Pure and Applied Mathematics Quarterly
Volume 17, Number 3, 1005–1024, 2021

The complex Monge-Ampère equation with a gradient term

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Dedicated to Professor D.H. Phong on the occasion of his 65th birthday

Abstract: We consider the complex Monge-Ampère equation with an additional linear gradient term inside the determinant. We prove existence and uniqueness of solutions to this equation on compact Hermitian manifolds.

1. Introduction

Let M be a compact complex manifold of complex dimension n. When M admits a Kähler metric $g = (g_{i\bar{j}})$, Yau [35] proved the now classic result that the complex Monge-Ampère equation

$$\det(g_{i\bar{j}} + u_{i\bar{j}}) = e^F \det(g_{i\bar{j}}), \quad (g_{i\bar{j}} + u_{i\bar{j}}) > 0, \tag{1.1}$$

admits a unique solution u with $\sup_M u = 0$, as long as F is normalized so that $(e^F - 1)$ has zero integral. Equivalently, one can prescribe the volume form of a Kähler metric within a given Kähler class.

Yau's result has been extended and built on in various ways. Modulo adding a constant to F, the equation (1.1) can be solved for g Hermitian (by work of Cherrier [6] and the authors [30], see also [16, 29]) and for g almost Hermitian (Chu-Tosatti-Weinkove [7]). Fu-Wang-Wu [11, 12] considered the Monge-Ampère equation obtained by taking the determinant of the $(n-1, n-1)$ form

$$\omega^{n-1} + \sqrt{-1}\partial\overline{\partial}u \wedge \omega^{n-2}.$$

Received May 11, 2019.

*Partially supported by NSF grants DMS-1610278 and DMS-1903147. Part of this work was done while the first-named author was visiting the Center for Mathematical Sciences and Applications at Harvard University, which he thanks for the hospitality.

†Partially supported by NSF grant DMS-1709544.

This is the natural equation on compact manifolds associated to Harvey-Lawson's notion of $(n-1)$-plurisubharmonicity [18], and was solved for ω Hermitian by the authors [31, 33]. Building on this work, Székelyhidi-Tosatti-Weinkove [27] proved existence of solutions for Monge-Ampère equation associated to

$$\omega^{n-1} + \sqrt{-1}\partial\overline{\partial}u \wedge \omega^{n-2} + L(x, \nabla u),$$

for the specific first order term

$$L(x, \nabla u) = \mathrm{Re}(\sqrt{-1}\partial u \wedge \overline{\partial}\omega^{n-2}) \tag{1.2}$$

introduced by Popovici [25] and independently in [33]. This yielded a solution of the Gauduchon conjecture [15] on the existence of Gauduchon metrics with prescribed volume form. The proof in [27] makes careful use of the specific form of this first order term term $L(x, \nabla u)$. See also [17, 10, 26, 38] for related follow-up work.

Other nonlinear equations involving gradient terms arise naturally by motivations from mathematical physics, including the Fu-Yau equation [13] and its extensions by Phong-Picard-Zhang [21, 22, 23]. In particular, the paper [21] considers the complex Hessian equations

$$(\chi(z, u) + \sqrt{-1}\partial\overline{\partial}u)^k \wedge \omega^{n-k} = \psi(z, u, \nabla u)\omega^n$$

where gradient terms appear on the right hand side.

In light of these results, it is natural to consider fully nonlinear equations in terms of the metric

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}u + L(x, \nabla u),$$

for L a linear term involving the gradient of u. Indeed, this study was initiated recently by R. Yuan [36]. However the family of equations he deals with includes the Monge-Ampère equation $\tilde{\omega}^n = e^F\omega^n$ only in the case of complex dimension $n = 2$ [36, Corollary 1.5]. The current paper settles the case $n > 2$ left open by Yuan.

More precisely, let (M, g) be a compact Hermitian manifold of complex dimension n. By analogy to (1.2), we consider the term

$$L(x, \nabla u) = \sqrt{-1}a \wedge \overline{\partial}u - \sqrt{-1}\overline{a} \wedge \partial u$$

where a is a smooth $(1, 0)$-form. Indeed, this is the most general term of the form $\alpha \wedge \partial u + \beta \wedge \overline{\partial}u$ for 1-forms α and β, which is also real and of type $(1, 1)$. In local coordinates, we may write $L(x, \nabla u) = \sqrt{-1}(a_i u_{\overline{j}} + a_{\overline{j}} u_i)dz^i \wedge d\overline{z}^j$, where $a = a_i dz^i$ and $a_{\overline{i}} = \overline{a_i}$.

We prove the following:

Theorem 1.1. *Given $F \in C^\infty(M)$ and a smooth $(1,0)$ form a on M, there exists a unique pair (u,b) with $u \in C^\infty(M)$ and $b \in \mathbb{R}$ satisfying the equation*

$$\begin{aligned} &\det(g_{i\overline{j}} + a_i u_{\overline{j}} + a_{\overline{j}} u_i + u_{i\overline{j}}) = e^{F+b} \det(g_{i\overline{j}}), \\ &\text{with} \quad (\tilde{g}_{i\overline{j}}) := (g_{i\overline{j}} + a_i u_{\overline{j}} + a_{\overline{j}} u_i + u_{i\overline{j}}) > 0, \quad \text{and } \sup_M u = 0. \end{aligned} \tag{1.3}$$

The case $n = 2$ is due to Yuan [36]. We also remark that Zhang [37] proved a uniform gradient estimate for a class of equations which includes (1.3).

We can rewrite (1.3) in coordinate-free notation by letting

$$\tilde{\omega} := \omega + \sqrt{-1} a \wedge \overline{\partial} u - \sqrt{-1}\overline{a} \wedge \partial u + \sqrt{-1}\partial\overline{\partial} u > 0,$$

be the new Hermitian metric whose volume form equals

$$\tilde{\omega}^n = e^{F+b} \omega^n.$$

Remark 1.2. As an aside, note that if we choose a to be a holomorphic 1-form, then we can write

$$\tilde{\omega} = \omega + \partial\overline{\gamma} + \overline{\partial}\gamma, \tag{1.4}$$

where γ is the $(1,0)$ form given by

$$\gamma = -\sqrt{-1}\left(ua + \frac{\partial u}{2}\right).$$

In this case, if we also have that $\partial\overline{\partial}\omega = 0$ (which when $n = 2$ is the Gauduchon condition [14]), then ω defines a cohomology class in Aeppli cohomology, and (1.4) shows that the metric $\tilde{\omega}$ also satisfies $\partial\overline{\partial}\tilde{\omega} = 0$ and lies in the same Aeppli cohomology class.

