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Twisted Kähler-Einstein metrics

Julius Ross* and Gábor Székelyhidi†

Dedicated to D. H. Phong on the occasion of his 65th birthday

Abstract: We prove an existence result for twisted Kähler-Einstein metrics, assuming an appropriate twisted K-stability condition. An improvement over earlier results is that certain non-negative twisting forms are allowed.

1. Introduction

Let M be a Fano manifold, together with a line bundle $T \to M$. Let $\beta \in c_1(T)$ be a smooth non-negative form that can be expressed as an average

$$\beta = \int_{|T|} [D]\, d\mu(D), \tag{1}$$

where $d\mu$ is a volume form on the linear system $|T|$. A typical example is obtained if $|T|$ is basepoint free, and β is the pullback of the Fubini-Study metric under the corresponding map $M \to \mathbf{P}^N$ (see [17, Theorem 19]). More generally we could allow the divisors D to be in the linear system $|kT|$ for some $k > 1$, but for simplicity of notation we will only consider the case $k = 1$.

Our goal is to study the existence of solutions to the equation

$$\mathrm{Ric}(\omega) = \omega + \beta \tag{2}$$

on M. We necessarily have $\omega \in c_1(L)$, where $L = K^{-1} \otimes T^{-1}$ in terms of the canonical bundle K of M. We call a solution ω of this equation a twisted Kähler-Einstein metric on (M, β). The main result is the following.

Theorem 1. *There exists a twisted Kähler-Einstein metric on (M, β) if (M, β) is K-stable.*

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We will define K-stability of the pair (M, β) in Section 2 below. Note that if T is trivial, so that $\beta = 0$, then $L = K^{-1}$, and we are seeking a Kähler-Einstein metric on M. In this case Theorem 1 was proven by Chen-Donaldson-Sun [4] in solving the Yau-Tian-Donaldson conjecture [29, 26, 14]. When $\beta \in c_1(M)$ is strictly positive, Datar and the second author [7] showed a slightly weaker statement, namely that if (M, β) is K-stable, then for any $\epsilon > 0$ there is a solution of the equation $\mathrm{Ric}(\omega) = \omega + (1+\epsilon)\beta$. This is more or less equivalent to replacing "K-stable" by "uniformly K-stable" in the statement of Theorem 1. In much more generality, allowing positive currents β, the result assuming uniform K-stability was also shown by Berman-Boucksom-Jonsson [23], using very different techniques. In the setting when $\beta \in c_1(M)$ is the current of integration along a smooth divisor, the statement of Theorem 1 was also shown by Chen-Donaldson-Sun [4], where instead of twisted Kähler-Einstein metrics, one considers Kähler-Einstein metrics with cone singularities along the divisor. Let us also remark that it would be natural to extend Theorem 1 to pairs (M, β) that admit automorphisms, using a suitable notion of K-polystability rather than K-stability. This would not introduce substantial new difficulties, however in this paper we focus on the case of no automorphisms to simplify the discussion.

In Section 2 we will give the definition of K-stability of a pair (M, β), which is similar to log-K-stability [18] and twisted K-stability [9]. In the case when β is the pullback of a positive form by a map, stability of the pair is related to the stability of the map in the sense of [10]. We then prove Theorem 1 in Section 3 along the lines of the argument in [7]. An important simplification of the prior arguments in Chen-Donaldson-Sun [4] as well as [25, 7] is provided by the work of the second author and Liu [19] on Gromov-Hausdorff limits of Kähler manifolds with only lower bounds on the Ricci curvature, rather than a two-sided bound as in Donaldson-Sun [16]. An additional observation, given in Corollary 9 below, allows us to obtain the existence of a twisted Kähler-Einstein metric under the assumption of K-stability, rather than the stronger uniform K-stability which would follow more directly from the methods of [7].

2. K-stability

Let M, T, β be as in the introduction, and $L = K^{-1} \otimes T^{-1}$. Note that since M is Fano, the line bundles T, L are uniquely determined by β, given that $\beta \in c_1(T)$. In this section we discuss K-stability of (M, β), and prove some basic properties. First we have the following definition, which agrees with that in Tian [26] when T is the trivial bundle so that $\beta = 0$.

