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Infinitesimal deformation of Deligne cycle class map

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Abstract: In this note, we study the infinitesimal forms of Deligne cycle class maps. As an application, we prove that the infinitesimal form of a conjecture by Beilinson [1] is true.

Keywords: Chow groups, Milnor K-theory, Deligne cohomology, infinitesimal deformation, Abel-Jacobi map.

1. Introduction

In [1], Beilinson made the following conjecture:

Conjecture 1.1 (Conjecture 2.4.2.1 [1])**.** *Let X be a smooth projective variety defined over a number field k, then for each positive integer p, the rational Chow group $CH^p(X)_{\mathbb{Q}}$ injects into Deligne cohomology of $X_{\mathbb{C}}$, where $X_{\mathbb{C}} := X \times_k \mathbb{C}$. Concretely, if a class in $CH^p(X)_{\mathbb{Q}}$ vanishes in Deligne cohomology $H^{2p}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(p))_{\mathbb{Q}}$ under the composition*

$$CH^p(X)_{\mathbb{Q}} \to CH^p(X_{\mathbb{C}})_{\mathbb{Q}} \xrightarrow{r} H^{2p}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(p))_{\mathbb{Q}},$$

where the right arrow r is the cycle class map for Deligne cohomology, then it is 0.

This conjecture is very difficult to approach, and up to now *there is not a single example with dimension $X \geq 2$ with large Chow ring $CH(X \times_k \mathbb{C})$*

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for which this conjecture has been verified.[1] Esnault and Harris [5] suggest a modest conjecture (see Theorem 0.1 [5]) which follows from Conjecture 1.1 and has been proved in a particular case (see Theorem 0.2 [5]) by using l-adic cohomology.

The main result of this note is to study the infinitesimal form of the Deligne cycles class map, see Lemma 2.7 and Theorem 2.9. As an application, we prove that the infinitesimal form of the Conjecture 1.1 is true, see Theorem 2.11.

As a further application, we show that the infinitesimal form of the following conjecture (due to Griffiths-Harris) is true, see Theorem 3.5,

Conjecture 1.2 ([13])**.** *Let $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a general hypersurface of degree $d \geq 6$, we use*

$$\psi : CH^2_{hom}(X) \to J^2(X)$$

to denote the Abel-Jacobi map from algebraic 1-cycles on X homologically equivalent to zero to the intermediate Jacobian $J^2(X)$, ψ is zero.

2. Deformation of Deligne cycle class map

Let X be a smooth projective variety defined over $\mathbb{C}$ and $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers, we use $X[\varepsilon] := X \times \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ to denote the first order infinitesimal deformation of X. The classical definition of Chow groups can not recognize nilpotents, so to overcome this deficiency, for each positive integer p, one uses the following Soulé's variant (see Theorem 5 in [17]) of Bloch-Quillen identification to study the infinitesimal deformations of Chow groups,

$$CH^p(X) = H^p(X, K_p^M(O_X)) \text{ modulo torsion,} \tag{2.1}$$

where $K_p^M(O_X)$ is the Milnor K-theory sheaf associated to the presheaf

$$U \to K_p^M(O_X(U)).$$

Using the identification (2.1), one considers $H^p(X, K_p^M(O_{X[\varepsilon]}))$ as the first order infinitesimal deformation of $CH^p(X)$ and defines,

Definition 2.1. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the formal tangent space to $CH^p(X)$, denoted $T_f CH^p(X)$, is defined to be the kernel of the natural map*

$$H^p(X, K_p^M(O_{X[\varepsilon]})) \xrightarrow{\varepsilon=0} H^p(X, K_p^M(O_X)).$$

[1] See the first paragraph of page 1 [5].

Deligne cohomology $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ is defined to be the hypercohomology of the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$ (in analytic topology):

$$\mathbb{Z}(p)_{\mathcal{D}}: \ 0 \to \mathbb{Z}(p) \to O_X \to \cdots \to \Omega^{p-1}_{X/\mathbb{C}} \to 0,$$

where $\mathbb{Z}(p) = (2\pi i)^p\mathbb{Z}$ is in degree 0. The infinitesimal deformation of this complex, denoted $\mathbb{Z}(p)_{\mathcal{D}}[\varepsilon]$, has the form,

$$\mathbb{Z}(p)_{\mathcal{D}}[\varepsilon]: \ 0 \to \mathbb{Z}(p) \to O_{X[\varepsilon]} \to \cdots \to \Omega^{p-1}_{X[\varepsilon]/\mathbb{C}[\varepsilon]} \to 0,$$

where $\mathbb{Z}(p)$ is still equal to $(2\pi i)^p\mathbb{Z}$.

