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Hilbert manifold structure for weakly asymptotically hyperbolic relativistic initial data

Erwann Delay and Jérémie Fougeirol

Abstract: We construct a Hilbert manifold structure à la Bartnik for the space of weakly asymptotically hyperbolic initial data for the vacuum constraint equations. The proofs requires new weighted Poincaré and Korn-type inequalities for asymptotically hyperbolic manifolds with inner boundary.

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1. Introduction

This work builds upon a paper of 2005 [4], in which Robert Bartnik constructed a Hilbert manifold structure for the space of asymptotically flat solutions of the Einstein equations (see also [18], [19], [20]). The result assumed rather weak regularity assumptions concerning the metric involved (constant curvature only up to weighted L^2 terms) and so fits naturally with to the a priori estimates for the Einstein equation with bounded L^2 curvature of S. Klainerman, I. Rodnianski and J. Szeftel [21]. Actually R. Bartnik showed in [3] that these assumptions on the regularity are the weakest possible to define the ADM mass of the manifold, thus justifying these regularity conditions. This work extends part of [12], where we also show that such assumptions allow us to rigorously define the mass of an asymptotically hyperbolic (AH) manifold, in a way compatible with previous definitions of the mass (see [8], [14] or [9]) and with the Hilbert manifold structure exposed here.

Before stating the main theorem, let us introduce some notation. The constraint operator $\mathbf{\Phi}$ we are studying, acts on pairs of the form (g, π) where g is Riemannian metric and π is a contravariant symmetric two tensor field valued in n-forms. It takes the form (mre on this in section 6)

$$\begin{aligned}
\mathbf{\Phi_0}(g,\pi) &:= (R(g)-2\Lambda)\sqrt{g} - \left(|\pi|_g^2 - \tfrac{1}{n-1}(\mathrm{tr}_g\pi)^2\right)/\sqrt{g},\\
\mathbf{\Phi_i}(g,\pi) &:= g_{ij}\nabla_k \pi^{jk},
\end{aligned}$$

where $R(g)$, $\sqrt{g}$, ∇, are respectively the scalar curvature, the volume form, and the connection of g, and Λ is the cosmological constant.

We work with a phase space $\mathcal{F}$ consisting of pairs (g, π) of $H^2 \times H^1$ local regularity which are asymptotic to a frozen smooth pair $(\mathring{g}, \mathring{\pi})$, with a decay parameter δ appropriate for AH initial data which will be made precise as needed (see section 6 for the definition of $\mathcal{F}$ and $\mathcal{L}^*$).

We are now going to state our main result:

Theorem 1.1. *Let $\delta \in\,]-2\,,-1[$ and $\tau \in \mathbb{R}$. Let $(\mathcal{M}, \mathring{g})$ be a smooth 3-dimensional asymptotically hyperbolic manifold with sectional curvatures equal to -1 plus corrections which are in L^2_δ. Let $\mathring{\pi} := -2\tau\mathring{g}^{-1}d\mu(\mathring{g})$ and let $\Lambda = 3(\tau^2-1)$. Let $\mathbf{\Phi} : \mathcal{F} \to \mathcal{L}^*$ be the constraint operator with the cosmological* *constant Λ. For every $\varepsilon \in \mathcal{L}^*$, the set of solutions of the constraint equations*

$$\mathcal{C}(\varepsilon) := \{(g,\pi) \in \mathcal{F} : \mathbf{\Phi}(g,\pi) = \varepsilon\} ,$$

is a submanifold of $\mathcal{F}$. In particular, the space of solutions of the vacuum constraint equations $\mathcal{C} = \mathcal{C}(0)$ has a Hilbert submanifold structure.

The tensor field $\mathring{\pi}$ together with the constant Λ above have been chosen so that $(\mathring{g}, \mathring{\pi})$ satisfies asymptotically the vacuum constraint equations. The model we have in mind is hyperbolic space, which can be used as part of initial data either for Minkowski, De Sitter or Anti De Sitter space time, depending on the choice of $\mathring{\pi}$ and Λ[1]. We emphasise that our initial AH manifolds $(\mathcal{M}, \mathring{g})$ may have any type of conformal infinity and does not need to have constant scalar curvature at infinity.

The low regularity of the metrics involved (the curvature may not be bounded), and the non linear characteristic of the constraint operator, forces us to a very precise analysis of the different steps of the proof. In the current context one often assumes more regularity, which allows much simpler arguments. For example, the operators at hand are not trivially bounded, similarly both elliptic estimates and Fredholm properties require careful justification.

In order to overcome difficulties arising in the asymptotically hyperbolic case, as compared to the asymptotically flat one covered by R. Bartnik, we had to create a Hessian-type operator $\mathring{T}$ and a differential operator of order two, called $\mathring{U}$, constructed using the first derivatives of the Killing operator $\mathring{S}$. Using these we were able to establish new Poincaré and Korn-type estimates of second order on an asymptotically hyperbolic manifold with boundary. These estimates provide the key tools to prove the triviality of the adjoint kernel, as needed for the proof.

2. Notations and conventions

Let $(\mathcal{M}, g)$ be a Riemannian manifold. We define $T^r_m(\mathcal{M})$ to be the bundle of tensors of covariant rank m and contravariant rank r. For all $u \in T^r_m(\mathcal{M})$, $|u|_g$ will denote the norm of u with respect to the metric g, and the notation $|u|_{g,x}$ is used to indicate the point x of the manifold under consideration. The symbol $d\mu(g)$ denotes the Riemannian measure determined by g. Riem $g, \mathrm{Ric}(g)$ and $R(g)$ are respectively the Riemann tensor, the Ricci tensor and the scalar

[1]For the non usual embedding of the 3D-hyperbolic space as a slice of De Sitter, one can take $\tau = \sqrt{2}$, so $\Lambda = 3$ and the second fundamental form is $\mathring{K} = \tau\mathring{g} = \sqrt{2}\mathring{g}$ (equivalently $\mathring{\pi} = -2\sqrt{2}\mathring{g}^{-1} d\mu(\mathring{g})$).

curvature of the metric g. Given a Riemannian metric g with connection ∇, we use the following notations for the Hessian and Laplacian of a function u:

$$\begin{aligned} \nabla^2_{ij}u &= \nabla_i\nabla_j u, \\ \Delta u &= \operatorname{tr}_g \nabla^2 u = g^{ij}\nabla^2_{ij}u. \end{aligned}$$

The abbreviation "w.r.t" stands for "with respect to".

Most of the work presented in this article is done on an n-dimensional manifold as often as possible and, unless explicitly stated, the results are valid in any dimension n. However, the Sobolev inequalities strongly constrain the dimension to $n = 3$ in several proofs; this is made clear wherever relevant.

The letter C will denote a constant depending on the background metric $\mathring{g}$ and the decay rate δ and its expression may change from line to line. The nature of the dependence of C will be specified most of the time.

2.1. Conformally compact manifold

Let $(\mathcal{M}, \mathring{g})$ be a $\mathscr{C}^\infty$ n-dimensional complete non-compact Riemannian manifold. The manifold $(\mathcal{M}, \mathring{g})$ will be said to be conformally compact if there exists a Riemannian metric $\mathring{h}$ such that $\mathring{g} = \rho^{-2}\mathring{h}$, where $(\bar{\mathcal{M}} \equiv \mathcal{M} \cup \partial_\infty\mathcal{M}, \mathring{h})$ is a $\mathscr{C}^\infty$ compact Riemannian manifold with boundary $\partial_\infty\mathcal{M}$ and ρ is a function on $\bar{\mathcal{M}}$, called defining function for $\partial_\infty\mathcal{M}$, satisfying

- $\rho \in \mathscr{C}^\infty(\bar{\mathcal{M}})$,
- $\rho \geqslant 0$ on $\bar{\mathcal{M}}$,
- $\partial_\infty\mathcal{M} = \{x \in \mathcal{M} : \rho(x) = 0\}$,
- $d\rho|_{\partial_\infty\mathcal{M}}$ never vanishes.

An asymptotically hyperbolic metric on $\mathcal{M}$ is, by definition, a Riemannian metric $\mathring{g}$ satisfying:

- $\mathring{g}$ is conformally compact,
- $|d\rho|^2_{\mathring{h},\partial_\infty\mathcal{M}} = 1$.

This terminology comes from the fact that the sectional curvatures of $\mathring{g}$ approach -1 at the conformal boundary $\partial_\infty\mathcal{M}$. The manifold $\mathcal{M}$ we consider in the main theorem does not have any boundary. But for some applications to the ends of $\mathcal{M}$, for some inequalities, we will allow $\mathcal{M}$ to have an inner boundary $\partial\mathcal{M}$ (see Lemma 4.1 for instance).

The existence of these two metrics on $\mathcal{M}$ can be source of confusions, to avoid these we have to introduce some further notations. The symbols

$\mathring{\nabla}, \mathring{\Delta}, \mathring{\Gamma}_{ij}^{\ k}, |u|_{\mathring{g}}$ (resp. ${}^{(\mathring{h})}\nabla, \Delta_{\mathring{h}}, {}^{(\mathring{h})}\Gamma_{ij}^{\ k}, |u|_{\mathring{h}}$) will respectively denote the connection, the Laplace operator, the Christoffel symbols and the tensor norm w.r.t. $\mathring{g}$ (resp. $\mathring{h}$). We have the following correspondences between these quantities:

For volume measures, in dimension n, clearly $d\mu(\mathring{g}) = \rho^{-n} d\mu(\mathring{h})$.
For Christoffel symbols, $\mathring{\Gamma}_{ij}^{\ k} = {}^{(\mathring{h})}\Gamma_{i\,j}^{\ k} - \frac{1}{\rho}\left(\delta_j^k \partial_i \rho + \delta_i^k \partial_j \rho - \mathring{h}^{kl}\mathring{h}_{ij}\partial_l \rho\right)$.

In particular, for the Hessian of ρ (see e.g. equation (C.12) from [5])

$$\mathring{\nabla}_{ij}^2 \rho = {}^{(\mathring{h})}\mathring{\nabla}_{ij}^2 \rho + \rho\left(2\frac{\partial_i \rho}{\rho}\frac{\partial_j \rho}{\rho} - \mathring{g}_{ij}|d\rho|_{\mathring{h}}^2\right). \tag{1}$$

Taking the trace of (1), we obtain the expression of the Laplace operator of the function ρ

$$\mathring{\Delta}\rho = \rho^2 \Delta_{\mathring{h}}\rho - (n-2)\rho|d\rho|_{\mathring{h}}^2. \tag{2}$$

More generally,

$$\forall u \in \mathscr{C}_c^\infty(\mathcal{M}), \quad \mathring{\Delta}u = \rho^2\left(\Delta_{\mathring{h}}u - \frac{n-2}{\rho}\, d\rho \cdot_{\mathring{h}} du\right), \tag{3}$$

where $\cdot_{\mathring{h}}$ is the scalar product w.r.t. the metric $\mathring{h}$.

For tensor norms, for all $u \in T_m^r(\mathcal{M})$, $|u|_{\mathring{g}} = \rho^{m-r}|u|_{\mathring{h}}$.

Definition 2.1 (A convenient cut-off function)**.** *Let $(\mathcal{M}, \mathring{g})$ be a conformally compact manifold with defining function ρ.*

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function such that:

- $\chi(\mathbb{R}) \subset [0, 1]$,
- *supp* $\chi \subset (-\infty, 2]$,
- $\chi_{|(-\infty,1]} = 1$,

then for R large enough, we can define a cut-off function on $\mathcal{M}$ by

$$\chi_R(x) = \chi(-\ln(\rho(x))/R).$$

Setting $\Omega_R = \{x \in \mathcal{M} : \rho(x) > e^{-2R}\}$, then χ_R verifies

$$\chi_R = \begin{cases} 1 \text{ on } \Omega_{R/2} \\ 0 \text{ on } \mathcal{M} \setminus \Omega_R\,. \end{cases}$$

In other words, χ_R is a cut-off function near the boundary at infinity of $\mathcal{M}$.

2.2. Weighted Sobolev and Hölder spaces

Thanks to the background metric $\mathring{g}$, we can define the following norm

$$\forall\, 1 \leqslant p < \infty \ \text{ and } \ \delta \in \mathbb{R} \ : \quad ||u||_{p,\delta} = \left(\int_{\mathcal{M}} |u|^p_{\mathring{g}}\, \rho^{p\delta} d\mu(\mathring{g})\right)^{1/p} .$$
$$\text{For } p = \infty \ \text{ and } \ \delta \in \mathbb{R} \ : \quad ||u||_{\infty,\delta} = \underset{\mathcal{M}}{sup}\left(\rho^{\delta}|u|_{\mathring{g}}\right) .$$

The weighted Lebesgue space L^p_δ is now defined as the space of measurable functions in L^p_{loc} whose norm mentioned above is finite. The weighted Sobolev space $W^{k,p}_\delta$ is then the space of measurable functions of $W^{k,p}_{\text{loc}}$ whose following norm is finite:

$$||u||_{k,p,\delta} = \sum_{|\alpha|\leqslant k} ||\mathring{\nabla}^{\alpha} u||_{p,\delta},$$

where α is a multi-index of size n and $\mathring{\nabla}^{\alpha} u = \mathring{\nabla}^{\alpha_1}_{i_1} \dots \mathring{\nabla}^{\alpha_n}_{i_n} u$,

$$\alpha = (\alpha_1, \dots, \alpha_n) \ \text{ and } \ |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

NB: for $\delta = 0$, we recover the norms of the classic Lebesgue and Sobolev spaces.

$W^{k,p}_\delta(T^r_m\mathcal{M})$ will refer to Sobolev spaces of sections of the (r,m)-tensor bundle over $\mathcal{M}$. For a domain $U \subset \mathcal{M}$, $||u||_{k,p,\delta;U}$ will be the restriction to U of the $W^{k,p}_\delta$- norm of u. The weighted Hölder space $C^{s,\alpha}_\delta(\mathcal{M}, g)$, with $0 < \alpha < 1$ is endowed with the norm

$$||u||_{C^{s,\alpha}_\delta} = \max_{|k|\leqslant s} ||\mathring{\nabla}^{k} u||_{C^{0,\alpha}_\delta} ,$$

with

$$||u||_{C^{0,\alpha}_\delta} = \sup_{x\in\mathcal{M}} \rho^{\delta} |u|_{\mathring{g}} + \sup_{x\in\mathcal{M}} \rho^{\delta}\left(\sup_{d_{\mathring{g}}(x,y)\leqslant 1} \frac{|\widetilde{u}(x) - \widetilde{u}(y)|_{\widetilde{g}}}{d_{\mathring{g}}(x,y)^{\alpha}}\right),$$

where $\widetilde{u}$ and $\widetilde{g}$ represent tensors u and g in an appropriate orthonormal basis.

2.3. Elliptic operators

Here we recall some classic results on elliptic operators that can be found in e.g [1]. Let B_1 and B_2 be two tensor bundles over a conformally compact manifold $(\mathcal{M}, \mathring{g})$ with defining function ρ and let $A : \mathscr{C}^\infty(B_1) \to \mathscr{C}^\infty(B_2)$ be a partial differential linear operator of order m defined by

$$A = \sum_{|\alpha| \leqslant m} a_\alpha \mathring{\nabla}^\alpha. \tag{4}$$

For $s \in \mathbb{N}$, we say that the operator A of the form (4) has symbol in $\mathcal{OP}^m_s$ if

$$a_\alpha \in C^{s_\alpha}_{-|\alpha|} L(B_1, B_2), \quad \text{with } \; s_\alpha = \max(s, |\alpha| - m + 1).$$

We say that A is an elliptic operator if for all α such that $|\alpha| = m$, for all $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \neq 0$, $a_\alpha \xi^\alpha : B_1 \to B_2$ is a tensor bundles isomorphism and

$$||a_\alpha \xi^\alpha||_{\mathring{g}} < c_1 \, |\xi^\alpha|_{\mathring{g}} \qquad \text{and} \qquad ||(a_\alpha \xi^\alpha)^{-1}||_{\mathring{g}} < c_2 \, |\xi^\alpha|_{\mathring{g}}^{-1}.$$

for some positive constants c_1, c_2 and all $\xi^\alpha \neq 0$.

Let us recall the following classical elliptic estimate.

Lemma 2.2. *Set $s_0 \in \mathbb{N}$. For every elliptic operator A with symbol in $\mathcal{OP}^m_{s_0}$, there exists a positive constant $c = c\,(\mathring{g}, \delta, s_0)$ such that the following inequality is valid for all $s \leqslant s_0$:*

$$||u||_{2,s+m,\delta} \leqslant c\, \left(||Au||_{2,s,\delta} + ||u||_{2,0,\delta}\right). \tag{5}$$

A usual consequence of this together with an asymptotic inequality is a better estimate implying Fredholm properties.

Theorem 2.3. *For all $R > 0$, let Ω_R be as in Definition 2.1. Given $\delta \in \mathbb{R}$ and A an elliptic operator with symbol in $\mathcal{OP}^m_0$. Assume there exists a constant C such that for R large enough*

$$\forall u \in \mathscr{C}^\infty_c(\mathcal{M} \setminus \overline{\Omega}_R), \quad ||u||_{2,\delta} \leqslant C \, ||Au||_{2,\delta}. \tag{6}$$

Then we can choose R large enough so that the following inequality

$$||u||_{m,2,\delta} \leqslant C \, \left(||Au||_{2,\delta} + ||u||_{2,\delta;\Omega_R}\right) \,, \tag{7}$$

is valid for all $u \in \mathscr{C}^\infty_c(\mathcal{M})$. In particular, $A : W^{m,2}_\delta \to L^2_\delta$ is semi-Fredholm, i.e., A has finite dimensional kernel and closed range. If moreover W is a closed complementing subspace of the kernel of A in $W^{m,2}_\delta$, then there exists a constant C such that all $u \in W$

$$||u||_{m,2,\delta} \leqslant C \, ||Au||_{2,\delta}$$

Proof: This is a very classical argument, see for instance proof of proposition 2.6 and equation (2.10) in [1]. □

3. Preliminary analysis

In this section, we provide a list of useful inequalities (see [1], Theorem 2.3 for example):

Proposition 3.1. *Weighted Hölder inequalities (in any dimension):*

- *Set* $\delta \in \mathbb{R}$ *so that* $\delta = \delta_1 + \delta_2$*, let* $p, q, r \in \mathbb{N}$ *be such that*

$$1 \leqslant p \leqslant q \leqslant r \leqslant \infty \ \ \text{and} \ \ \frac{1}{p} = \frac{1}{q} + \frac{1}{r}\,,$$

then

$$||uv||_{p,\delta} \leqslant ||u||_{q,\delta_1}\ ||v||_{r,\delta_2}. \tag{8}$$

- *Set* $\delta \in \mathbb{R}$ *and* $\lambda \in [0,1]$*, let* $p, q, r \in \mathbb{N}$ *be such that*

$$1 \leqslant p \leqslant q \leqslant r \leqslant \infty \ \ \text{and} \ \ \frac{1}{p} = \frac{\lambda}{q} + \frac{1-\lambda}{r}\,,$$

then

$$||u||_{p,\delta} \leqslant ||u||^{\lambda}_{q,\delta}\ ||u||^{1-\lambda}_{r,\delta}. \tag{9}$$

Theorem 3.2. *Let* $(\mathcal{M}, \mathring{g})$ *be a conformally compact* n*-dimensional manifold.*

Weighted Sobolev inequalities/embeddings:
For all $1 \leqslant p \leqslant q < \infty$*, for all* $k \geqslant k'$*, if* $\delta \leqslant \delta' - \frac{n}{p}\frac{(q-p)}{q}$*, then* $W^{k,q}_{\delta} \subset W^{k',p}_{\delta'}$*, and there exists a positive constant* $c = c\,(\mathring{g}, \delta, \delta', k, k', n, p, q)$ *such that*

$$||u||_{k',p,\delta'} \leqslant c\,||u||_{k,q,\delta}.$$

If $q = \infty$*, for all* $1 \leqslant p \leqslant \infty$*; for all* $k \geqslant k'$*, if* $\delta \leqslant \delta' - \frac{n}{p}$*, then* $W^{k,\infty}_{\delta} \subset W^{k',p}_{\delta'}$*, and there exists a positive constant* $c = c\,(\mathring{g}, \delta, \delta', k, k', n, p)$ *such that*

$$||u||_{k',p,\delta'} \leqslant c\,||u||_{k,\infty,\delta}.$$

Set $1 \leqslant p < \infty$ *and let* k, j *be integers. In each of the following cases, there exists a positive constant* $c = c\,(\mathring{g}, \delta, k, n, j, p, q)$ *such that for all* $u \in W^{j+k,p}_{\delta}(\mathcal{M})$*,*

- *If* $pk < n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q \leqslant \frac{np}{n-kp}$.
- *If* $pk = n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q < \infty$.
- *If* $pk > n$, $||u||_{j,q,\delta} \leqslant c\,||u||_{j+k,p,\delta}$, $\forall\, p \leqslant q \leqslant \infty$.

Weighted Hölder inclusion:
Set $k \in \mathbb{N}$, $n = 3$ *and* $p = 2$, *then for all* $0 < \alpha \leqslant \frac{1}{2}$, *for all* $\delta \leqslant \delta'$, *there exists a positive constant* $c = c\,(\mathring{g}, \delta, \delta', k, k', \alpha)$ *such that*

$$\forall u \in W^{2,2}_{\delta}(\mathcal{M}), \quad ||u||_{C^{0,\alpha}_{\delta'}} \leqslant c\,||u||_{2,2,\delta}. \tag{10}$$

Ehrling inequality:
Let j *and* k *be two integers such that* $0 < j < k$. *For all* $\varepsilon > 0$, *there exists a positive constant* $C(\varepsilon)$ *such that*

$$\forall u \in W^{k,p}_{\delta}, \quad ||u||_{j,p,\delta} \leqslant \varepsilon||u||_{k,p,\delta} + C(\varepsilon)||u||_{p,\delta}. \tag{11}$$

Rellich Theorem:
For all $k > k'$ *and* $\delta < \delta'$, *the inclusion* $W^{k,2}_{\delta} \subset W^{k',2}_{\delta'}$ *is compact.*

A consequence of the Sobolev inequalities (cf. [5] for example) is the following behaviour at infinity.

Proposition 3.3. *In dimension* $n = 3$, *for all* $k > \frac{3}{2}$,

$$u \in W^{k,2}_{\delta} \Rightarrow u = o(\rho^{-\delta}). \tag{12}$$

We will also need to control the norm of some products.

Lemma 3.4. *In dimension 3, for all* $\delta_1, \delta_2 \in \mathbb{R}$, *there exists a positive constant* $c = c\,(\mathring{g}, \delta_1, \delta_2)$ *such that*

$$||uv||_{2,\delta} \leqslant c\;||u||_{1,2,\delta_1}\;||v||_{1,2,\delta_2}, \quad \text{where } \delta = \delta_1 + \delta_2. \tag{13}$$

Remark. In the particular case where δ_1, δ_2 and δ are non positive, then δ_1 and δ_2 are both greater than δ. The weighted Sobolev embedding leads to

$$||uv||_{2,\delta} \leqslant c\;||u||_{1,2,\delta}\;||v||_{1,2,\delta}. \tag{14}$$

The following estimates will also be useful.