Definition 2. A special degeneration for (M, L) of exponent $r > 0$ consists of an embedding $M \subset \mathbf{P}^N$ using a basis of sections of L^r, together with a $\mathbf{C}^*$-action λ on $\mathbf{P}^N$, such that the limit $\lim_{t\to 0} \lambda(t) \cdot M$ is a normal variety.

We will refer to a special degeneration by the $\mathbf{C}^*$-action λ, leaving implicit the projective embedding of M that is also part of the data. Next, we define the Donaldson-Futaki invariant $DF(M, \lambda)$ in the same way as in Donaldson [14], in terms of the weights of the action on the spaces of sections $H^0(M, L^{kr})$ as $k \to \infty$. In addition we will need a differential geometric formula for the Donaldson-Futaki invariant. For this let $Z = \lim_{t\to 0} \lambda(t) \cdot M$. We can assume that the S^1-subgroup of λ acts through $SU(N + 1)$, and so we have a Hamiltonian function θ on $\mathbf{P}^N$ generating λ.

Proposition 3. *Let ω denote the restriction of the Fubini-Study metric to Z. We then have*

$$DF(M, \lambda) = -V^{-1} \int_Z \theta \left(n\mathrm{Ric}(\omega|_Z) - \hat{R}\omega\right) \wedge \omega^{n-1}, \tag{3}$$

where V is the volume of Z, and $\hat{R}$ is the average scalar curvature, so that the integral above is unchanged by adding a constant to θ.

Proof. Let us denote by ω_s the restriction of the Fubini-Study metric on $\lambda(e^{-s}) \cdot M$. We thus have a family of metrics $\omega_s = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_s$ on M in a fixed Kähler class. Since the central fiber Z of our degeneration is normal, the Donaldson-Futaki invariant $DF(M, \lambda)$ is given by the asymptotic derivative of the Mabuchi functional [20] along this family ω_s (see Paul-Tian [22, Corollary 1.3]). I.e. we have

$$DF(M, \lambda) = \lim_{s\to\infty} -V^{-1} \int_M \dot{\varphi}_s (n\mathrm{Ric}(\omega_s) - \hat{R}\omega_s) \wedge \omega_s^{n-1}. \tag{4}$$

In addition we have $\dot{\varphi}_s = \theta$ under identifying M with $\lambda(e^{-s}) \cdot M$. It therefore remains to show that these integrals on M converge to the corresponding integral on Z.

If Z were smooth, then this convergence would be immediate. It is thus enough to show that the singularities of Z do not contribute to the limit. For this, note first that we have a uniform upper bound $\mathrm{Ric}(\omega_s) < C\omega_s$ for the Ricci curvatures, where C depends on the curvature of the Fubini-Study metric, since curvature decreases in holomorphic subbundles. We can view $C\omega_s - \mathrm{Ric}(\omega_s)$ as a positive current of dimension $(n - 1, n - 1)$, supported on $\lambda(e^{-s}) \cdot M$. As $s \to \infty$, these currents converge (along a subsequence if necessary) weakly to a limit current T, supported on Z. On the regular part of Z, this limit current is necessarily given by $C\omega - \mathrm{Ric}(\omega)$ in terms of the Fubini-Study metric ω, and since the codimension of the singular set is at least 2, this determines T. □

We are now ready to define the twisted Futaki invariant of the special degeneration.

Definition 4. Suppose that we have a special degeneration λ for M with Hamiltonian θ as above, and $Z = \lim_{t\to 0}\lambda(t)\cdot M$. Under the assumption (1) we have an induced current $\gamma = \lim_{t\to 0}\lambda(t)_*\beta$ on Z. The twisted Futaki invariant of this special degeneration is then defined to be

$$\mathrm{Fut}_\beta(M,\lambda) = DF(M,\lambda) + nV^{-1}\int_Z \theta\,(\gamma - c\omega_{FS})\wedge\omega_{FS}^{n-1}, \tag{5}$$

where c is a constant so that the expression is invariant under adding a constant to θ.

Given this, we define K-stability of (M,β) as follows.

Definition 5. The pair (M,β) is K-stable, if $\mathrm{Fut}_\beta(M,\lambda)\geq 0$ for all special degenerations for (M,L), with equality only if λ is trivial.

