

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}\Big[e^{g_1 f_{j_1}} \Lambda^r e^{-g_1 f_{j_1}}\Big]^0. \tag{8.2}$$

Assume $j_1 \in \{1, \dots, n-1\}$. Then $e^{g_1 f_{j_1}} = 1 + g_1(e_{j_1,j_1} + e_{2n-j_1,2n-j_1})\Lambda^{-1}$.

For r odd and $r > 1$, the right-hand side of (8.2) is zero. Hence $\frac{\partial}{\partial t_r}|_{\mu^J(c_1)} = \Gamma_r = 0$. For $r = 1$ we have

$$\begin{aligned}\frac{\partial}{\partial t_1}\Big|_{\mu^J(c_1)} &= -\frac{\partial}{\partial x}\Big[e^{g_1 f_{j_1}} \Lambda e^{-g_1 f_{j_1}}\Big]^0 \\ &= \frac{\partial}{\partial x} g_1 h_{j_1} = -\frac{1}{(x+c_1)^2} h_{j_1} = -\frac{\partial \mu^J}{\partial c_1}(c_1).\end{aligned}$$

Hence $\Gamma_1 = -\frac{\partial}{\partial c_1}$.

Assume $j_1 = n$. By formula (7.12), we have

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}\Big[(1 + g_1 e_{n,n})\Lambda^{-1})\Lambda^r(1 - g_1 e_{n,n})\Lambda^{-1}\Big]^0. \tag{8.3}$$

For r odd and $r > 1$, we have $\frac{\partial}{\partial t_r}|_{\mu^J(c_1)} = 0$ by (8.3) and Lemma 2.4. Hence $\Gamma_r = 0$. For $r = 1$, we have

$$\begin{aligned}\frac{\partial}{\partial t_r}\Big|_{\mu^J(c_1)} &= -\frac{dg_1}{dx}(e_{n,n} - e_{n+1,n+1}) \\ &= \frac{dg_1}{dx} h_n = -\frac{1}{(x+c_1)^2} h_n = -\frac{\partial \mu^J}{\partial c_1}(c_1).\end{aligned}$$

Hence $\Gamma_1 = -\frac{\partial}{\partial c_1}$.

Assume $j_1 = 0$. By formula (7.12), we have

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c_1)} = -\frac{\partial}{\partial x}\Big[(1 + g_1 e_{2n,2n})\Lambda^{-1}\Lambda^r(1 - g_1 e_{2n,2n})\Lambda^{-1}\Big]^0. \tag{8.4}$$

For r odd and $r > 1$, we have $\frac{\partial}{\partial t_r}|_{\mu^J(c_1)} = 0$ by (8.4) and Lemma 2.4. Hence $\Gamma_r = 0$. For $r = 1$, we have

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(c_1)} = -\frac{dg_1}{dx}(e_{2n,2n} - e_{1,1}) = \frac{dg_1}{dx} h_0 = -\frac{1}{(x+c_1)^2} h_0 = -\frac{\partial \mu^J}{\partial c_1}(c_1).$$

Hence $\Gamma_1 = -\frac{\partial}{\partial c_1}$. Theorem 8.1 is proved for $m = 1$.

8.3. Beginning of proof of Theorem 8.1 for $m > 1$

We prove the first statement of Theorem 8.1 by induction on m. Let $J = (j_1, \dots, j_m)$. Assume that the statement is proved for $\tilde{J} = (j_1, \dots, j_{m-1})$. Let

$$Y^{\tilde{J}} \,:\, \mathbb{C}^{m-1} \to (\mathbb{C}[x])^{n+1}, \quad \tilde{c} = (c_1, \dots, c_{m-1}) \;\mapsto\; (y_0(x,\tilde{c}), \dots, y_n(x,\tilde{c}))$$

be the generation of tuples in the $\tilde{J}$-th direction. Then the generation of tuples in the J-th direction is

$$\begin{aligned} Y^J \,:\, \mathbb{C}^m \;&\to\; (\mathbb{C}[x])^{n+1}, \\ (\tilde{c}, c_m) \;&\mapsto\; (y_0(x,\tilde{c}), \dots, y_{j_m,0}(x,\tilde{c}) + c_m y_{j_m}(x,\tilde{c}), \dots, y_n(x,\tilde{c}), \end{aligned}$$

see (6.16) and (6.17). We have $g_m = \ln'(y_{j_m,0}(x,\tilde{c}) + c_m y_{j_m}(x,\tilde{c}))$ $- \ln'(y_{j_m}(x,\tilde{c}))$, see (7.8).

By the induction assumption, there exists a polynomial vector field $\Gamma_{r,\tilde{J}} = \sum_{i=1}^{m-1} \gamma_i(\tilde{c}) \frac{\partial}{\partial c_i}$ on $\mathbb{C}^{m-1}$ such that for all $\tilde{c} \in \mathbb{C}^{m-1}$ we have

$$\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde{J}}(\tilde{c})} = \mathfrak{L}_{\Gamma_{r,\tilde{J}}} \mu^{\tilde{J}}(\tilde{c}). \tag{8.5}$$

Proposition 8.3. *There exists a scalar polynomial $\gamma_m(\tilde{c}, c_m)$ on $\mathbb{C}^m$ such that the vector field $\Gamma_r = \Gamma_{r,\tilde{J}} + \gamma_m(\tilde{c}, c_m) \frac{\partial}{\partial c_m}$ satisfies (8.1) for all $(\tilde{c}, c_m) \in \mathbb{C}^m$.*

8.4. Proof of Proposition 8.3

Lemma 8.4. *Let $j_m \in \{1, 2, \dots, n-1\}$, then we have*

$$d\mathfrak{m}_i\Big|_{\mu^J(\tilde{c}, c_m)} \left(\frac{\partial}{\partial t_r}\Big|_{\mu^J(\tilde{c}, c_m)} - \mathfrak{L}_{\Gamma_{r,\tilde{J}}} \mu^J(\tilde{c}, c_m) \right) = 0, \tag{8.6}$$

for all $i \notin \{j_m, 2n - j_m\}$.

