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Areas of totally geodesic surfaces of hyperbolic 3–orbifolds

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Abstract: The geodesic length spectrum of a complete, finite volume, hyperbolic 3–orbifold M is a fundamental invariant of the topology of M via Mostow–Prasad Rigidity. Motivated by this, the second author and Reid defined a two-dimensional analogue of the geodesic length spectrum given by the multiset of isometry types of totally geodesic, immersed, finite-area surfaces of M called the geometric genus spectrum. They showed that if M is arithmetic and contains a totally geodesic surface, then the geometric genus spectrum of M determines its commensurability class. In this paper we define a coarser invariant called the totally geodesic area set given by the set of areas of surfaces in the geometric genus spectrum. We prove a number of results quantifying the extent to which non-commensurable arithmetic hyperbolic 3–orbifolds can have arbitrarily large overlaps in their totally geodesic area sets.

Keywords: Hyperbolic orbifolds, totally geodesic surfaces.

1. Introduction

Given a complete, hyperbolic 3–orbifold M with finite volume, the *geodesic length spectrum* of M is the multiset of lengths of closed geodesics on M. The length spectrum of M determines the Laplace eigenvalue spectrum of M (see [12, Thm. 2]) and thus determines spectral invariants like dimension, volume,

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and total scalar curvature. When M is arithmetic, Chinburg–Hamilton–Long–Reid [7] proved that the length spectrum determines the commensurability class of M. It is an open question whether this holds in the setting of non-arithmetic hyperbolic 3–manifolds (see [22, 23]). Motivated by this question, Futer–Millichap [11] constructed, for all sufficiently large V, pairs of non-commensurable non-arithmetic hyperbolic 3–manifolds with volume approximately V and whose length spectra agree up to length V.

In [19], a two-dimensional analogue of the geodesic length spectrum of M was introduced. The *geometric genus spectrum* is the set of isometry types of immersed, totally geodesic hyperbolic surfaces in M considered with multiplicity. That every hyperbolic surface occurs with finite multiplicity follows from Thurston's work on pleated surfaces [25, Cor. 8.8.6]. Unlike the geodesic length spectrum, the geometric genus spectrum can be finite or even empty. Indeed, many hyperbolic 3–manifolds do not contain any immersed totally geodesic hyperbolic surfaces. It is here that arithmeticity becomes relevant, as an arithmetic hyperbolic 3–manifold that contains a single totally geodesic surface must contain infinitely many pairwise non-commensurable totally geodesic surfaces (see [18, Ch. 9]). In [19], the second author and Reid considered the class of arithmetic hyperbolic 3–manifolds and showed that if M_1, M_2 are two such manifolds with equal (nonempty) geometric genus spectra then M_1, M_2 are commensurable.

In this paper we will consider the case in which M is arithmetic and define the coarser invariant of all areas of immersed, totally geodesic hyperbolic 2–orbifolds of M whose fundamental groups are maximal arithmetic Fuchsian groups in the sense of Maclachlan–Reid [18]. We will call this invariant the *totally geodesic area set* and denote it by $\mathrm{TGA}(M)$. The set $\mathrm{TGA}(M)$ is a discrete subset of the positive real numbers. In the case that $\mathrm{TGA}(M)$ is nonempty, we do not in general expect that $\mathrm{TGA}(M)$ will determine the commensurability class of M. In §5, for instance, we construct non-commensurable arithmetic Kleinian groups whose commensurators contain maximal arithmetic Fuchsian groups having exactly the same coareas. One may therefore ask whether a two-dimensional analogue of [11] holds. That is, can the totally geodesic area sets of non-commensurable arithmetic hyperbolic 3–orbifolds have arbitrarily large overlap? In this paper we answer this question in the affirmative by proving a number of results which quantify the extent to which non-commensurable arithmetic hyperbolic 3–orbifolds may have arbitrarily large overlaps in their totally geodesic area sets.

Before stating our main results we briefly recall that the Gauss–Bonnet formula for closed hyperbolic surfaces implies that the area of such a surface S is

$$\mathrm{Area}(S) = -2\pi\chi(S) = -2\pi(2-2g)$$

where $g = \text{genus}(S)$ and $\chi(S)$ is the Euler characteristic of S. In the orbifold setting the analogous result states that if S is a hyperbolic 2–orbifold of signature $(g; p_1, \ldots, p_m)$ then

$$\text{Area}(S) = -2\pi\chi(S) = -2\pi\left((2-2g) - \sum_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)\right).$$

In particular it is in general not the case that the area A of a hyperbolic 2–orbifold determines the topological type of the orbifold.

Throughout this paper we will make use of standard asymptotic notation and will use the Vinogradov symbol $f \ll g$ to indicate that there exists a positive constant C such that $|f| < C\,|g|$.

Theorem 1.1. *Let M be an arithmetic hyperbolic 3–orbifold and $A_1 < A_2 < \cdots < A_s$ be positive real numbers contained in* $\text{TGA}(M)$*. Let $F(V)$ denote the number of commensurability classes of hyperbolic 3–orbifolds containing a representative M' with $\{A_1, \ldots, A_s\} \subset \text{TGA}(M')$ and $\text{vol}(M') < V$. Then for all sufficiently large V, we have*

$$F(V) \gg \frac{1}{\zeta(2)^{2n/3}} \cdot \frac{V^{2/3}}{\text{vol}(M)^{12}},$$

where the implicit constant depends only on the set $\{A_1, \ldots, A_s\}$ and n denotes the degree over $\mathbf{Q}$ of the invariant trace field of M.

As an immediate application of Theorem 1.1 we obtain the following corollary.

Corollary 1.2. *Let $A_1 < A_2 < \cdots < A_s$ be positive real numbers. If there exists an arithmetic hyperbolic 3–orbifold which contains immersed totally geodesic hyperbolic 2–orbifolds of areas $A_1, \ldots, A_s$ whose fundamental groups are maximal arithmetic Fuchsian groups then in fact there exist infinitely many pairwise non-commensurable arithmetic hyperbolic 3–orbifolds with this property.*

We are also able to obtain an upper bound for the maximum cardinality of the set of pairwise non-commensurable arithmetic hyperbolic 3–orbifolds having bounded volume and totally geodesic area sets containing $\{A_1, \ldots, A_s\}$. In fact, we will prove a stronger result and give an upper bound for the number of maximal arithmetic hyperbolic 3–orbifolds containing immersed totally geodesic hyperbolic 2–orbifolds with areas $A_1, \ldots, A_s$ and fundamental groups that are maximal arithmetic Fuchsian groups. Our result has the added benefit of making the dependence upon $A_1, \ldots, A_s$ explicit.

Theorem 1.3. *Let $A_1 < A_2 < \cdots < A_s$ be positive real numbers and $G(V)$ denote the number of isometry classes of maximal arithmetic hyperbolic 3–orbifolds M with $\{A_1, \dots, A_s\} \subset \mathrm{TGA}(M)$ and $\mathrm{vol}(M) < V$. Then for any $\epsilon > 0$ and all sufficiently large V, we have $G(V) < e^{c\log(A_1)^{1+\epsilon}} V^{26}$, where c is a positive constant which depends only on ϵ.*

Theorems 1.1 and 1.3 both concern the behavior of totally geodesic area sets across commensurability classes and the set of maximal arithmetic hyperbolic 3–orbifolds in a commensurability class. In §6 we take a different perspective and fix an arithmetic hyperbolic 3–orbifold M whose totally geodesic area set contains a fixed set of real numbers $\{A_1, \dots, A_s\}$. The main result of §6 is a lower bound for the number of covers of M which have bounded volume and whose totally geodesic area sets also contain $\{A_1, \dots, A_s\}$.

Theorem 1.4. *Let M be an arithmetic hyperbolic 3–orbifold and $A_1 < A_2 < \cdots < A_s$ be positive real numbers contained in $\mathrm{TGA}(M)$. Let $H(V)$ denote the number of covers M' of M such that $\{A_1, \dots, A_s\} \subset \mathrm{TGA}(M')$ and $\mathrm{vol}(M') \leq V$. Then for all sufficiently large V*

$$H(V) \gg \frac{V^{1/3}}{\mathrm{vol}(M)^{1/3}\left[\log(V) - \log(\mathrm{vol}(M))\right]^{1/2}},$$

where the implicit constant depends on K, the field of definition of M, and s.