It will be important for us to replace the smooth form β with currents of integration along divisors. The definition of the twisted Futaki invariant above applies in this case too, leading to log-K-stability (see Donaldson [15], Li [18]), and we will need to compare these two notions. As in [7], the twisted Futaki invariant with a smooth form β is the same as the twisted Futaki invariant using a generic divisor in the same class. This follows from the decomposition (1), together with the following result from Wang [27, Theorem 26].

Proposition 6. *Let $D\subset \mathbf{P}^N$ have dimension $n-1$, and λ a $\mathbf{C}^*$-action with Hamiltonian θ as above. Suppose that θ is normalized to have zero average on $\mathbf{P}^N$. Let $D_0 = \lim_{t\to 0}\lambda(t)\cdot D$, and denote by $w(D_0,\lambda)$ the weight of the induced action on the Chow line over D_0. Then (up to a multiplicative normalization constant)*

$$w(D_0,\lambda) = -\int_{D_0}\theta\,\omega^{n-1}. \tag{6}$$

Under a projective embedding of the Chow variety, we can view each D as in this proposition as a line in a vector space V spanned by a vector v_D. The weight $w(D_0,\lambda)$ is determined by the lowest weight in the weight decomposition of v_D under the $\mathbf{C}^*$-action λ. It follows that as D varies in a linear system as in (1), there will be a hyperplane section $H \subset |T|$ such that the corresponding weights will all be equal for $D \notin H$. More precisely we have the following.

Proposition 7. *Given any $\mathbf{C}^*$-action λ with Hamiltonian θ on $\mathbf{P}^N$, there is a hyperplane $H \subset |T|$ such that for all $D \in |T|$ we have*

$$\lim_{t\to 0}\int_{\lambda(t)\cdot D}\theta\omega^{n-1} \leq \lim_{t\to 0}\int_{\lambda(t)\cdot M}\theta\,(\lambda(t))_*\beta\wedge\omega^{n-1}, \tag{7}$$

with equality for $D \in |T| \setminus H$. In addition, given an action of a torus $\mathbf{T}$, we can choose a $D \in |T|$ such that equality holds above for all $\lambda \subset \mathbf{T}$.

Proof. (Compare [17, Lemma 9].) Using (1) the equation (7) is true when averaged over $|T|$, i.e. we have

$$\int_{|T|}\lim_{t\to 0}\int_{\lambda(t)\cdot D}\theta\omega^{n-1}\,d\mu(D) = \lim_{t\to 0}\int_{\lambda(t)\cdot M}(\lambda(t))_*\beta\wedge\omega^{n-1}. \tag{8}$$

At the same time by Proposition 6, up to a normalizing constant, the limit on the left hand side of (7) is a Chow weight in geometric invariant theory. In particular it is given by the minimal weight under the weight decomposition of the vector corresponding to D in the Chow variety, under the $\mathbf{C}^*$-action λ. Generically, i.e. on the complement of a hyperplane (corresponding to the vanishing of the lowest weight component), this weight will achieve its minimum and is independent of D.

For the second statement in the Proposition, we can take a generic D that has a non-zero component in all the weight spaces which appear under the action of $\mathbf{T}$ on elements in $|T|$. ☐

This result leads to an important finiteness property of special degenerations inside a fixed projective space. We first have the following (that is essentially a standard piece of Geometric Invariant Theory).

Lemma 8. *Fix $r > 0$. There is a finite set $\mathcal{F} \subset \mathbf{R}$ with the following property. Suppose that we have a special degeneration λ of exponent r for M, and a divisor $D \in |T|$ on M such that the limit (M_0, D_0) of the pair (M, D) under λ is not fixed by any $\mathbf{C}^*$ subgroup of $SL(N+1)$ commuting with λ, apart from λ itself (i.e. the centralizer of λ in the stabilizer group is just λ). Let θ be the Hamiltonian for λ normalized to have zero average on $\mathbf{P}^N$, and let $\|\lambda\|$ denote the L^2-norm of θ on $\mathbf{P}^N$. Then the normalized twisted Futaki invariant $\|\lambda\|^{-1}\mathrm{Fut}_D(M,\lambda)$ lies in $\mathcal{F}$.*

Proof. Note first of all that since any $\mathbf{C}^*$-subgroup can be conjugated into a maximal torus of $SL(N+1)$, up to moving the pair (M,D) in its orbit, we can assume that λ is in a fixed maximal torus $\mathbf{T}$. Then if (M_0,D_0) is as in the statement of the Lemma, the normalized twisted Futaki invariant is determined by the pair (M_0,D_0), since the induced $\mathbf{C}^*$-action is uniquely determined up to scaling.