Lemma 3.5. *In dimension* 3*, there exists a positive constant* $c = c(\mathring{g}, \delta)$ *such that*

$$\forall u \in W^{2,2}_\delta(\mathcal{M}), \quad ||u||_{\infty,\delta} \leqslant \varepsilon\, ||u||_{2,2,\delta} + c\varepsilon^{-3}||u||_{1,2,\delta}. \tag{15}$$

$$\forall u \in W^{1,2}_\delta(\mathcal{M}), \quad ||u||_{3,\delta} \leqslant \varepsilon\, ||u||_{1,2,\delta} + c\varepsilon^{-1}||u||_{2,\delta}. \tag{16}$$

We conclude this section by an easy exercise.

Lemma 3.6. *In all dimension* n*, we define for any real* R *large enough,*

$$E_R := \mathcal{M} \setminus \Omega_R = \{\rho \leqslant e^{-2R}\}.$$

Then $\forall\, \delta \in \mathbb{R}$, $\forall u \in L^p_\delta(E_R)$,

$$||u||_{p,\eta;E_R} \leqslant e^{2R(\delta-\eta)}||u||_{p,\delta;E_R}, \quad \forall\, \delta \leqslant \eta\,,\ \forall\, 1 \leqslant p \leqslant \infty. \tag{17}$$

4. The Hessian type operator $\mathring{T}$

In this section, we prove some preliminary results of Poincaré-inequality type, concerning a shifted Hessian operator $\mathring{T}$ acting on a function N by:

$$\mathring{T} = \mathring{T}(N) := \mathring{\nabla}^2 N - N\mathring{g}. \tag{18}$$

This operator is natural because $\mathring{T}(N) - \operatorname{tr}_{\mathring{g}} \mathring{T}(N)\mathring{g}$ is, up to a zero order term in a weighted L^2 space, the adjoint of the linearised scalar curvature operator at the AH metric $\mathring{g}$. It will appear in the the study of the kernel of the adjoint of the linearised constraint operator (see e.g. equation (67)).

We will need the next three Lemmas which provide general equalities on $(\mathcal{M}, \mathring{g})$. Here we allow a possible inner boundary $\partial\mathcal{M}$. From now on, $d\sigma(\mathring{g})$ will be the measure induced by $\mathring{g}$ on $\partial\mathcal{M}$ and η is the exterior unit normal to $\partial\mathcal{M}$. The term $o(1)$ will tend to zero when approaching $\partial_\infty\mathcal{M}$.

Lemma 4.1. *Let* $(\mathcal{M}, \mathring{g})$ *be an* n*-dimensional asymptotically hyperbolic manifold and* $N \in \mathscr{C}^\infty(\mathcal{M})$ *with a compact support on* $\mathcal{M}$. $\forall\delta \in \mathbb{R}$,

$$\int_{\mathcal{M}} 2N\langle dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\mu(\mathring{g}) = -\int_{\mathcal{M}} [2\delta + 1 - n + o(1)]N^2\rho^{2\delta}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} N^2\langle \frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g})\,.$$

Proof: Integration by parts gives

$$\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\,d\mu(\mathring{g}) = \int_{\mathcal{M}} 2N\langle\,dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\mu(\mathring{g}) - \int_{\mathcal{M}} N^2\mathring{\Delta}(\rho^{-1})\rho^{2\delta+1}\,d\mu(\mathring{g}) + \int_{\mathcal{M}}(2\delta+1)N^2|d\rho|^2_{\mathring{h}}\rho^{2\delta}\,d\mu(\mathring{g}). \tag{19}$$

Let us compute (see [5], (D.4) for instance)

$$\mathring{\nabla}^2(\rho^{-1}) = \rho^{-1}|d\rho|^2_{\mathring{h}}\mathring{g} - \rho^{-2\,(\mathring{h})}\nabla^2\rho. \tag{20}$$

Taking the $\mathring{g}$-trace,

$$\mathring{\Delta}(\rho^{-1}) = n\rho^{-1}|d\rho|^2_{\mathring{h}} - \Delta_{\mathring{h}}\rho.$$

The metric $\mathring{h}$ being defined (and so bounded) up to $\partial_\infty\mathcal{M}$ and ρ being a smooth function on $\bar{\mathcal{M}}$, $\Delta_{\mathring{h}}\rho$ is a smooth function bounded on $\bar{\mathcal{M}}$, and so we can write $\Delta_{\mathring{h}}\rho = O(1) = o(\rho^{-1})$ near $\partial_\infty\mathcal{M}$. We obtain

$$\mathring{\Delta}(\rho^{-1}) = \rho^{-1}\big(n|d\rho|^2_{\mathring{h}} + o(1)\big). \tag{21}$$

According to $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ near the boundary at infinity, (19) becomes

$$\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\,d\mu(\mathring{g}) = \int_{\mathcal{M}} 2N\langle dN, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\mu(\mathring{g}) + \int_{\mathcal{M}}[2\delta+1-n+o(1)]N^2\rho^{2\delta}\,d\mu(\mathring{g}). \tag{22}$$

From the Divergence Theorem,

$$\int_{\mathcal{M}} \mathring{\nabla}_i(-N^2\mathring{\nabla}^i(\rho^{-1})\rho^{2\delta+1})\,d\mu(\mathring{g}) = \int_{\partial\mathcal{M}} N^2\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g}). \tag{23}$$

We end the proof replacing the left-hand side of (22) by its expression (23). □

Lemma 4.2. *Let $(M,\mathring{g})$ be an n-dimensional asymptotically hyperbolic manifold and $N \in \mathscr{C}^\infty(\mathcal{M})$ have compact support on $\mathcal{M}$. $\forall\delta\in\mathbb{R}$,*

$$-2\int_{\mathcal{M}} \mathring{T}(N)(dN, \frac{d\rho}{\rho})\rho^{2\delta}\,d\mu(\mathring{g}) = \int_{\mathcal{M}}\{2\delta+1-n+o(1)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\,d\mu(\mathring{g})$$

$$-\int_{\mathcal{M}}[2\delta+1-n+o(1)]N^2\rho^{2\delta}\,d\mu(\mathring{g})$$
$$+\int_{\partial\mathcal{M}}(N^2-|dN|^2_{\mathring{g}})\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g}), \tag{24}$$

Proof: We integrate by parts the term $\mathring{\nabla}_i(|dN|^2_{\mathring{g}}\,\mathring{\nabla}^i(\rho^{-1})\,\rho^{2\delta+1})$ and the result follows from the Divergence Theorem and Lemma 4.1. ☐

Lemma 4.3. *Let $(M,\mathring{g})$ be an n-dimensional asymptotically hyperbolic manifold and $N\in\mathscr{C}^\infty(\mathcal{M})$ with a compact support on $\mathcal{M}$. $\forall\delta\in\mathbb{R}$,*

$$-\int_{\mathcal{M}}\operatorname{tr}_{\mathring{g}}\mathring{T}(N)\,N\,\rho^{2\delta}\,d\mu(\mathring{g}) = -\int_{\mathcal{M}}[\,\delta(2\delta+1-n)-n+o(1)]N^2\rho^{2\delta}\,d\mu(\mathring{g})$$
$$+\int_{\mathcal{M}}|dN|^2_{\mathring{g}}\rho^{2\delta}\,d\mu(\mathring{g})-\int_{\partial\mathcal{M}}N\langle dN,\eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g})$$
$$+\int_{\partial\mathcal{M}}\delta N^2\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,\rho^{2\delta}\,d\sigma(\mathring{g}), \tag{25}$$

Proof: We integrate by parts the term $\mathring{\nabla}_i(N\mathring{\nabla}^iN\rho^{2\delta})$ and the result follows from the Divergence Theorem and Lemma 4.1. ☐

The next proposition stems from the two previous lemmas and will play an important role to prove the triviality of the kernel of the adjoint:

Proposition 4.4. *For every $\varepsilon>0$, for all $\delta\in\,]-(n+1)/2,-1[\,$, there exists $R_{\varepsilon,\delta}>0$ such that for all $R>R_{\varepsilon,\delta}$, there exists a positive constant c such that*

$$\forall N\in\mathscr{C}^\infty_c(E_R),\quad ||N||_{2,2,-\delta;E_R}\leqslant c\,||\mathring{T}(N)||_{2,-\delta;E_R}. \tag{26}$$

Proof: Let $\mathring{\nabla}_nN$ (resp. $\mathring{\nabla}_TN$) be the component of dN normal (resp. tangential) to $\partial\mathcal{M}$, so $\mathring{\nabla}_nN:=\langle dN,\eta\rangle_{\mathring{g}}$ and $|dN|^2_{\mathring{g}}=|\mathring{\nabla}_nN|^2_{\mathring{g}}+|\mathring{\nabla}_TN|^2_{\mathring{g}}$.

For $\upsilon>0$, the combination (25) $-$ $(\frac{1}{2}+\upsilon)$(24) gives

$$(1+2\upsilon)\int_{\mathcal{M}}\mathring{T}(N)(dN,\frac{d\rho}{\rho})\rho^{2\delta}\,d\mu(\mathring{g})-\int_{\mathcal{M}}N\operatorname{tr}_{\mathring{g}}\mathring{T}\rho^{2\delta}\,d\mu(\mathring{g})$$
$$=\int_{\mathcal{M}}\{\tfrac{n+1}{2}-\delta+o(1)+O(\upsilon)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\,d\mu(\mathring{g})$$
$$+\int_{\mathcal{M}}[-2\delta^2+n\delta+\tfrac{n+1}{2}+o(1)+O(\upsilon)]N^2\rho^{2\delta}\,d\mu(\mathring{g})$$
$$+\int_{\partial\mathcal{M}}\{(\delta-\tfrac{1}{2}-\upsilon)N^2+(\tfrac{1}{2}+\upsilon)|dN|^2_{\mathring{g}}\}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\rho^{2\delta}\,d\sigma(\mathring{g})$$
$$-\int_{\partial\mathcal{M}}N\mathring{\nabla}_nN\,\rho^{2\delta}\,d\sigma(\mathring{g}). \tag{27}$$

Application on E_R: E_R possesses two disjoint boundary components. A boundary at infinity, denoted by $\partial E_\infty = \partial_\infty \mathcal{M}$, and an inner boundary $\partial E_R = \partial\Omega_R = \{\rho = \mathrm{e}^{-2R}\}$. Since $N \in \mathscr{C}_c^\infty(E_R)$, N vanishes near ∂E_∞ but not necessarily on $\partial\Omega_R$ and this is the reason why boundary terms in (27) will only concern $\partial\Omega_R$. If η_R is the normal to $\partial\Omega_R$ exterior to E_R and note that when $R \to +\infty$ we have $\eta_R - \frac{d\rho}{\rho} \to 0$, so that $\langle \frac{d\rho}{\rho}, \eta_R\rangle_{\mathring{g}} = \frac{|d\rho|^2_{\mathring{g}}}{\rho^2} + o(1) = 1 + o(1)$. We deduce that

$$
\begin{aligned}
&(1+2\upsilon)\int_{E_R} \mathring{T}(N)(dN, \frac{d\rho}{\rho})\rho^{2\delta}\, d\mu(\mathring{g}) - \int_{E_R} N\, \mathrm{tr}_{\mathring{g}}\mathring{T}\rho^{2\delta}\, d\mu(\mathring{g})\\
= &\int_{E_R}\{\tfrac{n+1}{2} - \delta + o(1) + O(\upsilon)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g})\\
&+ \int_{E_R}[-2\delta^2 + n\delta + \tfrac{n+1}{2} + o(1) + O(\upsilon)]N^2\rho^{2\delta}\, d\mu(\mathring{g})\\
&+ \int_{\partial E_R}\{(\delta - \tfrac{1}{2} + o(1) - \upsilon)N^2 + (\tfrac{1}{2} + o(1) + \upsilon)|dN|^2_{\mathring{g}} - N\mathring{\nabla}_n N\}\rho^{2\delta}\, d\sigma(\mathring{g}).
\end{aligned}
$$

Now, according to the following inequalities

$$
\begin{cases}
|\mathrm{tr}_{\mathring{g}}\mathring{T}|^2_{\mathring{g}} \leqslant n\,|\mathring{T}|^2_{\mathring{g}},\\
\mathring{T}(dN, \frac{d\rho}{\rho}) \leqslant \frac{a}{2}|\mathring{T}|^2_{\mathring{g}} + \frac{1}{2a}|dN|^2_{\mathring{g}}|d\rho|^2_{\mathring{h}}, \qquad \forall a > 0\,,\\
-N\,\mathrm{tr}_{\mathring{g}}\mathring{T} \leqslant \frac{b}{2}\,|\mathrm{tr}_{\mathring{g}}\mathring{T}|^2_{\mathring{g}} + \frac{1}{2b}N^2|d\rho|^2_{\mathring{h}}, \qquad \forall b > 0\,,
\end{cases}
$$

we obtain that $\forall\varepsilon > 0, \exists R_\varepsilon > 0$ and some positive constants a and b such that $\forall R > R_\varepsilon$ ($o(1) > -\varepsilon$ on E_R so)

$$
\begin{aligned}
&\Big(\frac{a(1+2\upsilon) + bn}{2}\Big)\int_{E_R}|\mathring{T}(N)|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g})\\
\geqslant\ &\int_{E_R}\{\tfrac{n+1}{2} - \delta - \varepsilon + O(\upsilon)\}|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g})\\
&+ \int_{E_R}[-2\delta^2 + n\delta + \tfrac{n+1}{2} - \varepsilon + O(\upsilon)]N^2\rho^{2\delta}\, d\mu(\mathring{g})\\
&+ \int_{\partial E_R}(\delta - 1 - \varepsilon - \upsilon)N^2\rho^{2\delta}\, d\sigma(\mathring{g})\\
&+ \int_{\partial E_R}\left\{\tfrac{1}{2}|\mathring{\nabla}_T N|^2_{\mathring{g}} + \tfrac{1}{2}(N - \mathring{\nabla}_n N)^2\right\}\rho^{2\delta}\, d\sigma(\mathring{g}).\\
&+ \int_{\partial E_R}(-\varepsilon + \upsilon)|dN|^2_{\mathring{g}}\rho^{2\delta}\, d\sigma(\mathring{g}).
\end{aligned}
$$

- The N^2 interior term is non negative if $\delta \in \left]-\frac{1}{2}\,;\frac{n+1}{2}\right[$ and ε and υ are small.

- The $|dN|^2_{\mathring{g}}$ interior term is non negative if $\delta < (n+1)/2$ and ε and υ are small.
- The boundary term is non negative if $\delta > 1$ and $\varepsilon \leq \upsilon$ are small.

Moreover, a quick calculation shows that on the interval $[0\,;\frac{n+1}{2}[$,

$$\tfrac{n+1}{2} - \delta \leqslant -2\delta^2 + n\delta + \tfrac{n+1}{2}.$$

Consequently, for $\delta \in]1\,;\frac{n+1}{2}[$, and $\varepsilon \leq \upsilon$ are small,

$$\begin{aligned}&\Big(\frac{a(1+2\upsilon)+bn}{2}\Big)\int_{E_R} |\mathring{T}(N)|^2_{\mathring{g}}\rho^{2\delta}\,d\mu(\mathring{g})\\ \geqslant\ &\int_{E_R}\{\tfrac{n+1}{2}-\delta-\varepsilon+O(\upsilon)\}\Big(N^2+|dN|^2_{\mathring{g}}\Big)\rho^{2\delta}\,d\mu(\mathring{g}),\end{aligned}$$

namely

$$||N||_{1,2,\delta;E_R} \leqslant c\,||\mathring{T}(N)||_{2,\delta;E_R}.$$

Combining this inequality with the triangle inequality (as in Lemma 4.6 below) on E_R, the proof of (26) is completed. □

Remark 4.5. It is well known that the kernel of $\mathring{T}$, if non trivial, is finite dimensional and generated by functions of order exactly ρ^{-1} (see eg. [6]). But $\rho^{-1} \in W^{2,2}_{-\delta}(\mathcal{M})$ iff $\delta \leq -\frac{n+1}{2}$ so the condition $\delta > -\frac{n+1}{2}$ is necessary in proposition 4.4. We have not attempted to improve the condition $\delta < -1$, because for the mass of an AH manifold to be well defined, one needs $\delta < -\frac{n}{2}$.

We note the following, which is a simple consequence of the triangle inequality.

Lemma 4.6. *For all $\delta \in \mathbb{R}$, there exists a positive constant $c > 0$ depending on $\mathring{g}$ such that for all $N \in W^{2;2}_{-\delta}(\mathcal{M})$,*

$$||\mathring{T}(N)||_{2,-\delta} \geqslant ||\mathring{\nabla}^2 N||_{2,-\delta} - c\,||N||_{2,-\delta}. \tag{28}$$

Let us now show that $\mathring{T}(N)$ control N and its first derivatives.

Lemma 4.7. *For all $\delta \in]-(n+1)/2\,,0]$, there exists a positive constant $c = c\,(\mathring{g},\delta)$ such that for all $N \in W^{2,2}_{-\delta}(\mathcal{M})$,*

$$||N||_{1,2,-\delta} \leqslant c\,||\mathring{T}(N)||_{2,-\delta}. \tag{29}$$

Proof: By density, we may assume $N \in \mathscr{C}^\infty_c(\mathcal{M})$. We use the proof of Proposition 4.4 which establishes (29) if the support of N is in a neighborhood of the boundary at infinity. We obtain the result near the boundary for $\delta \in]-(n+1)/2\,;0]$ and conclude using triviality of the kernel of $\mathring{T}$ for $-\delta < \frac{n+1}{2}$ (see remark 4.5) thanks to a proof similar to the one of Theorem 2.3. □

Lemmas 4.6 and 4.7 together imply that $\mathring{T}(N)$ also controls the second derivatives of N:

Proposition 4.8. *For all $\delta \in]-(n+1)/2\,,0]$, there exists a positive constant $c = c\,(\mathring{g},\delta)$ such that*

$$||N||_{2,2,-\delta} \leqslant c\,||\mathring{T}(N)||_{2,-\delta}. \tag{30}$$

5. The Killing operator $\mathring{S}$

We study the Killing operator $\mathring{S}$ defined on 1-forms by

$$\mathring{S}(Y)_{ij} = \frac{1}{2}(\mathring{\nabla}_i Y_j + \mathring{\nabla}_j Y_i) = \mathring{\nabla}_{(i} Y_{j)}. \tag{31}$$

It is well known that this operator plays an important role when analysing the kernel of the adjoint. The goal of this section is to establish a weighted Korn-type inequality for this operator.

The trace of the Killing operator is

$$\mathrm{tr}_{\mathring{g}}\mathring{S}(Y) = \mathring{g}^{ij}\mathring{S}(Y)_{ij} = \mathring{\nabla}^i Y_i =: \mathrm{div}Y. \tag{32}$$

The next three lemmas are respectively versions of Lemma D.1 and Propositions D.2 and D.3 from [5] with inner boundary:

Lemma 5.1. *Let V be a vector field and Y a 1-form both compactly supported on $\mathcal{M}$. Then,*

$$\begin{aligned}
&\int_{\mathcal{M}} (\mathring{S}(Y) + \tfrac{1}{2}\mathrm{tr}_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,V)\, d\mu(\mathring{g}) \\
&= -\frac{1}{2}\int_{\mathcal{M}} \{\mathring{\nabla}V(Y,Y) + \tfrac{1}{2}\mathrm{div}V|Y|^2_{\mathring{g}}\}\, d\mu(\mathring{g}) + \frac{1}{2}\int_{\partial\mathcal{M}} \langle Y,V\rangle_{\mathring{g}}\, \langle Y,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) \\
&\qquad + \frac{1}{4}\int_{\partial\mathcal{M}} |Y|^2_{\mathring{g}}\, \langle V,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}),
\end{aligned}$$

Lemma 5.2. *Let u be a function, V a vector field and Y a 1-form all compactly supported on $\mathcal{M}$. Then,*

$$\begin{aligned}\int_{\mathcal{M}} \mathrm{e}^{2u}(\mathring{S}(Y) + \tfrac{1}{2}\mathrm{tr}_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,V)\, d\mu(\mathring{g}) = \\ -\frac{1}{2}\int_{\mathcal{M}} \mathrm{e}^{2u}\Big\{\mathring{\nabla}V(Y,Y) + \tfrac{1}{2}\mathrm{div}V|Y|^2_{\mathring{g}}\Big\}\, d\mu(\mathring{g}) \\ +\frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{e}^{2u}\langle Y,V\rangle_{\mathring{g}}\ \langle Y,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) \\ -\frac{1}{2}\int_{\mathcal{M}} \mathrm{e}^{2u}\Big\{2\langle du,Y\rangle_{\mathring{g}}\ \langle Y,V\rangle_{\mathring{g}} + \langle du,V\rangle_{\mathring{g}}|Y|^2_{\mathring{g}}\Big\}\, d\mu(\mathring{g}) \\ +\frac{1}{4}\int_{\partial\mathcal{M}} \mathrm{e}^{2u}|Y|^2_{\mathring{g}}\ \langle V,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}),\end{aligned}$$

Lemma 5.3. *Let Y be a 1-form compactly supported $\mathcal{M}$ and $u, v \in \mathscr{C}^\infty(\mathcal{M})$ two functions defined in a neighborhood of the support of Y. Then,*

$$\begin{aligned}&-2\int_{\mathcal{M}} v\,\mathrm{e}^{2u}\mathring{S}(Y)(dv,dv)\langle dv,Y\rangle_{\mathring{g}}\, d\mu(\mathring{g}) = \\ &\int_{\mathcal{M}} \mathrm{e}^{2u}\langle dv,Y\rangle_{\mathring{g}}\Big\{\langle dv,Y\rangle_{\mathring{g}}\Big[|dv|^2_{\mathring{g}} + v\mathring{\Delta}v + 2v\langle dv,du\rangle_{\mathring{g}}\Big] + 2v\,\mathring{\nabla}^2 v(Y,dv)\Big\}\, d\mu(\mathring{g}) \\ &-\int_{\partial\mathcal{M}} v\,\mathrm{e}^{2u}\langle dv,Y\rangle^2_{\mathring{g}}\langle dv,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}),\end{aligned}$$

As a consequence of the above lemmas we obtain versions of Corollaries D.4 and D.5 from [5] where now an inner boundary is allowed:

Corollary 5.4. *Let $(M,\mathring{g})$ be an asymptotically hyperbolic manifold and Y a 1-form compactly supported on $\mathcal{M}$. $\forall\delta\in\mathbb{R}$,*

$$\begin{aligned}&2\int_{\mathcal{M}} \rho^{2\delta}(\mathring{S}(Y) + \tfrac{1}{2}\mathrm{tr}_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y,\frac{d\rho}{\rho})\, d\mu(\mathring{g}) \\ &= \int_{\mathcal{M}} \rho^{2\delta}\Big\{(\tfrac{n+1}{2} - \delta + o(1))|Y|^2_{\mathring{g}} - (2\delta+1)\langle\frac{d\rho}{\rho},Y\rangle^2_{\mathring{g}}\Big\}\, d\mu(\mathring{g}) \\ &+\frac{1}{2}\int_{\partial\mathcal{M}} \rho^{2\delta}|Y|^2_{\mathring{g}}\ \langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\langle Y,\frac{d\rho}{\rho}\rangle_{\mathring{g}}\ \langle Y,\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}).\end{aligned} \tag{33}$$