The outline of our proof is as follows. We begin by proving *a priori* estimates for solutions of (1.3). In Section 2, we establish a uniform L^∞ bound for u, with an approach that uses the Aleksandrov-Bakelman-Pucci estimate. In Section 3 we give an estimate on the second derivatives $\sqrt{-1}\partial\overline{\partial}u$ of u in terms of the first derivatives, using a maximum principle argument involving the largest eigenvalue λ_1 of the metric $\tilde{g}$. The particular quantity we use for the maximum principle is

$$Q = \log \lambda_1 + \frac{|\partial u|_g^2}{\sup_M |\partial u|_g^2 + 1} + e^{-Au},$$

for a large constant A. This differs (and in many cases is simpler) than the quantities used in the literature mentioned above. To overcome the fact that the eigenvalue λ_1 is not differentiable in general, we choose to use a viscosity argument (adapted from [5], and hinted to in [26]), which to our knowledge is new in this Hermitian setting. Finally, in Section 4, we complete the proof of Theorem 1.1: we apply a standard blow-up argument to obtain the first order estimate and then standard theory gives the higher order estimates. Given the C^∞ *a priori* estimates, the existence follows from a fairly standard continuity argument and uniqueness is a consequence of the maximum principle.

Instead of using a blow-up argument, the gradient estimate can be obtained directly by a maximum principle argument, as shown in an earlier work of Zhang [37, Remark 2] (see also the related works [4, 10, 36]). We thank the referee for pointing out the reference [37], of which we were not aware when we completed the first version of this article.

2. Zero order estimate

Let $u, F \in C^\infty(M)$ and $a \in \Lambda^{1,0}M$ satisfy

$$\begin{aligned} \det(g_{i\bar{j}} + a_i u_{\bar{j}} + a_{\bar{j}} u_i + u_{i\bar{j}}) &= e^F \det(g_{i\bar{j}}) \\ (\tilde{g}_{i\bar{j}}) := (g_{i\bar{j}} + a_i u_{\bar{j}} + a_{\bar{j}} u_i + u_{i\bar{j}}) &> 0, \end{aligned} \tag{2.1}$$

with $\sup_M u = 0$. We will write $\tilde{\omega}$ for the $(1,1)$ form associated to the metric $\tilde{g}_{i\bar{j}}$.

We prove a uniform estimate for u.

Theorem 2.1. *There is a constant C that depends only on $\sup_M |F|$, $\sup_M |a|_g$, and on the geometry of (M, g) such that*

$$\sup_M |u| \le C. \tag{2.2}$$

Proof. We employ the Aleksandrov-Bakelman-Pucci estimate, whose usage for the complex Monge-Ampère equation originated in work of Cheng-Yau (see [1]), and was more recently revisited by Błocki [2, 3] and Székelyhidi [26]. We follow [7, 26, 32].

First, we observe that

$$\int_M (-u)\omega^n \le C, \tag{2.3}$$

for a uniform constant C. Indeed, let

$$H(u)=\Delta_g u+\mathrm{tr}_\omega(\sqrt{-1}a\wedge\overline{\partial}u-\sqrt{-1}\overline{a}\wedge\partial u)=\mathrm{tr}_g\tilde{g}-n\geq -n,$$

where $\Delta_g u=\mathrm{tr}_\omega\sqrt{-1}\partial\overline{\partial}u=\frac{n\sqrt{-1}\partial\overline{\partial}u\wedge\omega^{n-1}}{\omega^n}$ is the complex Laplacian of g. Since the kernel of H consists of just constants, a classical argument of Gauduchon [14] (cf. [7, Theorem 2.2]) shows that there is a smooth function v such that

$$\int_M H(\psi)e^v\omega^n=0, \tag{2.4}$$

for all smooth functions ψ. We then define a new Hermitian metric $\hat{\omega}=e^{v/(n-1)}\omega$. Its operator $\hat{H}$, defined in the same way

$$\hat{H}(\psi)=\Delta_{\hat{g}}\psi+\mathrm{tr}_{\hat{\omega}}(\sqrt{-1}a\wedge\overline{\partial}\psi-\sqrt{-1}\overline{a}\wedge\partial\psi), \tag{2.5}$$

satisfies

$$\hat{H}(u)=e^{-v/(n-1)}H(u)\geq -C, \tag{2.6}$$

and now we have

$$\int_M \hat{H}(\psi)\hat{\omega}^n=0, \tag{2.7}$$

for all ψ. We may then use the Green's function for $\hat{H}$ (with respect to the metric $\hat{\omega}$), to deduce the uniform L^1 bound for u in (2.3) by the exact same argument as in [33, Proof of Theorem 2.1]. Briefly, standard theory gives us a Green's function $G(x,y)$, normalized to have zero integral, which has a uniform lower bound and such that

$$\psi(x)=\frac{1}{\int_M\hat{\omega}^n}\int_M\psi\hat{\omega}^n-\int_M\hat{H}(\psi)(y)G(x,y)\hat{\omega}^n(y),$$

holds for all ψ and all $x\in M$. Thanks to (2.7) we can add a uniform constant to G to make it nonnegative, while preserving the same Green's formula, and we then apply this to u with x a point where $u(x)=0$, so that from (2.6) and the lower bound for G we easily deduce (2.3).

