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A method in deformation theory

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This paper is dedicated to David Mumford, in gratitude for everything I learned from him

Abstract: We describe a method in deformation theory that David Mumford and the present author developed in 1966.

Keywords: Deformation theory, finite group schemes, abelian varieties, Newton polygons, automorphisms of algebraic curves.

Introduction

0.1. A method in deformation theory

Suppose we study a problem in lifting theory, or we want a deformation with specific properties of a generic fibre. If the universal deformation theory as in [30] applies, which is the case in many situations, it seems the problem is (almost) solved: "just" inspect properties of all fibers; in several cases this works well, for examples if obstructions vanish and the lifting problem hence is formally smooth. However in more difficult problems we can encounter situations not easily solved in this way. This was exactly what David Mumford and I experienced in 1966/1967. In our discussions then an idea came up. Now, half a century later we do not know how this originated. I am used to indicate this idea by "*the Mumford method*"; I cannot find any earlier case of this construction.

0.2. Two steps

The basic question is to start with N_0 and find a lifting or a deformation with prescribed properties; please see our four examples for specific cases. It might be that just an abstract deformation theory does not give an obvious

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solution. We discuss a method that solves such a problem in several cases. It consists of two steps:
(I) deform N_0, the central fiber studied, to a "better" situation M_0; we have to define what better means; usually we have make a non-canonical choice of such a deformation; in most cases this is the hard part of the proof;
(II) apply general theory to solve the problem at hand for this "good" situation where lifting of M_0 is known for abstract reasons.

0.3. We reproduce the basic idea explained in [27], Lemma 2.1

Suppose given rings and homomorphisms $\kappa \leftarrow R \subset k \overset{\rho}{\longleftarrow} \Lambda$ and N_0 over κ, and M_0 over R, and M over Λ such that

$$N_0 = M_0 \otimes_R \kappa, \quad M_0 \otimes k = M \otimes_\Lambda k :$$

$$\begin{array}{ccccccc}
N_0 & \longrightarrow & M_0 & \longleftarrow & M_0 \otimes k & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\kappa) & \longrightarrow & \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(\Lambda).
\end{array}$$

Let $\Lambda' := \{x \in \Lambda \mid x_0 = \rho(x) \in R\}$; in this situation N_0 lifts to $\Lambda' \to R \to \kappa$.

Usually κ is a field (of positive characteristic in our situations, preferably perfect), M_0/R is a deformation to an integral domain R (of equal characteristic in our situations), k is an algebraic closure of the field of fractions of R, and the integral domain $\Lambda \to k$ is for example (a slightly ramified extension of) the ring of infinite Witt vectors (in three of our four examples). A ring of mixed characteristic will be an integral domain Λ of characteristic zero with a homomorphism $\Lambda \to K \supset \mathbb{F}_p$.

0.4. We discuss this method in the following situations

- *Lifting finite group schemes,* 1968, see Section 1.
- *Lifting polarized abelian varieties,* 1980, see Section 2.
- *A conjecture by Grothendieck,* 1970, 2000-2001, see Section 3.
- *Lifting an algebraic curve with an automorphism,* 1985, 1995, 1989, 1999, 2014, see Section 4.

This note does not contain new material.

In each of the situations it was not clear how abstract considerations, as in deformation theory, could give an answer to the problem posed in general. Particular situations are solved by just pure thought, Step 2. In Step 1 we deform to such a situation, where the next step applies. The combination of the two steps can give insight and an answer asked for.

0.5. Lifting questions

Definition. If we have an object N_0/κ (with certain properties) over $\kappa \supset \mathbb{F}_p$. We say N/Γ is a lifting of N_0 if $\Gamma \to \kappa$ is a mixed characteristic integral domain, and N/Γ (with certain properties) satisfies $N_0 \cong N \otimes_\Gamma \kappa$. Compare with 5.1.3.

1. Lifting finite group schemes

1.1. We start with a question

Question. Let $\kappa \supset \mathbb{F}_p$ be a field and let N_0/κ be a *finite group scheme.* Does there exist a lifting to characteristic zero?

1.1.1. Example. Here is an example of a non-commutative finite group scheme of order p^2. We define N_0 over $R \supset \mathbb{F}_p$ by

$$N_0(C) = \begin{pmatrix} \rho & \tau \\ 0 & 1 \end{pmatrix}, \quad \rho^p = 1, \quad \tau^p = 0$$

for any commutative R-algebra C. We can easily write out the coordinate ring of G, and the group axioms. As a group scheme in characteristic zero is reduced, see [1], [17], we see that any flat lifting N/Γ would give a non-commutative constant group scheme over $k = \overline{k} \supset \Gamma \supset \mathbb{Z}$. However by elementary group theory we know there does not exist a non-commutative group of order p^2.