Proof: We apply Lemma 5.2 with

$$\begin{cases} V = d(\rho^{-1}) = -\rho^{-2}d\rho \\ u = (\delta+\frac{1}{2})\ln\rho \end{cases}$$

and

$$\begin{cases} \mathring{\nabla} V = \mathring{\nabla}^2(\rho^{-1}) = \rho^{-1}|d\rho|^2_{\mathring{h}}\mathring{g} - \rho^{-2}\,{}^{(\mathring{h})}\nabla^2\rho \quad \text{from } (20) \\ \operatorname{div} V = \mathring{\Delta}(\rho^{-1}) = \rho^{-1}(n|d\rho|^2_{\mathring{h}} + o(1)) \quad \text{from } (21) \end{cases},$$

$$du = (\delta + \tfrac{1}{2})\frac{d\rho}{\rho} \text{ and } \mathrm{e}^{2u} = \rho^{2\delta+1}.$$

The metric $\mathring{h}$ being defined (and so bounded) up-to $\partial_\infty\mathcal{M}$ and ρ being a smooth function on $\bar{\mathcal{M}}$, ${}^{(\mathring{h})}\nabla^2\rho$ is a smooth function bounded on $\bar{\mathcal{M}}$, and so we can write ${}^{(\mathring{h})}\nabla^2\rho = o(\rho^{-1})$ near $\partial_\infty\mathcal{M}$. Thus,

$$\begin{cases} \mathring{\nabla} V(Y,Y) = \rho^{-1}(|d\rho|^2_{\mathring{h}}|Y|^2_{\mathring{g}} + o(1)|Y|^2_{\mathring{g}}) \\ \operatorname{div} V|Y|^2_{\mathring{g}} = \rho^{-1}(n|d\rho|^2_{\mathring{h}} + o(1))|Y|^2_{\mathring{g}} \end{cases}.$$

We obtain

$$\begin{aligned} &\int_{\mathcal{M}} \rho^{2\delta}(\mathring{S}(Y) + \tfrac{1}{2}\mathrm{tr}_{\mathring{g}}(\mathring{S}(Y))\mathring{g})(Y, \frac{d\rho}{\rho})\, d\mu(\mathring{g}) = \\ &\frac{1}{2}\int_{\mathcal{M}} \rho^{2\delta}\Big\{ \left[1 - (\delta + \tfrac{1}{2}) + \tfrac{n}{2}\right] |d\rho|^2_{\mathring{h}}|Y|^2_{\mathring{g}} - (2\delta+1)\langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}} + o(1)|Y|^2_{\mathring{g}}\Big\}\, d\mu(\mathring{g}) \\ &+ \frac{1}{4}\int_{\partial\mathcal{M}} \rho^{2\delta}|Y|^2_{\mathring{g}}\, \langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) + \frac{1}{2}\int_{\partial\mathcal{M}} \rho^{2\delta}\langle Y, \frac{d\rho}{\rho}\rangle_{\mathring{g}}\, \langle Y, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}). \end{aligned}$$

We end the proof using that $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ on an asymptotically hyperbolic manifold. □

Corollary 5.5. *Let $(M, \mathring{g})$ be an asymptotically hyperbolic manifold and Y a 1-form compactly supported on $\mathcal{M}$. Then,*

$$\begin{aligned} &2\int_{\mathcal{M}} \rho^{2\delta}\mathring{S}(Y)(\frac{d\rho}{\rho}, \frac{d\rho}{\rho})\, \langle\frac{d\rho}{\rho}, Y\rangle_{\mathring{g}}\, d\mu(\mathring{g}) = \\ &\int_{\mathcal{M}} \rho^{2\delta}\,(n - 1 - 2\delta + o(1))\, \langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\, \langle\frac{d\rho}{\rho}, Y\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}), \end{aligned} \tag{34}$$

Proof: We apply Lemma 5.3 with

$$\begin{cases} v = \rho^{-1} \\ u = (\delta+2)\ln\rho \end{cases}, \quad \begin{cases} dv = d(\rho^{-1}) = -\rho^{-1}\frac{d\rho}{\rho} \\ du = (\delta+2)\frac{d\rho}{\rho} \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{e}^{2u} = \rho^{2\delta+4} \\ |dv|^2_{\mathring{g}} = \rho^{-2}|d\rho|^2_{\mathring{h}} \end{cases}$$

together with

$$\begin{cases} \mathring{\nabla}^2 v = \mathring{\nabla}^2(\rho^{-1}) = \rho^{-1}\left(|d\rho|^2_{\mathring{h}}\mathring{g} - \rho^{-1}\,{}^{(\mathring{h})}\nabla^2\rho\right) \text{ from (20)} \\ \mathring{\Delta} v = \mathring{\Delta}(\rho^{-1}) = \rho^{-1}\left(n|d\rho|^2_{\mathring{h}} + o(1)\right) \text{ from (21)} \end{cases}.$$

Since $|d\rho|^2_{\mathring{h}} = 1 + o(1)$ on an asymptotically hyperbolic manifold, we end up with

$$2\int_{\mathcal{M}} \rho^{2\delta}\mathring{S}(Y)(\frac{d\rho}{\rho},\frac{d\rho}{\rho})\,\langle\frac{d\rho}{\rho},Y\rangle_{\mathring{g}}\,d\mu(\mathring{g}) = \int_{\mathcal{M}}\rho^{2\delta}\Big\{n+3-2(\delta+2)+o(1)\Big\}\langle\frac{d\rho}{\rho},Y\rangle^2_{\mathring{g}}\,d\mu(\mathring{g}) + \int_{\partial\mathcal{M}}\rho^{2\delta}\,\langle\frac{d\rho}{\rho},Y\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\,d\sigma(\mathring{g}).$$

□

The following lemma establishes a Korn-type inequality for the Killing operator $\mathring{S}$:

Lemma 5.6. *Assume $\mathcal{M}$ has no inner boundary. Then for all $\delta > -(n+1)/2$ and $\delta \neq -(n-1)/2$, there exists a positive constant $c = c(\mathring{g},\delta)$ such that for all 1-form $Y \in W^{1,2}_{-\delta}(T^*\mathcal{M})$,*

$$||Y||_{1,2,-\delta} \leqslant c\,||\mathring{S}(Y)||_{2,-\delta}.$$

Proof: Here again we use the scheme of the proof of Theorem 2.3, but now for the operator $\mathring{S}$. Lemma 2.8 from [5] (for $\mathring{g}$ with $N = 0$) replaces Lemma 2.2, in order to get a Korn-type inequality

$$\begin{aligned} ||Y||_{1,2,-\delta} &\leqslant c\,(||\mathring{S}(Y) - \operatorname{tr}_{\mathring{g}}(S(Y))\mathring{g}||_{2,-\delta} + ||Y||_{2,-\delta}) \\ &\leqslant c\,(||\mathring{S}(Y)||_{2,-\delta} + ||Y||_{2,-\delta})\,, \end{aligned}$$

where $c = c(\mathring{g},\delta)$ is a positive constant.

We now use Proposition D12 from [5]: Let $(\mathcal{M},\mathring{g})$ be a conformally compact manifold with $\mathring{g} = \rho^{-2}\mathring{h}$. For all $\delta \neq -(n+1)/2$ and $\delta \neq -(n-1)/2$, there exists two constants $c_\delta > 0$ and $\rho_{\varepsilon,\delta} > 0$ such that for all differentiable vector field Y compactly supported in $\{\rho < \rho_{\varepsilon,\delta}\}$:

$$||Y||_{2,-\delta} \leqslant c_\delta\,||\mathring{S}(Y)||_{2,-\delta}. \tag{35}$$

Defining R such that $\rho_{\varepsilon,\delta} = e^{-2R}$, we set Ω_R as in Definition 2.1 and we have (35) on $\mathcal{M} \setminus \Omega_R$, as in hypothesis of Theorem 2.3. The rest of the proof is analogous to the one of Theorem 2.3 and we can choose R large enough so that for all $Y \in \mathscr{C}_c^\infty(T^*\mathcal{M})$,

$$||Y||_{1,2,-\delta} \leqslant c \left(||\mathring{S}(Y)||_{2,-\delta} + ||Y||_{2,-\delta;\Omega_R} \right).$$

Hence, the operator $\mathring{S} : W^{1,2}_{-\delta}(T^*\mathcal{M}) \to L^2_{-\delta}(T^*\mathcal{M})$ has a finite dimensional kernel. So we can write $W^{1,2}_{-\delta}(T^*\mathcal{M}) = \ker\mathring{S} \oplus (\ker\mathring{S})^\perp$. From then, there exists a positive constant c such that for all $Y \in (\ker\mathring{S})^\perp$,

$$||Y||_{1,2,-\delta} \leqslant c\, ||\mathring{S}(Y)||_{2,-\delta}.$$

It remains to show that for any $\delta > -(n+1)/2$, we have $\ker \mathring{S} \cap W^{1,2}_{-\delta} = \{0\}$.

For that matter, we use the coordinate system $(x^1 = \rho, x^2, \ldots, x^n) = (\rho, \theta)$ on a neighborhood of the boundary $[0, \varepsilon] \times \partial_\infty\mathcal{M}$ that we may find in [5]. From the expression of the defining function ρ, the metric $\mathring{g}$ can be written:

$$\mathring{g} = \rho^{-2}\mathring{h} = \rho^{-2}(d\rho^2 + \hat{g}(\rho)) \quad \text{with} \;\; \hat{g}(\rho)(\partial_\rho, .) = 0.$$

The same conventions and notations as in that last paper will be used: the index ρ will be the radial coordinate one whereas indices relative to tangential coordinates will be designated by Latin capital indices. Finally lower case Latin indices designate any components. Christoffel symbols, in this coordinate system, are given in [5], just as the equation

$$\mathring{\nabla}_i X_j + \mathring{\nabla}_j X_i = 0 \tag{36}$$

which becomes the system:

$$\partial_\rho X_\rho + \rho^{-1} X_\rho = 0 \tag{37}$$

$$\partial_\rho X_A + \partial_A X_\rho + 2\rho^{-1} X_A - \hat{g}^{CD}(\rho)\hat{g}'_{DA}(\rho) X_C = 0 \tag{38}$$

$$\partial_A X_B + \partial_B X_A - 2\hat{\Gamma}^{\;C}_{A\;B}(\rho) X_C + (\hat{g}'_{AB}(\rho) - 2\rho^{-1}\hat{g}_{AB}(\rho)) X_\rho = 0$$

where $f' := \partial_\rho f$.

Solving equation (37) gives

$$X_\rho = \rho^{-1} K(\theta).$$

As the metric $\mathring{g}$ is polyhomogeneous (recall that $\mathring{g}$ is a frozen "model" metric smoothly conformally compact), $\hat{g}$ can be written as a development of powers of ρ and $\ln\rho$, and the first terms contain only powers of ρ. We set

$$\hat{g}^{CD}\hat{g}'_{DA} := T^C_{\ A},$$

where T is and order two tensor whose development is

$$T^C_{\ A}(\rho,\theta) = \rho\, {}^1T^C_{\ A}(\theta) + o(1).$$

Andersson and Chruściel have shown in [2] that the solution X of

$$(\Delta + Ric(\mathring{g}))X = 0\ ,$$

is also polyhomogeneous, thus there exists a one form Z at the boundary such that

$$X_A(\rho,\theta) = \rho^s Z_A(\theta) + o(\rho^s).$$

Inserting this in equation (38) we find $s = -2$ and $X_A = \rho^{-2}Z_A(\theta) + o(\rho^{-2})$.

We obtain the form of the solution of (36) near the boundary

$$X = \rho^{-1}K(\theta)\, d\rho + [\rho^{-2}Z_A(\theta) + o(\rho^{-2})]\, dx^A.$$

Moreover, $X \in W^{1,2}_{-\delta}$, leading to $Z = 0$. Indeed, assume $Z \neq 0$. The $\mathring{g}$-norm of X is, near infinity,

$$|X|^2_{\mathring{g}} \simeq K^2 + \rho^{-2}|Z|^2_{\hat{g}} + o(\rho^{-2}),$$

so

$$|X|^2_{\mathring{g}} \simeq \rho^{-2}\,|Z|^2_{\hat{g}}.$$

Consequently we obtain

$$\begin{aligned} ||X||_{2,-\delta} < \infty \quad &\Leftrightarrow \quad \int_0^{\varepsilon} \rho^{-2}\rho^{-2\delta}\rho^{-n}d\rho < \infty \\ &\Leftrightarrow \quad -2\delta - n - 2 > -1 \\ &\Leftrightarrow \quad \delta < -(n+1)/2. \end{aligned}$$

Given that $\delta > -(n+1)/2$, then $X \notin W^{1,2}_{-\delta}$. Hence, Z is necessarily zero at the boundary. Analysing the expansion coefficients of X using (36), we realize $Z = 0$ leads to $|X|_{\mathring{g}} = O(\rho^{\infty})$ by induction (see appendix of [6] for a similar calculation). We conclude the proof using the unique continuation theorem from [17]. □

Remark 5.7. Unfortunately, for the natural range of weight, we can not obtain a Korn inequalities on the end E_R like for the operator T in the Lemma 4.4 and we suspect this can not be done. This problem is solved in Section 9 by a Korn type inequality of second order.

6. The constraint operator Φ

Let $\mathcal{M}$ be an n-dimensional connected non compact oriented manifold. We consider $\mathcal{M}$ as a spacelike hypersurface of a $(n+1)$-dimensional Lorentzian manifold $(\mathcal{N}, \gamma)$, from now on referred to as spacetime. We will distinguish the two manifolds by different indices: Latin indices will take values from 1 to n and are spatial indices whereas Greek indices will take values from 0 to n and are spacetime indices. K is the second fundamental form of $\mathcal{M}$ in $\mathcal{N}$ defined by

$$K(X,Y) = \gamma(X, {}^{(\gamma)}\nabla_Y \vec{n}), \tag{39}$$

where ${}^{(\gamma)}\nabla$ is the spacetime connection on $T\mathcal{N}$, $X, Y \in T\mathcal{M}$ and $\vec{n}$ is the future-directed unit normal to $\mathcal{M}$ in $\mathcal{N}$. It is convenient to consider the conjugate momentum π as a reparametrisation of K:

$$\text{Let } \pi^{ij} := \widetilde{\pi}^{ij}\sqrt{g} \quad \text{with } \widetilde{\pi}^{ij} := K^{ij} - \mathrm{tr}_g K g^{ij}, \tag{40}$$

where $\sqrt{g}$ is the volume measure of the metric g, defined by

$$\sqrt{g} := \frac{\sqrt{det(g)}}{\sqrt{det(\mathring{g})}}\, d\mu(\mathring{g}).$$

Note that $\nabla(\sqrt{g}) = 0$ since the covariant derivative of the volume form vanishes.

$\widetilde{\pi}$ is a section of the bundle $S^2T\mathcal{M}$ whereas π is a section of the bundle $\widetilde{S} = S^2T\mathcal{M} \otimes \Lambda^3 T^*\mathcal{M}$. We consider a smooth and polyhomogeneous asymptotically hyperbolic metric $\mathring{g}$ on $\mathcal{M}$ as model. We require the reference metric $\mathring{g}$ to satisfy the following integrability condition[2]

$$Riem(\mathring{g})_{ijkl} - \mathring{g}_{il}\mathring{g}_{jk} + \mathring{g}_{ik}\mathring{g}_{jl} \in L^2_\delta, \tag{41}$$

in particular,

$$Ric(\mathring{g})_{jl} + (n-1)\mathring{g}_{jl} \in L^2_\delta. \tag{42}$$

[2]The L^2_δ conditions on tensors have to be understood in terms of their norm. The conditions are trivially true for the hyperbolic metric.

For any sufficiently regular Riemannian metric g on $\mathcal{M}$, we define the constraint operator $\mathbf{\Phi} = (\mathbf{\Phi_0}, \mathbf{\Phi_i}) = \mathbf{\Phi}(g, \pi)$ as follows:

$$\begin{aligned} \mathbf{\Phi_0}(g,\pi) &:= \left(R(g) - 2\Lambda - |K|_g^2 + (\mathrm{tr}_g K)^2\right)\sqrt{g} \\ &= (R(g) - 2\Lambda)\sqrt{g} - \left(|\pi|_g^2 - \tfrac{1}{n-1}(\mathrm{tr}_g \pi)^2\right)/\sqrt{g}. \end{aligned} \tag{43}$$

$$\begin{aligned} \mathbf{\Phi_i}(g,\pi) &:= 2(\nabla^j K_{ij} - \nabla_i(\mathrm{tr}_g K))\sqrt{g} \\ &= 2g_{ij}\nabla_k \pi^{jk} = 2g_{ij}\nabla_k \widetilde{\pi}^{jk}\sqrt{g}. \end{aligned} \tag{44}$$

We fix a real parameter τ and we set

$$\mathring{K} = \tau \mathring{g}\,. \tag{45}$$

The cosmological constant Λ is normalized here in dimension n by

$$2\Lambda = n(n-1)(\tau^2 - 1), \tag{46}$$

so that $\mathbf{\Phi}(\mathring{g}, \mathring{K})$ tends to zero at infinity. Taking the $\mathring{g}$-trace of (42) and using (46), we end up with the integrability condition

$$R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2 \in L^2_\delta. \tag{47}$$

From the choice of $\mathring{K}$, the conjugate momentum $\mathring{\pi}$ is then

$$\mathring{\pi}^{ij} = (\mathring{K}^{ij} - \mathrm{tr}_{\mathring{g}}\mathring{K}\mathring{g}^{ij})\, d\mu(\mathring{g}) = \tau(1-n)\mathring{g}^{ij}\, d\mu(\mathring{g}). \tag{48}$$

Note that we have $\mathring{\nabla}\mathring{\pi} = \mathring{\nabla}\mathring{K} = 0$.

If the spacetime satisfies Einstein's equations, the normalisation chosen insures that the constraint operator and the energy-momentum tensor are related by

$$\mathbf{\Phi}_\alpha = 16\pi G T_{\vec{n}\alpha}\sqrt{g}\,,$$

where G is Newton's gravitational constant. The pair $\xi := (N, X^i)$ is the *lapse-shift* associated to the spacetime foliation. We study the constraint operator $\mathbf{\Phi}$ for Riemannian metrics of the form $g = \mathring{g} + h$ with g is asymptotic to $\mathring{g}$, *i.e.* $|g - \mathring{g}|_{\mathring{g}} = |h|_{\mathring{g}} \underset{\partial_\infty \mathcal{M}}{\longrightarrow} 0$.

Let $\mathcal{S} := S^2 T^*\mathcal{M}$ be the bundle of symmetric bilinear forms on $\mathcal{M}$ and denote by $\widetilde{\mathcal{S}} := S^2 T\mathcal{M} \otimes \Lambda^n T^*\mathcal{M}$ is the bundle of symmetric 2-tensors-valued densities (n-forms) on $\mathcal{M}$. We also define $\mathcal{T} := T\mathcal{N}_{|\mathcal{M}}$ is the spacetime tangent bundle. The following spaces will be of particular interest in the sequel:

$$\begin{array}{rcl}
\mathcal{G} & := & W^{2,2}_\delta(\mathcal{S}).\\
\mathcal{K} & := & \{\pi : \pi - \mathring{\pi} \in W^{1,2}_\delta(\widetilde{\mathcal{S}})\}.\\
\mathcal{G}^+ & := & \{g : g - \mathring{g} \in \mathcal{G}, g > 0\}.\\
\mathcal{G}^+_\lambda & := & \{g \in \mathcal{G}^+ : \lambda\mathring{g} < g < \lambda^{-1}\mathring{g}\},\ \ 0 < \lambda < 1.\\
\mathcal{L}^* & := & L^2_\delta(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}) \text{ is the dual space of } \mathcal{L} := L^2_{-\delta}(\mathcal{T}).
\end{array}$$

From (10), tensors in $\mathcal{G}$ are Hölder-continuous and thus, the matrix inequalities in the spaces $\mathcal{G}^+$ and $\mathcal{G}^+_\lambda$ are satisfied pointwise. In particular, for all metrics $g \in \mathcal{G}^+_\lambda$, the metrics g and $\mathring{g}$ are equivalent in the following sense

$$\lambda\mathring{g}_{ij}(x)\, v^i v^j < g_{ij}(x)\, v^i v^j < \lambda^{-1}\mathring{g}_{ij}(x)\, v^i v^j, \quad \forall x \in \mathcal{M}\,, \forall v \in T\mathcal{M}. \tag{49}$$

We will write $|g|_{\mathring{g}} \simeq c\,|\mathring{g}|_{\mathring{g}} \simeq c\,|g|_g \simeq \sqrt{n}$.

$\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}$ will be the phase space of the constraint operator $\mathbf{\Phi}$. We will use (g, π) as well as (g, K) to express coordinates on $\mathcal{F}$.

Let $\mathring{\Gamma}$ and $\mathring{\nabla}$ (resp. Γ and ∇) be the Christoffel symbols and the Levi-Civita connection for $\mathring{g}$ (resp. g). We define

$$A_{i\ j}^{\ k} = \Gamma_{i\ j}^{\ k} - \mathring{\Gamma}_{i\ j}^{\ k}. \tag{50}$$

It is well known that

$$A_{i\ j}^{\ k} = \frac{1}{2} g^{kl}(\mathring{\nabla}_i g_{jl} + \mathring{\nabla}_j g_{il} - \mathring{\nabla}_l g_{ij}). \tag{51}$$

The scalar curvature of g can be expressed in terms of $\mathring{\nabla}$ and $A_{i\ k}^{\ j}$ (see eq. (3.7) of [4]):

Lemma 6.1.

$$\begin{aligned}
R(g) &= g^{jk} Ric(\mathring{g})_{jk} + g^{jk}(\mathring{\nabla}_i A_{j\ k}^{\ i} - \mathring{\nabla}_j A_{i\ k}^{\ i} + A_{j\ k}^{\ l} A_{i\ l}^{\ i} - A_{j\ l}^{\ i} A_{k\ i}^{\ l})\\
&= g^{jk} Ric(\mathring{g})_{jk} + Q(g^{-1}, \mathring{\nabla} g) + g^{ik} g^{jl}(\mathring{\nabla}^2_{ij} g_{kl} - \mathring{\nabla}^2_{ik} g_{jl})
\end{aligned} \tag{52}$$

where Q is a sum of quadratic terms in $g^{-1}, \mathring{\nabla} g$.