Definition 2.2. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the tangent complex to the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$, denoted $\overline{\mathbb{Z}}(p)_{\mathcal{D}}$, is defined to be the kernel of the natural map*

$$\mathbb{Z}(p)_{\mathcal{D}}[\varepsilon] \xrightarrow{\varepsilon=0} \mathbb{Z}(p)_{\mathcal{D}}.$$

Since the map $X \to X[\varepsilon]$ has a retraction $X[\varepsilon] \to X$, $\Omega^i_{X[\varepsilon]/\mathbb{C}[\varepsilon]} = \Omega^i_{X/\mathbb{C}} \oplus \varepsilon\Omega^i_{X/\mathbb{C}}$, where $i = 0, \cdots, p-1$. The tangent complex to the Deligne complex is a direct summand of the thickened Deligne complex

$$\mathbb{Z}(p)_{\mathcal{D}}[\varepsilon] = \mathbb{Z}(p)_{\mathcal{D}} \oplus \overline{\mathbb{Z}}(p)_{\mathcal{D}}.$$

One easily sees that

Lemma 2.3. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the tangent complex (to the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$) $\overline{\mathbb{Z}}(p)_{\mathcal{D}}$ is of the form*

$$0 \to O_X \to \cdots \to \Omega^{p-1}_{X/\mathbb{C}} \to 0,$$

where O_X is in degree 1 and $\Omega^{p-1}_{X/\mathbb{C}}$ is in degree p.

We consider the hypercohomology $\mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}[\varepsilon])$ of the complex $\mathbb{Z}(p)_{\mathcal{D}}[\varepsilon]$ as the infinitesimal deformation of the Deligne cohomology $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ and define,

Definition 2.4. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the formal tangent space to the Deligne cohomology $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$, denoted $T_f H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$, is defined to be the kernel of the natural map*

$$\mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}[\varepsilon]) \xrightarrow{\varepsilon=0} \mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}).$$

By the definition, the formal tangent space to the Deligne cohomology is the hypercohomology of the tangent complex $\overline{\mathbb{Z}}(p)_{\mathcal{D}}$:

$$T_f H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) = \mathbb{H}^{2p}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}}) = \mathbb{H}^{2p-1}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}}[-1]).$$

The following isomorphism is a standard fact in complex geometry,

Lemma 2.5 (cf. Proposition on page 17 of [9]). *With the notations above, one has the isomorphism*

$$\mathbb{H}^{2p}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}}) \cong H^{2p-1}(O_X) \oplus H^{2p-2}(\Omega^1_{X/\mathbb{C}}) \oplus \cdots \oplus H^p(\Omega^{p-1}_{X/\mathbb{C}}).$$

Let $\overline{K}^M_p(O_X)$ denote the kernel of the natural map

$$K^M_p(O_{X[\varepsilon]}) \xrightarrow{\varepsilon=0} K^M_p(O_X).$$

Since the map $X \to X[\varepsilon]$ has a retraction $X[\varepsilon] \to X$, $K^M_p(O_{X[\varepsilon]}) = K^M_p(O_X) \oplus \overline{K}^M_p(O_X)$. By Definition 2.1, the formal tangent space $T_f CH^p(X)$ is $H^p(X, \overline{K}^M_p(O_X))$. Next, we would like to construct a map between tangent spaces

$$H^p(X, \overline{K}^M_p(O_X)) \to \mathbb{H}^{2p}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}}),$$

which is the infinitesimal form of the Deligne cycle class map $CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$.

An element of $\overline{K}^M_p(O_X)$ is of the form $\sum_i \{f^i_1 + \varepsilon g^i_1, \cdots, f^i_p + \varepsilon g^i_p\} \in K^M_p(O_{X[\varepsilon]})$ such that $\sum_i \{f^i_1, \cdots, f^i_p\} = 0 \in K^M_p(O_X)$. We are reduced to looking at $\{f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p\} \in K^M_p(O_{X[\varepsilon]})$ such that $\{f_1, \cdots, f_p\} = 0 \in K^M_p(O_X)$. One sees that[2]

$$\begin{aligned}
(2.2) \quad & \{f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p\} \\
&= \left\{1 + \varepsilon \frac{g_1}{f_1}, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p\right\} + \{f_1, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p\} \\
&= \cdots\cdots\cdots \\
&= \sum_{j=1}^{j=p} \left\{f_1, \cdots, f_{j-1}, 1 + \varepsilon \frac{g_j}{f_j}, f_{j+1} + \varepsilon g_{j+1} \cdots, f_p + \varepsilon g_p\right\},
\end{aligned}$$

where we have used for $j = p + 1$ that $\{f_1, \cdots, f_p\} = 0$.