Next, we promote the L^1 bound (2.3) to the L^∞ bound (2.2) using ABP, as in [7, Proposition 3.1] and [26, 32]. Let $x_0\in M$ be a point where u achieves its infimum $I=\inf_M u$, and fix a coordinate unit ball B centered at x_0. In this ball, let $v = u + \varepsilon|x|^2$, where $\varepsilon > 0$ will be a uniform constant to be chosen later. We have $\inf_{\partial B} v \geq v(0) + \varepsilon$, so [26, Proposition 10] gives us that

$$\varepsilon^{2n} \leq C \int_P \det(D^2 v), \tag{2.8}$$

for a universal constant C, where

$$P = \{x \in B \mid |Dv(x)| < \varepsilon/2, \text{ and } v(y) \geq v(x) + Dv(x) \cdot (y - x)\ \forall y \in B\}.$$

Given now any $x \in P$, we have $D^2 v(x) \geq 0$ and $|Du(x)| \leq 5\varepsilon/2$ so at x

$$\sqrt{-1} a \wedge \overline{\partial} u - \sqrt{-1}\overline{a} \wedge \partial u + \sqrt{-1}\partial\overline{\partial} u \geq -C\varepsilon\omega,$$

for a uniform constant C, therefore if we choose ε sufficiently small (but uniformly bounded away from zero), we get

$$\tilde{\omega}(x) \geq \frac{1}{2}\omega(x),$$

and from the Monge-Ampère equation (2.1) we deduce

$$\tilde{\omega}(x) \leq C\omega(x),$$

from which

$$\sqrt{-1}\partial\overline{\partial} u(x) \leq C\omega(x),$$

and so $0 \leq \sqrt{-1}\partial\overline{\partial} v(x) \leq C\omega(x)$. But a simple linear algebra inequality (using that $(D^2 v(x)) \geq 0$) gives

$$\det(D^2 v(x)) \leq C \det(v_{i\overline{j}})^2(x) \leq C,$$

which together with (2.8) gives

$$|P| \geq C^{-1},$$

where $|P|$ denotes the Lebesgue measure. For all $x \in P$ we have

$$v(x) \leq v(0) + \frac{\varepsilon}{2} = I + \frac{\varepsilon}{2},$$

and we may assume that $I + \frac{\varepsilon}{2} < 0$, so

$$C^{-1} \leq |P| \leq \frac{\int_P(-v)}{|I + \frac{\varepsilon}{2}|} \leq \frac{C}{|I + \frac{\varepsilon}{2}|},$$

using the L^1 bound (2.3), which proves (2.2). □

3. Second order estimate

In this section we prove a bound on $\sqrt{-1}\partial\overline{\partial}u$ in terms of a bound on the square of the first derivative of u. This estimate takes the same form as the Hou-Ma-Wu estimate [19] for the complex Hessian equations (see also the later works [7, 26, 27, 31, 33]) although here the quantity to which we apply the maximum principle is slightly simpler.

Theorem 3.1. *Let $u, F \in C^\infty(M)$ and $a \in \Lambda^{1,0}M$ satisfy (2.1), with $\sup_M u = 0$. Then there is a constant C that depends only on $\sup_M |u|$, $\|a\|_{C^2(M)}$, $\|F\|_{C^2(M)}$ and on the geometry of (M, g) such that*

$$\sup_M |\sqrt{-1}\partial\overline{\partial}u|_g \leq C(1 + \sup_M |\partial u|_g^2).$$

Proof. Define the linearized operator L by

$$Lv = \tilde{g}^{i\overline{j}}(v_{i\overline{j}} + a_i v_{\overline{j}} + a_{\overline{j}} v_i) = \tilde{g}^{i\overline{j}} v_{i\overline{j}} + 2\mathrm{Re}\left(\tilde{g}^{i\overline{j}} a_{\overline{j}} v_i\right). \tag{3.1}$$

Observe that

$$Lu = \tilde{g}^{i\overline{j}}(\tilde{g}_{i\overline{j}} - g_{i\overline{j}}) = n - \mathrm{tr}_{\tilde{g}}g. \tag{3.2}$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of $\tilde{g}_{i\overline{j}}$ with respect to g. We consider the quantity

$$Q = \log \lambda_1 + \varphi(|\partial u|_g^2) + \psi(u),$$

where we define

$$\varphi(s) = \frac{s}{K}, \ s \geq 0, \quad \text{and} \quad \psi(t) = e^{-At}, \ t \leq 0,$$

with

$$K = \sup_M |\partial u|_g^2 + 1,$$

and $A > 0$ to be determined. Note that we have

$$-\psi' \geq A > 0, \quad \psi'' = -A\psi'.$$

We assume that Q achieves its maximum at $x_0 \in M$. It suffices to show that at x_0, we have $\lambda_1 \leq CK$ for a uniform C. Hence in what follows we may assume without loss of generality that λ_1 is large compared to K. We will calculate at the point x_0 using coordinates for which g is the identity and $\tilde{g}$ is diagonal with entries $\tilde{g}_{i\bar{i}} = \lambda_i$ for $i = 1, \ldots, n$.

Since λ_1 may not be smooth at x_0, we define a smooth function f on M by (cf. [5, Proof of Theorem 6])

$$Q(x_0) = \log f + \varphi(|\partial u|_g^2) + \psi(u), \tag{3.3}$$

where the right hand side of (3.3) is evaluated at a general point of M. Observe that f satisfies

$$f \geq \lambda_1 \quad \text{on } M, \quad f = \lambda_1 \quad \text{at } x_0. \tag{3.4}$$

We have the following lemma, which is a complex version of [5, Lemma 5]. Here and in the sequel, we use ∇_i or simply lower indices (after commas, when needed to avoid confusion) to denote covariant derivatives with respect to the Chern connection of g.

Lemma 3.2. *Let μ denote the multiplicity of the largest eigenvalue of $\tilde{g}$ at x_0, so that $\lambda_1 = \cdots = \lambda_\mu > \lambda_{\mu+1} \geq \cdots \geq \lambda_n$. Then at x_0, for each i with $1 \leq i \leq n$,*

$$\tilde{g}_{k\bar{\ell},i} = f_i g_{k\bar{\ell}}, \quad \textit{for } 1 \leq k, \ell \leq \mu, \tag{3.5}$$

and

$$f_{i\bar{i}} \geq \tilde{g}_{1\bar{1},i\bar{i}} + \sum_{q>\mu} \frac{|\tilde{g}_{q\bar{1},i}|^2 + |\tilde{g}_{q\bar{1},\bar{i}}|^2}{\lambda_1 - \lambda_q}. \tag{3.6}$$

Proof. The proof only uses the fact that f is smooth and satisfies (3.4). For a smooth vector field $V = V^k \frac{\partial}{\partial z^k}$ defined in a neighborhood of x_0, we consider the function

$$h = \tilde{g}_{k\bar{\ell}} V^k \overline{V^\ell} - f g_{k\bar{\ell}} V^k \overline{V^\ell},$$

which is nonpositive. For any choice of V with $V^k(x_0) = 0$ for $k > \mu$ we have $h(x_0) = 0$ and hence h has a local maximum at x_0.