2. Background

2.1. Notation

Throughout this article $\log(x)$ will denote the natural logarithm function, k will denote a number field, and K/k will be a relative quadratic extension. The degree of k will be denoted by n_k and the ring of k–integers of k will be denoted by $\mathcal{O}_k$. Furthermore, we will denote by $d_k, h_k, \zeta_k(s)$ the absolute discriminant, class number, and Dedekind zeta function of k. The signature of k will be denoted (r_1, r_2) where r_1 is the number of real places of k and r_2 is the number of complex places of k. Given an ideal $\mathfrak{a} \subset \mathcal{O}_k$ we will use $\mathrm{N}(\mathfrak{a})$, without any subscripts, to denote the norm of $\mathfrak{a}$ down to $\mathbf{Q}$. We will always use subscripts to denote the norm of a relative extension of number fields (e.g., $\mathrm{N}_{K/k}(\mathfrak{a})$). We denote the set of places/primes of k by V_k and the set of infinite places by V_k^∞. For each prime $\mathfrak{p} \in V_k$, we denote the associated complete local field by $k_\mathfrak{p}$. Given a quaternion algebra B over k, we will denote by $\mathrm{Ram}(B)$ the set of all places of k at which B is ramified. We will denote the set of finite (resp. infinite) places of k at which B is ramified by $\mathrm{Ram}_f(B)$ (resp. $\mathrm{Ram}_\infty(B)$). For future reference, define $|\mathrm{Ram}_\infty(B)| = r_{B,\infty}$ and $|\mathrm{Ram}_f(B)| = r_{B,f}$. The discriminant $\mathscr{D}_B$ of B is defined to be the product of all finite primes ramifying in B. We will denote by k_B the class field associated to B. More explicitly, k_B is the maximal abelian extension of k which has 2–elementary Galois group, is unramified outside of $\mathrm{Ram}_\infty(B)$, and in which every finite prime of k which ramifies in B splits completely. On the geometric side, we will denote by $\mathbf{H}^2$ and $\mathbf{H}^3$ real hyperbolic 2– and 3–space. We will use M to denote an arithmetic hyperbolic 3–orbifold and N to denote an arithmetic hyperbolic 2–orbifold. That is, $M = \mathbf{H}^3/\Gamma_M$ and $N = \mathbf{H}^2/\Gamma_N$ where Γ_M and Γ_N are arithmetic lattices in $\mathrm{PSL}(2,\mathbf{C})$ and $\mathrm{PSL}(2,\mathbf{R})$. We will refer to lattices in $\mathrm{PSL}(2,\mathbf{C})$ and $\mathrm{PSL}(2,\mathbf{R})$ as being Kleinian and Fuchsian respectively.

2.2. Number theoretic preliminaries

Given a square-free ideal $\mathfrak{a} \subset \mathcal{O}_k$, define two functions

$$\Phi_1(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \frac{\mathrm{N}(\mathfrak{p})-1}{2}, \quad \Phi_2(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} (\mathrm{N}(\mathfrak{p})+1)\,.$$

In this section we will record two results about these functions which we will later use to analyze the covolume of certain arithmetic groups. Both of these lemmas will make use of the following lemma of Belolipetsky–Gelander–Lubotzky–Shalev [2, Lemma 3.4].

Lemma 2.1 (Belolipetsky–Gelander–Lubotzky–Shalev)**.** *Let $I_k(X)$ denote the number of ideals of $\mathcal{O}_k$ of norm less than X, then $I_k(X) < \zeta_k(2)X^2$.*

Lemma 2.2. *The number of square-free ideals $\mathfrak{a} \subset \mathcal{O}_k$ with $\Phi_1(\mathfrak{a}) \leq X$ is at most $10^{4n_k}\zeta_k(2)X^4$.*

Proof. We will show that $\mathrm{N}(\mathfrak{a}) < 100^{n_k}\Phi_1(\mathfrak{a})^2$. The result will then follow from Lemma 2.1. Observe that

$$\frac{\Phi_1(\mathfrak{a})^2}{\mathrm{N}(\mathfrak{a})} = \prod_{\mathfrak{p}|\mathfrak{a}} \mathrm{N}(\mathfrak{p})^{-1}\left(\frac{\mathrm{N}(\mathfrak{p})-1}{2}\right)^2 = \prod_{\mathfrak{p}|\mathfrak{a}} \frac{\mathrm{N}(\mathfrak{p})-2+\mathrm{N}(\mathfrak{p})^{-1}}{4}.$$

The terms in the latter product are strictly greater than 1 whenever $\mathrm{N}(\mathfrak{p}) \geq 7$, and an easy calculation of the values for $\mathrm{N}(\mathfrak{p}) \in \{2,3,4,5\}$, along with the fact that k contains at most n_k primes of any fixed norm, shows that $\frac{\Phi_1(\mathfrak{a})^2}{\mathrm{N}(\mathfrak{a})} > \left(\frac{1}{100}\right)^{n_k}$, which is what we wanted to show. ☐

Lemma 2.3. *The number of square-free ideals* $\mathfrak{a} \subset \mathcal{O}_k$ *with* $\Phi_2(\mathfrak{a}) \leq X$ *is at most* $\zeta_k(2)X^2$.

Proof. This is an immediate consequence of Lemma 2.1 and the trivial bound $\Phi_2(\mathfrak{a}) > \mathrm{N}(\mathfrak{a})$. □

2.3. Arithmetic lattices

In this subsection we give a brief review of the construction of arithmetic lattices acting on $(\mathbf{H}^2)^a \times (\mathbf{H}^3)^b$ and refer the reader interested in a more detailed discussion to the text of Maclachlan–Reid [18]. To begin, we fix a number field k of signature (r_1, r_2) and a k–quaternion algebra B which is not totally definite (i.e. $V_k^\infty \not\subset \mathrm{Ram}(B)$). Under these assumptions, we have an isomorphism

$$B \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{M}(2, \mathbf{R})^{r_1 - r_{B,\infty}} \times \mathbf{H}^{r_{B,\infty}} \times \mathrm{M}(2, \mathbf{C})^{r_2}.$$

This isomorphism induces an injective homomorphism

$$B^\times \hookrightarrow \prod_{\mathfrak{p} \notin \mathrm{Ram}_\infty(B)} (B \otimes_k k_{\mathfrak{p}})^\times \longrightarrow \mathrm{GL}(2, \mathbf{R})^{r_1 - r_{B,\infty}} \times \mathrm{GL}(2, \mathbf{C})^{r_2}.$$

Restricting to the elements B^1 of $B^\times$ with reduced norm 1 gives us an injective homomorphism

$$\pi \colon B^1 \hookrightarrow \mathrm{SL}(2, \mathbf{R})^{r_1 - r_{B,\infty}} \times \mathrm{SL}(2, \mathbf{C})^{r_2}.$$

Given a maximal order $\mathcal{O}$ of B, by work of Vignéras [26], $\mathrm{P}(\pi(\mathcal{O}^1))$ is a lattice in

$$\mathrm{PSL}(2, \mathbf{R})^{r_1 - r_{B,\infty}} \times \mathrm{PSL}(2, \mathbf{C})^{r_2}.$$

Finally, we say that an irreducible lattice $\Gamma \subset \mathrm{PSL}(2, \mathbf{R})^{r_1 - r_{B,\infty}} \times \mathrm{PSL}(2, \mathbf{C})^{r_2}$ is *arithmetic* if Γ is commensurable with a lattice of the form $\mathrm{P}(\pi(\mathcal{O}^1))$. When $r_1 + r_2 - r_{B,\infty} = 1$, Γ is an arithmetic Fuchsian group or an arithmetic Kleinian group.

For a discrete, finitely generated subgroup Γ of either $\mathrm{PSL}(2, \mathbf{R})$ or $\mathrm{PSL}(2, \mathbf{C})$, the *trace field* of Γ is the field given by $\mathbf{Q}(\mathrm{tr}\,\gamma : \gamma \in \Gamma)$. Although the trace field of Γ is not an invariant of the commensurability class, it turns out that the trace field of the subgroup $\Gamma^{(2)} = \{\gamma^2 : \gamma \in \Gamma\}$ is a commensurability class invariant. We denote the trace field of $\Gamma^{(2)}$ by $k\Gamma$ and call it the *invariant trace field* of Γ. We may also define an algebra over the invariant trace field $k\Gamma$ by $B\Gamma := \left\{\sum b_i \gamma_i : b_i \in k\Gamma, \gamma_i \in \Gamma^{(2)}\right\}$, where each sum is finite. Multiplication in $B\Gamma$ is defined in the obvious manner: $(b_1\gamma_1)\cdot(b_2\gamma_2) := (b_1b_2)(\gamma_1\gamma_2)$. The algebra $B\Gamma$ is a quaternion algebra which is also an invariant of the commensurability class of Γ. We call $B\Gamma$ the *invariant quaternion algebra* of Γ. The invariant trace field of $\mathrm{P}(\pi(\mathcal{O}^1))$ is k and the invariant quaternion algebra of $\mathrm{P}(\pi(\mathcal{O}^1))$ is B. Note that the invariant trace field and invariant quaternion algebras are complete commensurability class invariants in the sense that if Γ_1 and Γ_2 are arithmetic lattices then they are commensurable (in the wide sense) if and only if $k\Gamma_1 \cong k\Gamma_2$ and $B\Gamma_1 \cong B\Gamma_2$ (see [18, Thms. 8.4.1 and 8.4.6]).

2.4. Maximal arithmetic lattices

We now briefly describe the construction of maximal arithmetic lattices in the commensurability class given by the arithmetic data (k, B). This construction is given in more detail in Borel [3], Chinburg–Friedman [6, p. 41], and Maclachlan–Reid [18, Ch. 11].