[2]This presentation (2.2) follows from the referee's suggestion.

Using $\varepsilon^2 = 0$, one sees that

$$dlog\{f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p\} = \varepsilon d\Big(\sum_{j=1}^{j=p}(-1)^{j-1}\Big(\frac{g_j}{f_j}\Big)\frac{df_1}{f_1}\cdots\frac{df_{j-1}}{f_{j-1}}\cdot\frac{df_{j+1}}{f_{j+1}}\cdots\frac{df_p}{f_p}\Big), \tag{2.3}$$

where $d = d_{\mathbb{C}[\varepsilon]}$.

The following commutative diagram, which gives a quasi-isomorphism from the upper complex to the bottom one, is the tangent to the commutative diagram in Section 2.7 [7] (page 56),

$$\begin{array}{ccccccccc} O_X & \longrightarrow & \cdots & \longrightarrow & \Omega^{p-2}_{X/\mathbb{C}} & \longrightarrow & \Omega^{p-1}_{X/\mathbb{C}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow & & \downarrow{\scriptstyle \alpha_{p-1}} & & \downarrow{\scriptstyle \alpha_p} & & \downarrow \\ O_X & \xrightarrow{-\delta_1} & \cdots & \xrightarrow{-\delta_{p-2}} & \Omega^{p-2}_{X/\mathbb{C}} & \xrightarrow{-\delta_{p-1}} & \Omega^{p}_{X/\mathbb{C}} \oplus \Omega^{p-1}_{X/\mathbb{C}} & \xrightarrow{-\delta_p} & \Omega^{p+1}_{X/\mathbb{C}} \oplus \Omega^{p}_{X/\mathbb{C}} \cdots, \end{array}$$

where $\alpha_i(x) = (-1)^{i-1}(x)$ for $1 \le i \le p-1$ and $\alpha_p(x) = (-1)^{p-1}(d_{\mathbb{C}}x, x)$; $\delta_i(x) = dx$ for $1 \le i \le p-2$, $\delta_{p-1}(x) = (0, dx)$ and $\delta_p(x, y) = (-dx, -x+dy)$. Let $\omega = \sum_{j=1}^{j=p}(-1)^{j-1}(\frac{g_j}{f_j})\frac{d_{\mathbb{C}}f_1}{f_1}\cdots\frac{d_{\mathbb{C}}f_{j-1}}{f_{j-1}}\cdot\frac{d_{\mathbb{C}}f_{j+1}}{f_{j+1}}\cdots\frac{d_{\mathbb{C}}f_p}{f_p}$, for $(d_{\mathbb{C}}\omega, \omega) \in \Omega^p_{X/\mathbb{C}} \oplus \Omega^{p-1}_{X/\mathbb{C}}$, there exists the unique $(-1)^{p-1}\omega \in \Omega^{p-1}_{X/\mathbb{C}}$ such that $\alpha_p((-1)^{p-1}\omega) = (d_{\mathbb{C}}\omega, \omega)$.

Definition 2.6. *One defines a map* $\beta : \overline{K}^M_p(O_X) \longrightarrow \Omega^{p-1}_{X/\mathbb{C}}$ *by*

$$\{f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p\} \to (-1)^{p-1}\omega, \tag{2.4}$$

Where $\{f_1, \cdots, f_p\} = 0$ *and*

$$\omega = \sum_{j=1}^{j=p}(-1)^{j-1}\Big(\frac{g_j}{f_j}\Big)\frac{d_{\mathbb{C}}f_1}{f_1}\cdots\frac{d_{\mathbb{C}}f_{j-1}}{f_{j-1}}\cdot\frac{d_{\mathbb{C}}f_{j+1}}{f_{j+1}}\cdots\frac{d_{\mathbb{C}}f_p}{f_p}.$$

We can see the map β (2.4) in an alternative way.[3] In (2.2),

$$\{f_1+\varepsilon g_1, \cdots, f_p+\varepsilon g_p\} = \sum_{j=1}^{j=p}\Big\{f_1, \cdots, f_{j-1}, 1+\varepsilon\frac{g_j}{f_j}, f_{j+1}+\varepsilon g_{j+1}\cdots, f_p+\varepsilon g_p\Big\}.$$

[3]We thank Spencer Bloch and Jerome William Hoffman for comments.