The pair (M_0,D_0) is represented by a point in a product of Chow varieties, i.e. under a projective embedding by a line spanned by a vector v in a vector space V admitting a $\mathbf{T}$-action. Under the decomposition of V into weight spaces for the $\mathbf{T}$-action, the weights appearing in the decomposition of v must lie in a codimension-one affine subspace of $\mathfrak{t}^*$ by the assumption that (M_0,D_0) has a one dimensional stabilizer in $\mathbf{T}$. The normalized twisted Futaki invariant is determined by this affine subspace rather than the components of v in each corresponding weight space. Since there are only a finite number of possible such affine subspaces, we can have only finitely many different normalized twisted Futaki invariants. □

Corollary 9. *Fix $r>0$. Suppose that for any $\epsilon>0$ we have a special degeneration λ of exponent r for (M,L) such that $\|\lambda\|^{-1}\mathrm{Fut}_\beta(M,\beta)<\epsilon$. Then (M,β) is not K-stable.*

Proof. Given a special degeneration λ, we will show that we can either find another special degeneration with non-positive twisted Futaki invariant, or we can find a special degeneration λ' to which Lemma 8 applies, and which has smaller normalized twisted Futaki invariant than λ. If ϵ is sufficiently small, this will necessarily be non-positive.

By conjugating, we can assume that λ is in a fixed maximal torus $\mathbf{T}$. By Proposition 7, we can choose a $D\in|T|$, such that the twisted Futaki invariant $\mathrm{Fut}_\beta(M,\tau)=\mathrm{Fut}_D(M,\tau)$ for any $\mathbf{C}^*$ subgroup τ in $\mathbf{T}$. Let us consider the effect of varying the $\mathbf{C}^*$-action on the central fiber and the normalized twisted Futaki invariant.

As above, we can view the pair (M,D) as a line spanned by a vector v in a vector space V with an action of $\mathbf{T}$. We decompose $v=\sum v_{\alpha_i}$ into components on which the torus acts by weights $\alpha_i\in\mathfrak{t}^*$. Let us denote by $\mathcal{W}\subset\mathfrak{t}^*$ the weights that appear in this decomposition. For any $\mathbf{C}^*$-subgroup $\tau\subset\mathbf{T}$, we will also denote by $\tau\in\mathfrak{t}$ its generator. The central fiber (M_0,D_0) under this $\mathbf{C}^*$ is determined by the sum of those components v_α for which $\langle\alpha,\tau\rangle$ is minimal, i.e. $\langle\alpha,\tau\rangle\leq\langle\beta,\tau\rangle$ for all $\beta\in\mathcal{W}$. Let us denote by $\mathcal{W}_\tau\subset\mathcal{W}$ the set of these minimal weights. The stabilizer of (M_0,D_0) in $\mathbf{T}$ is then the subgroup with Lie algebra

$$\{\eta\in\mathfrak{t}\,|\,\eta\text{ is constant on }\mathcal{W}_\tau\},\tag{9}$$

where we can view any $\eta \in \mathfrak{t}$ as a function on $\mathfrak{t}^*$. In particular the stabilizer of (M_0, D_0) is τ precisely when $\mathcal{W}_\tau$ spans a codimension-one affine subspace in $\mathfrak{t}^*$.