Let S be a finite set of primes of k which is disjoint from $\mathrm{Ram}_f(B)$. Given a prime $\mathfrak{p}\in S$, fix an edge $\{M_\mathfrak{p}^1, M_\mathfrak{p}^2\}$ in the tree of maximal orders of $B\otimes_k k_\mathfrak{p} \cong \mathrm{M}(2, k_\mathfrak{p})$ (i.e., in the affine building associated to $\mathrm{SL}(2, k_\mathfrak{p})$, which in this case has the structure of a tree). More algebraically, let $\pi_\mathfrak{p}$ be a uniformizer for $k_\mathfrak{p}$. Then we are fixing two maximal orders $\{M_\mathfrak{p}^1, M_\mathfrak{p}^2\}$ with the property that as $\mathcal{O}_{k_\mathfrak{p}}$–modules, $M_\mathfrak{p}^1/M_\mathfrak{p}^1\cap M_\mathfrak{p}^2 = \mathcal{O}_{k_\mathfrak{p}}/\pi_\mathfrak{p}\mathcal{O}_{k_\mathfrak{p}}$. Given a maximal order $\mathcal{O} < B$ and a prime $\mathfrak{p}$ of k, denote by $\mathcal{O}_\mathfrak{p}$ the maximal order $\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k_\mathfrak{p}}$ of $B\otimes_k k_\mathfrak{p}$. Define a subgroup $\Gamma_{S,\mathcal{O}} \subset \mathrm{PSL}(2,\mathbf{R})^{r_1-r_{B,\infty}}\times\mathrm{PSL}(2,\mathbf{C})^{r_2}$ by intersecting the preimage in $\mathrm{PGL}(2,\mathbf{R})^{r_1-r_{B,\infty}}\times\mathrm{PGL}(2,\mathbf{C})^{r_2}$ of

$$\{\overline{x}\in B^\times/k^\times : x\mathcal{O}_\mathfrak{p}x^{-1} = \mathcal{O}_\mathfrak{p} \text{ for } \mathfrak{p}\notin S \text{ and } x \text{ fixes } \{M_\mathfrak{p}^1, M_\mathfrak{p}^2\} \text{ for } \mathfrak{p}\in S\},$$

with $\mathrm{PSL}(2,\mathbf{R})^{r_1-r_{B,\infty}}\times\mathrm{PSL}(2,\mathbf{C})^{r_2}$.

By [3] every maximal arithmetic subgroup of $\mathrm{PSL}(2,\mathbf{R})^a\times\mathrm{PSL}(2,\mathbf{C})^b$ in the commensurability class defined by (k, B) is of the form $\Gamma_{S,\mathcal{O}}$. We note however, that the converse is false. Not every group of the form $\Gamma_{S,\mathcal{O}}$ is maximal. In the case that $S=\emptyset$ it is clear that $\Gamma_{S,\mathcal{O}}$ simply corresponds to the normalizer of $\mathcal{O}$. We will denote this group by $\Gamma_\mathcal{O}$ and call it a *minimal covolume group* because, as will be seen in §2.5, $\Gamma_\mathcal{O}$ has minimal covolume amongst all arithmetic lattices in the commensurability class given by (k, B).

2.5. The volume formula

In this section we give a formula of Borel [3] for the covolume of maximal arithmetic lattices arising from quaternion algebras and use the formula to prove two analytic results which will be needed in the proofs of Theorem 1.1 and Theorem 1.3. It was shown in [3] (see also [5, Prop. 2.1]) that if $\Gamma_{S,\mathcal{O}}$ is a maximal arithmetic subgroup of $\mathrm{PSL}(2,\mathbf{R})^{r_1-r_{B,\infty}} \times \mathrm{PSL}(2,\mathbf{C})^{r_2}$ then

$$\text{(2.4)} \qquad \mathrm{covol}(\Gamma_{S,\mathcal{O}}) = \frac{2(4\pi)^{r_1-r_{B,\infty}} d_k^{3/2}\zeta_k(2)}{(4\pi^2)^{r_{B,\infty}}(8\pi^2)^{r_2}} \cdot \frac{\Phi_1(\mathscr{D}_B)\Phi_2(\mathscr{D}_S)}{2^m[k_B:k]},$$

where Φ_1, Φ_2 are as defined in Section 2.2, $0 \leq m \leq |S|$, and $\mathscr{D}_S = \prod_{\mathfrak{p}\in S}\mathfrak{p}$. Note that the integer m can be explicitly determined (c.f. [18, pp. 355–356]).

Lemma 2.5. *Suppose that Γ is an arithmetic Fuchsian group of coarea X arising from a quaternion algebra B/k. Then there is a positive constant c, depending only on k, such that $|\mathrm{Ram}_f(B)| = r_{B,f} < c\log(X)$.*

Proof. Let $\mathcal{O}$ be a maximal order of B. As $\Gamma_{\mathcal{O}}$ has minimal coarea amongst the arithmetic Fuchsian groups commensurable with Γ, we obtain from (2.4) that

$$X = \mathrm{coarea}(\Gamma) \geq \mathrm{coarea}(\Gamma_{\mathcal{O}}) = \frac{8\pi d_k^{3/2}\zeta_k(2)}{(4\pi^2)^{n_k-1}[k_B:k]} \cdot \Phi_1(\mathscr{D}_B).$$

Note that by definition, k_B is contained in the narrow class field of k, which has degree $2^{n_k}h_k$ over k and therefore $[k_B:k] \leq 2^{n_k}h_k$. Hence we deduce from the inequality above that there exists a constant C, depending only on k, such that $\Phi_1(\mathscr{D}_B) < CX$. As $r_{B,f} = |\mathrm{Ram}_f(B)|$, we have

$$\frac{\mathrm{N}(\mathscr{D}_B)}{4^{r_{B,f}}} = \prod_{\mathfrak{p}|\mathscr{D}_B} \frac{\mathrm{N}(\mathfrak{p})}{4} \leq \prod_{\mathfrak{p}|\mathscr{D}_B} \frac{\mathrm{N}(\mathfrak{p})-1}{2} = \Phi_1(\mathscr{D}_B) < CX.$$

We now have two cases to consider. Suppose first that $\mathrm{N}(\mathscr{D}_B) < 5^{r_{B,f}}$, then

$$\mathrm{N}(\mathscr{D}_B) = \prod_{\substack{\mathfrak{p}|\mathscr{D}_B \\ \mathrm{N}(\mathfrak{p})\leq 9}} \mathrm{N}(\mathfrak{p}) \cdot \prod_{\substack{\mathfrak{p}|\mathscr{D}_B \\ \mathrm{N}(\mathfrak{p})\geq 11}} \mathrm{N}(\mathfrak{p}) < 5^{r_{B,f}}.$$

We note that the first product is trivially bounded below by 1, whereas the second product is bounded below by $10^{r_{B,f}-4n_k}$. Indeed for the bound on the second product, we note that each term in the product is greater than 10 and the total number of terms is $r_{B,f}-x$ where $x = |\{\mathfrak{p}\in\mathrm{Ram}_f(B) : \mathrm{N}(\mathfrak{p}) \leq 9\}|$. As there are at most n_k primes of k lying above a fixed rational prime, we have $x \leq 4n_k$. The bound for the second product follows. This shows that $10^{r_{B,f}-4n_k} < 5^{r_{B,f}}$. Straightforward manipulations now show that $r_{B,f} <$

$14n_k$, giving us an upper bound depending only on the degree of k. We now consider the second case: $\mathrm{N}(\mathscr{D}_B) \geq 5^{r_{B,f}}$. Combining this inequality with our previous inequality $\frac{\mathrm{N}(\mathscr{D}_B)}{4^{r_{B,f}}} < CX$, we easily obtain $r_{B,f} < c\log(X)$ where c is a positive constant. The lemma now follows from this and the previous case. □

Lemma 2.6. *Let $\Gamma_{S,\mathcal{O}}$ be a maximal arithmetic Kleinian group arising from a quaternion algebra A/K. Then there is a positive constant c such that $\Phi_1(\mathcal{D}_A)\Phi_2(\mathcal{D}_S) < c \cdot \mathrm{covol}(\Gamma_{S,\mathcal{O}})^{18}$. In fact, one may take $c = 5^{180}647^3$.*

Proof. By (2.4), we have

$$\mathrm{covol}(\Gamma_{S,\mathcal{O}}) = \frac{d_K^{3/2}\zeta_K(2)}{(4\pi^2)^{n_K-1}} \cdot \frac{\Phi_1(\mathscr{D}_A)\Phi_2(\mathscr{D}_S)}{2^m[K_A : K]}, \tag{2.7}$$

for some positive integer $m \leq |S|$. As K_A is unramified at all finite primes it is contained in the narrow class field of K, whose degree (over K) is bounded above by $h_K \cdot 2^{n_K-2}$. In [14, Lemma 3.1] it was shown that $h_K \leq 242 \cdot d_K^{3/4}/(1.64)^{n_K-2}$. Combining this inequality with the estimates $2^m \leq 2^{|S|}$ and $\zeta_K(2) \geq 1$, we obtain from (2.7) that

$$\frac{\Phi_1(\mathscr{D}_A)\Phi_2(\mathscr{D}_S)}{2^{|S|}} \leq \frac{242 \cdot 49^{n_K-1} \cdot \mathrm{covol}(\Gamma_{S,\mathcal{O}})}{d_K^{3/4}}. \tag{2.8}$$