Firstly, applying $dlog$ to $\sum_{j=1}^{j=p}\{f_1,\cdots,f_{j-1},1+\varepsilon\frac{g_j}{f_j},f_{j+1}+\varepsilon g_{j+1}\cdots,f_p+\varepsilon g_p\}$, where $d=d_{\mathbb{C}}$, one obtains

$$
\begin{aligned}
(2.5)\qquad & dlog\{f_1+\varepsilon g_1,\cdots,f_p+\varepsilon g_p\}\\
&= \varepsilon d\Bigg(\sum_{j=1}^{j=p}(-1)^{j-1}\Big(\frac{g_j}{f_j}\Big)\frac{df_1}{f_1}\cdots\frac{df_{j-1}}{f_{j-1}}\cdot\frac{df_{j+1}}{f_{j+1}}\cdots\frac{df_p}{f_p}\Bigg)\\
&+ d\varepsilon\Bigg(\sum_{j=1}^{j=p}(-1)^{j-1}\Big(\frac{g_j}{f_j}\Big)\frac{df_1}{f_1}\cdots\frac{df_{j-1}}{f_{j-1}}\cdot\frac{df_{j+1}}{f_{j+1}}\cdots\frac{df_p}{f_p}\Bigg).
\end{aligned}
$$

Secondly, applying the truncation map $\rfloor\frac{\partial}{\partial\varepsilon}\mid_{\varepsilon=0}:\Omega^p_{X[\varepsilon]/\mathbb{C}}\to\Omega^{p-1}_{X/\mathbb{C}}$ to (2.5), one obtains $(-1)^{p-1}(\sum_{j=1}^{j=p}(-1)^{j-1}(\frac{g_j}{f_j})\frac{df_1}{f_1}\cdots\frac{df_{j-1}}{f_{j-1}}\cdot\frac{df_{j+1}}{f_{j+1}}\cdots\frac{df_p}{f_p})$. This shows that the composition

$$\overline{K}^M_p(O_X)\xrightarrow{dlog}\Omega^p_{X[\varepsilon]/\mathbb{C}}\xrightarrow{\rfloor\frac{\partial}{\partial\varepsilon}\mid_{\varepsilon=0}}\Omega^{p-1}_{X/\mathbb{C}},$$

agrees with the map β (2.4).

The map β (2.4) induces a map from the complex

$$0\to 0\to\cdots\to 0\to\overline{K}^M_p(O_X),$$

where $\overline{K}^M_p(O_X)$ is in degree p, to the complex $\overline{\mathbb{Z}}(p)_{\mathcal{D}}$,

$$
(2.6)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \overline{K}^M_p(O_X)\\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle\beta}\downarrow{\scriptstyle (2.4)}\\
0 & \longrightarrow & O_X & \longrightarrow & \cdots & \longrightarrow & \Omega^{p-2}_{X/\mathbb{C}} & \longrightarrow & \Omega^{p-1}_{X/\mathbb{C}}.
\end{array}
$$

This induces a map between (hyper)cohomology groups

$$(2.7)\qquad \lambda: H^p(X,\overline{K}^M_p(O_X))\to\mathbb{H}^{2p}(X,\overline{\mathbb{Z}}(p)_{\mathcal{D}}).$$

By Lemma 2.5, $\mathbb{H}^{2p}(X,\overline{\mathbb{Z}}(p)_{\mathcal{D}})\cong H^{2p-1}(O_X)\oplus\cdots\oplus H^p(\Omega^{p-1}_{X/\mathbb{C}})$ and by the diagram (2.6), we note the image of the map (2.7) lies in $H^p(\Omega^{p-1}_{X/\mathbb{C}})$. So the map (2.7) is indeed the composition:

$$H^p(X,\overline{K}^M_p(O_X))\to H^p(\Omega^{p-1}_{X/\mathbb{C}})\hookrightarrow\mathbb{H}^{2p}(X,\overline{\mathbb{Z}}(p)_{\mathcal{D}}).$$

According to the known construction [4, 7, 14] of the Deligne cycle class, the Deligne cycle class comes from two parts: the fundamental cycle class (which is induced by the topological homology class) and the de Rham cycle class. Since the fundamental cycle class remains unchanged in the infinitesimal deformation, it does not contribute to the infinitesimal form of the Deligne cycle class map. Only the de Rham cycle class map is relevant with the infinitesimal form of the Deligne cycle class map. To study the infinitesimal form of the Deligne cycle class map, we need to see a construction of de Rham cycle class by using Milnor K-theory. This has been done by Esnault-Parajape [6].

There is a *dlog* map

$$dlog : O_X^* \to \Omega^{\geq 1}_{X/\mathbb{C}}[1]$$

inducing

$$dlog : K_p^M(O_X) \to \Omega^{\geq p}_{X/\mathbb{C}}[p],$$

where $K_p^M(O_X)$ is the Milnor K-theory sheaf. By using both the Soulé's variant of Bloch-Quillen identification (see (2.1) on page 2)

$$CH^p(X) = H^p(X, K_p^M(O_X)) \text{ modulo torsion} \tag{2.8}$$

and the *dlog* map, one obtains the de Rham cycle class map

$$CH^p(X) \to \mathbb{H}^{2p}(X, \Omega^{\geq p}_{X/\mathbb{C}}).$$

See Esnault-Parajape [6] (line 8–24 on page 68) for discussions.