Conclusion. *There exist* (non-commutative) *finite group schemes that cannot be lifted to characteristic zero.*

1.2. Commutative finite group schemes can be lifted

Theorem. *Any* commutative *finite group scheme N_0/κ can be lifted to a commutative finite flat group scheme in mixed characteristics.* See [27].

Example. Consider $N_0 = \alpha_p$. This finite group scheme does not admit a lift to an unramified mixed characteristics domain. For example this can be seen by using the classification as explained in [31]: finding N/Γ, with $\rho : \Gamma \to \kappa$ is equivalent by finding $a, b \in \Gamma$ with $ab = (\text{unit})\cdot p$ and $\rho(a) = 0 = \rho(b)$. Also see 5.1.3.

1.3. A sketch of the proof

We study deformations and liftings $N = \mathrm{Spec}(E)$ of $N_0 = \mathrm{Spec}(E_0)$ by fixing a base for the algebra E, free over a local base ring. In this way we obtain a prorepresentable functor. We try to see whether this prorepresenting ring has a characteristic zero fiber.

Step (I) (the hard part). Choose a deformation to $M_0 = \mathrm{Spec}(E_1)$ over an integral characteristic p domain R such that over the perfection of field of fractions $K \supset Q(R)$ we know that $M_0 \otimes K$ is a direct sum of a local-etale M_0' and an etale-local finite group scheme M_0''; see [27, pp. 319–331],

$$M_0 \otimes K = M_0' \times M_0''; \quad M_0' = (M_0 \otimes K)_{\mathrm{loc,et}} \quad M_0'' = (M_0 \otimes K)_{\mathrm{et,loc}}.$$

Step (II) (pure thought). As finite etale algebras admit a unique lifting, "une équivalence remarkable", EGA4_4.18.1, see [3], we see that M_0'' can be lifted to $W_\infty(K)$. Cartier duality gives $(M_0')^D$, an etale-local group scheme; we see this can be lifted to $W_\infty(K)$, and dualizing back we obtain a lift of M_0'. We see $M_0' \times M_0'' = M_0 \otimes K$ can be lifted. Hence Step(I) and Step(II) give a proof of 1.2.

2. Lifting polarized abelian varieties

2.1. Deformations

Question. *Suppose (A_0, μ) is a polarized abelian variety over $\kappa \supset \mathbb{F}_p$. Does there exist a lifting to characteristic zero?*

Typically this is a question, where the deformation theory is clear, but where in general deciding whether the deformation space does contain a characteristic zero fiber is something not easily seen by abstract methods in the most general case.

Classically the problem of (deformations in characteristic zero) was fully answered in the Kuranishi and in the Kodaira-Spencer theory. Then Schlessinger, Grothendieck and Mumford showed the way to formulated these problems in terms of algebraic geometry over an arbitrary base scheme. Some results (we start with a perfect field κ or an algebraically closed field k):

- Serre-Tate theory gives an equivalence between formal deformations of abelian varieties and of their p-divisible groups, see [9], [11].
- As ordinary p-divisible groups can be lifted, we conclude any polarized *ordinary* abelian variety can be lifted, and see [8].

- Grothendieck showed obstructions vanish: the deformation space of an abelian variety of dimension g (obtaining formal abelian schemes) is formally smooth,
- lifting along a polarization, this space is given by $g(g-1)/2$ equations, as Mumford showed,
- by Chow-Grotendieck we know a polarized formal abelian scheme can be algebraized, and
- Grothendieck proved that *principally* polarized abelian varieties have a formally smooth deformation space, hence liftability in this case; for all this and for references, see [18], 2.2 $\sim$ 2.4.
- What can be said about liftability of a polarized abelian variety (and hence about the dimension of components of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$)?

2.2. Lifting polarized abelian varieties

Theorem (Mumford, Norman-Oort)**.** *For any polarized abelian variety (A_0, μ) there exists a lifting to characteristic zero.*

See [14]. Also see [13] for the case $p > 2$.

Corollary. *Any irreducible component of $\mathcal{A}_g \otimes \mathbb{Z}_p$ is a complete intersection of relative dimension $g(g+1)/2$.*

As so often we encounter in mathematics that a good question asks for a general idea in order to obtain a solution. Mumford invented the technique of *displays*: instead of the step-by-step method of Schlessinger,

write down an equi-characteristic-p deformation of the matrix of the Frobenius morphism of a p-divisible group;

require this deformation still divides p;

conclude the deformation indeed gives a p-divisible group. This results (after a difficult computation) in the fact that

(Step 1) any polarized abelian variety (A_0, μ) admits a deformation in equi-characteristic-p to an ordinary polarized abelian variety; see [14, pp. 423–430].
(Step 2) By Serre-Tate theory we know the ordinary case is solved.