For (3.5), choose V with $V^k(x_0) = 0$ for $k > \mu$ and

$$\nabla_i V^k(x_0) = 0 = \nabla_{\bar{i}} V^k(x_0), \quad \text{for } k \leq \mu.$$

Then at x_0,

$$0 = h_i = \tilde{g}_{k\bar{\ell},i} V^k \overline{V^\ell} - f_i g_{k\bar{\ell}} V^k \overline{V^\ell},$$

and (3.5) follows since we can choose $V^k(x_0)$ for $k \leq \mu$ to be whatever we like.

For (3.6) we choose V with $V(x_0) = \frac{\partial}{\partial z^1}$ and

$$\nabla_i V^q(x_0) = \begin{cases} 0, & q \leq \mu \\ \frac{\tilde{g}_{1\bar{q},i}}{\lambda_1 - \lambda_q}, & q > \mu \end{cases}$$

and

$$\nabla_{\bar{i}} V^q(x_0) = \begin{cases} 0, & q \leq \mu \\ \frac{\tilde{g}_{1\bar{q},\bar{i}}}{\lambda_1 - \lambda_q}, & q > \mu. \end{cases}$$

Then at x_0,

$$\begin{aligned} 0 \geq h_{i\bar{i}} = {} & \tilde{g}_{1\bar{1},i\bar{i}} - f_{i\bar{i}} + \tilde{g}_{k\bar{\ell},i}(\nabla_{\bar{i}} V^k)\overline{V^\ell} + \tilde{g}_{k\bar{\ell},i} V^k \overline{\nabla_i V^\ell} + \tilde{g}_{k\bar{\ell},\bar{i}}(\nabla_i V^k)\overline{V^\ell} \\ & + \tilde{g}_{k\bar{\ell},\bar{i}} V^k \overline{\nabla_{\bar{i}} V^\ell} + \tilde{g}_{k\bar{\ell}} \nabla_i V^k \overline{\nabla_i V^\ell} + \tilde{g}_{k\bar{\ell}} \nabla_{\bar{i}} V^k \overline{\nabla_{\bar{i}} V^\ell} \\ & - f g_{k\bar{\ell}} \nabla_i V^k \overline{\nabla_i V^\ell} - f g_{k\bar{\ell}} \nabla_{\bar{i}} V^k \overline{\nabla_{\bar{i}} V^\ell}, \end{aligned} \tag{3.7}$$

noting that terms of the type $f_i g_{k\bar{\ell}}(\nabla_{\bar{i}} V^k)\overline{V^\ell}$ vanish by definition of V and

$$\tilde{g}_{k\bar{\ell}}(\nabla_{\bar{i}}\nabla_i V^k)\overline{V^\ell} - f g_{k\bar{\ell}}(\nabla_{\bar{i}}\nabla_i V^k)\overline{V^\ell} = 0 = \tilde{g}_{k\bar{\ell}} V^k \overline{\nabla_i \nabla_{\bar{i}} V^\ell} - f g_{k\bar{\ell}} V^k \overline{\nabla_i \nabla_{\bar{i}} V^\ell}$$

since $f g_{1\bar{1}} = \lambda_1 = \tilde{g}_{1\bar{1}}$ at x_0. Continuing from (3.7), using the definition of V,

$$\begin{aligned} 0 \geq {} & \tilde{g}_{1\bar{1},i\bar{i}} - f_{i\bar{i}} + 2\sum_{q>\mu} \frac{|\tilde{g}_{q\bar{1},i}|^2}{\lambda_1 - \lambda_q} + 2\sum_{q>\mu} \frac{|\tilde{g}_{q\bar{1},\bar{i}}|^2}{\lambda_1 - \lambda_q} \\ & + \sum_{q>\mu} \lambda_q \frac{|\tilde{g}_{1\bar{q},i}|^2}{(\lambda_1 - \lambda_q)^2} + \sum_{q>\mu} \lambda_q \frac{|\tilde{g}_{1\bar{q},\bar{i}}|^2}{(\lambda_1 - \lambda_q)^2} \\ & - \lambda_1 \sum_{q>\mu} \frac{|\tilde{g}_{1\bar{q},i}|^2}{(\lambda_1 - \lambda_q)^2} - \lambda_1 \sum_{q>\mu} \frac{|\tilde{g}_{1\bar{q},\bar{i}}|^2}{(\lambda_1 - \lambda_q)^2} \\ = {} & \tilde{g}_{1\bar{1},i\bar{i}} - f_{i\bar{i}} + \sum_{q>\mu} \frac{|\tilde{g}_{q\bar{1},i}|^2 + |\tilde{g}_{q\bar{1},\bar{i}}|^2}{\lambda_1 - \lambda_q}, \end{aligned}$$

as required. □

Differentiating (2.1) we obtain

$$\tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{i},k} = \tilde{g}^{i\bar{i}}(u_{i\bar{i}k} + a_{i,k}u_{\bar{i}} + a_i u_{k\bar{i}} + a_{\bar{i},k}u_i + a_{\bar{i}}u_{ik}) = F_k, \tag{3.8}$$

where here and henceforth we are computing at the point x_0. Differentiating again, and setting $k = 1$,

$$\tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{i},1\bar{1}} - \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j},1}\tilde{g}_{j\bar{i},\bar{1}} = F_{1\bar{1}}. \tag{3.9}$$

Now apply ∇_i to the defining equation (3.3) of f to obtain

$$0 = \frac{f_i}{\lambda_1} + \varphi'\left(u_p u_{\overline{p}i} + u_{pi}u_{\overline{p}}\right) + \psi' u_i. \tag{3.10}$$

Next apply the operator L, as defined in (3.1), to the defining equation of f to obtain,

$$\begin{aligned} 0 = {} & \frac{\tilde{g}^{i\bar{i}} f_{i\bar{i}}}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}|f_i|^2}{\lambda_1^2} + \varphi' \sum_p \tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right) \\ & + \varphi'\tilde{g}^{i\bar{i}}(u_{pi\bar{i}}u_{\overline{p}} + u_{\overline{p}i\bar{i}}u_p) + \psi''\tilde{g}^{i\bar{i}}|u_i|^2 + \psi'(n - \mathrm{tr}_{\tilde{g}} g) \\ & + 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}} a_{\bar{i}} \frac{f_i}{\lambda_1}\right) + 2\varphi'\mathrm{Re}\left(\tilde{g}^{i\bar{i}} a_{\bar{i}}\left(u_p u_{\overline{p}i} + u_{pi}u_{\overline{p}}\right)\right), \end{aligned} \tag{3.11}$$

where we have made use of (3.2). We wish to compare $\sum_i \tilde{g}^{i\bar{i}} f_{i\bar{i}}$ and $\sum_i \tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{i},1\bar{1}}$. From Lemma 3.2,

$$f_i = \tilde{g}_{11,\bar{i}}, \text{ and } f_{i\bar{i}} \geq \tilde{g}_{1\bar{1},i\bar{i}} + \sum_{q>\mu}\frac{|\tilde{g}_{1\bar{q},i}|^2 + |\tilde{g}_{q\bar{1},i}|^2}{\lambda_1 - \lambda_q}. \tag{3.12}$$