Although one has the trivial estimate $d_K \geq 1$, which could be applied to (2.8), one can obtain a much stronger bound by employing the Odlyzko bounds [21] (see also [4, §2]), which in our context imply that $d_K \geq e^{4n_K-6.5}$. Substituting this bound into (2.8) and simplifying now gives us

$$\frac{\Phi_1(\mathscr{D}_A)\Phi_2(\mathscr{D}_S)}{2^{|S|}} \leq 647 \cdot (5/2)^{n_K} \cdot \mathrm{covol}(\Gamma_{S,\mathcal{O}}). \tag{2.9}$$

Note that

$$\frac{\Phi_2(\mathscr{D}_S)}{2^{|S|}} = \prod_{\mathfrak{p}|\mathscr{D}_S} \frac{\mathrm{N}(\mathfrak{p})+1}{2},$$

hence $\Phi_2(\mathscr{D}_S)^{1/3} \leq \Phi_2(\mathscr{D}_S)/2^{|S|}$. Indeed, after an easy algebraic manipulation this follows from the inequality $(\mathrm{N}(\mathfrak{p})+1)^{2/3} \geq 2$, which in turn follows from the fact that $3^{2/3} > 2$. Similarly, we have the bound $\Phi_1(\mathscr{D}_A) \leq 2^{3n_K}\Phi_1(\mathscr{D}_A)^3$. Combining these with (2.9) yields

$$\Phi_1(\mathscr{D}_A)\Phi_2(\mathscr{D}_S) \leq 647^3 \cdot 5^{3n_K} \cdot \mathrm{covol}(\Gamma_{S,\mathcal{O}})^3. \tag{2.10}$$

The proof now follows from [5, Lemma 4.3], which implies that $n_K < 60 + 3\log(\mathrm{covol}(\Gamma_{S,\mathcal{O}}))$. □

2.6. Totally geodesic surfaces in arithmetic hyperbolic 3–orbifolds

Let M be an arithmetic hyperbolic 3–orbifold and N be an immersed totally geodesic hyperbolic 2–orbifold of M. In this brief section we will review some of the ways that the arithmetic invariants of M and N are related. We begin by stating a required result from [18, Cor. 9.5.3].

Proposition 2.11. *Let (A, K) be the invariant quaternion algebra and invariant trace field of M and (B, k) be the invariant quaternion algebra and invariant trace field of N. Then*

(i) $[K : k] = 2$ *and* $k = K \cap \mathbf{R}$,
(ii) $A \cong B \otimes_k K$.

As an application of Proposition 2.11 we show that every totally geodesic hyperbolic 2–orbifold of M has the same invariant trace field.

Lemma 2.12. *Let M be an arithmetic hyperbolic 3–orbifold with invariant trace field K and N an immersed totally geodesic hyperbolic 2–orbifold. The invariant trace field k of N is the maximal totally real subfield of K. In particular, if N, N' are totally geodesic hyperbolic 2–orbifolds of M then the invariant trace fields of N and N' coincide.*

Proof. The field k is a totally real number field which, by Proposition 2.11, satisfies $[K : k] = 2$. Let $F \subset K$ be a totally real subfield of K. The compositum kF of k and F is a totally real subfield of K which contains k, hence $kF = k$ or $kF = K$. As K is not totally real, $kF = k$ and $F \subset k$. It follows that k is the maximal totally real subfield of K. □

In light of Proposition 2.11(ii), it is of interest to determine when a quaternion algebra A over K is of the form $A \cong B \otimes_k K$. This is given by the following theorem [18, Thm. 9.5.5] (see [15, Lemma 3.2] for a more general result valid over arbitrary number fields).

Theorem 2.13. *Let K be a number field with a unique complex place and suppose that the maximal totally real subfield k of K satisfies $[K : k] = 2$. Suppose B is a quaternion algebra over k ramified at all real places of k except at the place lying under the complex place of K. Then $A \cong B \otimes_k K$ if and only if $\mathrm{Ram}_f(A)$ consists of $2r$ places $\{\mathfrak{P}_{i,j}\}_{1\leq i\leq r,\, 1\leq j\leq 2}$ satisfying $\mathfrak{P}_{1,j} \cap \mathcal{O}_k = \mathfrak{P}_{2,i} \cap \mathcal{O}_k = \mathfrak{p}_i$, where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \subset \mathrm{Ram}_f(B)$ with $\mathrm{Ram}_f(B) \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ consisting of primes in $\mathcal{O}_k$ which are inert or ramified in K/k.*

When the above conditions on $\mathrm{Ram}(A)$ are satisfied there will be infinitely many isomorphism classes of quaternion algebras B over k such that $A \cong B \otimes_k K$. In particular an arithmetic hyperbolic 3–orbifold which contains a single immersed totally geodesic hyperbolic 2–orbifold contains infinitely many primitive, totally geodesic, incommensurable hyperbolic 2–orbifolds.

Corollary 2.14. *Let K be a number field with a unique complex place and suppose that the maximal totally real subfield k of K satisfies $[K : k] = 2$. Let $B_1, \ldots, B_s$ be quaternion algebras over k such that*

$$B_1 \otimes_k K \cong B_2 \otimes_k K \cong \cdots \cong B_s \otimes_k K.$$

If S is a finite set of primes of k, then the number of number fields K' with $d_{K'} < x$ and which satisfy the following conditions:

(i) *K' has a unique complex place, and this place lies above the real place of k which splits in all the B_i,*
(ii) *$[K' : k] = 2$,*
(iii) *every prime $\mathfrak{p} \in S$ decomposes the same way in K'/k as it does in K/k and moreover the corresponding local fields are isomorphic,*
(iv) *$B_1 \otimes_k K' \cong B_2 \otimes_k K' \cong \cdots \cong B_s \otimes_k K'$,*

is greater than cx as $x \to \infty$, where c is a positive constant which depends only on k and $t = |S| + \sum_{i=1}^{s} |\mathrm{Ram}_f(B_i)|$.

Proof. We begin by noting that if we enlarge S so that it contains $\bigcup_{i=1}^{s} \mathrm{Ram}_f(B_i)$, then by Theorem 2.13, any relative quadratic extension K' of k satisfying conditions (i)–(iii) must also satisfy condition (iv). The corollary now follows from [8, Cor. 3.14] and the remark immediately following its proof. □

Proposition 2.15. *Let Γ be a maximal arithmetic Kleinian group arising from a quaternion algebra A/K and let k be the maximal totally real subfield of K. Let $\Gamma_1, \ldots, \Gamma_s$ be maximal arithmetic Fuchsian groups contained in Γ which arise from quaternion algebras $B_1/k, \ldots, B_s/k$. If K' is a number field satisfying conditions (i)–(iv) of Corollary 2.14 then there exists a maximal arithmetic subgroup arising from $A' = B_1 \otimes_k K'$ containing arithmetic Fuchsian subgroups with coareas* $\mathrm{coarea}(\Gamma_1), \ldots, \mathrm{coarea}(\Gamma_s)$.

Proof. For each $i \in \{1, \ldots, s\}$, let $\mathcal{O}_i$ be a maximal order contained in B_i. Given a prime $\mathfrak{p}$ of k not ramifying in B_i, fix an isomorphism $f^i_{\mathfrak{p}} \colon B_i \otimes_k k_{\mathfrak{p}} \to \mathrm{M}(2, k_{\mathfrak{p}})$ such that $f^i_{\mathfrak{p}}(\mathcal{O}_i) = \mathrm{M}(2, \mathcal{O}_{k_{\mathfrak{p}}})$. Let S be a finite set of primes of k containing

(1) $\bigcup_{i=1}^{s} \mathrm{Ram}_f(B_i)$, and

(2) all primes $\mathfrak{p} \notin \mathrm{Ram}_f(B_i)$ of k for which the closure of Γ_i in $\mathrm{P}((B_i \otimes_k k_\mathfrak{p})^*)$ does not have image in $\mathrm{PGL}(2, k_\mathfrak{p})$ coinciding with $\mathrm{PGL}(2, \mathcal{O}_{k_\mathfrak{p}})$ for some $i \in \{1, \ldots, s\}$.

Let K' be a quadratic extension of k which has a unique complex place (which lies above the real place of k splitting in all of the B_i) and in which every prime $\mathfrak{p} \in S$ has the same splitting behavior as it does in K/k, as in Corollary 2.14. By Theorem 2.13 there exist primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of k such that the $2r$ primes of K which ramify in A are precisely the primes of K lying above $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. By construction of K', the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ split in K'/k. Let A' be the quaternion algebra over K' which is ramified at all real places of K' and the $2r$ primes lying above $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. By Theorem 2.13, $B_i \otimes_k K' \cong A'$ for $1 \leq i \leq s$.

Let S_K (resp. $S_{K'}$) denote the set of primes of K (resp. K') lying above the primes of S. By the conditions listed in Corollary 2.14, there exists a bijection $\Psi\colon S_K \to S_{K'}$ such that $K_\mathfrak{P} \cong K'_{\Psi(\mathfrak{P})}$ for all $\mathfrak{P} \in S_K$. As $\Psi(\mathrm{Ram}_f(A)) = \mathrm{Ram}_f(A')$ and as over a non-archimedean local field there is a unique isomorphism class of quaternion division algebras, we may extend these field isomorphisms so as to obtain isomorphisms $A \otimes_K K_\mathfrak{P} \cong A' \otimes_{K'} K'_{\Psi(\mathfrak{P})}$ for all $\mathfrak{P} \in S_K$.