Proof. The proof is the same as the proof of Lemma 5.5 in [17]. Namely, by Theorem 4.7 we have $\mathfrak{m}_i \circ \mu^J(\tilde{c}, c_m) = \mathfrak{m}_i \circ \mu^{\tilde{J}}(\tilde{c})$ for all $i \notin \{j_m, 2n - j_m\}$. Hence,

$$\begin{aligned} d\mathfrak{m}_i\Big|_{\mu^J(\tilde{c}, c_m)} \left(\mathfrak{L}_{\Gamma_{r,\tilde{J}}} \mu^J(\tilde{c}, c_m) \right) \;&=\; \mathfrak{L}_{\Gamma_{r,\tilde{J}}} (\mathfrak{m}_i \circ \mu^J)(\tilde{c}, c_m) \\ &=\; \mathfrak{L}_{\Gamma_{r,\tilde{J}}} (\mathfrak{m}_i \circ \mu^{\tilde{J}})(\tilde{c}). \end{aligned} \tag{8.7}$$

By Theorems 4.6 and 4.7, we have

$$dm_i\Big|_{\mu^J(\tilde c,c_m)}\left(\frac{\partial}{\partial t_r}\Big|_{\mu^J(\tilde c,c_m)}\right)=\frac{\partial}{\partial t_r}\Big|_{\mathfrak{m}_i\circ\mu^J(\tilde c,c_m)}=\frac{\partial}{\partial t_r}\Big|_{\mathfrak{m}_i\circ\mu^{\tilde J}(\tilde c)}. \tag{8.8}$$

By the induction assumption, we have

$$\frac{\partial}{\partial t_r}\Big|_{\mathfrak{m}_i\circ\mu^{\tilde J}(\tilde c)}=\mathfrak{L}_{\Gamma_{r,\tilde J}}(\mathfrak{m}_i\circ\mu^{\tilde J})(\tilde c). \tag{8.9}$$

These three formulas prove the lemma. The other two cases are proved similarly. □

Lemma 8.5. *For $j_m\in\{1,\dots,n-1\}$, the difference $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)}-\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^J(\tilde c,c_m)$ has the form indicated in the right-hand side of formula (7.13). For $j_m=0$, the difference has the form indicated in the right-hand side of formula (7.14). For $j_m=n$, the difference has the form indicated in the right-hand side of formula (7.15).*

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}&-\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^J(\tilde c,c_m)\\&=\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}-\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}+\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}-\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^J(\tilde c,c_m)\\&=\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}-\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}+\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^{\tilde J}(\tilde c)-\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^J(\tilde c,c_m)\\&=\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}-\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}+\mathfrak{L}_{\Gamma_{r,\tilde J}}g_m(x,\tilde c,c_m)\,h_{j_m},\end{aligned}$$

see formula (7.11). If $j_m\in\{1,\dots,n-1\}$, then $\frac{\partial}{\partial t_r}\Big|_{\mu^J(c)}-\frac{\partial}{\partial t_r}\Big|_{\mu^{\tilde J}(\tilde c)}$ has the form indicated in the right-hand side of formula (7.13) by Proposition 7.6 and $\mathfrak{L}_{\Gamma_{r,\tilde J}}g_m(x,\tilde c,c_m)\,h_{j_m}$ has that form since $h_{j_m}=-e_{j_m,j_m}+e_{j_m+1,j_m+1}-e_{2n-j_m,2n-j_m}+e_{2n+1-j_m,2n+1-j_m}$. This proves the lemma for $j_m\in\{1,\dots,n-1\}$. The other two cases of the lemma are proved similarly. □

Let us finish the proof of Proposition 8.3. By Lemmas 8.4 and 8.5 the difference $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)}-\mathfrak{L}_{\Gamma_{r,\tilde J}}\mu^J(\tilde c,c_m)$ has the form indicated in the right-hand side of one of the formulas (7.13)–(7.15) and lies in the kernels of the differentials of Miura maps $\mathfrak{m}_i$ for all $i\notin\{j_m,2n-j_m\}$. By Propositions 7.9, 7.10, 7.11

we conclude that the difference has the form $\gamma_m(\tilde{c}, c_m)\frac{\partial \mu^J}{\partial c_m}$ for some scalar function $\gamma_m(\tilde{c}, c_m)$ on $\mathbb{C}^m$. Therefore,

$$\frac{\partial}{\partial t_r}\Big|_{\mu^J(\tilde{c},c_m)} = \mathfrak{L}_{\Gamma_{r,\tilde{J}}}\mu^J(\tilde{c}, c_m) + \gamma_m(\tilde{c}, c_m)\frac{\partial \mu^J}{\partial c_m}(\tilde{c}, c_m).$$

If we set $\Gamma_r = \Gamma_{r,\tilde{J}} + \gamma_m(\tilde{c}, c_m)\frac{\partial}{\partial c_m}$, then the vector field Γ_r will satisfy formula (8.1).

We need to prove that $\gamma_m(\tilde{c}, c_m)$ is a polynomial. The proof of that fact is the same as the proof of [17, Proposition 5.9]. Proposition 8.3 is proved.

8.5. End of proof Theorem 8.1 for $m > 1$

Proposition 8.3 implies the first statement of Theorem 8.1. The second statement says that if $r > 2m$, then $\frac{\partial}{\partial t_r}\big|_{\mu^J(c)} = 0$. But that follows from Corollary 7.5 and Lemma 2.3.

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