To compare $\tilde{g}_{11,i\bar{i}}$ and $\tilde{g}_{i\bar{i},1\bar{1}}$ we first compute, using T_{ij}^k and $R_{k\bar{\ell}i}{}^p$ to denote the torsion and Chern curvature tensors of g respectively (see for example [33]),

$$\begin{aligned} u_{i\bar{i}1\bar{1}} &= u_{i\bar{i}\bar{1}1} + R_{1\bar{1}i}{}^p u_{p\bar{i}} - R_{1\bar{1}}{}^{\bar{q}}{}_{\bar{i}} u_{i\bar{q}} \\ &= u_{i\bar{1}\bar{i}1} + R_{1\bar{1}i}{}^p u_{p\bar{i}} - R_{1\bar{1}}{}^{\bar{q}}{}_{\bar{i}} u_{i\bar{q}} + \nabla_1\overline{T_{i1}^q}u_{i\bar{q}} + \overline{T_{i1}^q}u_{i\bar{q}1} \\ &= u_{\bar{1}i1\bar{i}} + R_{1\bar{1}i}{}^p u_{p\bar{i}} - R_{1\bar{1}}{}^{\bar{q}}{}_{\bar{i}} u_{i\bar{q}} + \nabla_1\overline{T_{i1}^q}u_{i\bar{q}} + \overline{T_{i1}^q}u_{i\bar{q}1} \\ &\quad + R_{1\bar{i}}{}^{\bar{q}}{}_{\bar{1}} u_{\bar{q}i} - R_{1\bar{i}i}{}^p u_{\bar{1}p} \\ &= u_{1\bar{1}i\bar{i}} + R_{1\bar{1}i}{}^p u_{p\bar{i}} - R_{1\bar{1}}{}^{\bar{q}}{}_{\bar{i}} u_{i\bar{q}} + \nabla_1\overline{T_{i1}^q}u_{i\bar{q}} + \overline{T_{i1}^q}u_{i\bar{q}1} \\ &\quad + R_{1\bar{i}}{}^{\bar{q}}{}_{\bar{1}} u_{\bar{q}i} - R_{1\bar{i}i}{}^p u_{\bar{1}p} + \nabla_{\bar{i}} T_{i1}^q u_{\bar{1}q} + T_{i1}^q u_{\bar{1}q\bar{i}}, \end{aligned} \tag{3.13}$$

where for the second inequality and fourth inequalities, we used the formulae

$$u_{j\overline{\ell k}} - u_{j\overline{k\ell}} = \overline{T_{\ell k}^q}u_{j\bar{q}}, \quad u_{\bar{j}\ell k} - u_{\bar{j}k\ell} = T_{\ell k}^q u_{\bar{j}q}. \tag{3.14}$$

From (3.13) and the definition of $\tilde{g}_{i\bar{j}}$,

$$\begin{aligned}
\tilde{g}^{i\bar{i}}\tilde{g}_{1\bar{1},i\bar{i}} = {} & \tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{i},1\bar{1}} + \tilde{g}^{i\bar{i}}\{u_{1\bar{1}i\bar{i}} - u_{i\bar{i}1\bar{1}} + a_{1,i\bar{i}}u_{\bar{1}} - a_{i,1\bar{1}}u_{\bar{i}} \\
& + a_{\bar{1},i\bar{i}}u_1 - a_{\bar{i},1\bar{1}}u_i + a_{1,i}u_{\bar{1}\bar{i}} - a_{i,1}u_{\bar{i}\bar{1}} + a_{1,\bar{i}}u_{\bar{1}i} - a_{i,\bar{1}}u_{\bar{i}1} \\
& + a_{\bar{1},i}u_{1\bar{i}} - a_{\bar{i},1}u_{i\bar{1}} + a_{\bar{1},\bar{i}}u_{1i} - a_{\bar{i},\bar{1}}u_{i1} + a_1u_{\bar{1}i\bar{i}} - a_iu_{\bar{i}1\bar{1}} \\
& + a_{\bar{1}}u_{1i\bar{i}} - a_{\bar{i}}u_{i1\bar{1}}\} \\
\geq {} & \tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{i},1\bar{1}} + \tilde{g}^{i\bar{i}}\left(\overline{T^q_{1i}}u_{i\bar{q}1} + T^q_{1i}u_{\bar{1}q\bar{i}} + a_1u_{\bar{1}i\bar{i}} - a_iu_{\bar{i}1\bar{1}} + a_{\bar{1}}u_{1i\bar{i}} - a_{\bar{i}}u_{i1\bar{1}}\right) \\
& - \sum_p \tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right) - C(\mathrm{tr}_{\tilde{g}}g)(\mathrm{tr}_g\tilde{g}),
\end{aligned}$$

where for the last line we used the assumption that $K \leq \lambda_1 \leq \mathrm{tr}_g\tilde{g}$, and the uniform lower bound of $\mathrm{tr}_g\tilde{g}$ which follows from our equation (2.1).