The *dlog* map

$$dlog : K_p^M(O_X) \to \Omega^{\geq p}_{X/\mathbb{C}}[p],$$

is in the same way as (2.3). In this sense, we consider λ (2.7) as the infinitesimal form of the Deligne cycle class map. In summary,

Lemma 2.7. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the infinitesimal form of the Deligne cycle class map*

$$r : CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)),$$

is given by (2.7) $\lambda : H^p(X, \overline{K}_p^M(O_X)) \to \mathbb{H}^{2p}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}})$, which is the composition:

$$H^p(X, \overline{K}_p^M(O_X)) \to H^p(\Omega^{p-1}_{X/\mathbb{C}}) \hookrightarrow \mathbb{H}^{2p}(X, \overline{\mathbb{Z}}(p)_{\mathcal{D}}).$$

Remark 2.8. *When we consider the infinitesimal deformation of the Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$, $\mathbb{Z}(p)$ is fixed so that it does not appear in the tangent complex $\overline{\mathbb{Z}}(p)_{\mathcal{D}}$. This explains why the construction of λ (2.7) is simpler than the known construction [4, 7, 14] of the Deligne cycle class map.*

In the following, we consider $H^p(X, \overline{K}_p^M(O_X)) \to H^p(\Omega^{p-1}_{X/\mathbb{C}})$ as the infinitesimal form of the Deligne cycle class map $r : CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$, and describe it explicitly.

It is well known that $\overline{K}_p^M(O_X) \xrightarrow{\cong} \Omega^{p-1}_{X/\mathbb{Q}}$, where $\Omega^{p-1}_{X/\mathbb{Q}}$ is the absolute differential. For $p = 2$, this was computed by van der Kallen [18], Maazen and Stienstra did further computation in [16]. The isomorphism $\overline{K}_2^M(O_X) \xrightarrow{\cong} \Omega^1_{X/\mathbb{Q}}$ is given by

$$\{f_1 + \varepsilon g_1, f_2 + \varepsilon g_2\} \to \frac{g_2}{f_2}\frac{d_{\mathbb{Q}} f_1}{f_1} - \frac{g_1}{f_1}\frac{d_{\mathbb{Q}} f_2}{f_2}, \tag{2.9}$$

see Green-Griffiths [11] (page 130) and see also [3] (Lemma 3.1 on page 216) for the discussions of two methods: the formula of Maazen-Stienstra and the Chern character from K-theory to negative cyclic homology.

For general p, the isomorphism $\overline{K}_p^M(O_X) \xrightarrow{\cong} \Omega^{p-1}_{X/\mathbb{Q}}$ is given by

$$\{f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p\} \to (-1)^{p-1}\omega_{\mathbb{Q}}, \tag{2.10}$$

where $\omega_{\mathbb{Q}} = \sum_{j=1}^{j=p}(-1)^{j-1}(\frac{g_j}{f_j})\frac{d_{\mathbb{Q}} f_1}{f_1} \cdots \frac{d_{\mathbb{Q}} f_{j-1}}{f_{j-1}} \cdot \frac{d_{\mathbb{Q}} f_{j+1}}{f_{j+1}} \cdots \frac{d_{\mathbb{Q}} f_p}{f_p}$. Consequently, the formal tangent space $T_f CH^p(X)$ can be identified with $H^p(X, \Omega^{p-1}_{X/\mathbb{Q}})$.

Let $\omega = \sum_{j=1}^{j=p}(-1)^{j-1}(\frac{g_j}{f_j})\frac{d_{\mathbb{C}} f_1}{f_1} \cdots \frac{d_{\mathbb{C}} f_{j-1}}{f_{j-1}} \cdot \frac{d_{\mathbb{C}} f_{j+1}}{f_{j+1}} \cdots \frac{d_{\mathbb{C}} f_p}{f_p}$ as on page 5, there exists the following commutative diagram,

$$\begin{array}{ccc}
\{f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p\} & \xrightarrow[(2.4)]{\beta} & (-1)^{p-1}\omega \\
\cong \downarrow (2.10) & & \downarrow = \\
(-1)^{p-1}\omega_{\mathbb{Q}} & \xrightarrow{d_{\mathbb{Q}} \to d_{\mathbb{C}}} & (-1)^{p-1}\omega.
\end{array}$$