By 0.3 we conclude a proof of this theorem.

3. A conjecture by Grothendieck

In this section we describe deformations in characteristic p. In the influential paper [10] by Manin access is obtained to theory of abelian varieties in characteristic p via the study of p-divisible groups. A classification theory via modules over a certain ring, initiated by Dieudonné and further developed by Manin, classifies p-divisible groups over a perfect group. For coprime integers $m, n \in \mathbb{Z}_{\geq 0}$ a p-divisible group $G_{m,n}$ is defined, it is simple, of dimension m and the dual $G^t_{m,n} \cong G_{n,m}$ is of dimension n; the "Frobenius slopes" on this p-divisible group $G_{m,n}$ are equal to $m/(m+n)$.

3.1. A classification of *p*-divisible groups

Theorem/Notation (Manin)**.** Any p-divisible group X over an algebraically closed field $k \supset \mathbb{F}_p$ is isogenous with a product

$$X \sim_k \prod_i \; G_{m_i,n_i}, \quad \gcd(m_i, n_i) = 1.$$

Notation. The Newton polygon $\mathcal{N}(X)$ in this case is the lower convex polygon consisting of slopes $m_i/(m_i+n_i)$ with multiplicities $m_i + n_i$, the slopes arranged in non-decreasing order; these are called "the Frobenius slopes" of X.

We obtain a bijective map

$$\{X \mid d(X) = d, \quad h(X) = h\}/\sim_k \;\stackrel{\sim}{\longrightarrow}\; \{\mathrm{NP} \mid d(\mathcal{N}) = d, h(\mathcal{N}) = h\}.$$

For a p-divisible group Y over a field $K \supset \mathbb{F}_p$ we define $\mathcal{N}(Y)$ as the Newton polygon of $Y \otimes k$ for any $k \supset K$. We say a Newton Polygon is *isoclinic* if all slopes are equal, i.e. the polygon is straight line segment.

3.2. A partial ordering

We write $\zeta' \prec \zeta$ if these Newton Polygons have the same end points, i.e. the same height and dimension, and every point on ζ' is on or above ζ; in this case we say "ζ' *is above* ζ. In 3.3 we see an explanation for this choice of terminology and partial ordering.

For $d(\zeta) = d$ and $c(\zeta) = c = h - d$ the isoclinic Newton Polygon of slope $d/(d+c)$ is the minimal in this ordering. For symmetric Newton Polygons the supersingular $\sigma = \sigma_g = \mathcal{N}((G_1, 1)^g)$, isoclinic of slope $1/2$, is the minimal one appearing for abelian varieties of dimension g.

For an abelian variety A we write $\mathcal{N}(A) := \mathcal{N}(X[p^\infty])$. This is a symmetric Newton Polygon: for every slope s appearing in $\xi = \mathcal{N}(A)$, the slope $1-s$ appears with the same multiplicity.

3.3. Newton Polygons go up under specialization

Theorem (Grothendieck, Katz). *If $\mathcal{X} \to S$ is a p-divisible group over en irreducible scheme $S/\mathbb{F}_p$, with $0 \in S$ and generic point $\eta \in S$. Then*

$$\mathcal{N}(\mathcal{X}_0) \prec \mathcal{N}(\mathcal{X}_\eta);$$

see [5, page 150; [7], Th. 2.3.1, page 143].

Grothendieck asked whether the converse holds:

3.4. A conjecture by Grothendieck

Conjecture/Theorem. *Work in characteristic p. Suppose given a p-divisible group X_0/κ with Newton Polygon $\mathcal{N}(X_0) = \zeta'$ and suppose given a Newton Polygon $\zeta \succ \zeta'$*, i.e. ζ' is "above" ζ. *There exists a p-divisible group over en irreducible scheme $S/\mathbb{F}_p$, with $0 \in S$ and generic point $\eta \in S$ with $\mathcal{X}_0 = X_0$ and $\mathcal{N}(\mathcal{X}_\eta) = \zeta$*, i.e. the partial ordering is realized by a deformation of X_0,

$$\zeta' = \mathcal{N}(\mathcal{X}_0) \prec \mathcal{N}(\mathcal{X}_\eta) = \zeta.$$

See [6], [21], [23]. An analogous result for *principally* polarized p-divisible groups or principally polarized abelian varieties holds.