This result relies on the following fact:

Lemma 6.2.

$$Ric(g)_{jk} - Ric(\mathring{g})_{jk} = \mathring{\nabla}_i A_{j\ k}^{\ i} - \mathring{\nabla}_j A_{i\ k}^{\ i} + A_{j\ k}^{\ \mu} A_{i\ \mu}^{\ i} - A_{j\ \mu}^{\ i} A_{k\ i}^{\ \mu} \tag{53}$$

Proof:

$$\begin{aligned} Ric(g)_{jk} - Ric(\mathring{g})_{jk} &= \partial_i A^{\;i}_{j\;k} - \partial_j A^{\;i}_{i\;k} + [\Gamma^{\;i}_{l\;i}\Gamma^{\;l}_{j\;k} - \mathring{\Gamma}^{\;i}_{l\;i}\mathring{\Gamma}^{\;l}_{j\;k}] \\ &\quad -[\Gamma^{\;l}_{k\;i}\Gamma^{\;i}_{j\;l} - \mathring{\Gamma}^{\;l}_{k\;i}\mathring{\Gamma}^{\;i}_{j\;l}] \\ &= \partial_i A^{\;i}_{j\;k} - \partial_j A^{\;i}_{i\;k} + [A^{\;i}_{l\;i}A^{\;l}_{j\;k} + \mathring{\Gamma}^{\;i}_{l\;i}A^{\;l}_{j\;k} + \mathring{\Gamma}^{\;l}_{j\;k}A^{\;i}_{l\;i}] \\ &\quad -[A^{\;l}_{k\;i}A^{\;i}_{j\;l} + \mathring{\Gamma}^{\;l}_{k\;i}A^{\;i}_{j\;l} + \mathring{\Gamma}^{\;i}_{j\;l}A^{\;l}_{k\;i}] \end{aligned}$$

We end the proof adding and subtracting $\mathring{\Gamma}^{\;l}_{j\;i}A^{\;i}_{l\;k}$. □

Here we show $\mathbf{\Phi}$ is a well-defined mapping between the Hilbert spaces $\mathcal{F}$ and $\mathcal{L}^*$.

Proposition 6.3. *Set $(g,\pi) \in \mathcal{G}^+_\lambda \times \mathcal{K}$, with $0 < \lambda < 1$. Then in dimension $n = 3$, for all $\delta \leqslant 0$, there exists a positive constant $c = c(\lambda, \mathring{g}, \delta)$ such that*

$$||\mathbf{\Phi_0}(g,\pi)||_{2,\delta} \leqslant c\,(1 + ||g-\mathring{g}||^2_{2,2,\delta} + ||\pi - \mathring{\pi}||^2_{1,2,\delta}) \tag{54}$$

$$||\mathbf{\Phi_i}(g,\pi)||_{2,\delta} \leqslant c\left(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta} + ||\mathring{\nabla} g||_{1,2,\delta}(1 + ||\pi-\mathring{\pi}||_{1,2,\delta})\right) \tag{55}$$

Proof: Using (52) we can write

$$\begin{aligned} \mathbf{\Phi_0}(g,\pi) &= (R(g) - 2\Lambda)\sqrt{g} - (|\pi|^2_g - \tfrac{1}{n-1}(\mathrm{tr}_g \pi)^2]/\sqrt{g} \\ &= [R(g) - \mathbf{R}(\mathring{\mathbf{g}}) + \mathbf{R}(\mathring{\mathbf{g}}) - 2\Lambda + \mathbf{n}(\mathbf{n}-\mathbf{1})\tau^{\mathbf{2}} - \mathbf{n}(\mathbf{n}-\mathbf{1})\tau^{\mathbf{2}}]\sqrt{g} \\ &\quad -[|\pi - \mathring{\pi}|^2_g - |\mathring{\pi}|^2_g + 2(\pi-\mathring{\pi})_{ij}\mathring{\pi}^{ij} + 2\,|\mathring{\pi}|^2_g]/\sqrt{g} \\ &\quad + \tfrac{1}{n-1}[(\mathrm{tr}_g(\pi-\mathring{\pi}))^2 + (\mathrm{tr}_g\mathring{\pi})^2 + 2\,\mathrm{tr}_g(\pi-\mathring{\pi})\,\mathrm{tr}_g\mathring{\pi}]/\sqrt{g} \\ &= [R(g) - \mathbf{R}(\mathring{\mathbf{g}}) + \mathbf{R}(\mathring{\mathbf{g}}) - 2\Lambda + \mathbf{n}(\mathbf{n}-\mathbf{1})\tau^{\mathbf{2}} - \mathbf{n}(\mathbf{n}-\mathbf{1})\tau^{\mathbf{2}}]\sqrt{g} \\ &\quad -[|\pi-\mathring{\pi}|^2_g + 2g_{ik}g_{jl}(\pi-\mathring{\pi})^{kl}\mathring{\pi}^{ij} + |\mathring{\pi}|^2_g - \tfrac{1}{n-1}(\mathrm{tr}_{\mathring{g}}\mathring{\pi})^2]/\sqrt{g} \\ &\quad + \tfrac{1}{n-1}[(\mathrm{tr}_g(\pi-\mathring{\pi}))^2 + ((g-\mathring{g})_{ij}\mathring{\pi}^{ij})^2 + 2(g-\mathring{g})_{ij}\mathring{\pi}^{ij}\mathrm{tr}_{\mathring{g}}\mathring{\pi} \\ &\quad + 2\,\mathrm{tr}_g(\pi-\mathring{\pi})\,\mathrm{tr}_g\mathring{\pi}]/\sqrt{g} \\ &= [R(g) - R(\mathring{g}) + R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2]\sqrt{g} \\ &\quad -[|\pi-\mathring{\pi}|^2_g + 2g_{ik}g_{jl}(\pi-\mathring{\pi})^{kl}\mathring{\pi}^{ij}]/\sqrt{g} \\ &\quad + \tfrac{1}{n-1}[(\mathrm{tr}_g(\pi-\mathring{\pi}))^2 + ((g-\mathring{g})_{ij}\mathring{\pi}^{ij})^2 \\ &\quad + 2(g-\mathring{g})_{ij}\mathring{\pi}^{ij}\mathrm{tr}_{\mathring{g}}\mathring{\pi} + 2\,\mathrm{tr}_g(\pi-\mathring{\pi})\,\mathrm{tr}_g\mathring{\pi}]/\sqrt{g}. \end{aligned}$$

Since $g \in \mathcal{G}^+_\lambda$, we can use (49) and from Cauchy-Schwarz inequality and $2ab \leqslant a^2 + b^2$

$$|\mathbf{\Phi_0}(g,\pi)|_{\mathring{g}} \;\leqslant\; [|R(g) - R(\mathring{g})|_{\mathring{g}} + |R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2|_{\mathring{g}}]\sqrt{g}$$

$$+c\,[1+|\pi-\mathring{\pi}|_g^2+|g-\mathring{g}|_g^2]/\sqrt{g}.$$

From (53), $Ric(g)-Ric(\mathring{g})\simeq\mathring{\nabla}A+A^2\simeq(\mathring{\nabla}g)^2+g\mathring{\nabla}^2g+g^{-2}(\mathring{\nabla}g)^2$.

Using (13),

$$\begin{aligned}||Ric(g)-Ric(\mathring{g})||_{2,\delta} &\leqslant (c\,||\mathring{\nabla}g||_{1,2,\delta}^2+c\,||\mathring{\nabla}^2g||_{2,\delta})\\ &\leqslant c\,||g-\mathring{g}||_{2,2,-\delta}.\end{aligned}$$

In particular, we have the following integrability properties:

$$Ric(g)-Ric(\mathring{g})\in L^2_\delta, \tag{56}$$
$$R(g)-R(\mathring{g})\in L^2_\delta. \tag{57}$$

Thanks to (57), (47) and (14), we obtain the estimate

$$\begin{aligned}||\mathbf{\Phi_0}(g,\pi)||_{2,\delta} &\leqslant c\left(1+||(\pi-\mathring{\pi})^2||_{2,\delta}+||(g-\mathring{g})^2||_{2,\delta}\right)\\ &\leqslant c\left(1+||\pi-\mathring{\pi}||_{1,2,\delta}^2+||g-\mathring{g}||_{2,2,\delta}^2\right),\end{aligned}$$

hence $\mathbf{\Phi_0}(g,\pi)\in\mathcal{L}^*$.

Concerning $\mathbf{\Phi_i}(g,\pi)$, using (50) we have

$$\mathbf{\Phi_i}(g,\pi)=2g_{ij}(\mathring{\nabla}_k(\pi-\mathring{\pi})^{jk}+A^{\;j}_{k\;l}(\pi-\mathring{\pi})^{kl})+A^{\;j}_{k\;l}\,\mathring{\pi}^{kl}).$$

Considering (51), $\mathbf{\Phi_i}(g,\pi)$ is of the form

$$\mathbf{\Phi_i}(g,\pi)\simeq g(\mathring{\nabla}(\pi-\mathring{\pi})+g^{-1}\mathring{\nabla}g\,(\pi-\mathring{\pi})+g^{-1}\mathring{\nabla}g\,\mathring{\pi}), \tag{58}$$

thus

$$\begin{aligned}||\mathbf{\Phi_i}(g,\pi)||_{2,\delta} &\leqslant c\,(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g\,(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g\,\mathring{\pi}||_{2,\delta})\\ &\leqslant c\,(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g||_{1,2,\delta}||\pi-\mathring{\pi}||_{1,2,\delta}+||\mathring{\nabla}g||_{2,\delta}||\mathring{\pi}||_{\infty,0})\\ &\leqslant c\,(||\mathring{\nabla}(\pi-\mathring{\pi})||_{2,\delta}+||\mathring{\nabla}g||_{1,2,\delta}(1+||\pi-\mathring{\pi}||_{1,2,\delta})).\end{aligned}$$

□

We are ready now to prove the main result of this section.

Proposition 6.4. *In dimension $n=3$, for all $\delta\leqslant 0$, the map $\mathbf{\Phi}:\mathcal{F}\to\mathcal{L}^*$ is a smooth map between Hilbert spaces.*

Proof: We follow [4], we repeat the argument for completeness. From Proposition 6.3,

$$||\mathbf{\Phi}(g,\pi)||_{\mathcal{L}^*} \leqslant c(1+||g-\mathring{g}||^2_{\mathcal{G}}+||\pi-\mathring{\pi}||^2_{\mathcal{K}}),$$

i.e. $\mathbf{\Phi}$ is locally bounded on $\mathcal{F}$. The polynomial structure of the constraint operator allows us to show $\mathbf{\Phi}$ is smooth, *i.e.* indefinitely differentiable in a Fréchet sense. From the expression (52) of scalar curvature and given (58), $\mathbf{\Phi}$ can be expressed as

$$\mathbf{\Phi}(g,\pi)=F(g,g^{-1},\sqrt{g},1/\sqrt{g},\mathring{\nabla}g,\mathring{\nabla}^2g,\pi,\mathring{\nabla}\pi)\,,$$

where $F = F(a_1,\ldots,a_8)$ is a polynomial function quadratic in a_5 and a_7 and linear in the remaining arguments. The map $g \mapsto (g,g^{-1},\sqrt{g},1/\sqrt{g})$ is analytic on the space of positive definite matrices and the maps $g\mapsto \mathring{\nabla}g$, $g\mapsto \mathring{\nabla}^2 g$ and $\pi\mapsto\mathring{\nabla}\pi$ are bounded linear, thus smooth, from $\mathcal{F}$ to $\mathcal{L}^*$, which are Hilbert spaces. A result from Hille [15] on locally bounded polynomial functionals shows $\mathbf{\Phi}$ admit continuous Fréchet-derivatives of all orders. □

7. Expressions of the linearisation of $\mathbf{\Phi}$ and its adjoint

The set $\mathcal{C}=\{(g,\pi)\in\mathcal{G}^+\times\mathcal{K} : \mathbf{\Phi}(g,\pi)=0\} := \mathbf{\Phi}^{-1}(\{0\})\subset\mathcal{F}$ is the set of initial data for the vacuum Einstein's equations. To prove that $\mathcal{C}$ is a submanifold of $\mathcal{F}$, we show that 0 is a regular value of $\mathbf{\Phi}$, so we are interested in the surjectivity of the differential of $\mathbf{\Phi}$, also related to the injectivity of its adjoint. In this section we recall the expression of the linearisation of the constraint operator $\mathbf{\Phi}$ and its adjoint that we may find in [4] or [11] for example.

Proposition 7.1. *The differential of the constraint operator* $\mathbf{\Phi}$ *at* (g,π) *in the direction* (h,p) *is*

$$\begin{aligned}D\mathbf{\Phi_0}(g,\pi).(h,p) &= (\nabla^i\nabla^j h_{ij}-\Delta_g \mathrm{tr}_g h)\sqrt{g}-h_{ij}[R^{ij}-\tfrac{1}{2}(R(g)-2\Lambda)g^{ij}]\sqrt{g}\\ &\quad + h_{ij}(\tfrac{2}{n-1}\mathrm{tr}_g\pi\pi^{ij}-2\pi^i{}_k\pi^{kj}+\tfrac12|\pi|^2_g\,g^{ij}-\tfrac{1}{2(n-1)}(\mathrm{tr}_g\pi)^2g^{ij})/\sqrt{g}\\ &\quad + p^{ij}(\tfrac{2}{n-1}\mathrm{tr}_g\pi g_{ij}-2\pi_{ij})/\sqrt{g}.\end{aligned} \tag{59}$$

$$D\mathbf{\Phi_i}(g,\pi).(h,p)=\pi^{jk}(2\nabla_k h_{ij}-\nabla_i h_{jk})+2h_{ij}\nabla_k\pi^{jk}+2g_{ik}\nabla_j p^{jk}. \tag{60}$$

Using notations of [4], we write

$$\begin{aligned} \delta_g\delta_g h &= \nabla^i\nabla^j h_{ij}, \\ E^{ij} &= R^{ij} - \tfrac{1}{2}(R(g)-2\Lambda)g^{ij}, \\ \Pi^{ij} &= \left(\tfrac{2}{n-1}\mathrm{tr}_g\pi\pi^{ij} - 2\pi^i_{\ k}\pi^{kj} + \tfrac{1}{2}|\pi|^2_g\, g^{ij} - \tfrac{1}{2(n-1)}(\mathrm{tr}_g\pi)^2 g^{ij}\right)/(\sqrt{g})^2. \end{aligned}$$

We can express $D\mathbf{\Phi}$ in the following matricial form

$$D\mathbf{\Phi}(g,\pi).(h,p) = \begin{bmatrix} \sqrt{g}(\delta_g\delta_g - \Delta_g\mathrm{tr}_g + \Pi - E) & -2K \\ \hat{\pi}\nabla + 2\delta_g\pi & 2\delta_g \end{bmatrix}\begin{bmatrix} h \\ p \end{bmatrix}, \tag{61}$$

with $\hat{\pi}\nabla h = \hat{\pi}_i^{jkl}\nabla_j h_{kl} = (\pi^{jk}\delta_i^l + \pi^{jl}\delta_i^k - \pi^{kl}\delta_i^j)\nabla_j h_{kl}$.

To prove surjectivity of the differential of $\mathbf{\Phi}$, we investigate injectivity of the adjoint operator. Integrating by parts and ignoring boundary terms leads (cf. [11] for example) to an explicit formula of the formal $L^2(d\mu(\mathring{g}))$-adjoint of $D\mathbf{\Phi}(g,\pi)$:

$$\int_{\mathcal{M}} D\mathbf{\Phi}(g,\pi).(h,p)\,(N,X) = \int_{\mathcal{M}} (h,p)\bullet D\mathbf{\Phi}(g,\pi)^*(N,X).$$

The detail of the product is given by the following equality

$$\begin{aligned} (h,p)\bullet D\mathbf{\Phi_0}(g,\pi)^*N = {}& h_{ij}[\nabla^i\nabla^j N - g^{ij}\Delta_g N - [R^{ij} - \tfrac{1}{2}(R(g)-2\Lambda)g^{ij}]N]\sqrt{g} \\ &+ N h_{ij}\big(\tfrac{2}{n-1}\mathrm{tr}_g\pi\pi^{ij} - 2\pi^i_{\ k}\pi^{kj} \\ &\quad + \tfrac{1}{2}|\pi|^2_g\, g^{ij} - \tfrac{1}{2(n-1)}(\mathrm{tr}_g\pi)^2 g^{ij}\big)/\sqrt{g} \\ &+ N p^{ij}\big(\tfrac{2}{n-1}\mathrm{tr}_g\pi g_{ij} - 2\pi_{ij}\big)/\sqrt{g}. \\ (h,p)\bullet D\mathbf{\Phi_i}(g,\pi)^*X^i = {}& h_{ij}(X^k\nabla_k\pi^{ij} + \nabla_k X^k\pi^{ij} - 2\nabla_k X^{(i}\pi^{j)k}) - 2p^{ij}\nabla_{(i}X_{j)}. \end{aligned}$$

Then we can put $D\mathbf{\Phi}^*$ in the matricial form

$$D\mathbf{\Phi}(g,\pi)^*.(N,X) = \begin{bmatrix} \sqrt{g}(\nabla^2 - g\Delta_g + \Pi - E) & \nabla\pi - \hat{\pi}\nabla \\ -2K & -\mathcal{L}_g \end{bmatrix}\begin{bmatrix} N \\ X \end{bmatrix}, \tag{62}$$

with

$$\begin{aligned} (\nabla\pi - \hat{\pi}\nabla)X &= \mathcal{L}_X\pi = \nabla_X\pi^{ij} - \hat{\pi}_l^{kij}\nabla_k X^l, \\ \mathcal{L}_g(X) &= \mathcal{L}_X g = 2\,\nabla_{(i}X_{j)} = 2\,S(X). \end{aligned}$$

$D\mathbf{\Phi}(g,\pi)^*_1.\xi$ and $D\mathbf{\Phi}(g,\pi)^*_2.\xi$ will denote the two components of $D\mathbf{\Phi}(g,\pi)^*$ in (62).

$L^2_\delta\xi$ (resp. $W^{1,2}_\delta\mathring{\nabla}\xi$) is the set of terms of the form $u\,\xi$ (resp. $u\,\mathring{\nabla}\xi$) such that $||u||_{2,\delta}\leqslant C$ (resp. $||u||_{1,2,\delta}\leqslant C$), where C is a constant depending on $\mathring{g},\delta$ and $||(g,\pi)||_{\mathcal{F}}$. We have

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_1.\xi &= [\nabla_i\nabla_j N - g_{ij}\Delta_g N + (\Pi_{ij}-E_{ij})]N]\sqrt{g} + (\nabla\pi - \hat{\pi}\nabla)X \\ &= D\mathbf{\Phi}(g,0)^*\,(N,0) + \Pi_{ij}N\sqrt{g} + (\nabla\pi-\hat{\pi}\nabla)X, \end{aligned} \tag{63}$$

$$\begin{aligned} (\nabla\pi-\hat{\pi}\nabla)X &= X^k\nabla_k\pi_{ij} - (\pi^k_i\delta_{lj}+\pi^k_j\delta_{li}-\pi_{ij}\delta^k_l)\nabla_k X^l \\ &= L^2_\delta X + W^{1,2}_\delta\mathring{\nabla}X + (n-1)\tau(2\mathring{S}(X) - \mathring{g}\,\mathrm{tr}_{\mathring{g}}\mathring{S}(X)), \end{aligned}$$

$$\begin{aligned} \Pi(g,\pi)N &= L^2_\delta N + \Pi(\mathring{g},\mathring{\pi})N \\ &= L^2_\delta N - \tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g}N. \end{aligned}$$

We deduce

$$\Pi(g,\pi) + \tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g} \in L^2_\delta. \tag{64}$$

Taking into account (42) and (47), we arrive at

$$E + (n-1)\mathring{g} - \tfrac{1}{2}n(n-1)\tau^2\mathring{g} \in L^2_\delta. \tag{65}$$

On one hand, we find

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} =& \nabla^2 N - g\Delta_g N + (n-1)\mathring{g}N + [\Pi + \tfrac{1}{2}(n-1)(n-4)\tau^2\mathring{g}]N \\ & +(n-1)\tau(2\mathring{S}(X) - \mathring{g}\,\mathrm{tr}_{\mathring{g}}\mathring{S}(X)) \\ & -[E+(n-1)\mathring{g} - \tfrac{1}{2}n(n-1)\tau^2\mathring{g}]N] \\ & -(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X \\ =& \nabla^2 N - g\Delta_g N + (n-1)gN + (n-1)\tau(2\mathring{S}(X) - \mathring{g}\,\mathrm{tr}_{\mathring{g}}\mathring{S}(X)) \\ & -(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \end{aligned} \tag{66}$$

On the other hand, we can write

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_2.\xi &= -2KN - 2S(X) \\ &= -2(\mathring{S}(X) + \tau\mathring{g}N) + W^{1,2}_\delta\xi. \end{aligned}$$

From the definition of the operator $T = \nabla^2 N - Ng$ and the expression of $\mathring{S}$ as a function of $D\mathbf{\Phi}(g,\pi)^*_2.\xi$, we obtain

$$D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} = T - g\,\mathrm{tr}_g T + (n-1)\tau(2\mathring{S}(X) - \mathring{g}\,\mathrm{tr}_{\mathring{g}}\mathring{S}(X))$$

$$-(n-1)(n-2)\tau^2\mathring{g}N + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \tag{67}$$

$$D\mathbf{\Phi}(g,\pi)^*_2.\xi \;=\; -2(\mathring{S}(X) + \tau\mathring{g}N) + W^{1,2}_\delta\xi. \tag{68}$$

In odrer to put the adjoint equation in a hessian form, it is useful to restructure $D\mathbf{\Phi}^*$ into the operator P^* defined by

$$\begin{aligned} P^*(\xi) = P^*_{(g,\pi)}(\xi) &= \left[\begin{array}{c} g^{1/4}\left(\nabla^i\nabla_j N - \delta^i_j\Delta_g N + (\Pi^i_j - E^i_j)N\right) + g^{-1/4}\mathcal{L}_X\pi^i_j \\ -2g^{-1/4}\nabla_l(K^i_j N + S(X)^i_j) \end{array}\right] \\ &= \zeta\circ\left[\begin{array}{cc} 1 & 0 \\ 0 & \nabla \end{array}\right]\circ D\mathbf{\Phi}(g,\pi)^*\xi\,, \end{aligned} \tag{69}$$

where $g^{1/4} = (det(g)/det(\mathring{g}))^{1/4}$ is a density of weight $\frac{1}{2}$ and

$$\zeta = \zeta(g) = \left[\begin{array}{cc} g^{-1/4}g_{jk} & 0 \\ 0 & g^{1/4}g^{ik} \end{array}\right]. \tag{70}$$

Finally, we can put $P^*_{(g,\pi)}(\xi)$ into the form

$$P^*_{(g,\pi)}(\xi) = \left(\begin{array}{c} g^{-1/4}\, D\mathbf{\Phi}(g,\pi)^*_1.\xi \\ g^{1/4}\,\nabla D\mathbf{\Phi}(g,\pi)^*_2.\xi \end{array}\right). \tag{71}$$

Expression (69) of P^* allows us to rewrite the $L^2(d\mu(\mathring{g}))$-adjoint of P^* as follows

$$P_{(g,\pi)} = D\mathbf{\Phi}(g,\pi)\circ\left[\begin{array}{cc} 1 & 0 \\ 0 & -\delta_g \end{array}\right]\circ\zeta\,, \tag{72}$$

with $\delta_g q = \nabla^l(q^{ij}_l)$ so that $P(f^i_j, q^j_{li}) = D\mathbf{\Phi}(f_{ij}, -\nabla^l q^{ij}_l)$ and so the composition PP^* is well defined.