Next, observe that

$$u_{i\bar{j}k} = u_{k\bar{j}i} + T^p_{ik}u_{\bar{j}p} = u_{ki\bar{j}} + T^p_{ik}u_{\bar{j}p} - u_pR_{i\bar{j}k}{}^p. \tag{3.15}$$

Then, using this and (3.8),

$$\begin{aligned}
\tilde{g}^{i\bar{i}}(a_1u_{\bar{1}i\bar{i}} + a_{\bar{1}}u_{1i\bar{i}}) &= 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}a_{\bar{1}}u_{1i\bar{i}}\right) - a_1u_{\bar{q}}\tilde{g}^{i\bar{i}}R_{i\bar{i}}{}^{\bar{q}}{}_{\bar{1}} \\
&= 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}a_{\bar{1}}\left(u_{i\bar{i}1} - T^p_{i1}u_{\bar{i}p} + u_pR_{i\bar{i}1}{}^p\right)\right) - a_1u_{\bar{q}}\tilde{g}^{i\bar{i}}R_{i\bar{i}}{}^{\bar{q}}{}_{\bar{1}} \\
&= 2\mathrm{Re}\Big(a_{\bar{1}}F_1 - \tilde{g}^{i\bar{i}}a_{\bar{1}}(T^p_{i1}u_{\bar{i}p} - u_pR_{i\bar{i}1}{}^p \\
&\qquad + a_{i,1}u_{\bar{i}} + a_iu_{1\bar{i}} + a_{\bar{i},1}u_i + a_{\bar{i}}u_{i1})\Big) - a_1u_{\bar{q}}\tilde{g}^{i\bar{i}}R_{i\bar{i}}{}^{\bar{q}}{}_{\bar{1}}.
\end{aligned}$$

We also have

$$\begin{aligned}
& \tilde{g}^{i\bar{i}}(\overline{T^q_{1i}}u_{i\bar{q}1} + T^q_{1i}u_{\bar{1}q\bar{i}}) \\
= {} & 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}u_{1\bar{q}i}\right) + \tilde{g}^{i\bar{i}}\overline{T^q_{1i}}T^p_{1i}u_{\bar{q}p} \\
= {} & 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\left(\tilde{g}_{1\bar{q},i} - a_{1,i}u_{\bar{q}} - a_1u_{\bar{q}i} - a_{\bar{q},i}u_1 - a_{\bar{q}}u_{1i}\right)\right) \\
& + \tilde{g}^{i\bar{i}}\overline{T^q_{1i}}T^p_{i1}u_{\bar{q}p}.
\end{aligned}$$

Combining the above with (3.9) gives

$$\begin{aligned}
\tilde{g}^{i\bar{i}}\tilde{g}_{1\bar{1},i\bar{i}} \geq {} & \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j},1}\tilde{g}_{j\bar{i},\bar{1}} + 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\tilde{g}_{1\bar{q},i}\right) - \tilde{g}^{i\bar{i}}\{a_iu_{\bar{i}1\bar{1}} + a_{\bar{i}}u_{i1\bar{1}}\} \\
& - 2\sum_{i,p}\tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right) - C(\mathrm{tr}_{\tilde{g}}g)(\mathrm{tr}_g\tilde{g}).
\end{aligned} \tag{3.16}$$

Next, using again Lemma 3.2,

$$
\begin{aligned}
(3.17)\quad 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}a_{\bar{i}}\frac{f_i}{\lambda_1}\right) &= 2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}a_{\bar{i}}\frac{\tilde{g}_{1\bar{1},i}}{\lambda_1}\right)\\
&= \frac{\tilde{g}^{i\bar{i}}}{\lambda_1}\left(a_{\bar{i}}u_{i1\bar{1}} + a_i u_{\bar{i}1\bar{1}} + a_{\bar{i}}T^p_{1i}u_{\bar{1}p} - a_{\bar{i}}u_p R_{1\bar{1}i}{}^p + a_i\overline{T^q_{1i}}u_{1\bar{q}}\right)\\
&\quad + 2\mathrm{Re}\left(\frac{\tilde{g}^{i\bar{i}}}{\lambda_1}a_{\bar{i}}\{a_{1,i}u_{\bar{1}} + a_1 u_{\bar{1}i} + a_{\bar{1},i}u_1 + a_{\bar{1}}u_{1i}\}\right)\\
&\geq \frac{\tilde{g}^{i\bar{i}}}{\lambda_1}\left(a_{\bar{i}}u_{i1\bar{1}} + a_i u_{\bar{i}1\bar{1}}\right) - \frac{1}{\lambda_1}\sum_p \tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right)\\
&\quad - C\mathrm{tr}_{\tilde{g}}g,
\end{aligned}
$$

and note that the terms involving three derivatives of u exactly match those from (3.16), after multiplying by $-1/\lambda_1$.

Now from (3.8) we have,

$$
\tilde{g}^{i\bar{i}}u_{i\bar{i}p}u_{\bar{p}} = F_p u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{i,p}u_{\bar{i}}u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_i u_{\bar{i}p}u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{\bar{i},p}u_i u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{\bar{i}}u_{ip}u_{\bar{p}}.
$$

Hence, making use of (3.15), and recalling that $\varphi' = 1/K$,

$$
\begin{aligned}
&\varphi'\tilde{g}^{i\bar{i}}(u_{pi\bar{i}}u_{\bar{p}} + u_{\bar{p}i\bar{i}}u_p)\\
&= \varphi'\tilde{g}^{i\bar{i}}\left(u_{i\bar{i}p}u_{\bar{p}} + u_{i\bar{i}\bar{p}}u_p + u_r u_{\bar{p}}R_{i\bar{i}p}{}^r - T^r_{ip}u_{\bar{p}}u_{\bar{i}r} + \overline{T^q_{pi}}u_p u_{i\bar{q}}\right)\\
(3.18)\quad &= 2\varphi'\mathrm{Re}\left(F_p u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{i,p}u_{\bar{i}}u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_i u_{\bar{i}p}u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{\bar{i},p}u_i u_{\bar{p}} - \tilde{g}^{i\bar{i}}a_{\bar{i}}u_{ip}u_{\bar{p}}\right)\\
&\quad + \varphi'\tilde{g}^{i\bar{i}}u_r u_{\bar{p}}R_{i\bar{i}p}{}^r - 2\varphi'\mathrm{Re}\left(\tilde{g}^{i\bar{i}}T^r_{ip}u_{\bar{p}}u_{\bar{i}r}\right)\\
&\geq -\frac{\varphi'}{4}\sum_p \tilde{g}^{i\bar{i}}(|u_{p\bar{i}}|^2 + |u_{pi}|^2) - C\mathrm{tr}_{\tilde{g}}g.
\end{aligned}
$$

We also have

$$
(3.19)\quad 2\varphi'\mathrm{Re}\left(\tilde{g}^{i\bar{i}}a_{\bar{i}}\left(u_p u_{\bar{p}i} + u_{pi}u_{\bar{p}}\right)\right) \geq -\frac{\varphi'}{4}\sum_p \tilde{g}^{i\bar{i}}(|u_{p\bar{i}}|^2 + |u_{pi}|^2) - C\mathrm{tr}_{\tilde{g}}g.
$$