This shows that the following diagram is commutative,

$$\begin{array}{ccc}
\overline{K}_p^M(O_X) & \xrightarrow[(2.4)]{\beta} & \Omega^{p-1}_{X/\mathbb{C}} \\
\cong \downarrow (2.10) & & \downarrow = \\
\Omega^{p-1}_{X/\mathbb{Q}} & \xrightarrow{d_{\mathbb{Q}} \to d_{\mathbb{C}}} & \Omega^{p-1}_{X/\mathbb{C}}.
\end{array}$$

Passing to cohomology groups, one has the following commutative diagram,

$$\begin{array}{ccc} H^p(X, \overline{K}_p^M(O_X)) & \longrightarrow & H^p(\Omega^{p-1}_{X/\mathbb{C}}) \\ \cong\downarrow & & \downarrow = \\ H^p(\Omega^{p-1}_{X/\mathbb{Q}}) & \xrightarrow{d_{\mathbb{Q}} \to d_{\mathbb{C}}} & H^p(\Omega^{p-1}_{X/\mathbb{C}}). \end{array}$$

To summarize,

Theorem 2.9. *Let X be a smooth projective variety defined over $\mathbb{C}$, for each positive integer p, the infinitesimal form of the Deligne cycle class map*

$$r : CH^p(X) \to H^{2p}_D(X, \mathbb{Z}(p)),$$

is given by

$$\delta r : H^p(\Omega^{p-1}_{X/\mathbb{Q}}) \to H^p(\Omega^{p-1}_{X/\mathbb{C}}),$$

where δr is induced by the natural map $\Omega^{p-1}_{X/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{C}}$.

Let X be a smooth projective variety over k, where k is a field of characteristic 0. For each positive integer p, one still has the Soulé's variant of Bloch-Quillen identification

$$CH^p(X) = H^p(X, K_p^M(O_X)) \text{ modulo torsion.}$$

Corollary 2.10. *Let X be a smooth projective variety defined over k, where k is a field of characteristic 0, for each positive integer p, the infinitesimal form of the composition*

$$CH^p(X)_{\mathbb{Q}} \to CH^p(X_{\mathbb{C}})_{\mathbb{Q}} \xrightarrow{r} H^{2p}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(p))_{\mathbb{Q}},$$

has the form

$$H^p(\Omega^{p-1}_{X/\mathbb{Q}}) \to H^p(\Omega^{p-1}_{X_{\mathbb{C}}/\mathbb{Q}}) \xrightarrow{\delta r} H^p(\Omega^{p-1}_{X_{\mathbb{C}}/\mathbb{C}}),$$

which is

$$H^p(\Omega^{p-1}_{X/\mathbb{Q}}) \to H^p(\Omega^{p-1}_{X_{\mathbb{C}}/\mathbb{C}}). \tag{2.11}$$

If k is a number field, $\Omega_{X/\mathbb{Q}} = \Omega_{X/k}$ and $H^p(\Omega^{p-1}_{X/\mathbb{Q}}) = H^p(\Omega^{p-1}_{X/k})$. By base change, $H^p(\Omega^{p-1}_{X_{\mathbb{C}}/\mathbb{C}}) \cong H^p(\Omega^{p-1}_{X/k}) \otimes_k \mathbb{C}$. The map (2.11) can be rewritten as

$$H^p(\Omega^{p-1}_{X/k}) \to H^p(\Omega^{p-1}_{X/k}) \otimes_k \mathbb{C},$$

which is obviously injective. In summary,

Theorem 2.11. *The infinitesimal form of Conjecture 1.1 is true. To be precise, let X be a smooth projective variety defined over a number field k, for each positive integer p, the infinitesimal form of the composition*

$$CH^p(X)_{\mathbb{Q}} \to CH^p(X_{\mathbb{C}})_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(p))_{\mathbb{Q}}$$

in Conjecture 1.1 is of the form,

$$H^p(\Omega^{p-1}_{X/k}) \to H^p(\Omega^{p-1}_{X/k}) \otimes_k \mathbb{C},$$

which is injective.

Conjecture 1.1 is part of the Bloch-Beilinson conjecture. Let X be a smooth projective variety over $\mathbb{C}$, for each positive integer p, Bloch-Beilinson conjecture predicts that there is a filtration which has the form

$$\begin{aligned} CH^p(X)_{\mathbb{Q}} = F^0CH^p(X)_{\mathbb{Q}} \supset F^1CH^p(X)_{\mathbb{Q}} \\ \supset \cdots \supset F^pCH^p(X)_{\mathbb{Q}} \supset F^{p+1}CH^p(X)_{\mathbb{Q}} = 0. \end{aligned}$$

The first two steps are known and $F^2CH^p(X)_{\mathbb{Q}}$ is the kernel of the Deligne cycle class map

$$r : CH^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))_{\mathbb{Q}}.$$

An alternative way to state Conjecture 1.1 is,

Conjecture 2.12 (cf. Implication 1.2 [10] page 478)**.** *If X is a smooth projective variety over a number field k, then*

$$F^2CH^p(X)_{\mathbb{Q}} = 0,$$

where $F^2CH^p(X)_{\mathbb{Q}}$ is the filtration induced from $F^2CH^p(X_{\mathbb{C}})_{\mathbb{Q}}$ under the natural map $CH^p(X) \to CH^p(X_{\mathbb{C}})$.