A systematic way of finding counterexamples in the non-principally polarized cases is described in [24].

Comment. For a symmetric Newton Polygon ξ we write

$$W_\xi = \{[(A,\lambda)] \in \mathcal{A}_g \mid \mathcal{N}(A) = \xi\} \subset \mathcal{A}_g := \mathcal{A}_{g,1} \otimes \mathbb{F}_p.$$

The conjecture by Grothendieck asks which Newton polygons appear in

$$\partial(W_\xi) = \left((W_\xi)^{\mathrm{Zar}} \setminus W_\xi\right) \subset \mathcal{A}_g.$$

The answer is that all $\xi' \precneqq \xi$ appear:

Corollary.

$$(W_\xi)^{\mathrm{Zar}} \quad = \quad \cup_{\xi' \prec \xi} \; W_{\xi'} \quad \subset \quad \mathcal{A}_{g,1} \otimes \mathbb{F}_p.$$

3.5. Two steps in proving this conjecture by Grothendieck

For a group scheme G over a field $\kappa \supset \mathbb{F}_p$ we write

$$a(G) := \dim_K (\mathrm{Hom}(\alpha_p, G \otimes K))$$

where K is a perfect field containing κ. For a p-divisible group over a perfect field this is the number of generators of the local-local part of the Dieudonné module. An abelian variety is ordinary if and only if $a(A) = 0$.

Step 1. For a p-divisible group X_0 there exists a deformation with *constant Newton Polygon* with generic fiber $a(X_0) \leq 1$.

Comments. This result we find in [6], 5.12 + [23], 2.8; the proof, a combination of an abstract method "purity" plus combinatorial arguments, after several years has not been simplified.

An analogous results holds for principally polarized p-divisible groups and for principally polarized abelian varieties. For polarized p-divisible group and for polarized abelian varieties the analogous statement is not correct.

In many situations $a(X) > 1$ gives a singular moduli point, and $a(X) \leq 1$ gives a regular moduli point; the statement in Step 1 can be seen as a method to "move out of the singularity".

It does not come as a surprise that in general there are many deformations achieving what we want, in many cases the deformation chosen is not unique nor canonical.

Step 2. For a p-divisible group X_0 with $a(X_0) = 1$ the deformation theory is easier to handle. A variant of the Cayley-Hamilton theorem, "a matrix is a zero of its own characteristic polynomial" can be formulated for the display-matrix of Frobenius, proved for the $a = 1$ case, and equations for all Newton Polygon strata in the deformation space can be read off easily in this situation, see [21]. We conclude: this Grothendieck conjecture holds for the $a(X_0) = 1$ situation. Combining these two steps, as in 0.3, gives a proof for Theorem 3.4.

4. Lifting an algebraic curve with an automorphism

Let C_0 be an algebraic curve (non-singular, absolutely irreducible and proper) over a field $\kappa \subset \mathbb{F}_p$ of genus $g > 1$, and $H \subset \mathrm{Aut}(C)$ a subgroup. In general there is no chance that the pair (C_0, H) can be lifted to characteristic zero. Indeed, if $\#(H) > 2g + 2$ the Hurwitz bound in characteristic zero shows this case cannot be lifted. More subtle counterexamples can be given, e.g. see [19, I.2].

However:

Conjecture. Let C_0 be an algebraic curve over a field $\kappa \subset \mathbb{F}_p$, and $\varphi \in \mathrm{Aut}(C_0)$, i.e. consider the case H is a cyclic subgroup. In [20], 4_A it is conjectured that (C_0, φ) can be lifted to characteristic zero.

This is clear for automorphisms of order prime to p. The case that the order of φ is not divisible by p^3 was proved earlier, see [29], [4]; for survey see [15]. Finally:

4.1. An answer to the question lifting an automorphism of an algebraic curve

Theorem (Obus-Wewers, Pop)**.** *Any* (C_0, φ) *can be lifted to characteristic zero.*

See [16], [28]. For a survey see [15]: Andrew Obus – *Lifting curves with automorphisms.*

The proof starts with (C_0, φ) defined over an algebraically closed field. In [16] a class of coverings $C_0 \to D_0 = C_0/\langle\varphi\rangle$ was shown to liftable (Step 2); here Step 2 is far from easy. In [28] it was shown that any (C_0, φ) can be deformed (Step 1) to a situation as in [16]. Hence the general case follows by 0.3.

5. Some questions

5.1. Normalizing local rings

The method sketched in this note proves some results. However because of limitations other aspects can remain unclear. We describe two types of such questions. At first two easy examples.