We now define a maximal arithmetic subgroup Γ' of $P(A'^*)$ by specifying that its closure in $\mathrm{P}((A' \otimes_{K'} K'_{\mathfrak{P}'})^*)$ be $\mathrm{PGL}(2, \mathcal{O}_{k_{\mathfrak{P}'}})$ if $\mathfrak{P}' \notin S_{K'}$ and that its closure corresponds to the image of Γ under the identification $A \otimes_K K_\mathfrak{P} \cong A' \otimes_{K'} K'_{\Psi(\mathfrak{P})}$ otherwise. Note that this local-global correspondence is valid for, and in fact uniquely characterizes, congruence arithmetic subgroups of quaternion algebras. All maximal arithmetic subgroups are congruence (see for instance [16, Lemma 4.2]).

By definition of S, the group Γ' is a maximal arithmetic Kleinian group arising from A' which contains maximal arithmetic subgroups isomorphic to $\Gamma_1, \ldots, \Gamma_s$. As coareas may be computed locally using local Tamagawa volumes (see [3, §6] or [18, Ch. 11]), the proposition follows. □

Remark 2.16. We note that the set S appearing in the proof of Proposition 2.15 can be taken so that its cardinality depends only on the coareas $\{\mathrm{coarea}(\Gamma_1), \ldots, \mathrm{coarea}(\Gamma_s)\}$. Indeed, Lemma 2.5 gives us a logarithmic bound for $\sum_{i=1}^{s} |\mathrm{Ram}_f(B_i)|$. To give a bound on the second condition defining the set S, we note that the work of Borel [3] shows that there are only finitely many arithmetic Fuchsian groups of bounded coarea. It follows that there exists a finite set of primes outside of which every arithmetic Fuchsian group with bounded coarea has local closure coinciding with $\mathrm{PGL}(2, \mathcal{O}_{k_\mathfrak{p}})$.

3. Proof of Theorem 1.1

We now prove Theorem 1.1. Since M is an arithmetic hyperbolic 3–orbifold containing totally geodesic hyperbolic 2–orbifolds of areas $A_1, \dots, A_s$ whose fundamental groups are maximal arithmetic Fuchsian groups, $\pi_1(M)$ contains maximal arithmetic Fuchsian subgroups $\Gamma_1, \dots, \Gamma_s$ such that the coarea of Γ_i is A_i. Denote by B_i the invariant quaternion algebra of Γ_i and by k_i the invariant trace field of Γ_i. Proposition 2.11 and Lemma 2.12 show that all of the k_i are equal and in fact are the maximal totally real subfield of the invariant trace field K of M. Denote by k the maximal totally real subfield of K and note that by Proposition 2.11, $[K : k] = 2$. Let K' be a number field which satisfies conditions (i)–(iv) in the statement of Corollary 2.14.

We may assume without loss of generality that $\pi_1(M) = \Gamma_{S,\mathcal{O}}$ is a maximal arithmetic Kleinian group arising from a quaternion algebra A over K. By Proposition 2.15, the quaternion algebra $A' = B_1 \otimes_k K'$ gives rise to a maximal arithmetic Kleinian group $\Gamma_{S',\mathcal{O}'}$ which contains arithmetic Fuchsian subgroups with coareas $A_1, \dots, A_s$. Moreover, the proof of Proposition 2.15 shows that $\Phi_1(\mathscr{D}_{A'}) = \Phi_1(\mathscr{D}_A)$ and $\Phi_2(\mathscr{D}_{S'}) \leq \Phi_2(\mathscr{D}_S)$. Because of this, (2.4) and Lemma 2.6 show that there is a positive constant c such that

$$\mathrm{covol}(\Gamma_{S',\mathcal{O}'}) \leq c \cdot \zeta(2)^{n_{K'}} \cdot \mathrm{vol}(M)^{18} \cdot d_{K'}^{3/2}.$$

The lower bound now follows from Corollary 2.14, which shows that the number of choices of K' with $d_{K'} < X$ is at least $c'X$ as $X \to \infty$ for some positive constant c'.

4. Proof of Theorem 1.3

We begin our proof of Theorem 1.3 by noting that if there do not exist any arithmetic hyperbolic 3–orbifolds with totally geodesic area sets containing $\{A_1, \dots, A_s\}$ then the statement of the theorem is trivially satisfied. Suppose therefore that M is an arithmetic hyperbolic 3–orbifold containing totally geodesic hyperbolic 2–orbifolds of areas $A_1, \dots, A_s$ whose fundamental groups are maximal arithmetic Fuchsian groups. Let $\Gamma_1, \dots, \Gamma_s$ be the maximal arithmetic Fuchsian groups contained in $\pi_1(M)$ whose coareas are $A_1, \dots, A_s$. As was pointed out in the proof of Theorem 1.1, all of the Γ_i have the same invariant trace field, which we denote by k. The proof of [14, Thm. 4.1] shows that the absolute discriminant d_k of k satisfies $d_k < A_1^{22}$. We now employ a theorem of Ellenberg–Venkatesh [10] in order to count the number of possibilities for the field k (see also [1, Appendix]).

Theorem 4.1 (Ellenberg–Venkatesh)**.** *Let $N(X)$ denote the number of isomorphism classes of number fields with absolute value of discriminant less than X. Then for any $\epsilon > 0$ there is a constant $c(\epsilon)$ such that $\log N(X) \leq c(\epsilon)(\log X)^{1+\epsilon}$ for all $X \geq 2$.*

Combining Theorem 4.1 with the bound $d_k \leq A_1^{22}$ we conclude that there are at most $e^{c\log(A_1)^{1+\epsilon}}$ possibilities for k, where the constant c is allowed to depend on ϵ. This shows that there are at most $e^{c\log(A_1)^{1+\epsilon}}$ number fields which may serve as the invariant trace field of arithmetic Fuchsian groups with coareas $A_1, \dots, A_s$. Fix one such field k. We now obtain an upper bound for the number of quadratic extensions of k which may serve as the invariant trace field of an arithmetic Kleinian group with covolume at most V. Let K/k be a quadratic extension and suppose that $\mathscr{C}$ is a commensurability class of arithmetic Kleinian groups defined over K such that the minimal covolume group $\Gamma_{\mathcal{O}} \in \mathscr{C}$ satisfies $\mathrm{covol}(\Gamma_{\mathcal{O}}) < V$. We now have

$$\mathrm{covol}(\Gamma_{\mathcal{O}}) = \frac{d_K^{3/2}\zeta_K(2)\Phi_1(\mathscr{D}_A)}{(4\pi^2)^{2n_k-1}[K_A : K]} < V, \tag{4.2}$$

where A is the invariant quaternion algebra of $\Gamma_{\mathcal{O}}$. By employing the bounds $\zeta_k(2) \geq 1$ and $\Phi_1(\mathscr{D}_A) \geq \frac{1}{2^{2n_k}}$ along with the bound

$$[K_A : K] \leq h_K 2^{n_k-2} \leq 242(1.220)^{n_k-2} d_K^{3/4}$$

from the proof of Lemma 2.6 (in the paragraph following (2.7)), we obtain

$$d_K < \left[5(79\pi^4)^{n_k}\right]^{4/3} \cdot V^{4/3}. \tag{4.3}$$

In order to estimate the number of quadratic extensions of k which satisfy (4.3) we will employ the following result of Cohen, Diaz y Diaz, and Olivier [8, Cor. 3.14].

Theorem 4.4 (Cohen–Diaz y Diaz–Olivier)**.** *Let k be a number field of signature (r_1, r_2) and $\mathfrak{m}_\infty$ be a subset of the real places of k. The number of quadratic extensions K/k in which the real places of k ramified in K/k is equal to $\mathfrak{m}_\infty$ and such that $d_K < X$ is asymptotic to $\frac{d_k^2}{2^{r_1+r_2}} \cdot \frac{\kappa}{\zeta_k(2)} \cdot X$, where κ is the residue at $s = 1$ of the Dedekind zeta function $\zeta_k(s)$ of k.*

Remark 4.5. Two comments about Theorem 4.4 are in order. The first is to point out that the asymptotic expression in the theorem turns out to be independent of the set $\mathfrak{m}_\infty$ of real places of k which will ramify in the quadratic extension K/k. The second comment is that in [8], they prove their result by counting quadratic extensions K/k satisfying the conditions in the theorem such that $\mathrm{N}_{k/\mathbf{Q}}(\Delta_{K/k}) < X$, not such that $d_K < X$. Here $\Delta_{K/k}$ is the relative discriminant of the quadratic extension K/k. However since $d_K = \mathrm{N}_{k/\mathbf{Q}}(\Delta_{K/k})d_k^2$, our statement is equivalent.