8. Elliptic estimates relative to the adjoint

In this section, we gather elliptic estimates satisfied by the adjoint operator $D\mathbf{\Phi}^*$. We start with:

Proposition 8.1. *Set $\delta \in\ ]-(n+1)/2\,, 0]$, with $n=3$, and $\delta \neq -(n-1)/2$. There exists a positive constant $C = C\,(\mathring{g}, \lambda, \delta, ||(g,\pi)||_{\mathcal{F}})$ such that the following elliptic estimate hold: $\forall\xi\in W^{2,2}_{-\delta}(\mathcal{T})$,*

$$||\xi||_{2,2,-\delta} \leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{1,2,-2\delta}\,, \tag{73}$$

Proof: Considering expression (68) of $\mathring{S}$ as a function of $D\mathbf{\Phi}(g,\pi)^*_2.\xi$,

$$T - g\,\mathrm{tr}_g T = \quad D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} + (n-1)\tau\big(D\mathbf{\Phi}(g,\pi)^*_2.\xi - \tfrac{1}{2}\mathring{g}\,\mathrm{tr}_{\mathring{g}} D\mathbf{\Phi}(g,\pi)^*_2.\xi\big) + L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X. \tag{74}$$

From Proposition 4.8 and equation (8), we have

$$||N||_{2,2,-\delta} \leqslant c\,(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + (n-1)|\tau|\,(1+\tfrac{n^2}{4})^{\frac{1}{2}}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{2,-\delta}) + C\,(||\xi||_{\infty,-2\delta} + ||\mathring{\nabla}\xi||_{3,-2\delta}).$$

Using (15), (16) and the Sobolev embedding ($\delta \leqslant 0$), there exists a positive constant $C = C\,(\mathring{g},\lambda,\delta,||(g,\pi)||_{\mathcal{F}})$ such that

$$||N||_{2,2,-\delta} \leqslant c\,(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + (n-1)|\tau|\,(1+\tfrac{n^2}{4})^{\frac{1}{2}}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{2,-\delta}) + \varepsilon\,||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}. \tag{75}$$

Besides, for every sufficiently regular 1-form X on $\mathcal{M}$, we have the following identity for the metric $\mathring{g}$ (see e.g. equation (3.15) of [4] for example, with a different convention for the Riemann tensor)

$$\mathring{\nabla}^2_{kj}X_i := \mathring{\nabla}_k\mathring{\nabla}_j X_i = Riem(\mathring{g})_{ijkl}X^l + \mathring{\nabla}_k\mathring{S}(X)_{ij} + \mathring{\nabla}_j\mathring{S}(X)_{ik} - \mathring{\nabla}_i\mathring{S}(X)_{jk}. \tag{76}$$

This leads to

$$\begin{aligned} ||\mathring{\nabla}^2 X||_{2,-\delta} &\leqslant ||Riem(\mathring{g})\,X||_{2,-\delta} + c\,||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta} \\ &\leqslant c\,||X||_{2,-\delta} + c\,||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta}. \end{aligned} \tag{77}$$

A consequence of Lemma 5.6 is

$$||X||_{2,-\delta} \leqslant ||X||_{1,2,-\delta} \leqslant c\,||\mathring{S}(X)||_{2,-\delta}, \tag{78}$$

which, together with (77), implies

$$||\mathring{\nabla}^2 X||_{2,-\delta} \leqslant c\,||\mathring{S}(X)||_{1,2,-\delta}, \tag{79}$$

and using (78),

$$||X||_{2,2,-\delta} \leqslant c_1\,||\mathring{S}(X)||_{1,2,-\delta}.$$

Now from (68) and (15), we also have the estimate

$$||\mathring{S}(X)||_{1,2,-\delta} \leqslant \tfrac{1}{4}||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta} + n\tau\,||N||_{1,2,-\delta}$$

$$+\,\varepsilon||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}. \tag{80}$$

Consequently, there exists a constant C depending on $\mathring{g}, \lambda, \varepsilon, \delta$ and $||(g,\pi)||_{\mathcal{F}}$ such that

$$\begin{aligned} ||X||_{2,2,-\delta} - nc_1\,\tau\,||N||_{1,2,-\delta} \leqslant \tfrac{c_1}{4}\,||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta} \\ + \varepsilon||\xi||_{2,2,-\delta} + C\,||\xi||_{1,2,-2\delta}. \end{aligned} \tag{81}$$

We can choose a small positive constant ε_0 so that (75) $+\,\varepsilon_0$(81) combine to yield (73). ☐

Remark 8.2. Although we have not tried, it might be possible to extend the result of proposition 8.1 to $\delta = -(n-1)/2$ using the operator $\mathring{U}$ introduced below.

Combining Proposition 8.1 and Ehrling's inequality (11) we get

Corollary 8.3. *In dimension $n = 3$, let δ be a real in $]-(n+1)/2\,, 0]$, with $\delta \neq -(n-1)/2$. Then the following estimate is satisfied: $\forall \xi \in W^{2,2}_{-\delta}(\mathcal{T})$,*

$$||\xi||_{2,2,-\delta} \leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{2,-2\delta}\,, \tag{82}$$

where C depends on $\mathring{g}, \lambda, \delta$ and $||(g,\pi)||_{\mathcal{F}}$.

The next lemma will be very useful in the proof of proposition 8.5, it is in fact the "time-symmetric" version thereof.

Lemma 8.4. *In dimension $n = 3$, let $\delta \leqslant 0$, then the operator $D\mathbf{\Phi}(g,0)^*\,(.,0) : W^{2,2}_{-\delta}(\mathcal{M}) \longrightarrow L^2_{-\delta}(\widetilde{\mathcal{S}})$ is bounded and depends on g in a Lipschitz way,*

$$\left\|\left[D\mathbf{\Phi}(g,0)^* - D\mathbf{\Phi}(\tilde{g},0)^*\right](N,0)\right\|_{2,-\delta} \leqslant C||g-\tilde{g}||_{\mathcal{F}}\,||N||_{2,2,-\delta}\,, \tag{83}$$

where the constant C depends on $\mathring{g}, \delta, ||g||_{\mathcal{F}}$ and $||\tilde{g}||_{\mathcal{F}}$.

Proof: Let us recall the expression of $D\mathbf{\Phi}(g,0)^*$:

$$D\mathbf{\Phi}(g,0)^*.(N,0) = [\nabla_i\nabla_j N - g_{ij}\Delta_g N - [R_{ij} - \tfrac{1}{2}(R(g) - 2\Lambda)g_{ij}]N]\sqrt{g}. \tag{84}$$

We begin by showing $D\mathbf{\Phi}(g,0)^*$ is bounded. In order to do this, we introduce the operator O acting on functions

$$O(N) = \nabla^2 N - g\,\Delta_g N \tag{85}$$

and we notice that $O(N) = \tilde{L}(\nabla^2 N)$ where $\tilde{L}$ is a linear invertible operator. Thus

$$\begin{aligned} ||O(N)||_{2,-\delta} &\leqslant c\,||\nabla^2 N||_{2,-\delta} \leqslant c\left(||\mathring{\nabla}^2 N||_{2,-\delta} + ||A\,dN||_{2,-\delta}\right) \\ &\leqslant C\,||N||_{2,2,-\delta}\,. \end{aligned}$$

Indeed, $A\,dN \simeq g^{-1}\mathring{\nabla} g\,dN$ and using the Hölder inequality (8), (13) and the Sobolev embedding,

$$\begin{aligned} ||A\,dN||_{2,-\delta} &\leqslant ||g^{-1}||_{\infty,0}||\mathring{\nabla} g\,dN||_{2,-\delta} \\ &\leqslant c\,||\mathring{\nabla} g||_{1,2,\delta}||dN||_{1,2,-2\delta} \\ &\leqslant C\,||N||_{2,2,-\delta}. \end{aligned} \tag{86}$$

We continue with

$$\begin{aligned} &||D\mathbf{\Phi}(g,0)^*.(N,0)/\sqrt{g}||_{2,-\delta} \\ &\qquad \leqslant ||O(N)||_{2,-\delta} + ||(Ric(g) - Ric(\mathring{g}))\,N||_{2,-\delta} + ||(n-1)\mathring{g}N||_{2,-\delta} \\ &\qquad + ||[Ric(\mathring{g}) + (n-1)\mathring{g}]\,N||_{2,-\delta} + ||\tfrac{1}{2}n(n-1)\tau^2\,g\,N||_{2,-\delta} \\ &\qquad + \tfrac{1}{2}\left\|\left[R(g) - 2\Lambda + n(n-1)\tau^2\right]\,g\,N\right\|_{2,-\delta}. \end{aligned}$$

Considering (56), (42), we have

$$\begin{aligned} ||(Ric(g) - Ric(\mathring{g}))N||_{2,-\delta} &\leqslant C\,||N||_{2,2,-\delta} \\ ||[Ric(\mathring{g}) + (n-1)\mathring{g}]N||_{2,-\delta} &\leqslant c\,||N||_{2,2,-\delta}. \end{aligned}$$

For the scalar curvature term, using the Hölder inequality (8) and the Sobolev embedding together with (47) and (57),

$$\begin{aligned} ||(R(g) - 2\Lambda + n(n-1)\tau^2)\,g\,N||_{2,-\delta} &\leqslant ||(R(\mathring{g}) - 2\Lambda + n(n-1)\tau^2)\,g\,N||_{2,-\delta} \\ &\quad + ||(R(g) - R(\mathring{g}))\,g\,N||_{2,-\delta} \\ &\leqslant C\,||N||_{2,2,-\delta}. \end{aligned}$$

Now we trivially estimate

$$\begin{aligned} ||(n-1)\mathring{g}N||_{2,-\delta} &\leqslant (n-1)||\mathring{g}||_{\infty,0}||N||_{2,-\delta} \\ &\leqslant c\,||N||_{2,2,-\delta}, \end{aligned}$$

and similarly,

$$||\tfrac{1}{2}n(n-1)\tau^2 gN||_{2,-\delta} \leqslant c\,||N||_{2,2,-\delta}.$$

We end up with

$$||D\mathbf{\Phi}(g,0)^*.(N,0)\big/\sqrt{g}||_{2,-\delta} \leqslant C\,||N||_{2,2,-\delta},$$

and finally

$$||D\mathbf{\Phi}(g,0)^*.(N,0)||_{2,-\delta} \leqslant C\,||\sqrt{g}||_{\infty,0}||N||_{2,2,-\delta} \leqslant C\,||N||_{2,2,-\delta}, \tag{87}$$

where C is a constant depending upon $\mathring{g},\delta$ and $||g||_{\mathcal{F}}$. The boundedness of the map is thus proved.

We now proceed to the proof of equation (83). Let us denote respectively by $\widetilde{\nabla}\,,\widetilde{\Delta}\,,Ric(\tilde{g})$ and $R(\tilde{g})$ the Levi-Civita connection, the Laplacian, the Ricci tensor and the scalar curvature of the Riemannian metric $\tilde{g}$. In order to lighten notations, we also set

$$D\mathbf{\Phi_0}(g)^*N := D\mathbf{\Phi}(g,0)^*\,(N,0) \;\text{ and }\; D\mathbf{\Phi_0}(\tilde{g})^*N := D\mathbf{\Phi}(\tilde{g},0)^*\,(N,0).$$

We decompose

$$\begin{aligned}[][D\mathbf{\Phi_0}(g)^* - D\mathbf{\Phi_0}(\tilde{g})^*]N = \quad & (\sqrt{g}-\sqrt{\tilde{g}})\,\tfrac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} \\ & +\sqrt{\tilde{g}}\left[\tfrac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \tfrac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\right].\end{aligned}$$

It directly implies

$$\begin{aligned}\Big\|[D\mathbf{\Phi_0}(g)^* - D\mathbf{\Phi_0}(\tilde{g})^*]N\Big\|_{2,-\delta} \quad \leqslant \quad & ||g-\tilde{g}||_{\mathcal{F}}\Big\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}}\Big\|_{2,-2\delta} \\ & +c\,\Big\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\Big\|_{2,-\delta}.\end{aligned} \tag{88}$$

Now, because

$$\begin{aligned}\Big(\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\Big) \quad = \quad & (\nabla-\widetilde{\nabla})\,dN + g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta}N \\ & -[Ric(g)-Ric(\tilde{g})]N \\ & +\tfrac{1}{2}\left[(R(g)-2\Lambda)g - (R(\tilde{g})-2\Lambda)\tilde{g}\right]N,\end{aligned}$$

we have

$$\Big\|\frac{D\mathbf{\Phi_0}(g)^*N}{\sqrt{g}} - \frac{D\mathbf{\Phi_0}(\tilde{g})^*N}{\sqrt{\tilde{g}}}\Big\|_{2,-\delta} \leqslant ||(\nabla-\widetilde{\nabla})\,dN||_{2,-\delta} + ||g\,\Delta_g N - \tilde{g}\,\widetilde{\Delta}N||_{2,-\delta}$$

$$+\tfrac{1}{2}\left\|[(R(g)-2\Lambda)g-(R(\tilde g)-2\Lambda)\tilde g]\,N\right\|_{2,-\delta}$$
$$-\,||[Ric(g)-Ric(\tilde g)]N||_{2,-\delta}.$$

We will estimate each terms of the right hand side of the above inequality.

- For the Hessian term, we write

$$\nabla-\widetilde{\nabla}=(g^{-1}-\tilde g^{-1})\mathring{\nabla}g+\tilde g^{-1}\mathring{\nabla}(g-\tilde g). \tag{89}$$

  Using the Hölder and Sobolev weighted inequalities, and (13), we obtain

$$||(\nabla-\widetilde{\nabla})\,dN||_{2,-\delta}\leqslant C\,||g-\tilde g||_{2,2,\delta}||N||_{2,2,-\delta}. \tag{90}$$

- For the Laplacians term, we decompose

$$\begin{aligned} g\,\Delta_g N-\tilde g\,\widetilde{\Delta}N &= g\,\Delta_g N-\tilde g\,\Delta_g N+\tilde g\,\Delta_g N-\tilde g\,\widetilde{\Delta}N\\ &=(g-\tilde g)\Delta_g N+\tilde g(\Delta_g N-\widetilde{\Delta}N)\\ &=(g-\tilde g)g^{-1}\nabla dN+\tilde g(g^{-1}-\tilde g^{-1})\nabla dN\\ &\quad+\tilde g\tilde g^{-1}(\nabla-\widetilde{\nabla})dN. \end{aligned}$$

  Using the Hölder inequality (8), the Sobolev embedding, we deduce

$$||g\,\Delta_g N-\tilde g\,\widetilde{\Delta}N||_{2,-\delta}\leqslant c\,||g-\tilde g||_{2,2,\delta}||\nabla dN||_{2,-\delta}+c\,||(\nabla-\widetilde{\nabla})dN||_{2,-\delta}.$$

  Now, considering (86) and given that $\nabla\simeq A+\mathring{\nabla}$, we have

$$||\nabla dN||_{2,-\delta}\leqslant C\,||N||_{2,2,-\delta}\,. \tag{91}$$

  Using (90) and (91), we finally get

$$||g\,\Delta_g N-\tilde g\,\widetilde{\Delta}N||_{2,\delta}\leqslant C\,||g-\tilde g||_{2,2,\delta}||N||_{2,2,-\delta}.$$

- For the Ricci tensors term, we define $\widetilde{A}_{i\,j}^{\,k}=\widetilde{\Gamma}_{i\,j}^{\,k}-\mathring{\Gamma}_{i\,j}^{\,k}$ and we set

$$\widetilde{T}:=\widetilde{\nabla}-\nabla=\tilde g^{-1}\mathring{\nabla}\tilde g-g^{-1}\mathring{\nabla}g=(g^{-1}-\tilde g^{-1})\mathring{\nabla}g+\tilde g^{-1}\mathring{\nabla}(g-\tilde g).$$

  Using the Hölder inequality (8), the Sobolev embedding,

$$||\widetilde{T}||_{1,2,\delta}\quad\leqslant\quad C||g-\tilde g||_{2,2,\delta}. \tag{92}$$

  We can show, adding and substracting $Ric(\mathring{g})$ and using (53), that

$$[Ric(g)-Ric(\tilde g)]N\simeq(\mathring{\nabla}\widetilde{T}+\widetilde{A}\widetilde{T}+\widetilde{T}^2)N\,,$$

which leads to

(93)
$$||[Ric(g) - Ric(\tilde{g})]N||_{2,-\delta} \leqslant ||\mathring{\nabla}\widetilde{T}N||_{2,-\delta} + ||\widetilde{A}\widetilde{T}N||_{2,-\delta} + ||\widetilde{T}^2 N||_{2,-\delta}.$$

Using the Hölder and Sobolev inequalities and (92)

$$||\mathring{\nabla}\widetilde{T}N||_{2,-\delta} \leqslant C\,||g - \tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}$$

The same method for the term $\widetilde{A}\widetilde{T}N$ gives, in view of (13)

$$||\widetilde{A}\widetilde{T}N||_{2,-\delta} \leqslant C\,||g - \tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}.$$

Using (8), (14), the Sobolev embedding together with (92),

$$||\widetilde{T}^2 N||_{2,-\delta} \leqslant C\,||g - \tilde{g}||^2_{2,2,\delta}||N||_{2,2,-\delta}.$$

Replacing in (93), we obtain

(94) $$||[Ric(g) - Ric(\tilde{g})]N||_{2,-\delta} \leqslant C\,||g - \tilde{g}||_{2,2,\delta}||N||_{2,2,-\delta}.$$

- For the scalar curvature term, we write

$$\begin{aligned}(R(g) - 2\Lambda)g\ &- (R(\tilde{g}) - 2\Lambda)\tilde{g}\\ &= (g - \tilde{g})(R(g) - 2\Lambda) + \tilde{g}\tilde{g}^{-1}(Ric(g) - Ric\,\tilde{g})\\ &\quad + \tilde{g}(g^{-1} - \tilde{g}^{-1})Ric(g)\\ &= (g - \tilde{g})\left[R(g) - 2\Lambda + n(n-1)\tau^2\right]\\ &\quad - n(n-1)\tau^2(g - \tilde{g}) + \tilde{g}\tilde{g}^{-1}(Ric(g) - Ric\,\tilde{g})\\ &\quad + \tilde{g}(g^{-1} - \tilde{g}^{-1})\{Ric(g) - Ric(\mathring{g})\}.\end{aligned}$$

the Hölder and Sobolev inequalities and (94) yield

$$||[(R(g) - 2\Lambda)g\ - (R(\tilde{g}) - 2\Lambda)\tilde{g}]N||_{2,-\delta} \leqslant C\,||g - \tilde{g}||_{2,2,\delta}||N||_{2,2,\delta}\,,$$

given that $\forall u \in L^\infty_0, \forall v \in L^2_\delta$ such that $||vN||_{2,-\delta} \leqslant C\,||N||_{2,2,-\delta}$,

$$\begin{aligned}||(g - \tilde{g})\,u\,v\,N||_{2,-\delta} &\leqslant ||g - \tilde{g}||_{\infty,0}||u||_{\infty,0}||vN||_{2,-\delta}\\ &\leqslant C\,||g - \tilde{g}||_{2,2,\delta}||v||_{2,\delta}||N||_{2,2,-\delta},\end{aligned}$$

where C is a positive constant depending on $\mathring{g}, \delta$ and $||g||_{\mathcal{F}}$.

Putting all the pieces together in (88) and taking (87) into account leads to

$$||[D\mathbf{\Phi_0}(g)^* - D\mathbf{\Phi_0}(\tilde{g})^*]N||_{2,-\delta} \leqslant C\,||g-\tilde{g}||_{2,2,\delta}||N||_{2,2,\delta},$$

and closes the proof. □

The dependence in (g,π) of P^* is controlled as follows:

Proposition 8.5. *Let $\delta \leqslant 0$, then in dimension 3 the operator*

$$P^* : W^{2,2}_{-\delta}(\mathcal{T}) \longrightarrow L^2_{-\delta}$$

is bounded and satisfies

$$||\xi||_{2,2,-\delta} \leqslant c\,||P^*\xi||_{2,-\delta} + C\,||\xi||_{1,2,-2\delta}\,, \tag{95}$$

where C depends on $\mathring{g}, \delta$ and $||(g,\pi)||_{\mathcal{F}}$.

Moreover, $P^*_{(g,\pi)}$ depends on $(g,\pi)\in\mathcal{F}$ in a Lipschitz way,

$$||(P^*_{(g,\pi)} - P^*_{(\tilde{g},\tilde{\pi})})\,\xi||_{2,-\delta} \leqslant C_1||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}\;||\xi||_{2,2,-\delta}\,, \tag{96}$$

where the constant C_1 depends on $\mathring{g}, \delta, ||(g,\pi)||_{\mathcal{F}}$ and $||(\tilde{g},\tilde{\pi})||_{\mathcal{F}}$.