Theorem 2.11 suggests that this Conjecture (and Conjecture 1.1) looks reasonable at the infinitesimal level.

Remark 2.13. *The assumption"k is a number field" in Theorem 2.11 is crucial, it guarantees $\Omega_{X/\mathbb{Q}} = \Omega_{X/k}$. This suggests that the assumption"k is a number field" in Conjecture 1.1 can not be loosened. In fact, for the ground field of transcendental degree one, Green-Griffiths-Paranjape [12] has found counterexamples, extending earlier examples by Bloch, Nori and Schoen.*

To understand algebraic cycles, the transcendental degree of the ground field does matter.

3. Deformation of Abel-Jacobi map

As a further application of Theorem 2.9, we study the infinitesimal form of the following conjecture of Griffiths and Harris:

Conjecture 3.1 ([13]). *Let $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a general hypersurface of degree $d \geq 6$, we use*

$$\psi : CH^2_{hom}(X) \to J^2(X)$$

to denote the Abel-Jacobi map from algebraic 1-cycles on X homologically equivalent to zero to the intermediate Jacobian $J^2(X)$, ψ is zero.

There are many known results showing that the image of the Abel-Jacobi map is torsion, or even that a Chow group is torsion. For example, Green and Voisin studied this conjecture and showed that

Theorem 3.2 (Theorem 0.1 of [8][4], 1.5 of [19]). *For $X \subset \mathbb{P}^4_{\mathbb{C}}$ a general hypersurface of degree $d \geq 6$, the image of the Abel-Jacobi map on algebraic 1-cycles on X homologically equivalent to zero has image contained in the torsion points of the intermediate Jacobian.*

In the following, we give an illustration of the use of tangent spaces to Chow groups to prove that the infinitesimal version of Conjecture 3.1 is true, see Theorem 3.5.

Let X be a smooth projective variety over $\mathbb{C}$, for each positive integer p, there exists the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & CH^p_{hom}(X) & \longrightarrow & CH^p(X) & \longrightarrow & \dfrac{CH^p(X)}{CH^p_{hom}(X)} & \longrightarrow & 0 \\
 & & \psi\downarrow & & r\downarrow & & cl\downarrow & & \\
0 & \longrightarrow & J^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) & \longrightarrow & Hg^p(X) & \longrightarrow & 0.
\end{array}$$

We briefly explain the terminologies in this diagram,

- $J^p(X)$ is Griffiths intermediate Jacobian and ψ is the Abel-Jacobi map.
- $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$ is Deligne cohomology and r is the Deligne cycle class map. For the construction of r, we refer to [4, 7, 14].
- $Hg^p(X)$ is the Hodge group, defined as $f^{-1}(H^{p,p}(X))$, where $f : H^{2p}(X, \mathbb{Z}(p)) \to H^{2p}(X,\mathbb{C})$ is induced by the inclusion $\mathbb{Z} \to \mathbb{C}$. The map *cl* is the cycle class map, see [15] for a survey.

Since the Hodge group $Hg^p(X)$ ($\subseteq H^{2p}(X,\mathbb{Z}(p))$) is discrete, so $J^p(X)$ and $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$ have the same tangent space. Similarly, $\frac{CH^p(X)}{CH^p_{hom}(X)}$ ($\subseteq H^{2p}(X,\mathbb{Z})$)

[4]In fact, Green [8] proved analogous results in all dimension.

is discrete so that $CH^p(X)$ and $CH^p_{hom}(X)$ have the same tangent space. So we have,

Theorem 3.3 (known to experts, see [19] (line 9–11 on page 707)). *Let X be a smooth projective variety over $\mathbb{C}$, for each positive integer p, the infinitesimal form of the Abel-Jacobi map,*

$$\psi : CH^p_{hom}(X) \to J^p(X),$$

agrees with that of the Deligne cycles class map,

$$r : CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Combining Theorem 3.3 with Theorem 2.9, one has,

Corollary 3.4. *Let X be a smooth projective variety over $\mathbb{C}$, for each positive integer p, the infinitesimal form of the Abel-Jacobi map,*

$$\psi : CH^p_{hom}(X) \to J^p(X)$$

is given by

$$\delta r : H^P(\Omega^{p-1}_{X/\mathbb{Q}}) \to H^p(\Omega^{p-1}_{X/\mathbb{C}}),$$

where δr is induced by the natural map $\Omega^{p-1}_{X/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{C}}$.