Recall that k is a totally real number field of degree n_k and that we are interested in counting the number of quadratic extensions K of k which may serve as the invariant trace field of an arithmetic Kleinian group of covolume less than V. Because such a number field K must have a unique complex place, its signature is $(2n_k - 2, 1)$. It now follows from Theorem 4.4 that the number of extensions K/k satisfying the bound in (4.3) is at most $\kappa d_k^2 \cdot 9 \cdot 10^{5n_k} \cdot V^{4/3}$ for sufficiently large V, where κ is the residue at $s = 1$ of $\zeta_k(s)$. By [13, p. 322] there exists an absolute constant $c > 0$ such that $\kappa \leq c^{n_k} d_k^{1/2}$. Combining this with our previous estimate we see that the number of extensions K/k satisfying the bound in (4.3) is at most $C^{n_k} \cdot d_k^{5/2} \cdot V^{4/3}$, where $C > 0$ is an absolute constant. We have already seen that $d_k < A_1^{22}$ and $n_k < 60+3\log(A_1)$ by [5, Lemma 4.3]. Putting all of this together we see that there exists an absolute constant C_1 such that our bound for the number of extensions K/k being considered is of the form $A_1^{C_1} V^{4/3}$.

Having given a bound on the number of quadratic extensions of k which satisfy the bound in (4.3), fix one such extension K. We will now bound the number of quaternion algebras A over K which could give rise to a minimal covolume group $\Gamma_{\mathcal{O}}$ satisfying (4.2). Bounding $[K_A : K]$ as above, we obtain from (4.2) that

$$\Phi_1(\mathscr{D}_A) < 5(5\pi^2)^{2n_k} V. \tag{4.6}$$

As $\mathscr{D}_A$ is a square-free integral ideal of $\mathcal{O}_K$, it follows from Lemma 2.2 and (4.6) that there are at most $5^4(250\pi^4)^{4n_k}\zeta_K(2)V^4$ many choices for $\mathscr{D}_A$. It is well-known that $\zeta_K(s) < \zeta(s)^{n_K}$ for all $s > 1$. Since $\zeta(2) = \pi^2/6$ we may simplify our upper bound on the number of choices for $\mathscr{D}_A$ to $C_2^{n_k} V^4$ where C_2 is an absolute constant. By [5, Lemma 4.3] we have $n_k < 60 + 3\log(A_1)$, therefore there exists an absolute constant C_3 such that our upper bound on the number of choices for $\mathscr{D}_A$ is of the form $A_1^{C_3} V^4$. Recall that any quaternion algebra over K which gives rise to a Kleinian group must be ramified at all real places of K [18, Ch. 8]. As isomorphism classes of quaternion algebras A over K are in one to one correspondence with the sets $\mathrm{Ram}(A)$ of places of K ramifying in A, it follows that the isomorphism class of any quaternion algebra over K which gives rise to a Kleinian group is given by $\mathrm{Ram}_f(A)$, the set of finite primes of K which ramify in A. As $\mathscr{D}_A$ is simply the product of all primes lying in $\mathrm{Ram}_f(A)$, there are at most $A_1^{C_3}V^4$ many isomorphism classes of quaternion algebras over K which could give rise to a minimum covolume group satisfying (4.2). These isomorphism classes correspond, by [18, Ch. 8], to the commensurability classes of arithmetic Kleinian groups with invariant trace field K and invariant quaternion algebra A which contain a representative with covolume less than V.

We may now regard K and A as being fixed. Then [14, Thm. 5.1] shows that the number of maximal arithmetic Kleinian groups arising from A with covolume at most V is less than $242V^{20}$. We have now exhibited all of the bounds needed to prove Theorem 1.3 and need only assemble them into our final bound.

We have shown that there are at most $e^{c\log(A_1)^{1+\epsilon}}$ possibilities for the totally real field k over which our arithmetic Fuchsian groups will be defined, $A_1^{C_1}V^{4/3}$ possibilities for the quadratic extension K/k over which the fundamental group of our maximal arithmetic hyperbolic 3–orbifold will be defined, $A_1^{C_3}V^4$ possibilities for the quaternion algebra A, and $242V^{20}$ possibilities for the fundamental group of our maximal arithmetic hyperbolic 3–orbifold. By enlarging the constant c as necessary, we see that for any $\epsilon > 0$ the number of isometry classes of maximal arithmetic hyperbolic 3–orbifolds containing totally geodesic hyperbolic 2–orbifolds of areas $A_1, \dots, A_s$ is at most $e^{c\log(A_1)^{1+\epsilon}}V^{26}$, where c is a constant depending only on ϵ.

5. Commensurability classes of arithmetic Kleinian groups whose maximal arithmetic Fuchsian subgroups have the same areas

In this section we will prove the following theorem.

Theorem 5.1. *There exist distinct commensurability classes of arithmetic Kleinian groups with the property that their maximal arithmetic Fuchsian subgroups have precisely the same areas.*

Let k be a totally real quartic field with narrow class number one and p, q be distinct rational primes which both split completely in the extension $k/\mathbf{Q}$. As an example of such a field, consider the field obtained by adjoining to $\mathbf{Q}$ a root of the polynomial $x^4 - x^3 - 3x^2 + x + 1$. This is a totally real quartic field with narrow class number 1 and discriminant $5^2 \cdot 29$. Let $\mathfrak{p}, \mathfrak{p}'$ be primes of k lying above p and $\mathfrak{q}, \mathfrak{q}'$ be primes of k lying above q. Let K be a quadratic extension of k which has signature $(6, 1)$ and in which $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{q}'$ all split. Let $\mathfrak{P}_1, \mathfrak{P}_2$ (respectively $\mathfrak{P}_1', \mathfrak{P}_2'$) be primes of K lying above $\mathfrak{p}$ (respectively $\mathfrak{p}'$). Similarly, let $\mathfrak{Q}_1, \mathfrak{Q}_2$ (respectively $\mathfrak{Q}_1', \mathfrak{Q}_2'$) be primes of K lying above $\mathfrak{q}$

(respectively $\mathfrak{q}'$). Let A_1 be the quaternion algebra over K which is ramified at every real place of K and satisfies $\mathrm{Ram}_f(A_1) = \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{Q}_1, \mathfrak{Q}_2\}$. Let A_2 be the quaternion algebra over K which is ramified at every real place of K and satisfies $\mathrm{Ram}_f(A_2) = \{\mathfrak{P}_1', \mathfrak{P}_2', \mathfrak{Q}_1', \mathfrak{Q}_2'\}$.

Let S_1 (respectively S_2) be the set of all quaternion algebras B over k such that $B \otimes_k K \cong A_1$ (respectively $B \otimes_k K \cong A_2$). We will now define a bijection $f\colon S_1 \to S_2$. Suppose that $B \in S_1$; that is, $B \otimes_k K \cong A_1$. By Theorem 2.13, B is ramified at all real places of k not lying beneath the complex place of K. Furthermore, $\mathrm{Ram}_f(B)$ consists of $\mathfrak{p}, \mathfrak{q}$ and a set of primes of k none of which split in K/k. In particular neither $\mathfrak{p}'$ nor $\mathfrak{q}'$ lies in $\mathrm{Ram}_f(B)$. Define B' to be the quaternion algebra over k whose ramification is exactly the same as that of B though with $\mathfrak{p}, \mathfrak{q}$ replaced by $\mathfrak{p}', \mathfrak{q}'$. The classification of quaternion algebras over number fields shows that B' is uniquely defined (up to isomorphism). Moreover, Theorem 2.13 shows that $B \otimes_k K \cong A_2$ and therefore the map $f\colon S_1 \to S_2$ is easily seen to be a bijection.

For each $B \in S_1$, we will now show that the maximal arithmetic Fuchsian groups arising from B have the same areas as those arising from $f(B)$. Let T_1 be the set of maximal arithmetic Fuchsian subgroups of A_1^* arising from B and T_2 be the set of maximal arithmetic Fuchsian subgroups of A_2^* arising from $f(B)$. We will define a bijection $g\colon T_1 \to T_2$ which is area preserving. Using the notation developed in §2.4, let $\Gamma_{S,\mathcal{O}} \in T_1$. Recall that this means that $\mathcal{O}$ is a maximal order of B and S is a set of finite primes of k disjoint from $\mathrm{Ram}_f(B)$. Define $g(\Gamma_{S,\mathcal{O}}) = \Gamma_{S',\mathcal{O}'}$ where $\mathcal{O}'$ is a maximal order of $f(B)$ and S' consists of exactly the same primes as S, though with $\mathfrak{p}$ replaced by $\mathfrak{p}'$ and $\mathfrak{q}$ replaced by $\mathfrak{q}'$. We claim that $\mathrm{area}(\Gamma_{S,\mathcal{O}}) = \mathrm{area}(\Gamma_{S',\mathcal{O}'})$. Indeed, this follows from (2.4) as $N(\mathfrak{p}) = N(\mathfrak{p}')$, $N(\mathfrak{q}) = N(\mathfrak{q}')$, and $[k_B : k] = [k_{f(B)} : k] = 1$. The latter is due to the fact that k_B and $k_{f(B)}$ are both extensions of k which are contained in the narrow class field of k, which by hypothesis is simply k. Note that the number m appearing in (2.4) will coincide for both groups because of the way that m is defined and the fact that k has trivial narrow class group (see [18, pp. 355–356]). It follows that the commensurability classes defined by A_1 and A_2 have the desired property, concluding the proof of Theorem 5.1.

6. Counting covers

In this section, we will count certain congruence subgroups of a fixed arithmetic Kleinian group Γ that contains a set of Fuchsian subgroups with specified coareas. For this, we prefer to work in the quadratic form model of hyperbolic space. We therefore first recall some basic background about arithmetic lattices in $\mathrm{SO}_0(3,1) = \mathrm{Isom}^+(\mathbf{H}^3)$ and their finite quotients.