Consider our given special degeneration λ. If $\mathcal{W}_\lambda$ spans a codimension-one affine subspace, then we are already done. Otherwise, we can find another $\mathbf{C}^*$-action τ which is orthogonal to λ in $\mathfrak{t}$ (here we use the inner product on $\mathfrak{t}$ given by the L^2-product on $\mathbf{P}^N$ of the corresponding Hamiltonian functions), and is constant on $\mathcal{W}_\lambda$. For rational t let us consider the $\mathbf{C}^*$-actions $\lambda + t\tau$. We can find an interval (a_1, a_2) containing 0, such that if $t \in (a_1, a_2)$ then $\mathcal{W}_{\lambda+t\tau} = \mathcal{W}_\lambda$, however for $i = 1, 2$ we have $\mathcal{W}_{\lambda+a_i\tau} \supsetneq \mathcal{W}_\lambda$. For $t \in (a_1, a_2)$ the central fibers (M_0, D_0) of the degenerations given by $\lambda + t\tau$ will all be the same. As a result the twisted Futaki invariant varies linearly in t, while the norm is smallest when $t = 0$. It follows that the normalized twisted Futaki invariant of $\lambda + t\tau$ will be strictly smaller for either $t = a_1$ or $t = a_2$ than for $t = 0$. Moreover the original central fiber (M_0, D_0) will be a specialization of the new (M_0', D_0'), and so M_0' is also normal. The new central fiber has smaller stabilizer, and so after finitely many such steps the result follows. □

3. Proof of the main result

In this section we prove Theorem 1, along similar lines to the argument in [7]. Instead of the partial C^0-estimate in [25], we will use the main result in [19], which leads to substantial simplifications, and allows us to work with non-negative β rather than just those that are strictly positive. We first set up the relevant continuity method.

3.1. The continuity method

Let $\alpha \in c_1(L)$ be a Kähler form, and consider the equations

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha + \beta, \tag{10}$$

for $\omega_t \in c_1(L)$. For $t = 0$ the equation can be solved using Yau's theorem [28], and the set of $t \in [0, 1]$ for which the solution exists is open. Suppose that we can solve the equation for $t \in [0, T)$. If $t > t_0 > 0$, then by Myers' theorem we have a diameter bound, and since the volume is fixed, the Bishop-Gromov theorem implies that the manifolds (M, ω_t) are uniformly non-collapsed. Along a sequence $t_k \to T$, we can extract a Gromov-Hausdorff limit Z. Let us denote by M_k the metric spaces (M, ω_{t_k}), so $M_k \to Z$ in the Gromov-Hausdorff sense.

Theorem 1.1 in [19] (which is based on ideas of Donaldson-Sun [16]) implies that for a sufficiently large $\ell > 0$, we have a sequence of uniformly Lipschitz holomorphic maps $F_k : M_k \to \mathbf{P}^N$, using sections of L^ℓ. These converge to a Lipschitz map $F_\infty : Z \to \mathbf{P}^N$ that is a homeomorphism to its image. We will identify Z with its image $F_\infty(Z)$, which is a normal projective variety. Up to choosing a further subsequence we can assume that

$$(F_k)_*[(1-t_k)\alpha + \beta] \to \gamma \tag{11}$$

weakly for a positive current γ on Z. Note that since the F_k are all defined using sections of L^ℓ, we have a sequence $g_k \in PGL(N+1)$ such that $F_k = g_k \circ F_1$, so Z is in the closure of the $PGL(N+1)$-orbit of $F_1(M)$.

We next show that Z admits a twisted Kähler-Einstein metric, which we can formally view as a solution of the equation $\mathrm{Ric}(\omega_T) = T\omega_T + \gamma$. More precisely, let us denote by L the $\mathbf{Q}$-line bundle on Z such that $L^l = \mathcal{O}(1)$. We then have the following.

Proposition 10. *The $\mathbf{Q}$-line bundle L over Z admits a metric with locally bounded potentials with the following property. Locally on Z_{reg}, if the metric is given by $e^{-\varphi_T}$, then its curvature form ω_{φ_T} satisfies*

$$\omega^n_{\varphi_T} = e^{-T\varphi_T - \psi} \tag{12}$$

in the sense of measures, where $\sqrt{-1}\partial\overline{\partial}\psi = \gamma$. Here Z_{reg} denotes the regular set of Z in the complex analytic sense.

Proof. The metric on (a power of) L is obtained by the partial C^0-estimate, as a limit of metrics h_k on $L \to M_k$ that have curvature ω_{t_k}. More concretely, the partial C^0-estimate implies that under our embeddings $F_k : M_k \to \mathbf{P}^N$, the pullback of the Fubini-Study metric is uniformly equivalent to h_k. Using this we can extract a limit metric on $\mathcal{O}(1)|_Z$ which will also be uniformly equivalent to the restriction of the Fubini-Study metric.