Proof: Let us begin by showing P^* is bounded, i.e.

$$||P^*\,\xi||_{2,-\delta} \leqslant C\;||\xi||_{2,2,-\delta}. \tag{97}$$

We set

$$\begin{cases} P^* = P^*_{(g,\pi)}, \\ D\mathbf{\Phi}^*_1 = D\mathbf{\Phi}(g,\pi)^*_1, \\ D\mathbf{\Phi}^*_2 = D\mathbf{\Phi}(g,\pi)^*_2. \end{cases}$$

From (71), we have

$$\begin{aligned} ||P^*\,\xi||_{2,-\delta} &\leqslant c\,(||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta} + ||\nabla D\mathbf{\Phi}^*_2.\xi||_{2,-\delta}) \\ &\leqslant c\,(||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta} + ||\mathring{\nabla} D\mathbf{\Phi}^*_2.\xi||_{2,-\delta} + ||AD\mathbf{\Phi}^*_2.\xi||_{2,-\delta}). \end{aligned} \tag{98}$$

From (67), (8), (13), the Sobolev embedding ($\delta \leqslant 0$) we first estimate

$$\begin{aligned} ||D\mathbf{\Phi}^*_1.\xi||_{2,-\delta} \leqslant c\,\big(&||T||_{2,-\delta} + ||\mathring{S}(X)||_{2,-\delta} + ||N||_{2,-\delta}\big) \\ &+ C\,\big(||\xi||_{\infty,-2\delta} + ||\mathring{\nabla}X||_{1,2,-2\delta}\big) \end{aligned}$$

$$
\begin{aligned}
&\leqslant c\left(||N||_{2,2,-\delta} + ||X||_{1,2,-\delta} + ||N||_{2,-\delta}\right) \\
&\quad + C\left(||\xi||_{2,-\delta} + ||\mathring{\nabla} X||_{1,2,-\delta}\right) \\
&\leqslant C\,||\xi||_{2,2,-\delta}. \qquad (99)
\end{aligned}
$$

From (68) along with (8), (13), the Sobolev embedding we can also control

$$
\begin{aligned}
||D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} &\leqslant c\left(||\mathring{S}(X)||_{2,-\delta} + ||N||_{2,-\delta}\right) + C\,||\xi||_{1,2,-2\delta}, \qquad (100)\\
||AD\mathbf{\Phi}_2^*.\xi||_{2,-\delta} &\leqslant c\left(||A\mathring{S}(X)||_{2,-\delta} + ||AN||_{2,-\delta}\right) + ||\xi||_{\infty,-2\delta} \\
&\leqslant C\left(||\mathring{S}(X)||_{1,2,-\delta} + ||N||_{1,2,-\delta}\right) + ||\xi||_{2,2,-2\delta}, \\
||\mathring{\nabla} D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} &\leqslant c\left(||\mathring{\nabla}\mathring{S}(X)||_{2,-\delta} + ||\mathring{\nabla} N||_{2,-\delta}\right) + ||\xi||_{\infty,-2\delta} \\
&\leqslant C\,||\xi||_{2,2,-\delta}.
\end{aligned}
$$

Consequently,

$$
||D\mathbf{\Phi}_2^*.\xi||_{2,-\delta} \leqslant ||D\mathbf{\Phi}_2^*.\xi||_{1,2,-\delta} \leqslant C\,||\xi||_{2,2,-\delta}. \qquad (101)
$$

Every term of the right hand side of (98) is then controlled by $||\xi||_{2,2,-2\delta}$ leading to (97). The estimate (95) satisfied by P^* comes directly from (73). We now look into the Lipschitz behaviour of P^*. We set

$$
\begin{cases}
\widetilde{P}^* = P^*_{(\tilde{g},\tilde{\pi})}, \\
D\widetilde{\mathbf{\Phi}}_1^* = D\mathbf{\Phi}(\tilde{g},\tilde{\pi})_1^*, \\
D\widetilde{\mathbf{\Phi}}_2^* = D\mathbf{\Phi}(\tilde{g},\tilde{\pi})_2^*.
\end{cases}
$$

Let us write

$$
(P^* - \widetilde{P}^*)\,\xi \;=\; \begin{pmatrix} g^{-1/4}\, D\mathbf{\Phi}_1^*.\xi - \tilde{g}^{-1/4}\, D\widetilde{\mathbf{\Phi}}_1^*.\xi \\ g^{1/4}\, \nabla D\mathbf{\Phi}_2^*.\xi - \tilde{g}^{1/4}\, \tilde{\nabla} D\widetilde{\mathbf{\Phi}}_2^*.\xi \end{pmatrix} =: \begin{pmatrix} E \\ F \end{pmatrix},
$$

so

$$
||(P^* - \widetilde{P}^*)\,\xi||_{2,-\delta} \leqslant ||E||_{2,-\delta} + ||F||_{2,-\delta}. \qquad (102)
$$

We start to estimate

$$
\begin{aligned}
E &= g^{-1/4}\, D\mathbf{\Phi}_1^*.\xi - \tilde{g}^{-1/4}\, D\widetilde{\mathbf{\Phi}}_1^*.\xi \\
&= (g^{-1/4} - \tilde{g}^{-1/4})\, D\mathbf{\Phi}_1^*.\xi + \tilde{g}^{-1/4}\,(D\mathbf{\Phi}_1^*.\xi - D\widetilde{\mathbf{\Phi}}_1^*.\xi).
\end{aligned}
$$

Using (8) and the Sobolev inequality

$$\begin{aligned}||E||_{2,-\delta} &\leqslant ||(g^{-1/4}-\tilde{g}^{-1/4})\, D\mathbf{\Phi}_1^*.\xi||_{2,-\delta}+||\tilde{g}^{-1/4}\,(D\mathbf{\Phi}_1^*.\xi-D\widetilde{\mathbf{\Phi}}_1^*.\xi)||_{2,-\delta}\\ &\leqslant c\,||g-\tilde{g}||_{\mathcal{F}}||D\mathbf{\Phi}_1^*.\xi||_{2,-\delta}+c\,||D\mathbf{\Phi}_1^*.\xi-D\widetilde{\mathbf{\Phi}}_1^*.\xi||_{2,-\delta}.\end{aligned}$$

From (63), we expand

$$\begin{aligned}D\mathbf{\Phi}_1^*.\xi-D\widetilde{\mathbf{\Phi}}_1^*.\xi= &\ [D\mathbf{\Phi}(g,0)^*-D\mathbf{\Phi}(\tilde{g},0)^*]\,(N,0)+(\Pi\sqrt{g}-\tilde{\Pi}\sqrt{\tilde{g}})N\\ &+X\mathring{\nabla}(\pi-\tilde{\pi})+(\pi-\tilde{\pi})\mathring{\nabla}X+AX(\pi-\tilde{\pi}).\end{aligned}$$

Using (8), (13), the Sobolev embedding (Theorem 3.2) we already have

$$\begin{aligned}||(\pi-\tilde{\pi})\mathring{\nabla}X||_{2,-\delta}+||X\mathring{\nabla}(\pi-\tilde{\pi})||_{2,-\delta} &\leqslant ||\pi-\tilde{\pi}||_{1,2,\delta}||\mathring{\nabla}X||_{1,2,-2\delta}\\ &\quad +||\mathring{\nabla}(\pi-\tilde{\pi})||_{2,\delta}||X||_{\infty,-2\delta}\\ &\leqslant c\,||\pi-\tilde{\pi}||_{1,2,\delta}\,||X||_{2,2,-\delta},\end{aligned}$$

and

$$\begin{aligned}||AX(\pi-\tilde{\pi})||_{2,-\delta} &\leqslant ||A(\pi-\tilde{\pi})||_{2,\delta}||X||_{\infty,-2\delta}\\ &\leqslant C\,||\pi-\tilde{\pi}||_{1,2,\delta}\,||X||_{2,2,-\delta}.\end{aligned}$$

Now, we write formally

$$\begin{aligned}\Pi\sqrt{g}-\tilde{\Pi}\sqrt{\tilde{g}} &\sim \frac{1}{\sqrt{g}}g^{-1}\pi^2-\frac{1}{\sqrt{\tilde{g}}}\tilde{g}^{-1}\tilde{\pi}^2\\ &\sim \frac{1}{\sqrt{g}}(g^{-1}-\tilde{g}^{-1})\pi^2+\frac{1}{\sqrt{g}}\tilde{g}^{-1}(\pi^2-\tilde{\pi}^2)+(\frac{1}{\sqrt{g}}-\frac{1}{\sqrt{\tilde{g}}})\tilde{g}^{-1}\tilde{\pi}^2,\end{aligned}$$

leading to

$$||(\Pi\sqrt{g}-\tilde{\Pi}\sqrt{\tilde{g}})N||_{2,-\delta} \leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||N||_{2,2,-\delta}.$$

Given also (83), we obtain

$$||D\mathbf{\Phi}_1^*.\xi-D\widetilde{\mathbf{\Phi}}_1^*.\xi||_{2,-\delta}\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta},$$

and taking (99) into account,

$$||E||_{2,-\delta}\leqslant C\,||(g-\tilde{g},\pi-\tilde{\pi})||_{\mathcal{F}}||\xi||_{2,2,-\delta}. \tag{103}$$

We will now estimate the term

$$F = g^{1/4}\,\nabla D\mathbf{\Phi}_2^*.\xi - \tilde g^{1/4}\,\tilde\nabla D\widetilde{\mathbf{\Phi}}_2^*.\xi$$
$$= g^{1/4}\,(\nabla-\tilde\nabla)D\mathbf{\Phi}_2^*.\xi + (g^{1/4}-\tilde g^{1/4})\,\tilde\nabla D\widetilde{\mathbf{\Phi}}_2^*.\xi + g^{1/4}\,\tilde\nabla(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi).$$

Using (89), (8), (13), the Sobolev inequality (Theorem 3.2)

$$\begin{aligned}||F||_{2,-\delta} &\leqslant c\,||\nabla-\tilde\nabla||_{1,2,\delta}||D\mathbf{\Phi}_2^*.\xi||_{1,2,-2\delta} + ||g^{1/4}-\tilde g^{1/4}||_{\infty,-\delta}||\tilde\nabla D\widetilde{\mathbf{\Phi}}_2^*.\xi||_{2,-2\delta}\\ &\quad + c\,||\mathring\nabla(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} + c\,||A(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}\\ &\leqslant C\,||g-\tilde g||_{\mathcal F}||D\mathbf{\Phi}_2^*.\xi||_{1,2,-2\delta} + c\,||g-\tilde g||_{\mathcal F}||\tilde\nabla D\widetilde{\mathbf{\Phi}}_2^*.\xi||_{2,-2\delta}\\ &\quad + c\,||\mathring\nabla(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta} + c\,||A(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}.\end{aligned}$$

Considering (100) and (101), one has

$$\begin{aligned}||F||_{2,-\delta} &\leqslant C\,||g-\tilde g||_{\mathcal F}||\xi||_{2,2,-\delta} + c\,||\mathring\nabla(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}\\ &\quad + c\,||A(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta},\end{aligned} \tag{104}$$

with, formally,

$$\begin{aligned}D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi &\sim (K-\tilde K)N + (A-\tilde A)X\\ &\sim (\pi-\tilde\pi)N + (\nabla-\tilde\nabla)X.\end{aligned}$$

Using (89), (8), (13), the Sobolev embedding (Theorem 3.2) we deduce

$$\begin{aligned}&||\mathring\nabla(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}\\ &\qquad\leqslant c\,||\mathring\nabla(\pi-\tilde\pi)||_{2,\delta}||N||_{\infty,-2\delta} + c\,||\pi-\tilde\pi||_{1,2,\delta}||\mathring\nabla N||_{1,2,-2\delta}\\ &\qquad\quad + ||\mathring\nabla(\nabla-\tilde\nabla)||_{2,\delta}||X||_{\infty,-2\delta} + ||\nabla-\tilde\nabla||_{1,2,\delta}||\mathring\nabla X||_{1,2,-2\delta}\\ &\qquad\leqslant c\,||\pi-\tilde\pi||_{1,2,\delta}||N||_{2,2,-\delta} + c\,||\nabla-\tilde\nabla||_{1,2,\delta}||X||_{2,2,-\delta}\\ &\qquad\leqslant C\,||(g-\tilde g,\pi-\tilde\pi)||_{\mathcal F}||\xi||_{2,2,-\delta}.\end{aligned}$$

In the same way,

$$\begin{aligned}&||A(D\mathbf{\Phi}_2^*.\xi - D\widetilde{\mathbf{\Phi}}_2^*.\xi)||_{2,-\delta}\\ &\qquad\leqslant c\,||A(\pi-\tilde\pi)||_{2,\delta}||N||_{\infty,-2\delta} + ||A(\nabla-\tilde\nabla)||_{2,\delta}||X||_{\infty,-2\delta}\\ &\qquad\leqslant C\,||(g-\tilde g,\pi-\tilde\pi)||_{\mathcal F}||\xi||_{2,2,-\delta}.\end{aligned}$$

We deduce from (104)

$$||F||_{2,-\delta} \leqslant C\,||(g-\tilde g,\pi-\tilde\pi)||_{\mathcal F}||\xi||_{2,2,-\delta}. \tag{105}$$

The desired Lipschitz estimate (96) arises from (102), considering (103) and (105). □

We show in the following proposition that the estimate (82) of Corollary 8.3 is also verified by weak solutions ξ only in $L^2_{-\delta}(\mathcal{T})$. We say that $\xi \in \mathcal{L}$ is a weak solution of $D\mathbf{\Phi}(g,\pi)^*\xi = (f_1, f_2)$, with $(f_1, f_2) \in L^2_{-\delta}(\widetilde{\mathcal{S}}) \times W^{1,2}_{-\delta}(\mathcal{S})$ when

$$\int_{\mathcal{M}} \langle \xi, D\mathbf{\Phi}(g,\pi).(h,p)\rangle_{\mathring{g}} = \int_{\mathcal{M}} \langle (f_1, f_2), (h,p)\rangle_{\mathring{g}}, \ \forall (h,p) \in W^{2,2}_{\delta}(\mathcal{S}) \times W^{1,2}_{\delta}(\widetilde{\mathcal{S}}).$$

It suffices to test with $(h,p) \in \mathscr{C}^\infty_c(S) \times \mathscr{C}^\infty_c(\widetilde{\mathcal{S}})$ since this space is dense in $W^{2,2}_{\delta}(\mathcal{S}) \times W^{1,2}_{\delta}(\widetilde{\mathcal{S}})$.

Proposition 8.6. *Let $\delta \in \,]-(n+1)/2\,,0] \setminus \{-(n-1)/2\}$ with $n=3$, $(g,\pi) \in \mathcal{G}^+ \times \mathcal{K}$, and $(f_1, f_2) \in L^2_{-\delta}(\widetilde{\mathcal{S}}) \times W^{1,2}_{-\delta}(\mathcal{S})$. Let $\xi \in \mathcal{L}$ be a weak solution of $D\mathbf{\Phi}(g,\pi)^*\xi = (f_1, f_2)$. Then $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ is a strong solution and satisfies (82).*

Proof: In [4], Bartnik shows that $\xi \in W^{2,2}_{\text{loc}}$. We can find a cut-off function χ_R as in Definition 2.1 such that

- $\chi_R \in \mathscr{C}^\infty_c(\Omega_R)$.
- $\chi_R = 1$ on $\Omega_{R/2}$.

In particular, $\chi_R\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ and from Proposition 8.1, we can write:

$$\begin{aligned} ||\chi_R\xi||_{2,2,-\delta} \leqslant c\,(||D\mathbf{\Phi}(g,\pi)^*_1.(\chi_R\xi)||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.(\chi_R\xi)||_{1,2,-\delta}) \\ + C\,||\xi||_{2,-\delta}, \end{aligned} \tag{106}$$

using the Sobolev embedding ($\delta \leqslant 0$) and the Ehrling inequality. We have to show that $\chi_R\xi$ is uniformly bounded in $W^{2,2}_{-\delta}$, *i.e.* bounded independently of R. In order to do so, we adapt S. McCormick's method from [19]. From (66) and (68), we decompose formally

$$\begin{aligned} D\mathbf{\Phi}(g,\pi)^*_1.(\chi_R\xi) &\simeq \chi_R(D\mathbf{\Phi}(g,\pi)^*_1.(\xi)) + N\,\nabla^2\chi_R \\ &\quad + dN\,\nabla\chi_R + X\,\nabla\chi_R + \xi\,W^{1,2}_{\delta}\,\nabla\chi_R, \\ D\mathbf{\Phi}(g,\pi)^*_2.(\chi_R\xi) &\simeq \chi_R(D\mathbf{\Phi}(g,\pi)^*_2.(\xi)) + X\,\nabla\chi_R. \end{aligned}$$

As derivatives of χ_R are supported in $A_R := \Omega_R \setminus \Omega_{R/2}$, we can use (8), (15), (16), the Sobolev embedding and the Ehrling inequality to obtain

$$\begin{aligned} ||D\mathbf{\Phi}(g,\pi)^*_1.(\chi_R\xi)||_{2,-\delta} \leqslant c\,(||\chi_R(D\mathbf{\Phi}(g,\pi)^*_1.(\xi))||_{2,-\delta} + ||N\,\nabla^2\chi_R||_{2,-\delta} \\ + ||dN\,\nabla\chi_R||_{2,-\delta} + ||X\,\nabla\chi_R||_{2,-\delta} \end{aligned}$$

$$
\begin{aligned}
&+ ||\xi\, W^{1,2}_{\delta}\, \nabla\chi_R||_{2,-\delta})\\
&\leqslant c\,||D\mathbf{\Phi}(g,\pi)^*_1.(\xi)||_{2,-\delta} + C\,||\xi||_{2,-\delta;A_R}\\
&+ \varepsilon||\mathring{\nabla}^2\xi||_{2,-\delta;A_R}, \qquad (107)
\end{aligned}
$$

where C is a constant depending on $\mathring{g}, \lambda, \varepsilon, \delta$ and $||(g,\pi)||_{\mathcal{F}}$.

Likewise,

$$
\begin{aligned}
||D\mathbf{\Phi}(g,\pi)^*_2.(\chi_R\xi)||_{1,2,-\delta} &\leqslant c\,(||\chi_R(D\mathbf{\Phi}(g,\pi)^*_2.(\xi))||_{1,2,-\delta} + ||X\,\nabla\chi_R||_{1,2,-\delta})\\
&\leqslant c\,||D\mathbf{\Phi}(g,\pi)^*_2.(\xi)||_{1,2,-\delta} + C\,||X||_{2,-\delta;A_R}\\
&\quad +\varepsilon||\mathring{\nabla}^2 X||_{2,-\delta;A_R}. \qquad (108)
\end{aligned}
$$

Plugging (107) and (108) into (106) yields

$$
\begin{aligned}
||\chi_R\xi||_{2,2,-\delta} &\leqslant c\,||D\mathbf{\Phi}(g,\pi)^*_1.(\xi)||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.(\xi)||_{1,2,-\delta} + C\,||\xi||_{2,-\delta}\\
&\quad +\varepsilon||\mathring{\nabla}^2\xi||_{2,-\delta;A_R}. \qquad (109)
\end{aligned}
$$

Equation (85) allows us to determine the following link between operators O and T:

$$O(N) = T(N) - g\,\mathrm{tr}_g T(N) - (n-1)gN$$

and we deduce from (74)

$$
\begin{aligned}
O(N) &= D\mathbf{\Phi}(g,\pi)^*_1.\xi/\sqrt{g} + (n-1)\tau\big(D\mathbf{\Phi}(g,\pi)^*_2.\xi - \tfrac{1}{2}\mathring{g}\,\mathrm{tr}_{\mathring{g}} D\mathbf{\Phi}(g,\pi)^*_2.\xi\big)\\
&\quad +L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X - (n-1)gN. \qquad (110)
\end{aligned}
$$

Since $O(N) = \tilde{L}(\nabla^2 N)$ with $\tilde{L}$ a linear invertible operator, we also get

$$||\nabla^2 N||_{2,-\delta} \leqslant c\,||O(N)||_{2,-\delta},$$

leading to

$$||\mathring{\nabla}^2 N||_{2,-\delta} \leqslant c\,(||O(N)||_{2,-\delta} + ||A\,dN||_{2,-\delta})\,. \qquad (111)$$

From (111), (110) and using (8), (15), (16), the Sobolev embedding and the Ehrling inequality, there exists a constant C depending on $\mathring{g}, \lambda, \varepsilon, \delta$ and $||(g,\pi)||_{\mathcal{F}}$ such that

$$
\begin{aligned}
||\mathring{\nabla}^2 N||_{2,-\delta} &\leqslant c\,(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{2,-\delta})\\
&\quad +\varepsilon\,||\mathring{\nabla}^2\xi||_{2,-\delta} + C\,||\xi||_{2,-\delta}. \qquad (112)
\end{aligned}
$$

Moreover, combining (79) and (80) and using the Sobolev embedding and the Ehrling inequality, we get

$$||\mathring{\nabla}^2 X||_{2,-\delta} \leqslant c\,||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta} + \varepsilon||\mathring{\nabla}^2\xi||_{2,-\delta} + C\,||\xi||_{2,-\delta}. \tag{113}$$

Now combination of (112) and (113) gives

$$||\mathring{\nabla}^2\xi||_{2,-\delta} \leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{2,-\delta}. \tag{114}$$

Given that $\xi \in W^{2,2}_{-\delta}(A_R)$ and that all the inequalities used to obtain (114) are valid in particular on every compact set, we have the following local estimate on A_R

$$\begin{aligned} ||\mathring{\nabla}^2\xi||_{2,-\delta;A_R} &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta;A_R} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta;A_R}\right) \\ &\quad + C\,||\xi||_{2,-\delta;A_R} \\ &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.\xi||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.\xi||_{1,2,-\delta}\right) + C\,||\xi||_{2,-\delta}. \qquad (115) \end{aligned}$$

Finally, equation (109) leads to

$$\begin{aligned} ||\chi_R\xi||_{2,2,-\delta} &\leqslant c\left(||D\mathbf{\Phi}(g,\pi)^*_1.(\xi)||_{2,-\delta} + ||D\mathbf{\Phi}(g,\pi)^*_2.(\xi)||_{1,2,-\delta}\right) + C\,||\xi||_{2,-\delta} \\ &\leqslant c\left(||f_1||_{2,-\delta} + ||f_2||_{1,2,-\delta}\right) + C\,||\xi||_{2,-\delta}. \end{aligned}$$

$\chi_R\xi$ is then uniformly bounded in $W^{2,2}_{-\delta}$. As $\chi_R\xi$ converge to ξ in $L^2_{-\delta}$, classical arguments prove that $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ is a strong solution and verifies (82). □

9. The operator $\mathring{U}$

Here we introduce a second order operator $\mathring{U}$, acting on 1-forms, inspired by the formula (76) and so related to the covariant derivatives of Killing operator $\mathring{S}$. It will allow us to control the $W^{2,2}_\delta$-norm of a 1-form X with the L^2_δ-norms of $\mathring{S}(X)$ and $\mathring{U}(X)$, in other words with the $W^{1,2}_\delta$-norm of $\mathring{S}(X)$. The key estimate will arise from a succession of lemmas.

Let $\mathring{U}$ be the operator defined on 1-forms by

$$\mathring{U}_{kji}(X) = \mathring{\nabla}^2_{kj}X_i - \mathring{g}_{jk}X_i + \mathring{g}_{ik}X_j. \tag{116}$$

This readily implies

$$||\mathring{\nabla}^2 X||_{2,-\delta} - c\,||X||_{2,-\delta} \leqslant ||\mathring{U}(X)||_{2,-\delta}. \tag{117}$$

The next four lemmas are established on an asymptotically hyperbolic manifold $(M, \mathring{g})$ with eventually an inner boundary $\partial\mathcal{M}$.