6.1. Preliminaries on arithmetic groups of simplest type

Suppose that $k \subset \mathbf{R}$ is a totally real number field and (V, f) is a k–quadratic space of dimension 4, where f is a non-degenerate quadratic form with signature $(3, 1)$ over $\mathbf{R}$ and which has signature $(4, 0)$ over any non-identity Galois embedding $\sigma\colon k \hookrightarrow \mathbf{R}$. Then $\mathbf{G} = \mathrm{SO}(f)$ is a k–algebraic group. We can explicitly identify the k–points of $\mathbf{G}$ via

$$\mathbf{G}(k) = \{A \in \mathrm{SL}(4, k) \mid A^T F A = F\},$$

where F is a matrix representing f in a given basis of V. Moreover, in what follows we choose this basis so that F is diagonal with coefficients in the ring of integers $\mathcal{O}_k$. Such a choice is always possible.

Using this description of $\mathbf{G}(k)$, we define $\mathbf{G}(\mathcal{O}_k)$ as the group $\mathbf{G}(k) \cap \mathrm{SL}(4, \mathcal{O}_k)$. Restriction of scalars and projection to the first factor gives the composition

$$\pi\colon \mathbf{G}(k) \to \mathrm{SO}(3, 1) \times \prod \mathrm{SO}(4) \to \mathrm{SO}(3, 1),$$

which has the property that $\pi(\mathbf{G}(\mathcal{O}_k))$ is a lattice in $\mathrm{SO}(3, 1)$. A subgroup $\Gamma < \mathrm{SO}_0(3, 1)$ is called *arithmetic of simplest type* if Γ is commensurable with $\pi(\mathbf{G}(\mathcal{O}_k)) \cap \mathrm{SO}_0(3, 1)$ for some k and (V, f) as above. It is well known that all arithmetic Kleinian groups in $\mathrm{SO}_0(3, 1)$ which contain a geodesic surface are arithmetic of the simplest type, so we restrict ourselves to this class now (see [18]).

Recall that the *discriminant* of a quadratic form f is defined as the square class in $k^*/(k^*)^2$ of the determinant of any matrix representing f in $\mathrm{GL}(4, k)$. In what follows, we will abusively use the notation $\mathrm{disc}(f)$ to mean a fixed representative of this square class in k given by our choice of diagonal representative F for f above. In particular, due to our choices above, $\mathrm{disc}(f) \in \mathcal{O}_k$. Additionally, under the identity embedding of k into $\mathbf{R}$, f has signature $(3, 1)$ and hence $\mathrm{disc}(f) < 0$.

Fix $\Gamma < \mathrm{SO}_0(3, 1)$ an arithmetic Kleinian group of simplest type and retain the notation from above. Then we may form the hyperboloid model of hyperbolic space via

$$\mathbf{H}^3_f = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid f(x_1, x_2, x_3, x_4) = -1,\ x_4 > 0\} \subset V \otimes_k \mathbf{R} \cong \mathbf{R}^4.$$

The bilinear form associated to f defines an inner product on $\mathbf{H}^3_f$ which determines the usual hyperbolic metric. From here forward, we drop the subscript and identify $\mathbf{H}^3_f$ with $\mathbf{H}^3$. Geodesic surfaces in $\mathbf{H}^3/\Gamma$ then correspond to isometric embeddings $\iota\colon \mathbf{H}^2 \hookrightarrow \mathbf{H}^3$ such that the subgroup of Γ stabilizing $\iota(\mathbf{H}^2)$ is a lattice in the corresponding copy of $\mathrm{O}_0(2,1)$. The condition that this subgroup is a lattice implies that $\iota(\mathbf{H}^2)$ must be defined by $\mathbf{H}^3 \cap (W \otimes_k \mathbf{R})$ where $(W, f|_W)$ is a k–quadratic subspace of (V, f) such that $f|_W$ has signature $(2,1)$ under the identity embedding of k into $\mathbf{R}$. As W is k–rational, its orthogonal complement is k–rational and hence there is a k–rational normal vector n_W to W. By scaling, we may choose n_W so that n_W has coefficients in $\mathcal{O}_k$ under our preferred basis above. Moreover, any such choice of n_W has the property that, under the identity embedding of k into $\mathbf{R}$, $f(n_W) > 0$.

We also require some facts about finite groups of Lie type associated to quadratic forms. Let $q = p^r$ where p is an odd prime and $r \geq 1$. Recall that, up to isomorphism, there is exactly one orthogonal group $\mathrm{O}(3,q)$ in dimension 3 and two orthogonal groups $\mathrm{O}^{\pm}(4,q)$ in dimension 4 (see [24, p. 375–384]). Define $\Omega(3,q) = [\mathrm{O}(3,q), \mathrm{O}(3,q)]$ and $\Omega^{\pm}(4,q) = [\mathrm{O}^{\pm}(4,q), \mathrm{O}^{\pm}(4,q)]$ their commutator subgroups and $\mathrm{P}\,\Omega^{\pm}(4,q)$ the corresponding central quotient. If Γ is an arithmetic Kleinian group of simplest type then $\Gamma < \mathbf{G}(\mathcal{O}_{k_\mathfrak{p}})$ for all but finitely many finite places $\mathfrak{p}$ of k. It is then a consequence of strong approximation [27] (see also [17, Thm. 2.6] in this specific setting), that the reduction map

$$R_\mathfrak{p}\colon \Gamma \hookrightarrow \mathrm{SO}(f, \mathcal{O}_{k_\mathfrak{p}}) \to \mathrm{SO}(f, \mathcal{O}_{k_\mathfrak{p}}/\mathfrak{m}_\mathfrak{p}) \cong \mathrm{SO}^{\pm}(4,q),$$

composed with the central quotient has the property that

$$\mathrm{PSL}(2,q)^2 \cong \mathrm{P}\,\Omega^{+}(4,q) \leq \mathrm{P}\,R_\mathfrak{p}(\Gamma) \leq \mathrm{PSO}^{+}(4,q), \tag{6.1}$$

when $\mathrm{disc}(f)$ is a square in $\mathbf{F}_q \cong \mathcal{O}_{k_\mathfrak{p}}/\mathfrak{m}_\mathfrak{p}$, where $\mathfrak{m}_\mathfrak{p}$ is the unique maximal ideal in $\mathcal{O}_{k_\mathfrak{p}}$. Additionally,

$$\mathrm{PSL}(2,q^2) \cong \mathrm{P}\,\Omega^{-}(4,q) \leq \mathrm{P}\,R_\mathfrak{p}(\Gamma) \leq \mathrm{PSO}^{-}(4,q), \tag{6.2}$$

when $\mathrm{disc}(f)$ is not a square in $\mathbf{F}_q^*$ for $q = |\mathcal{O}_{k_\mathfrak{p}}/\mathfrak{m}_\mathfrak{p}|$. Moreover, again by [17, Thm. 2.6], if $\Delta < \Gamma$ is an arithmetic Fuchsian subgroup, then for almost every prime

$$\mathrm{P}\,\Omega(3,q) = \Omega(3,q) \leq \mathrm{P}\,R_\mathfrak{p}(\Delta) \leq \mathrm{P}\,\mathrm{O}(3,q). \tag{6.3}$$

The following lemma will be important in what follows.

Lemma 6.4. *Let G be either $\Omega^{-}(4,q)$ or $\mathrm{SO}^{-}(4,q)$ and let (V,h) be the corresponding 4–dimensional $\mathbf{F}_q$–quadratic space on which G acts. For any* *line ℓ in V, define $H_\ell = \{g \in G \mid g\ell = \ell\}$. Then there are exactly three conjugacy classes of subgroups of the form H_ℓ. Moreover, writing $\ell = \mathbf{F}_q\langle v\rangle$ for any choice of non-zero vector $v \in \ell$, then the value of $h(v)$, taken up to squares in $\mathbf{F}_q^*$, is a complete invariant of this conjugacy class.*

Proof. Fix any line $\ell \subset V$. If $v, w \in \ell - \{0\}$ then $v = \lambda w$ for some $\lambda \in \mathbf{F}_q^*$ and so $h(v) = \lambda^2 h(w)$. In particular, the restriction of the quadratic form h to any $\ell - \{0\}$ is either identically 0, takes only square values in $\mathbf{F}_q^*$, or takes only non-square values in $\mathbf{F}_q^*$.