Let us now consider a point $p \in Z_{reg}$ and a sequence $p_k \in M_k$ such that $p_k \to p$ under the Gromov-Hausdorff convergence. We have a holomorphic chart z_i on a neighborhood of p, and using the maps F_k this gives rise to charts z_{ki} on neighborhoods of $p_k \in M_k$ for large k, converging to z_i. Using these charts we can view the metrics ω_{t_k} as being defined on a fixed ball $B \subset \mathbf{C}^n$. By the gradient estimate for holomorphic functions, we have a uniform bound $\omega_{t_k} > C^{-1}\omega_{Euc}$. In addition, by [19, Proposition 3.1] we can assume (shrinking the charts if necessary) that we have uniformly bounded Kähler potentials φ_{t_k} for the ω_{t_k}. Let us denote by α_k, β_k the forms corresponding to α, β on

M. Equation (10) implies that α_k, β_k have potentials $\psi_{\alpha_k}, \psi_{\beta_k}$ satisfying the equation

$$\omega_{t_k}^n = e^{-t_k\varphi_{t_k}-(1-t_k)\psi_{\alpha_k}-\psi_{\beta_k}}, \tag{13}$$

i.e.

$$\mathrm{Ric}(\omega_{t_k}) = t_k\omega_{t_k} + (1-t_k)\alpha_k + \beta_k. \tag{14}$$

Our goal is to be able to pass this equation to the limit as $k \to \infty$, i.e. $t_k \to T$.

Let us observe first that since α, β are fixed forms on M, using the lower bound $\omega_{t_k} > C^{-1}\omega_{Euc}$, we have a uniform bound

$$\int_B [(1-t_k)\alpha_k + \beta_k] \wedge \omega_{Euc}^{n-1} < C. \tag{15}$$

It follows that we can take a weak limit

$$\gamma = \lim_{k\to\infty} (1-t_k)\alpha_k + \beta_k. \tag{16}$$

From (13), and the lower bound for ω_{t_k} we have uniform upper bounds for $(1-t_k)\psi_{\alpha_k}+\psi_{\beta_k}$. These psh functions can also not converge to $-\infty$ everywhere as $k\to\infty$, since the volume of B with respect to the metric ω_{t_k} is bounded above. It follows that up to choosing a subsequence we can extract a limit

$$(1-t_k)\psi_{\alpha_k} + \psi_{\beta_k} \to \psi, \text{ in } L^1_{loc}. \tag{17}$$

We then necessarily have $\gamma = \sqrt{-1}\partial\overline{\partial}\psi$.

Let $\kappa > 0$, and denote by E_κ the set where the Lelong numbers of γ are at least κ. By Siu's theorem [24] E_κ is a subvariety in B. From [19, Claim 4.3], and the subsequent argument, it follows that for any $q \notin E_\kappa$, we have $V_{2n} - \lim_{r\to 0} r^{-2n}\mathrm{vol}(B(q,r)) < \Psi(\kappa)$, where the volume is measured using the limit metric on Z. Here, and below, $\Psi(\kappa)$ denotes a function converging to zero as $\kappa\to 0$, which may change from line to line. In other words in the limit space Z the complement of E_κ is contained in the ϵ-regular set for $\epsilon = \Psi(\kappa)$.

Suppose now that $q \notin E_\kappa$, and δ is sufficiently small so that $V_{2n} - \delta^{-2n}\mathrm{vol}(B(q,\delta)) < \epsilon$, where V_{2n} is the volume of the Euclidean unit ball. Then we can apply Lemma 11 below to see that on $B(q,\delta)$ the metrics ω_{t_k} are bi-Hölder equivalent to ω_{Euc}. On these balls the Kähler potentials φ_{t_k} satisfy uniform gradient estimates with respect to ω_{t_k}, since $\Delta_{\omega_{t_k}}\varphi_{t_k} = n$, and so the φ_{t_k} satisfy uniform Hölder bounds with respect to ω_{Euc}. It follows from this that up to choosing a subsequence we can find a limit $\varphi_{t_k} \to \varphi_T$ in $C^\alpha_{loc}(B \setminus E_\kappa)$, and φ_T is uniformly bounded on B. In particular for $\omega_T = \sqrt{-1}\partial\overline{\partial}\varphi_T$, the measures $\omega_{t_k}^n$ converge weakly to ω_T^n on $B \setminus E_\kappa$.