5.1.1. Example. Suppose $L' \subset L$, an inclusion of fields, $\Lambda = L[[t]]$, with $\rho : L[[t]] \to L$ given by $\rho(t) = 0$, and

$$\Lambda' = \{\sum a_j t^i \mid a_j \in L, \quad \rho(t_0) \in L'\} \subset \Lambda.$$

This local ring has residue field L'; the normalization $(\Lambda')^{\sim}$ equals

$$(\Lambda')^{\sim} =: \Lambda'' = \{\sum a_j t^i \mid a_j \in L, \quad \rho(t_0) \in K\} \subset \Lambda,$$

where K is the algebraic closure of L' inside L. We see that normalizing the local ring Λ' may extend the residue class field in case $L' \neq K \subset L$.

5.1.2. Example. Suppose $\mathbb{F}_p \subset \kappa \subset \kappa[[t]] \subset K$, where K is perfect, $\rho : \Lambda := W_\infty(K) \to K$ the residue class map with $\tau : K \to W_\infty(K)$ the Teichmüller lift, and

$$\Lambda' = \{\sum a_j p^j \mid a_j \in K, \quad \rho(t_0) \in \kappa\}.$$

The residue class map $\Lambda' \to \kappa$ factors over

$$\Lambda' \longrightarrow \Lambda'/(\tau(t)^2 - p) \longrightarrow \kappa.$$

We see that the unramified situation $\Lambda \to K$ gives rise to a ramified situation $\Lambda'/(\tau(t)^2 - p) \to \kappa$.

5.1.3. Definition. Suppose given N_0/κ (with certain properties). We say $N \to \mathrm{Spec}(D)$ is a *strong lifting* if $D \to \kappa$ where D is a mixed characteristics integral domain that is a discrete valuation ring with $N \otimes_D \kappa = N_0$. Compare with 0.5.

5.2. Ramification

In general the method described in this note does not determine whether ramification is needed in the lifting process. We describe the simplest example illustrating this. Suppose given $\alpha_p = N_0$ over a finite field κ. The method gives

$$\kappa \twoheadleftarrow R = \kappa[[t]] \subset K \twoheadleftarrow \Lambda,$$

a deformation $\mathcal{N}/R$ with $\mathcal{N} \otimes \kappa = N_0$ and $\mathcal{N} \otimes K \cong \mu_{p,K} = M_0 \otimes K$, and a lifting to the mixed characteristics integral domain Λ, resulting in a lifting

$$N \to \mathrm{Spec}(\Lambda'), \quad N \otimes \kappa \cong N_0.$$

We see that the process does produce a lifting, as is the case for every commutative finite group scheme over any $\kappa \supset \mathbb{F}_p$, but we do not see how to derive information about ramification. In this particular case $N_0 = \alpha_p$ we know that we have a strong lifting to any ramified situation by the classification in [31].

5.2.1. Question. How much ramification is needed in case N_0 is a finite commutative group scheme?

The group scheme $N_{0,n} = \mathbb{G}_{a,\mathbb{F}_p}[F^{n-1}]$ can be lifted to $W[\sqrt[n]{p}]$ for every $n > 1$. We expect that that N_0 cannot be lifted to a mixed characteristics domain with smaller ramification.

We expect that the results in Section 1 *do not admit a bound on the ramification* if all finite commutative group schemes are considered.

5.2.2. In [13] we find that for $p > 2$ a polarized abelian variety (A_0, μ) can be lifted with ramification $e \leq p - 1$. Moreover we find there an example by Ogus of a polarized superspecial abelian surface that cannot be lifted to an unramified mixed characteristics ring.

5.2.4. Consider (C_0, φ) and ask how much ramification is needed. As an example: see [26], Chapter 2, Question 8.5: it seems unknown whether local $\mathbb{Z}/p$ coverings can be lifted to $W[\zeta_{p^n}]$.

We expect that results in Section 4 *do not admit a bound on the ramification* if all curves with an automorphism are considered

If we start with an algebraically closed field $\kappa = \overline{\kappa}$ a positive answer to the lifting question also gives a positive answer to the strong lifting question. What can be said if we start over an arbitrary field $\kappa \supset \mathbb{F}_p$?

5.3.

Question: residue class field extension. The method provides a positive answer to the lifting question in the four cases studied. However we have seen in 5.1.2 we have a priori no control whether a residue class field extension is necessary in order to obtain a strong lifting (a lifting to a normal domain). Can we determine in all four cases mentioned whether and how much ramification is needed?

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