Combining (3.11), (3.12), (3.16), (3.17), (3.18) and (3.19) gives

$$
0 \geq \frac{\tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j},1}\tilde{g}_{j\bar{i},\bar{1}}}{\lambda_1} + \sum_{q>\mu}\frac{\tilde{g}^{i\bar{i}}(|\tilde{g}_{1\bar{q},i}|^2 + |\tilde{g}_{q\bar{1},i}|^2)}{\lambda_1(\lambda_1 - \lambda_q)} - \frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\bar{1},i}|^2}{\lambda_1^2}
$$

$$(3.20)\qquad + \frac{2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\tilde{g}_{1\bar{q},i}\right)}{\lambda_1} + \left(\frac{1}{2}\varphi' - \frac{C}{\lambda_1}\right)\sum_p \tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right)$$
$$+ \psi''\tilde{g}^{i\bar{i}}|u_i|^2 + \psi'(n - \mathrm{tr}_{\tilde{g}}g) - C\mathrm{tr}_{\tilde{g}}g$$

for C a universal constant (depending on F, a etc).

We need to get a lower bound of

$$(3.21)\qquad \frac{\tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j},1}\tilde{g}_{j\bar{i},\bar{1}}}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\bar{1},i}|^2}{\lambda_1^2} \geq \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{1},1}\tilde{g}_{1\bar{i},\bar{1}}}{\lambda_1^2} - \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\bar{1},i}|^2}{\lambda_1^2},$$

where we have discarded the terms with $j \neq 1$. But note that

$$\tilde{g}_{i\bar{1},1} = \tilde{g}_{1\bar{1},i} + \lambda_1 X_{1\bar{1}i},$$

where $X_{1\bar{1}i}$ is defined by

$$X_{1\bar{1}i}$$
$$:= \frac{1}{\lambda_1}\left(T^p_{i1}u_{\bar{1}p} + a_{i,1}u_{\bar{1}} + a_i u_{1\bar{1}} + a_{\bar{1},1}u_i - a_{1,i}u_{\bar{1}} - a_1 u_{i\bar{1}} - a_{\bar{1},i}u_1 + a_{\bar{1}}T^k_{1i}u_k\right),$$

and satisfies $|X_{1\bar{1}i}| \leq C$ for a uniform C. In the above, we used (3.15) and the formula

$$u_{ij} - u_{ji} = T^k_{ji}u_k.$$

Then

$$(3.22)\qquad \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}\tilde{g}_{i\bar{1},1}\tilde{g}_{1\bar{i},\bar{1}}}{\lambda_1^2} \geq \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\bar{1},i}|^2}{\lambda_1^2} + 2\mathrm{Re}\left(\sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}g_{1\bar{1},i}\overline{X_{1\bar{1}i}}}{\lambda_1}\right).$$

To deal with the second term, we use (3.10) to compute

$$2\mathrm{Re}\left(\sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}}\tilde{g}_{1\bar{1},i}\overline{X_{1\bar{1}i}}}{\lambda_1}\right)$$
$$(3.23)\qquad = -2\mathrm{Re}\left(\sum_{i=2}^n \tilde{g}^{i\bar{i}}(\varphi'(u_p u_{\bar{p}i} + u_{pi}u_{\bar{p}}) + \psi' u_i)\overline{X_{1\bar{1}i}}\right)$$
$$\geq -\frac{\varphi'}{8}\sum_p \tilde{g}^{i\bar{i}}(|u_{p\bar{i}}|^2 + |u_{pi}|^2) - C\mathrm{tr}_{\tilde{g}}g + \psi'(C\tilde{g}^{i\bar{i}}|u_i|^2 + \frac{1}{4}\mathrm{tr}_{\tilde{g}}g),$$

where we recall that $\psi' < 0$.

Next we deal with the fourth term on the right hand side of (3.20). From Lemma 3.2 we have $\tilde{g}_{1\overline{q},i} = 0$ for $1 < q \leq \mu$ and hence

$$\frac{2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\tilde{g}_{1\overline{q},i}\right)}{\lambda_1} = \frac{2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^1_{1i}}\tilde{g}_{1\overline{1},i}\right)}{\lambda_1} + 2\sum_{q>\mu}\frac{\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\tilde{g}_{1\overline{q},i}\right)}{\lambda_1} \tag{3.24}$$

But using the same argument as in (3.23), replacing $|X_{1\overline{1}i}| \leq C$ by $|T^1_{1i}| \leq C$, we obtain

$$\begin{aligned}\frac{2\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^1_{1i}}\tilde{g}_{1\overline{1},i}\right)}{\lambda_1} \geq & -\frac{\varphi'}{8}\sum_p \tilde{g}^{i\bar{i}}(|u_{p\bar{i}}|^2 + |u_{pi}|^2) - C\mathrm{tr}_{\tilde{g}}g \\ & + \psi'(C\tilde{g}^{i\bar{i}}|u_i|^2 + \frac{1}{4}\mathrm{tr}_{\tilde{g}}g).\end{aligned} \tag{3.25}$$

On the other hand we have

$$\begin{aligned}2\sum_{q>\mu}\frac{\mathrm{Re}\left(\tilde{g}^{i\bar{i}}\overline{T^q_{1i}}\tilde{g}_{1\overline{q},i}\right)}{\lambda_1} &\geq -\sum_{q>\mu}\frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\overline{q},i}|^2}{\lambda_1(\lambda_1-\lambda_q)} - \sum_{q>\mu}\tilde{g}^{i\bar{i}}|T^q_{1i}|^2\frac{(\lambda_1-\lambda_q)}{\lambda_1} \\ &\geq -\sum_{q>\mu}\frac{\tilde{g}^{i\bar{i}}|\tilde{g}_{1\overline{q},i}|^2}{\lambda_1(\lambda_1-\lambda_q)} - C\mathrm{tr}_{\tilde{g}}g.\end{aligned} \tag{3.26}$$

Combining (3.20) with (3.21), (3.22), (3.23), (3.24), (3.25) and (3.26) we obtain for a uniform constant C,

$$\begin{aligned}0 \geq & \left(\frac{1}{4}\varphi' - \frac{C}{\lambda_1}\right)\sum_{i,p}\tilde{g}^{i\bar{i}}\left(|u_{p\bar{i}}|^2 + |u_{pi}|^2\right) + (-\psi'/2 - C)\,\mathrm{tr}_{\tilde{g}}g \\ & + (\psi'' + C\psi')\tilde{g}^{i\bar{i}}|u_i|^2 + \psi' n.\end{aligned}$$

But since we may assume that $\lambda_1 \geq 4CK$, the first term on the right hand side is nonnegative. Pick $A = 2(C+1)$ so that $-\psi'/2 - C \geq 1$ and $\psi'' + C\psi' \geq 0$. Then $\mathrm{tr}_{\tilde{g}}g$ and hence λ_1 is uniformly bounded from above at the maximum of Q, and the result follows. □

Remark. In the proof above we used a viscosity type argument to deal with the non-differentiability of the largest eigenvalue λ_1. There are other methods to deal with this issue: one is to use a perturbation argument as in [26, 27]; another is to replace λ_1 by a carefully chosen quadratic function of $\tilde{g}_{i\bar{j}}$ as in [33].