In particular, for X a smooth projective three-fold over $\mathbb{C}$, the infinitesimal form of the Abel-Jacobi map

$$\psi : CH^2_{hom}(X) \to J^2(X)$$

is given by

$$\delta r : H^2(\Omega^1_{X/\mathbb{Q}}) \to H^2(\Omega^1_{X/\mathbb{C}}).$$

This has been known to experts, see page 189 (line 21–27) of [11].

Now, we prove that the infinitesimal form of Conjecture 3.1 is true,

Theorem 3.5. *For $X \subset \mathbb{P}^4_{\mathbb{C}}$ a general*[5] *hypersurface of degree $d \geq 6$, the image of the infinitesimal form of the Abel-Jacobi map*

$$\delta r : H^2(\Omega^1_{X/\mathbb{Q}}) \to H^2(\Omega^1_{X/\mathbb{C}})$$

is zero.

[5] In Conjecture 3.1, *general* means *outside a countable union of proper subvarieties.* In Theorem 3.5, *general* means *all the coefficients of the defining equation of X are algebraically independent.*

Proof. There is a natural short exact sequence of sheaves

$$0 \to \Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/\mathbb{C}} \to 0.$$

The associated long exact sequence is of the form

$$\begin{aligned} 0 &\to H^0(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X) \to \cdots \to H^2(\Omega^1_{X/\mathbb{Q}}) \xrightarrow{\delta r} H^2(\Omega^1_{X/\mathbb{C}}) \\ &\xrightarrow{f} H^3(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X) \to \cdots . \end{aligned}$$

So the image of δr can be identified with the kernel of f,

$$\mathrm{Im}(\delta r) = \mathrm{Ker}(f).$$

The dual of $H^2(\Omega^1_{X/\mathbb{C}}) \xrightarrow{f} H^3(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X)$ ($\cong \Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes H^3(O_X)$), can be rewritten using the Poincaré residue representation as

$$\Omega^{1\,*}_{\mathbb{C}/\mathbb{Q}} \otimes R^{d-5} \to R^{2d-5},$$

where R^j is the Jacobian ring of the hypersurface at degree j (the assumption "$d \geq 6$" guarantees $d-5 \geq 1$).

Let $S^d \subset \mathbb{C}[z_0, z_1, z_2, z_3]$ denote homogeneous polynomials of degree d. The map $\Omega^{1\,*}_{\mathbb{C}/\mathbb{Q}} \to R^d$ is defined as

$$\frac{\partial}{\partial \alpha} \to \frac{\partial F}{\partial \alpha},$$

where $\alpha \in \mathbb{C}$ and F is the equation of the hypersurface X. Let α run through all the complex numbers, then $\frac{\partial F}{\partial \alpha}$ generates a subspace of S^d, denoted W.

Now, the map

$$\Omega^{1\,*}_{\mathbb{C}/\mathbb{Q}} \otimes R^{d-5} \to R^{2d-5}$$

can be described as a composition

$$\Omega^{1\,*}_{\mathbb{C}/\mathbb{Q}} \otimes R^{d-5} \to W \otimes R^{d-5} \to R^{2d-5},$$

where the right map $W \otimes R^{d-5} \to R^{2d-5}$ is polynomial multiplication.

Since X is general, all the coefficients of F are algebraically independent, the codimension of W in S^d is zero. By taking $p = 0$ and $k = 2d - 5$ in the Theorem on page 74 of [9] we see that $W \otimes S^{d-5} \to S^{2d-5}$ is surjective. Consequently, $W \otimes R^{d-5} \to R^{2d-5}$ is surjective because of the following commutative diagram (both of the vertical arrows are surjective),

$$\begin{array}{ccc} W \otimes S^{d-5} & \longrightarrow & S^{2d-5} \\ \downarrow & & \downarrow \\ W \otimes R^{d-5} & \longrightarrow & R^{2d-5}. \end{array}$$

So the map $\Omega^{1\,*}_{\mathbb{C}/\mathbb{Q}} \otimes R^{d-5} \to R^{2d-5}$ is surjective. Dually, the map $H^2(\Omega^1_{X/\mathbb{C}}) \xrightarrow{f} H^3(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X)$ is injective. In conclusion, $\mathrm{Im}(\delta r) = \mathrm{Ker}(f) = 0$. □

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