When $G = \mathrm{SO}^-(4, q)$, Witt's theorem implies that for any two non-zero vectors $v, w \in V$ with $h(v) = h(w)$, there exists some $g \in G$ such that $gv = w$. In particular G acts transitively on each of the three types of lines above and we conclude the lemma. When $G = \Omega^-(4, q)$, the conclusion of Witt's theorem is still valid, as shown in [20, Lemma 3.1], and so we conclude similarly. □

Corollary 6.5. *With the notation of Lemma 6.4, if $v \in \ell$ such that $h(v) \neq 0$, then H_ℓ is a maximal subgroup of G where $H_\ell = \mathrm{SO}(3, q)$ when $G = \Omega^-(4, q)$, or $H_\ell = \mathrm{O}(3, q)$ when $G = \mathrm{SO}^-(4, q)$.*

6.2. Proof of Theorem 1.4

Let Γ be an arithmetic Kleinian group of simplest type and k, (V, f) the number field and quadratic space associated to Γ from Section 6.1. Let $\mathrm{A} = \{A_1, \ldots, A_s\}$ be a finite set of real numbers such that $\mathrm{A} \subset \mathrm{TGA}(M)$, then by definition there exists a Fuchsian subgroup $\Delta_i < \Gamma$ with $\mathrm{coarea}(\Delta_i) = A_i$. From the previous subsection, each Δ_i lies in the stabilizer of a k–defined hyperplane W_i and we use v_i to denote choice of normal to this hyperplane. Recall that $f(v_i) > 0$.

Using strong approximation, we may find a cofinite set of prime ideals $\mathcal{Q}$ in $\mathcal{O}_k$ such that for each $\mathfrak{p} \in \mathcal{Q}$

1. $\mathfrak{p}$ is a non-dyadic prime,
2. The reduction $f_\mathfrak{p}$ of the quadratic form f over $\mathcal{O}_k/\mathfrak{p}$ is non-degenerate,
3. $f_\mathfrak{p}(v_i) \neq 0$ for each $1 \leq i \leq s$,
4. One of Equations (6.1) or (6.2) holds for the reduction map $R_\mathfrak{p}$,
5. When (3) holds, Equation (6.3) holds as well.

Lemma 6.6. *There is a positive density subset $\mathcal{I}$ of prime ideals in $\mathcal{O}_k$ such that for each $\mathfrak{p} \in \mathcal{I}$, there exists a fixed maximal subgroup $H < \mathrm{P}\,R_\mathfrak{p}(\Gamma)$ such that for each i, $\mathrm{P}\,R_\mathfrak{p}(\gamma_i \Delta_i \gamma_i^{-1}) \leq H$ for some $\gamma_i \in \Gamma$.*

Proof. We will first prove that there is a positive density set $\mathcal{E}$ of primes $\mathfrak{p}$ in $\mathcal{O}_k$ for which $\sqrt{\mathrm{disc}(f)}$ is not a square in $\mathbb{F}_q \cong \mathcal{O}_k/\mathfrak{p}$ and for which each $f(v_i)$, $f(v_j)$ are in the same square class in $\mathcal{O}_k/\mathfrak{p}$ for all $1 \leq i, j \leq s$.

Let $k_f = k(\sqrt{\mathrm{disc}(f)})$ and $k_v = k(\sqrt{f(v_1)}, \dots, \sqrt{f(v_s)})$. As $\mathrm{disc}(f) < 0$ and $f(v_i) > 0$ for each i, k_f is an imaginary quadratic extension of k and k_v is a totally real, multi-quadratic extension of k, where we allow for the possibility that $k_v = k$. The compositum $L = k_f k_v$ is then a multi-quadratic extension of k with degree 2^m where $1 \leq m \leq s+1$. Since $k_f \cap k_v = k$ and $\mathrm{Gal}(L/k) \cong (\mathbf{Z}/2\mathbf{Z})^m$, by the Chebotarev density theorem there exists a positive density set of prime ideals $\mathcal{E}$ in $\mathcal{O}_k$ such that each prime in $\mathcal{E}$ is inert in the extension k_f/k and splits completely in the extension k_v/k. Hence when $\mathfrak{p} \in \mathcal{E}$, $\mathrm{disc}(f)$ is not a square and for every $1 \leq i \leq s$, $f(v_i)$ is a square in $\mathbf{F}_q^*$. Letting $\mathcal{I} = \mathcal{Q} \cap \mathcal{E}$ and appealing to Lemma 6.4 and Corollary 6.5 completes the proof. ☐

We will say that an ideal $\mathfrak{a}$ of $\mathcal{O}_k$ is *square-free with level in* $\mathcal{I}$, if $\mathfrak{a}$ has primary decomposition $\mathfrak{a} = \mathfrak{p}_1 \dots \mathfrak{p}_n$ where each $\mathfrak{p}_i \in \mathcal{I}$ and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$. We denote by I the set of square-free ideals with level in $\mathcal{I}$. When $\mathfrak{a} \in \mathrm{I}$, the Chinese remainder theorem and Lemma 6.6 imply that there exist $\gamma_i \in \Gamma$ such that

$$\begin{aligned}\Omega_3(\mathcal{O}_k/\mathfrak{a}) &\leq \mathrm{P}\,R_{\mathfrak{a}}(\gamma_1 \Delta_1 \gamma_1^{-1}), \mathrm{P}\,R_{\mathfrak{a}}(\gamma_2 \Delta_2 \gamma_2^{-1}), \dots, \mathrm{P}\,R_{\mathfrak{a}}(\gamma_s \Delta_s \gamma_s^{-1}) \\ &\leq H \leq \mathrm{PO}_3(\mathcal{O}_k/\mathfrak{a}),\end{aligned}$$

where $R_{\mathfrak{a}}$ is the corresponding reduction map. We therefore see that $\Lambda_{\mathfrak{a}} = R_{\mathfrak{a}}^{-1}(H)$ is a congruence subgroup containing Fuchsian subgroups $\Delta_i' = \gamma_i \Delta_i \gamma_i^{-1}$ with $\mathrm{coarea}(\Delta_i') = A_i$. In particular, we obtain the following proposition.

Proposition 6.7. *There is an injection from* I *into the set of congruence subgroups* $\Lambda < \Gamma$ *which contain Fuchsian subgroups* $\Delta_i' < \Lambda$ *with* $\mathrm{coarea}(\Delta_i') = A_i$ *for* $i = 1, \dots, s$.

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. In light of Proposition 6.7, to give a lower bound on $H(V)$ it suffices to count the set I. For each prime ideal $\mathfrak{p} \in \mathrm{I}$, if $q = |\mathcal{O}_k/\mathfrak{p}|$ then

$$\begin{aligned}|\Omega(3,q)| &= \frac{|\mathrm{PO}(3,q)|}{2} = \frac{q(q-1)(q+1)}{2} \approx q^3, \\ |\mathrm{P}\,\Omega^-(4,q)| &= \frac{|\mathrm{PSO}^-(4,q)|}{2} = \frac{q^2(q^2-1)(q^2+1)}{2} \approx q^6.\end{aligned}$$

Hence, $[\Gamma:\Lambda_{\mathfrak{p}}] \approx \mathrm{N}(\mathfrak{p})^3$ and by the Chinese remainder theorem $[\Gamma:\Lambda_{\mathfrak{a}}] \approx \mathrm{N}(\mathfrak{a})^3$ when $\mathfrak{a} \in \mathrm{I}$. In particular, for each $X \in \mathbf{N}$ if

$$\mathrm{L}_{\Gamma,\mathrm{A}}(X) = \{\Lambda \;:\; \Lambda < \Gamma,\; [\Gamma:\Lambda] \leq X,\; \Lambda \text{ is congruence}\},$$

then Proposition 6.7 shows that $\mathrm{L}_{\Gamma,\mathrm{A}}(X) \geq \left|\left\{\mathfrak{a} \in \mathrm{I} \;:\; \mathrm{N}(\mathfrak{a}) \leq X^{1/3}\right\}\right|$.

Let S denote the set of ideals $\mathfrak{a}$ in $\mathcal{O}_k$ such that $\mathfrak{a}$ has primary decomposition $\mathfrak{a} = \mathfrak{p}_1 \dots \mathfrak{p}_n$ where each $\mathfrak{p}_i$ is inert in $k_f = k(\sqrt{\mathrm{disc}(f)})/k$. By [9],

$$|\{\mathfrak{a} \in S \;:\; \mathrm{N}(\mathfrak{a}) \leq X\}| \sim \frac{X}{(\log X)^{1/2}},$$

where $f(X) \sim g(X)$ if their ratio goes to 1 as X goes to infinity. In particular, restricting to the ideals in I gives that

$$|\{\mathfrak{a} \in \mathrm{I} \;:\; \mathrm{N}(\mathfrak{a}) \leq X\}| \sim \frac{cX}{(\log X)^{1/2}},$$

where c is some positive constant which depends only on the field k and the number of hyperbolic 2–orbifolds s. Hence, $\mathrm{L}_{\Gamma,\mathrm{A}}(X) \gg \frac{X^{1/3}}{(\log X)^{1/2}}$, where the implicit constant depends only on k and s. Now assume that $V \geq \mathrm{vol}(M)$. Letting $\mathrm{Cov}(M)$ be the set of all manifolds which finitely cover M, this gives that

$$\begin{aligned} H(V) &= |\{M' \in \mathrm{Cov}(M) \;:\; \mathrm{A} \subset \mathrm{TGA}(M'),\; \mathrm{vol}(M') \leq V\}| \\ &\geq \mathrm{L}_{\Gamma,\mathrm{A}}\left(\left\lfloor \frac{V}{\mathrm{vol}(M)} \right\rfloor\right) \\ &\gg \frac{(V/\mathrm{vol}(M))^{1/3}}{\log(V/\mathrm{vol}(M))^{1/2}}, \end{aligned}$$

where the implicit constant depends only on k and s. This completes the proof. □

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