4. Proof of the main theorem

4.1. Higher order estimates

First, we discuss the *a priori* higher order estimates, in the same setting as Theorems 2.1 and 3.1. Thanks to the estimates in these Theorems, a blowup argument can be employed exactly as in [8, 26, 27, 31] to obtain that $\sup_M |\partial u|_g \leq C$, and therefore also $\sup_M \mathrm{tr}_g \tilde{g} \leq C$. Here we use the classical Liouville Theorem stating that a bounded plurisubharmonic function on $\mathbb{C}^n$ is constant (indeed, by restricting to complex lines, this reduces to the well-known fact that a bounded subharmonic function in $\mathbb{C}$ is constant).

The PDE (2.1) then implies that $\tilde{g}$ is uniformly equivalent to g, at which point we can then apply the Evans-Krylov theory [9, 20, 34] (see also [28]) to obtain uniform *a priori* $C^{2,\alpha}$ bounds on u, for some uniform $0 < \alpha < 1$. Differentiating the equation and using Schauder theory, we then deduce uniform *a priori* C^k bounds for all $k \geq 0$.

4.2. Existence of a solution

We employ the continuity method. For $t \in [0,1]$ we consider the family of equations for (u_t, b_t)

$$\begin{aligned}&\det(g_{i\bar{j}} + a_i u_{t,\bar{j}} + a_{\bar{j}} u_{t,i} + u_{t,i\bar{j}}) = e^{tF+b_t}\det(g_{i\bar{j}}),\\ &\text{with } (g_{i\bar{j}} + a_i u_{t,\bar{j}} + a_{\bar{j}} u_{t,i} + u_{t,i\bar{j}}) > 0.\end{aligned} \tag{4.1}$$

Suppose we have a solution for $t = \hat{t}$ and write

$$\hat{\omega} = \omega + \sqrt{-1}a \wedge \overline{\partial} u_{\hat{t}} - \sqrt{-1}\overline{a} \wedge \partial u_{\hat{t}} + \sqrt{-1}\partial\overline{\partial} u_{\hat{t}},$$

and $\hat{H}$ for the linearized operator defined as in (2.5). By the same argument of Gauduchon [14] that was mentioned earlier, we may find a smooth function v, normalized by $\int_M e^v \hat{\omega}^n = 1$, such that

$$\int_M \hat{H}(\psi) e^v \hat{\omega}^n = 0,$$

for all smooth functions ψ, i.e. e^v generates the kernel of the adjoint $\hat{H}^*$ of $\hat{H}$ (with respect to the L^2 inner product with volume form $\hat{\omega}^n$). Fix $0 < \alpha < 1$ and consider the operator

$$\Upsilon(\psi) = \log \frac{(\hat{\omega} + \sqrt{-1}a \wedge \overline{\partial}\psi - \sqrt{-1}\overline{a} \wedge \partial\psi + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\hat{\omega}^n}$$

$$-\log\left(\int_M e^v(\hat{\omega}+\sqrt{-1}a\wedge\overline{\partial}\psi-\sqrt{-1}\overline{a}\wedge\partial\psi+\sqrt{-1}\partial\overline{\partial}\psi)^n\right),$$

mapping $C^{3,\alpha}$ functions ψ with zero average (and such that $\hat{\omega}+\sqrt{-1}a\wedge\overline{\partial}\psi-\sqrt{-1}\overline{a}\wedge\partial\psi+\sqrt{-1}\partial\overline{\partial}\psi>0$) to the space of $C^{1,\alpha}$ functions w satisfying $\int_M e^{w+v}\hat{\omega}^n=1$ (whose tangent space at 0 consists precisely of $C^{1,\alpha}$ functions orthogonal to the kernel of $\hat{H}^*$). For any $C^{3,\alpha}$ function ζ we have

$$\int_M e^v\hat{H}(\zeta)\hat{\omega}^n=\int_M \zeta\hat{H}^*(e^v)\hat{\omega}^n=0,$$

hence the linearization of Υ at 0 is $\hat{H}$. Thanks to the Fredholm alternative, $\hat{H}$ is an isomorphism of the tangent spaces, and so the Inverse Function Theorem provides us with $C^{3,\alpha}$ functions ψ_t for t near $\hat{t}$ which satisfy

$$\Upsilon(\psi_t)=(t-\hat{t})F-\log\left(\int_M e^{(t-\hat{t})F}e^v\hat{\omega}^n\right),$$

so that $u_t=u_{\hat{t}}+\psi_t$ solve (4.1) for some $b_t\in\mathbb{R}$. Lastly, differentiating (4.1) and using Schauder estimates and bootstrapping, we easily see that our $C^{3,\alpha}$ solutions are in fact smooth.

This establishes that the set of all $t\in[0,1]$ for which we have a solution (u_t,b_t) of (4.1) is open (and nonempty, since we can take $(u_0,b_0)=(0,0)$). At this point we can also impose that $\sup_M u_t=0$ by adding a t-dependent constant. To show that the set of such $t\in[0,1]$ is also closed, it suffices to prove *a priori* estimates for u_t (in C^k for all $k\geq 0$) and b_t. The bound $|b_t|\leq\sup_M|F|$ is elementary by the maximum principle, and then the estimates for u_t follow from section 4.1 above.

4.3. Uniqueness

In the setting of the main theorem 1.1, uniqueness of b and u follows from a simple maximum principle argument, see e.g. [7].

Acknowledgments

Both authors owe many thanks to Professor Phong, to whom this article is dedicated. His mathematical wisdom and insights are an inspiration to us. Happy birthday Phong!

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