Lemma 9.1. *Let $(M, \mathring{g})$ be an asymptotically hyperbolic manifold and $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ be compactly supported on $\mathcal{M}$. $\forall \delta \in \mathbb{R}$,*

$$\int_{\mathcal{M}} \mathring{\nabla} X(\frac{d\rho}{\rho}, X)\rho^{2\delta}\, d\mu(\mathring{g}) = \int_{\mathcal{M}} \rho^{2\delta}(\tfrac{n-1}{2} - \delta + o(1))|X|^2_{\mathring{g}}\, d\mu(\mathring{g}) + \frac{1}{2}\int_{\partial\mathcal{M}} |X|^2_{\mathring{g}}\langle \frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}}\, \rho^{2\delta}\, d\sigma(\mathring{g}) \tag{118}$$

$$\begin{aligned}\int_{\mathcal{M}} \mathring{\nabla} X(X, \frac{d\rho}{\rho})\rho^{2\delta}\, d\mu(\mathring{g}) = & -\int_{\mathcal{M}} \rho^{2\delta}\Big(\mathrm{div} X\langle X, \frac{d\rho}{\rho}\rangle \\ & \qquad +(2\delta+1)\langle X, \frac{d\rho}{\rho}\rangle^2\Big)\, d\mu(\mathring{g}) \\ & + \int_{\mathcal{M}} \rho^{2\delta}(1+o(1))|X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ & + \int_{\partial\mathcal{M}} \rho^{2\delta}\, \langle X, \frac{d\rho}{\rho}\rangle\langle X, \eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g})\,,\end{aligned} \tag{119}$$

Proof: To prove (118), we integrate by parts the term $\mathring{\nabla}_i(|X|^2_{\mathring{g}}\, \mathring{\nabla}^i\rho\, \rho^{2\delta-1})$ and the result follows from the Divergence Theorem, along with the definition (116) of $\mathring{U}$ and equation (1). To obtain (119), we integrate by parts the term $\mathring{\nabla}_i(X^i\, \langle X, d\rho\rangle\, \rho^{2\delta-1})$ and the result follows from the Divergence Theorem, along with the definition (116). □

Lemma 9.2. *Let $(M, \mathring{g})$ be an asymptotically hyperbolic manifold and $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ be compactly supported on $\mathcal{M}$. $\forall \delta \in \mathbb{R}$,*

$$\begin{aligned}\int_{\mathcal{M}} \mathring{U}_{kji}(X)\mathring{g}^{kj}X^i\rho^{2\delta}\, d\mu(\mathring{g}) = & -\int_{\mathcal{M}} \rho^{2\delta}\, |\mathring{\nabla} X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ & + \int_{\mathcal{M}} \rho^{2\delta}\, [2\delta^2 - \delta(n-1) \\ & \qquad -(n-1) + o(1)]|X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \\ & + \int_{\partial\mathcal{M}} \rho^{2\delta}\, (\mathring{\nabla} X(\eta, X) - \delta|X|^2_{\mathring{g}}\langle \frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}})\, d\sigma(\mathring{g}).\end{aligned} \tag{120}$$

Proof: From (116), we have $\mathring{U}_{kji}(X)\, \mathring{g}^{kj}X^i = \langle X, \mathring{\Delta} X\rangle - (n-1)|X|^2_{\mathring{g}}$. We integrate the term $\mathring{\nabla}_k(\mathring{\nabla}^k X_i X^i \rho^{2\delta})$ and the lemma stems from the Divergence Theorem and Lemma 9.1. □

Lemma 9.3. *Let* $(M, \mathring{g})$ *be an asymptotically hyperbolic manifold and* $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ *be compactly supported on* $\mathcal{M}$*.* $\forall \delta \in \mathbb{R}$*,*

$$\begin{aligned}
&\int_{\mathcal{M}} \rho^{2\delta}\,(\mathring{U}_{kji}(X)\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k \rho}{\rho} + \mathrm{div} X \langle X, \frac{d\rho}{\rho}\rangle)\, d\mu(\mathring{g}) = \\
&\int_{\mathcal{M}} \rho^{2\delta}\,(\tfrac{n-1}{2} - \delta + o(1))[|\mathring{\nabla} X|^2_{\mathring{g}} - |X|^2_{\mathring{g}}]\, d\mu(\mathring{g}) \\
&- \int_{\mathcal{M}} \rho^{2\delta}\,[(2\delta+1)\langle X, \frac{d\rho}{\rho}\rangle^2 - |X|^2_{\mathring{g}}]\, d\mu(\mathring{g}) \\
&+ \frac{1}{2}\int_{\partial\mathcal{M}} \rho^{2\delta}\,(|\mathring{\nabla} X|^2_{\mathring{g}} \langle \frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}} - |X|^2_{\mathring{g}} \langle \frac{d\rho}{\rho}, \eta\rangle_{\mathring{g}})\, d\sigma(\mathring{g}) \\
&+ \int_{\partial\mathcal{M}} \rho^{2\delta}\,\langle X, \frac{d\rho}{\rho}\rangle \langle X, \eta\rangle\, d\sigma(\mathring{g}). \qquad (121)
\end{aligned}$$

Proof: From the definition of $\mathring{U}$ in (116), we compute

$$\mathring{U}_{kji}(X)\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k \rho}{\rho} = \mathring{\nabla}^2_{kj} X_i \mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k \rho}{\rho} - \mathring{\nabla} X(\frac{d\rho}{\rho}, X) + \mathring{\nabla} X(X, \frac{d\rho}{\rho}).$$

We integrate the term $\mathring{\nabla}_k(|\mathring{\nabla} X|^2_{\mathring{g}} \frac{\mathring{\nabla}^k \rho}{\rho} \rho^{2\delta})$ and the lemma stems from the Divergence Theorem and Lemma 9.1. □

Lemma 9.4. *Let* $(M, \mathring{g})$ *be an asymptotically hyperbolic manifold and* $X \in \mathscr{C}^\infty(T^*\mathcal{M})$ *be compactly supported on* $\mathcal{M}$*.* $\forall \delta \in \mathbb{R}$*,*

$$\begin{aligned}
&\int_{\mathcal{M}} \rho^{2\delta}(\mathring{U}_{kji}(X)\mathring{g}^{ik} X^j + 2\,|\mathring{S}(X)|^2_{\mathring{g}} - 2\delta\, \mathrm{div} X \langle X, \frac{d\rho}{\rho}\rangle)\, d\mu(\mathring{g}) = \\
&\int_{\mathcal{M}} \rho^{2\delta}\,[n-1-2\delta+o(1)]|X|^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\mathcal{M}} \rho^{2\delta}\,|\mathring{\nabla} X|^2_{\mathring{g}}\, d\mu(\mathring{g}) \qquad (122) \\
&+ \int_{\mathcal{M}} \rho^{2\delta}\, 2\delta(2\delta+1)\langle X, \frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) + \int_{\partial\mathcal{M}} \rho^{2\delta}\,(\mathring{\nabla} X(X, \eta) \\
&- 2\delta\,\langle X, \frac{d\rho}{\rho}\rangle_{\mathring{g}} \langle X, \eta\rangle_{\mathring{g}})\, d\sigma(\mathring{g})\,,
\end{aligned}$$

Proof: From the definition (116), we deduce $\mathring{U}_{kji}(X)\,\mathring{g}^{ik} X^j = \mathring{\nabla}^2_{kj} X^k X^j + (n-1)|X|^2_{\mathring{g}}$.

We integrate by parts the term $\mathring{\nabla}_k(\mathring{\nabla}_j X^k X^j \rho^{2\delta})$ and the result follows from the Divergence Theorem along with (119) and the equality $2\,|\mathring{S}(X)|^2_{\mathring{g}} = |\mathring{\nabla} X|^2_{\mathring{g}} + \mathring{\nabla}_j X_k \mathring{\nabla}^k X^j$. □

We can now prove the following proposition, crucial in the demonstration of the triviality of the adjoint kernel.

Proposition 9.5. *For any $\varepsilon > 0$, for all $\delta \in \left]-2\,,-1\right[$, there exists $R_{\varepsilon,\delta} > 0$ such that for all $R > R_{\varepsilon,\delta}$, there exists a constant $c > 0$ such that*

(123)
$$\forall X \in \mathscr{C}^\infty_c(E_R), \quad ||X||_{1,2,-\delta;E_R} \leqslant c\,(||\mathring{U}(X)||_{2,-\delta;E_R} + ||\mathring{S}(X)||_{2,-\delta;E_R}).$$

Proof: The linear combination (121) $-\frac{1}{2}$(120) $+\frac{1}{2}$(122) $+$ (33) $-\frac{1}{2}$(34) yields

$$\begin{aligned}
&\int_{\mathcal{M}} \rho^{2\delta}\, \mathring{U}_{kji}(X)\Big(\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k\rho}{\rho} - \tfrac{1}{2}\mathring{g}^{kj}X^i + \tfrac{1}{2}\mathring{g}^{ik}X^j\Big)\, d\mu(\mathring{g}) \\
&+ \int_{\mathcal{M}} \rho^{2\delta}\Big(|\mathring{S}(X)|^2_{\mathring{g}} - \mathring{S}(X)(\frac{d\rho}{\rho},\frac{d\rho}{\rho})\langle X,\frac{d\rho}{\rho}\rangle_{\mathring{g}} + (2-\delta)\mathrm{div}X\langle X,\frac{d\rho}{\rho}\rangle\Big)\, d\mu(\mathring{g}) \\
= \ &\int_{\mathcal{M}} \{-\delta^2 + (\tfrac{n-3}{2})\delta + n + 1 + o(1)\}|X|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \\
&+ \int_{\mathcal{M}} \rho^{2\delta}\, \Big\{2\delta^2 - 2\delta - (\tfrac{n+3}{2}) + o(1)\Big\}\, \langle X,\frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\, d\mu(\mathring{g}) \\
&+ \int_{\mathcal{M}} \big[\tfrac{n+1}{2} - \delta + o(1)\big]\, |\mathring{\nabla}X|^2_{\mathring{g}}\rho^{2\delta}\, d\mu(\mathring{g}) \\
&+ \int_{\partial\mathcal{M}} \rho^{2\delta}\, \tfrac{1}{2}(\delta-1)|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\, d\sigma(\mathring{g}) \\
&+ \int_{\partial\mathcal{M}} \Big(\tfrac{1}{2}|\mathring{\nabla}X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}} + \tfrac{1}{2}|X|^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}} + \tfrac{1}{2}\mathring{\nabla}X(X,\frac{d\rho}{\rho}) \\
&\qquad\qquad - \tfrac{1}{2}\mathring{\nabla}X(\frac{d\rho}{\rho},X)\Big)\, d\sigma(\mathring{g}) \\
&+ \int_{\partial\mathcal{M}} \rho^{2\delta}\, \Big((2-\delta)\langle X,\frac{d\rho}{\rho}\rangle_{\mathring{g}}\langle X,\eta\rangle_{\mathring{g}} - \tfrac{1}{2}\langle X,\frac{d\rho}{\rho}\rangle^2_{\mathring{g}}\langle\frac{d\rho}{\rho},\eta\rangle_{\mathring{g}}\Big)\, d\sigma(\mathring{g}).
\end{aligned}$$
(124)

Application on E_R: We use the same notations as in Proposition 4.4. Since $X \in \mathscr{C}^\infty_c(E_R)$, the boundary terms in (124) will only concern ∂E_R. X_n (resp. X_T) being the component of X normal (resp. tangential) to the level set of ρ, we have $X_n := \langle X,\eta\rangle_{\mathring{g}}$ and $|X|^2_{\mathring{g}} = X_n^2 + X_T^2$. The equality (124) applied to E_R produces

$$\int_{E_R} \rho^{2\delta}\, \mathring{U}_{kji}(X)\Big(\mathring{\nabla}^j X^i \frac{\mathring{\nabla}^k\rho}{\rho} - \tfrac{1}{2}\mathring{g}^{kj}X^i + \tfrac{1}{2}\mathring{g}^{ik}X^j\Big)\, d\mu(\mathring{g})$$

$$+\int_{E_R}\rho^{2\delta}\left(|\mathring{S}(X)|^2_{\mathring{g}}-\mathring{S}(X)(\eta_R,\eta_R)\,X_n+(2-\delta)\mathrm{div}XX_n\right)d\mu(\mathring{g})=$$

$$\int_{E_R}\rho^{2\delta}\,\{-\delta^2+(\tfrac{n-3}{2})\delta+n+1+o(1)\}X_T^2\,d\mu(\mathring{g})$$
$$+\int_{E_R}\rho^{2\delta}\left\{[\tfrac{n+1}{2}-\delta+o(1)]\,|\mathring{\nabla}X|^2_{\mathring{g}}+\left[\delta^2+(\tfrac{n-7}{2})\delta+\tfrac{n-1}{2}+o(1)\right]X_n^2\right\}d\mu(\mathring{g})$$
$$+\int_{\partial E_R}\rho^{2\delta}\,\tfrac{1}{2}(\delta-1)\big(X_T^2+o(1)\big)\,d\sigma(\mathring{g})+\int_{\partial E_R}\rho^{2\delta}\,\tfrac{1}{2}(2-\delta)\big(X_n^2+o(1)\big)\,d\sigma(\mathring{g})$$
$$+\int_{\partial E_R}\Big(\tfrac{1}{2}|\mathring{\nabla}X|^2_{\mathring{g}}|\eta|^2_{\mathring{g}}+\tfrac{1}{2}|X|^2_{\mathring{g}}+\tfrac{1}{2}\mathring{\nabla}X(X,\eta)-\tfrac{1}{2}\mathring{\nabla}X(\eta,X)\Big)\,d\sigma(\mathring{g}).$$

Given the following equalities

$$\begin{aligned}0\leqslant\tfrac{1}{4}|\mathring{\nabla}_iX_j\eta^i-X_j|^2_{\mathring{g}}&=\tfrac{1}{4}|\mathring{\nabla}X|^2_{\mathring{g}}|\eta|^2_{\mathring{g}}+\tfrac{1}{4}|X|^2_{\mathring{g}}-\tfrac{1}{2}\mathring{\nabla}X(\eta,X),\\0\leqslant\tfrac{1}{4}|\mathring{\nabla}_iX_j\eta^j+X_i|^2_{\mathring{g}}&=\tfrac{1}{4}|\mathring{\nabla}X|^2_{\mathring{g}}|\eta|^2_{\mathring{g}}+\tfrac{1}{4}|X|^2_{\mathring{g}}+\tfrac{1}{2}\mathring{\nabla}X(X,\eta),\end{aligned}$$

we get that for all $\varepsilon>0$, there exists $R_\varepsilon>0$ and $c_\varepsilon>>1$ such that $\forall R>R_\varepsilon$,

$$\begin{aligned}c_\varepsilon\int_{E_R}\rho^{2\delta}\left(|\mathring{U}(X)|^2_{\mathring{g}}+|\mathring{S}(X)|^2_{\mathring{g}}\right)d\mu(\mathring{g})\\\geqslant\int_{E_R}\rho^{2\delta}\,\{-\delta^2+(\tfrac{n-3}{2})\delta+n+1-\varepsilon\}X_T^2\,d\mu(\mathring{g})\\+\int_{E_R}\rho^{2\delta}\,[\tfrac{n+1}{2}-\delta-\varepsilon]|\mathring{\nabla}X|^2_{\mathring{g}}\,d\mu(\mathring{g})\\+\int_{E_R}\rho^{2\delta}\,\{\delta^2+(\tfrac{n-7}{2})\delta+\tfrac{n-1}{2}-\varepsilon\}X_n^2\,d\mu(\mathring{g})\\+\frac{1}{2}\int_{\partial E_R}\rho^{2\delta}\,(2-\delta-\varepsilon)X_n^2\,d\sigma(\mathring{g})\\+\frac{1}{2}\int_{\partial E_R}\rho^{2\delta}\,(\delta-1-\varepsilon)X_T^2\,d\sigma(\mathring{g}).\end{aligned}\tag{125}$$

- The interior X_T^2 term is non negative if $\delta\in\,]-2\,;\frac{n+1}{2}[$ and ε sufficiently small.
- The interior X_n^2 term is non negative $\forall n\geqslant 3$, $\forall\delta\in\,]\frac{n-1}{2}\,;\frac{n+1}{2}[$ and ε sufficiently small.
- The $|\mathring{\nabla}X|^2_{\mathring{g}}$ term is non negative if $\delta<\frac{n+1}{2}$ and ε sufficiently small.
- The boundary term is non negative if $\delta\in\,]1,2[$ and ε sufficiently small.

Hence, $\forall\delta\in\,]1\,;2[$, $\forall\varepsilon>0,\exists R_\varepsilon>0$ and $c_\varepsilon>>1$ such that $\forall R>R_\varepsilon$,

$$c_\varepsilon\int_{E_R}\rho^{2\delta}\left(|\mathring{U}(X)|^2_{\mathring{g}}+|\mathring{S}(X)|^2_{\mathring{g}}\right)d\mu(\mathring{g})\geqslant$$

$$\int_{E_R} \rho^{2\delta}\,[\tfrac{n+1}{2}-\delta-\varepsilon]|\mathring{\nabla}X|^2_{\mathring{g}}\,d\mu(\mathring{g})$$
$$+\int_{E_R} \rho^{2\delta}\Big(\{-\delta^2+(\tfrac{n-3}{2})\delta+n+1-\varepsilon\}X_T^2$$
$$+\{\delta^2+(\tfrac{n-7}{2})\delta+\tfrac{n-1}{2}-\varepsilon\}X_n^2\,d\mu(\mathring{g})\Big)\,d\mu(\mathring{g}).$$

□

10. The triviality of the adjoint kernel

In this section we show that the kernel of $D\mathbf{\Phi}(g,\pi)^*$ is trivial. We start with a rewriting of the system associated to the kernel.

Lemma 10.1. *Let $\delta \in \mathbb{R}$ and $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$ be a solution of $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$. Then $\xi = (N,X)$ satisfies a system of the form*

$$\begin{cases}\mathring{T}(N) = b_0\xi + b_1\mathring{\nabla}\xi,\\ \mathring{S}(X) = -\mathring{g}\tau N + b_2\xi,\end{cases} \tag{126}$$

with $b_0 \in L^2_\delta$ and $b_1, b_2 \in W^{1,2}_\delta$.

Proof: From (74) and (68), $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$ leading to

$$\begin{aligned} T(N) - (\mathrm{tr}_g T(N))g &= L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}X,\\ \mathring{S}(X) &= -\mathring{g}\tau N + W^{1,2}_\delta\xi. \end{aligned} \tag{127}$$

Using the trace of (127), we deduce

$$\begin{aligned} \mathring{T}(N) &= L^2_\delta\xi + W^{1,2}_\delta\mathring{\nabla}\xi.\\ \mathring{S}(X) &= -\mathring{g}\tau N + W^{1,2}_\delta\xi. \end{aligned} \tag{128}$$

□

We are now in position to prove the triviality of $\ker D\mathbf{\Phi_0}(g,\pi)^*$.

Theorem 10.2. *Let $\Omega \subset \mathcal{M}$ be a connected open set such that $E_R \subset \Omega$. We fix $(g,\pi) \in \mathcal{F}$. Set $\delta \in\,]-2,-1[$ and $n = 3$. If $\xi \in L^2_{-\delta}(\mathcal{T})$ verifies $D\mathbf{\Phi_0}(g,\pi)^*\xi = 0$ on Ω, then $\xi \equiv 0$ on Ω.*

Proof: From Proposition 8.6, ξ is in $W^{2,2}_{-\delta}(\mathcal{T})$. According to Lemma 10.1 and since ξ is a solution of $D\mathbf{\Phi_0}(g)^*\xi = 0$, ξ satisfies the system (126).

From (41), and the definition of $\mathring{U}$, one has

$$\mathring{U}_{kji}(X) = \mathring{\nabla}^2_{kj}X_i - Riem(\mathring{g})_{ijkl}X^l + L^2_\delta X. \tag{129}$$

Concerning (76) and using (126), we obtain

$$\begin{aligned} \mathring{U}(X) &= c_1 \mathring{\nabla}\mathring{S}(X) + L^2_\delta X \\ &= -c_1 \tau \mathring{g} \mathring{\nabla} N + L^2_\delta \xi + W^{1,2}_\delta \mathring{\nabla}\xi. \end{aligned} \tag{130}$$

We must show that a solution ξ of (126) such that $\xi = o(\rho^\delta)$ (from (12)) vanishes. Before pursuing the proof of theorem, let us recall Proposition 3.9 of [4]:

Proposition 10.3. *In dimension $n = 3$, set $\delta \leqslant 0$ and $(g, \pi) \in \mathcal{F}$. Let Ω be a connected subset of $\mathcal{M}$. If ξ satisfy $D\mathbf{\Phi}(g, \pi)^*\xi = 0$ on Ω, and $\xi \equiv 0$ on an open set $U \subset \Omega$, then $\xi \equiv 0$ on all of Ω.*

Given the previous proposition, it remains to show ξ vanishes near infinity. As for (117), we have

$$||\mathring{\nabla}^2 X||_{2,-\delta;E_R} - c\,||X||_{2,-\delta;E_R} \leqslant ||\mathring{U}(X)||_{2,-\delta;E_R}. \tag{131}$$

By the combination of (123), together with (131), the Sobolev inequality and (8) and (130), we obtain

$$\begin{aligned} ||X||_{2,2,-\delta;E_R} &\leqslant c\,(||\mathring{U}(X)||_{2,-\delta;E_R} + ||\mathring{S}(X)||_{2,-\delta;E_R}) \\ &\leqslant c\,||N||_{1,2,-\delta;E_R} + C\,(||\xi||_{\infty,-2\delta;E_R} + ||\mathring{\nabla}\xi||_{3,-2\delta;E_R}). \end{aligned} \tag{132}$$

Using Proposition 4.4, we can choose a positive small ε_0 such that (26) + ε_0(132) give

$$||\xi||_{2,2,-\delta;E_R} \leqslant c\,||\mathring{T}(N)||_{2,-\delta;E_R} + C\,(||\xi||_{\infty,-2\delta;E_R} + ||\mathring{\nabla}\xi||_{3,-2\delta;E_R}).$$

Considering Lemma 10.1 along with Sobolev inequality and (17) with $\delta \leqslant 0$,

$$\begin{aligned} ||\xi||_{2,2,-\delta;E_R} &\leqslant C\,||\xi||_{2,2,-2\delta;E_R} \\ &\leqslant C\,e^{4R\delta}\,||\xi||_{2,2,-\delta;E_R}. \end{aligned}$$

We end up with the fact that when $\delta \in]-(n+1)/2\,,-1]$, we have the estimate

$$||\xi||_{2,2,-\delta;E_R} \leqslant C\,e^{4R\delta}\,||\xi||_{2,2,-\delta;E_R}.$$

For R large enough, this implies $||\xi||_{2,2,-\delta;E_R} = 0$ so ξ vanishes on E_R, for $R >> 1$. From Proposition 10.3 we conclude that $\xi \equiv 0$ on Ω. This ends the proof of Theorem 10.2. □

For futures reference we present a non-used corollary.

Corollary 10.4. *Set* $\delta \in \,]-(n+1)/2\,,0] \setminus \{-(n-1)/2\}$, *with* $n=3$. *There exists a constant* $C>0$ *depending on* $||(g,\pi)||_{\mathcal{F}}$ *such that for* $\xi \in W^{2,2}_{-\delta}(\mathcal{T})$,

$$||\xi||_{2,2,-\delta} \leqslant C\,||P^*\xi||_{2,-\delta}. \tag{133}$$

Proof: In order to show that the kernel of P^* is finite dimensional, we apply Riesz's theorem, that every bounded subset of ker P^* is $||\,||_{2,2,-\delta}-$compact. Let $\{\xi_k\}$ be a sequence of ker P^* such that $||\xi_k||_{2,2,-\delta}=1$. Rellich's theorem tells us we can extract from $\{\xi_k\}$ a sub-sequence, also denoted by $\{\xi_k\}$, converging in $W^{1,2}_{-2\delta}$ to a limit $\bar{\xi}$. Hence, $\{\xi_k\}$ is a Cauchy sequence in $W^{1,2}_{-2\delta}$. From (73), considering that $\{\xi_k\} \in \ker P^*$, $\{\xi_k\}$ is a Cauchy sequence in $W^{2,2}_{-\delta}$, and so converges to $\bar{\xi}$ in $W^{2,2}_{-\delta}$, from the uniqueness of limits. This ends the proof of the finite dimension of ker P^*. The space ker P^* is thus a closed subspace of the Hilbert vector space $W^{2,2}_{-\delta}$. Being a finite dimensional closed subspace of a normed vector space, it splits and if we set W to be the closed complement of ker P^*,

$$W^{2,2}_{-\delta} = \ker P^* \oplus W.$$

From the same argument as in the proof of Theorem 2.3, there exists a constant $C>0$ depending on $||(g,\pi)||_{\mathcal{F}}$ such that for all $\xi \in W$,

$$||\xi||_{2,2,-\delta} \leqslant C\,||P^*\xi||_{2,-\delta}. \tag{134}$$

We conclude thanks to the triviality of ker P^* from Theorem 10.2. ☐

11. The submanifold structure

The section is devoted to the main result of the paper, namely the smooth Hilbert submanifold structure of the set of solutions to the vacuum constraint equations.

We start with a well known fact.

Lemma 11.1. *Let* X, Y *be two Banach spaces and* T *a linear operator with closed range.*

$$\begin{aligned} T : X &\to Y\,, \\ T^* : Y^* &\to X^*\,. \end{aligned}$$

then $(\mathrm{Coker} T)^* \simeq \ker T^*$, *where* $\mathrm{Coker} T = {}^{Y}\!/\!_{\mathrm{Im}\, T}.$

Proof: We define

$$\begin{array}{rcl} \psi\colon \ker T^* & \to & (\operatorname{Coker} T)^* = \mathcal{L}\left({}^{Y}\!\big/_{\operatorname{Im} T}, \mathbb{R}\right) \\ \rho & \mapsto & (\lambda\colon y + TX \mapsto \rho(y)) \ . \end{array}$$

The map λ is well defined because $\forall x \in X\,, \rho(Tx) = T^*(\rho)(x) = 0$.

The map ψ is invertible and

$$\begin{array}{rcl} \psi^{-1}\colon (\operatorname{Coker} T)^* & \to & \ker T^* \\ \lambda & \mapsto & \rho \qquad\quad \text{where } \rho(y) := \lambda(y + TX). \end{array}$$

Note that $\rho \in \ker T^*$ because $T^*(\rho)(x) = \rho(Tx) = \lambda(\bar{0}) = 0$.

Of course, the closed range of T implies that Coker$T$ is a Banach space. □

In order to prove the Theorem 1.1, we will use the implicit function theorem, so we have to show:

- ker $D\mathbf{\Phi}(g, \pi)$ splits.
- $D\mathbf{\Phi}(g, \pi)$ is surjective.

$D\mathbf{\Phi}(g, \pi)$ being a bounded operator, its kernel is closed by continuity and $T_{(g,\pi)}\mathcal{F} = W^{2,2}_\delta(\mathcal{S}) \times W^{1,2}_\delta(\widetilde{\mathcal{S}})$ can be written as a direct sum of $\ker D\mathbf{\Phi}(g, \pi)$ and its orthogonal complement $(\ker D\mathbf{\Phi}(g, \pi))^\perp$, which is always closed. Hence $\ker D\mathbf{\Phi}(g, \pi)$ splits.

The triviality of ker $D\mathbf{\Phi}(g, \pi)^*$, established in Theorem 10.2, leads to

$$(\ker D\mathbf{\Phi}(g, \pi)^*)^\perp = \mathcal{L}^*.$$

Using the classical relation

$$(\ker D\mathbf{\Phi}(g, \pi)^*)^\perp = \overline{\operatorname{Im} D\mathbf{\Phi}(g, \pi)},$$

we get

$$\overline{\operatorname{Im} D\mathbf{\Phi}(g, \pi)} = \mathcal{L}^*.$$

Thus, in order to obtain the surjectivity of $D\mathbf{\Phi}(g, \pi)$, it suffices to prove it has closed range. For that we will prove the range is the direct sum of a closed space and a finite dimensional space. To do so, we consider particular variations (h, p) of (g, π) of the form

$$\begin{cases} h_{ij} = 2\, y\, g_{ij} \\ p^{ij} = \left(2\mathcal{S}(Y)^{ij} - g^{ij}\operatorname{tr}_g \mathcal{S}(Y) - (n-1)(n-2)\tau y\, g^{ij}\right)\sqrt{g} \end{cases}, \tag{135}$$

determined from fields (y, Y^i). We define the operator

$$F(y,Y^i) = [F_0(y,Y^i), F_i(y,Y^i)] = [D\mathbf{\Phi_0}(g,\pi)(h,p), D\mathbf{\Phi_i}(g,\pi)(h,p)]. \tag{136}$$

Equations (59) and (60) provide,

$$\begin{cases} F_0(y,Y^i) &=& 2(n-1)\sqrt{g}\,[-\Delta y + ny] + (4-n)\,\mathbf{\Phi_0}(g,\pi)\,y \\ & & +2(n-2)\tau\,\mathrm{div}Y\sqrt{g} + L^2_\delta\,[y+Y] + W^{1,2}_\delta\,\mathring{\nabla}Y, \\ F_i(y,Y^i) &=& -2\sqrt{g}\,[-\Delta Y^i + (n-1)Y^i] + 2\,\mathbf{\Phi_i}(g,\pi)\,y \\ & & +W^{1,2}_\delta\,\mathring{\nabla}y + L^2_\delta\,[y+Y]. \end{cases}$$

In order to prove Fredholm properties of F, we first study its behaviour at infinity. We will use the following notion of operator asymptotic to $\mathring{\Delta}$. We will justify later on the terminology "asymptotic" used in this definition.

Definition 11.2. *We say an operator P acting on functions, of the form*

$$Pu = a^{ij}(x)\mathring{\nabla}^2_{ij}u + b^i(x)\partial_i u + c(x)u\ ,$$

is asymptotic to $\mathring{\Delta}$ with a decaying rate τ if there exists $n < q < \infty$, $\tau \leqslant 0$ and two positive constants C_1, λ such that

$$\lambda|\xi|^2_{\mathring{g}} \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2_{\mathring{g}}, \forall x \in \mathcal{M},\ \ \xi \in T\mathcal{M}.$$
$$||a^{ij} - \mathring{g}^{ij}||_{1,q,\tau} + ||b^i||_{q,\tau} + ||c||_{q/2,\tau} \leqslant C_1.$$

In our situation, we are interested in the Laplacian relative to the metric g.

Proposition 11.3. *Let $g \in \mathcal{G}^+$ with $\delta \leqslant 0$. Then the Laplacian Δ acting on functions is asymptotic to $\mathring{\Delta}$ with a decaying rate δ.*

Proof: Let us write

$$\begin{aligned} \Delta = g^{ij}\nabla^2_{ij} &= g^{ij}\mathring{\nabla}^2_{ij} + g^{ij}(\nabla_i - \mathring{\nabla}_i)\mathring{\nabla}_j \\ &= g^{ij}\mathring{\nabla}^2_{ij} - g^{ij}A_i{}^k{}_j\,\mathring{\nabla}_k. \end{aligned} \tag{137}$$

The metrics g and $\mathring{g}$ being equivalent, equation (49) directly gives

$$\lambda|\xi|^2_{\mathring{g}} \leqslant g^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2_{\mathring{g}}, \forall x \in \mathcal{M},\ \ \xi \in T\mathcal{M}.$$

Setting

$$b^k = g^{ij}A_i{}^k{}_j,$$

then $b \in W^{1,2}_\delta$ from (51). Given the Sobolev inequality, there exists a constant $C_1 > 0$ such that

$$||g^{ij} - \mathring{g}^{ij}||_{1,6,\delta} + ||b^k||_{6,\delta} \leqslant c\,(||g^{ij} - \mathring{g}^{ij}||_{2,2,\delta} + ||b^k||_{1,2,\delta}) \leqslant C_1 .$$

□

The operator $\mathcal{A} = -\Delta + n$, acting on functions, will be of great interest. It satisfies a classical elliptic estimate, valid for any weight s.

Proposition 11.4. *Let $g \in \mathcal{G}^+$ with $\delta \leqslant 0$ and $\mathcal{A} = -\Delta + n$. Let $s \in \mathbb{R}$. There exists a constant $C = C(n, p, q, s, \delta, C_1, \lambda)$ such that if $u \in L^2_s$ and $\mathcal{A}u \in L^2_s$, then $u \in W^{2,2}_s$ and*

$$||u||_{2,2,s} \leqslant C\ (||\mathcal{A}u||_{2,s} + ||u||_{2,s})\,.$$

Proof: By elliptic regularity, $u \in W^{2,2}_{loc}$ and the estimate arises from interior estimates (see [13] for example) and scaling. □

For a special interval of weights we obtain better estimates and Fredholm properties.

Theorem 11.5. *Let $g \in \mathcal{G}^+$ with $\delta \leqslant 0$ so that Δ is asymptotic to $\mathring{\Delta}$. We consider the operator $\mathcal{A} = -\Delta + n$, with $n = 3$ and a real s with $|s| < (n+1)/2$. Then $\mathcal{A} : W^{2,2}_s(\mathcal{M}) \to L^2_s(\mathcal{M})$ is bounded. Moreover, for R large enough, it satisfies the following elliptic estimate*

$$||u||_{2,2,s} \leqslant C\ (||\mathcal{A}u||_{2,s} + ||u||_{2,s;\Omega_R})\,. \tag{138}$$

In particular, $\mathcal{A}$ is a semi-Fredholm operator.

Remark 11.6. We note that this result, with the low degree of regularity of the metric, does not seem to have been established so far.

Proof: We define the following operator norm:

$$||\Delta - \mathring{\Delta}||_{op} = \sup_{\mathcal{M}}\{||(\Delta - \mathring{\Delta})u||_{2,s} : u \in W^{2,2}_s, ||u||_{2,2,s} = 1\}\ ,$$

and $||\bullet||_{op,R}$ denotes the same norm restricted to functions supported in $E_R = \mathcal{M} \setminus \Omega_R$.

If $supp(u) \subset E_R$, then from the expression (137) of Δ,

$$||(\Delta - \mathring{\Delta})u||_{2,s} \quad \leqslant \quad ||(g^{ij} - \mathring{g}^{ij})\mathring{\nabla}^2_{ij}u||_{2,s} + ||b^k\mathring{\nabla}_k u||_{2,s}$$

$$\begin{aligned} &\leqslant \sup_{E_R}\{g^{ij}-\mathring{g}^{ij}\}||\mathring{\nabla}^2_{ij}u||_{2,s}+||b^k\mathring{\nabla}_k u||_{2,s} \\ &\leqslant c\,||g-\mathring{g}||_{\infty,0;E_R}||\mathring{\nabla}^2 u||_{2,s}+||b\,\mathring{\nabla}u||_{2,s}. \end{aligned}$$

Using (8), the Sobolev embedding ($\delta \leqslant 0$),

$$||(\Delta-\mathring{\Delta})u||_{2,s} \quad \leqslant \quad c\,(||g-\mathring{g}||_{2,2,\delta;E_R}+||b||_{1,2,\delta;E_R})||u||_{2,2,s}.$$

Recalling that $||g-\mathring{g}||_{2,2,\delta}+||b||_{1,2,\delta}$ is bounded because $g \in \mathcal{G}^+$, we obtain

$$||\Delta-\mathring{\Delta}||_{op,R} = o(1) \ \text{ when } \ R \to +\infty. \tag{139}$$

This justifies *a posteriori* the terminology used in Definition 11.2.

Let χ_R be a cut-off function as in Definition 2.1

$$\chi_R \quad = \quad \begin{cases} 1 \text{ on } \Omega_{R/2} \\ 0 \text{ on } \mathcal{M}\setminus\Omega_R\,. \end{cases}$$

We decompose the function u as $u = u_0 + u_\infty$, with $u_\infty = (1-\chi_R)u$.

We consider the operator $\mathring{\mathcal{A}} = -\mathring{\Delta}+n$ acting on functions. Using Corollary 3.13 of [1] with $\lambda = n$, we obtain that $\forall |s| < (n+1)/2$, $\mathring{\mathcal{A}} : W^{2,2}_s \to L^2_s$ is a Fredholm operator and an isomorphism. So there exists a positive constant $C = C(n,s)$ such that

$$||u||_{2,2,s} \leqslant C\,||\mathring{\mathcal{A}}u||_{2,s}. \tag{140}$$

Applying (140) to u_∞,

$$\begin{aligned} ||u_\infty||_{2,2,s} &\leqslant C\,||\mathring{\mathcal{A}}u_\infty||_{2,s} \\ &\leqslant C\,||\mathcal{A}u_\infty||_{2,s}+||\Delta-\mathring{\Delta}||_{op,R}||u_\infty||_{2,2,s}. \end{aligned} \tag{141}$$

Yet $\mathcal{A}u_\infty = \mathcal{A}u - \mathcal{A}u_0 = \mathcal{A}u - \chi_{\mathbf{R}}\mathcal{A}\mathbf{u} + \chi_{\mathbf{R}}\mathcal{A}\mathbf{u} - \mathcal{A}u_0$, thus, we have

$$\begin{aligned} ||\mathcal{A}u_\infty||_{2,s} &\leqslant ||\mathcal{A}u||_{2,s}+||\chi_R\mathcal{A}u||_{2,s}+||\chi_R\mathcal{A}u-\mathcal{A}u_0||_{2,s} \\ &\leqslant C\,||\mathcal{A}u||_{2,s}+||\chi_R\mathcal{A}u-\mathcal{A}u_0||_{2,s;\Omega_R}. \end{aligned}$$

From the expression (137) of Δ,

$$\begin{aligned} \chi_R\mathcal{A}u-\mathcal{A}u_0 &= -u\mathcal{A}\chi_R + n\chi_R u + 2g^{ij}\partial_i u\partial_j\chi_R \\ &= 2g^{ij}\partial_i u\,\partial_j\chi_R + (g^{ij}\mathring{\nabla}^2_{ij}\chi_R + b^i\partial_i\chi_R)u\,, \end{aligned}$$

leading to

$$||\chi_R\mathcal{A}u-\mathcal{A}u_0||_{2,s;\Omega_R}\leqslant c\,||u||_{1,2,s;\Omega_R}.$$

Finally, we arrive at

$$||\mathcal{A}u_\infty||_{2,s}\leqslant C\,\left(||\mathcal{A}u||_{2,s}+||u||_{1,2,s;\Omega_R}\right).$$

Replacing (141) and considering (139), we obtain for R large enough

$$(142)\qquad ||u_\infty||_{2,\delta}\leqslant||u_\infty||_{2,2,s}\leqslant C\,\left(||\mathcal{A}u||_{2,s}+||u||_{1,2,s;\Omega_R}\right).$$

Using (142) and the fact that on Ω_R, $|u_0|_{\mathring{g}}\leqslant|u|_{\mathring{g}}$

$$\begin{array}{rcl}||u||_{2,s}&\leqslant&||u_\infty||_{2,s}+||u_0||_{2,s}\\&\leqslant&C\,\left(||\mathcal{A}u||_{2,s}+||u||_{1,2,s;\Omega_R}\right)+||u||_{2,s;\Omega_R}.\end{array}$$

Thanks to the Ehrling inequality (11),

$$\begin{array}{rcl}||u||_{2,s}&\leqslant&C\,\left(||\mathcal{A}u||_{2,s}+||u||_{2,s;\Omega_R}\right)+\varepsilon||u||_{2,2,s;\Omega_R}\\&\leqslant&C\,\left(||\mathcal{A}u||_{2,s}+||u||_{2,s;\Omega_R}\right)+\varepsilon||u||_{2,2,s}.\end{array}$$

We conclude by combining this inequality with Proposition 11.4. □

We need a similar result for an operator acting on 1-forms. Let us define $\mathring{B}=-\mathring{\Delta}+n-1$, be a Laplacian acting on 1-forms.

Theorem 11.7. *We assume $\delta\leqslant 0$ and $g\in\mathcal{G}^+$. Set $B=-\Delta+n-1$ and suppose that $|s|<(n+1)/2$.*

Then $B:W^{2,2}_s(T^*\mathcal{M})\to L^2_s(\mathcal{T}^*\mathcal{M})$ is bounded. Furthermore, for R large enough it satisfies

$$(143)\qquad ||Y||_{2,2,s}\leqslant C\,\left(||BY||_{2,s}+||Y||_{2,s;\Omega_R}\right).$$

In particular, B is a semi-Fredholm operator.

Proof: From Proposition E of [16], the indicial radius of $\mathring{B}$ is $(n+1)/2$ and by Theorem C of [16], $\forall|s|<(n+1)/2$, $\mathring{B}:W^{2,2}_s\to L^2_s$ is a Fredholm operator. By Corollary 3.13 of [1], $\mathring{B}$ is an isomorphism for the same range of s. So there exists a positive constant $C=C(n,s)$ such that

$$(144)\qquad ||Y||_{2,2,s}\leqslant C\,||\mathring{B}Y||_{2,s}.$$

The proof is now nearly identical to the one of Theorem 11.5 with (144) replacing (140). □

We are ready now to prove the Fredholm property of the operator F associated with our special variations.

Theorem 11.8. *Let δ be a real in $]-(n+1)/2,0]$ with $n=3$. Then the operator $F : W^{2,2}_{\delta}(\mathcal{M}) \times W^{2,2}_{\delta}(\mathcal{TM}) \to L^2_\delta(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}) := \mathcal{L}^*$ is bounded. Furthermore, for R large, it satisfies*

$$||(y,Y)||_{2,2,\delta} \leqslant C\left(||F(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||(y,Y)||_{2,\delta;\Omega_R}\right). \tag{145}$$

In particular, F is a semi-Fredholm operator.

Proof: Starting from the definition of F, the triangle inequality together with (54) and the Sobolev embedding directly yield

$$||F(y,Y)||_{2,\delta} \leqslant C\, ||(y,Y)||_{2,2,\delta},$$

where C is a constant depending on $\mathring{g}$ and $||g||_{\mathcal{F}}$.

Hence F is a bounded (continuous) operator. Plugging the expression of $F_0(y,Y^i)$ in (138) and using the Hölder inequality (8), (15), the Ehrling inequality (11) along with the Sobolev embedding and $\mathbf{\Phi_0}(g,\pi) \in L^2_\delta$,

$$\begin{aligned} ||y||_{2,2,\delta} &\leqslant C\left(||-\Delta y + ny||_{2,\delta} + ||y||_{2,\delta;\Omega_R}\right) \\ &\leqslant C\left(||F_0(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||Y||_{2,2,\delta} + ||y||_{2,\delta;\Omega_R}\right). \end{aligned} \tag{146}$$

Plugging the expression of $F_i(y,Y^i)$ in (143) and using the Hölder inequality (8), (15),the Ehrling inequality (11) along with the Sobolev embedding and $\mathbf{\Phi_i}(g,\pi) \in L^2_\delta$,

$$\begin{aligned} ||Y||_{2,2,\delta} &\leqslant C\left(||-\Delta Y + (n-1)Y||_{2,\delta} + ||Y||_{2,\delta;\Omega_R}\right) \\ &\leqslant C\left(||F_i(y,Y)||_{2,\delta} + ||(y,Y)||_{2,0} + ||Y||_{2,\delta;\Omega_R}\right). \end{aligned} \tag{147}$$

Finally, combination of (146) and (147) gives (145). For all $\delta \in\,]-(n+1)/2,0]$, the estimate (145) verified by F is analogous to the one of Theorem 2.3 and by a similar proof, we show F is semi-Fredholm. □

Now F and its adjoint F^* have similar structure

$$F^* : L^2_{-\delta}(\mathcal{T}) \to W^{-2,2}_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}).$$

Let $\widetilde{F^*}$ be the restriction of F^* defined as follows

$$\widetilde{F^*} : W^{2,2}_{-\delta}(\mathcal{T}) \to L^2_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}).$$

The following is similar to Theorem 11.8 but for $\widetilde{F^*}$:

Theorem 11.9. *Let $\delta \in]-(n+1)/2, 0]$ with $n = 3$. Then the operator $\widetilde{F^*} : W^{2,2}_{-\delta}(\mathcal{T}) \to L^2_{-\delta}(\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M})$ is bounded. Furthermore, for large R, it satisfies*

(148)
$$||(y,Y)||_{2,2,-\delta} \leqslant C\left(||\widetilde{F^*}(y,Y)||_{2,-\delta} + ||(y,Y)||_{2,-2\delta} + ||(y,Y)||_{2,-\delta;\Omega_R}\right).$$

In particular, $\widetilde{F^*}$ is a semi-Fredholm operator.

Proof: The definition of $\widetilde{F^*}$, the Triangle inequality together with (54) and the Sobolev embedding directly yield

$$||\widetilde{F^*}(y,Y)||_{2,-\delta} \leqslant C\,||(y,Y)||_{2,2,-\delta}.$$

Plugging the expression of $\widetilde{F^*_0}(y,Y^i)$ (formally identical to $F_0(y,Y^i)$) in (138) and using the Hölder inequality (8), (15), the Ehrling inequality (11) along with the Sobolev embedding and $\mathbf{\Phi_0}(g,\pi) \in L^2_\delta$,

$$\begin{aligned} ||(y,Y)||_{2,2,-\delta} &\leqslant C\,(||-\Delta y + ny||_{2,-\delta} + ||N||_{2,-\delta;\Omega_R}) \\ &\leqslant C\left(||\widetilde{F^*_0}(y,Y)||_{2,\delta} + ||(y,Y)||_{2,-2\delta} + ||Y||_{2,2,-\delta} + ||y||_{2,-\delta;\Omega_R}\right). \end{aligned} \tag{149}$$

The same argument using equation (143) gives

$$\begin{aligned} ||Y||_{2,2,-\delta} &\leqslant C\,(||-\Delta Y + (n-1)Y||_{2,-\delta} + ||Y||_{2,-\delta;\Omega_R}) \\ &\leqslant C\left(||\widetilde{F^*_i}(y,Y)||_{2,-\delta} + ||(y,Y)||_{2,-2\delta} + ||Y||_{2,-\delta;\Omega_R}\right). \end{aligned} \tag{150}$$

Finally, combining (149) with (150) gives (148).

Similarly to F, for all $\delta \in]-(n+1)/2, 0]$, $\widetilde{F^*}$ is a semi-Fredholm operator. □

We are now in possession of all the tools necessary to finish the proof of Theorem 1.1.

By elliptic regularity, $\ker F^* = \ker \widetilde{F^*}$ is also finite dimensional. If we apply Lemma 11.1 to F, we get

$$(\mathrm{Coker} F)^* \simeq \ker F^*.$$

So $(\mathrm{Coker} F)^*$ is finite dimensional and then $(\mathrm{Coker} F)^* \simeq \mathrm{Coker} F$. Thus we have the isomorphism

$$\mathrm{Coker} F = \mathcal{L}^*/_{\mathrm{Im} F} \simeq \ker F^*.$$

The operator F satisfies

$$\mathrm{Im}F \subset \mathrm{Im}D\mathbf{\Phi}(g,\pi) \subset \mathcal{L}^*.$$

Let π be the canonical projection:

$$\pi : \mathcal{L}^* \to \mathcal{L}^*\!\big/_{\mathrm{Im}F}.$$

$\pi(\mathrm{Im}D\mathbf{\Phi}(g,\pi))$ is closed, being the subspace of a finite dimensional vector space. $\mathrm{Im}(D\mathbf{\Phi}(g,\pi))$ is closed, being the inverse image of a closed set by a continuous map. This ends the proof of the manifold structure of $\mathcal{C}$, as a smooth submanifold of $\mathcal{F}$. In fact, all level sets of $\mathbf{\Phi}(g,\pi)$ are smooth submanifolds of F. □

Conclusion

We have adapted here the Hilbert manifold structure of Bartnik [4] to the AH setting. Concerning the rest of this Bartnik paper, we also obtain some AH equivalent in [12] such as his Theorems 4.1, 5.1, 5.2, 6.1. We have the convergence for the definition of the mass but its geometric invariance is not complete actually (corresponding to his Theorem 4.7) and use a conjecture there. We either need an equivalent to the Theorem 3.1 of [3] or an adaptation of the proof of [7] using a poor regularity to prove that. Finally we mention an Hilbert structure on higher dimensional compact manifolds in [10], a first step before an AF or AH version.

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Erwann Delay
Avignon Université
Laboratoire de Mathématiques d'Avignon (EA 2151)
301 rue Baruch de Spinoza
F-84916 Avignon
France
E-mail: erwann.delay@univ-avignon.fr
url: `https://erwanndelay.wordpress.com`

Jérémie Fougeirol
Avignon Université
Laboratoire de Mathématiques d'Avignon (EA 2151)
301 rue Baruch de Spinoza
F-84916 Avignon
France
E-mail: jeremie.fougeirol@free.fr