To derive the required equation (12), we note that on $B \setminus E_\kappa$ we have

$$e^{-(1-t_k)\psi_{\alpha_k} - \psi_{\beta_k}} \to e^{-\psi} \text{ in } L^1_{loc}. \tag{18}$$

From the semicontinuity theorem of Demailly-Kollár [8] this follows if we bound the Lelong numbers of ψ, which will be the case if κ is sufficiently small. It follows that on $B \setminus E_\kappa$ we have an equality of measures $\omega_T^n = e^{-T\varphi_T - \psi}$, and since E_κ has zero measure with respect to ω_T^n, the equality holds on B as well. □

We used the following lemma in the argument.

Lemma 11. *Suppose that $B(p,1)$ is a unit ball in a Kähler manifold with* Ric ≥ 0, *together with holomorphic coordinates z_i that give an ϵ-Gromov-Hausdorff approximation of $B(p,1)$ to the Euclidean unit ball $B(0,1) \subset \mathbf{C}^n$. There exists an $\alpha > 1 - \Psi(\epsilon)$ and $C > 0$ such that for $q, q' \in B(p, 1/2)$ we have*

$$d(q,q') \leq C|z(q) - z(q')|^\alpha. \tag{19}$$

As above, $\Psi(\epsilon)$ denotes a function converging to zero as $\epsilon \to 0$, which may change from line to line.

Proof. We can assume that $z(p) = 0$. It is enough to prove that for any $\delta > 0$, if ϵ is sufficiently small, then for all $k > 0$ and $q \notin B(p, 2^{-k})$, we have $|z(q)| > (2+\delta)^{-k}$. We prove this by induction.

Suppose that we have shown that $|z| > (2+\delta)^{-k}$ outside of $B(p,2^{-k})$. Denote by $2^k B(p,2^{-k})$ the same ball scaled up to unit size. By Colding's volume convergence theorem [6] and the Bishop-Gromov monotonicity, together with [19, Theorem 2.1], we have holomorphic coordinates w on this ball, giving a $\Psi(\epsilon)$-Gromov-Hausdorff approximation to the Euclidean unit ball. We can assume that $w(p) = 0$. Let us also use the coordinates $z' = (2+\delta)^k z$, which map our ball onto a region containing the Euclidean unit ball. Viewing w as a function of z, the Schwarz lemma implies that $|w| \leq (1 + \Psi(\epsilon))|z'|$ on the unit z'-ball, and so in particular, using that w is a Gromov-Hausdorff approximation, we have $|z'| \geq (1 - \Psi(\epsilon))/2$ outside of the ball $2^k B(p, 2^{-k-1})$. Scaling back, this means that $|z| \geq (2 + \Psi(\epsilon))^{-1}(2+\delta)^{-k}$ outside of $B(p, 2^{-k-1})$. We then just need to choose ϵ small enough to make $\Psi(\epsilon) < \delta$, and the inductive step follows. □

3.2. The Ding functional and the Futaki invariant

We will next use the existence of a twisted Kähler-Einstein metric as in Proposition 10 to deduce the vanishing of the twisted Futaki invariant, and the reductivity of the automorphism group.

Let $Z \subset \mathbf{P}^N$ be a normal variety, together with the following additional data. We have a $\mathbf{Q}$-line bundle L on Z (a power of which is just $\mathcal{O}(1)$), and a locally bounded metric $e^{-\varphi_0}$ on L. In addition we have a closed positive current γ on Z. We say that these define a twisted Kähler-Einstein metric if the conclusion of Proposition 10 holds, i.e. locally on Z_{reg} we have the equation $\omega_{\varphi_0}^n = e^{-T\varphi_0-\psi}$, where $\sqrt{-1}\partial\overline{\partial}\psi = \gamma$. In terms of this we can define the twisted Ding functional on the space of all metrics $e^{-\varphi}$ with locally bounded potentials. Abusing notation slightly, we will denote by $e^{-T\varphi-\psi}$ the measure

$$e^{-T\varphi-\psi} = e^{-T(\varphi-\varphi_0)}\omega_{\varphi_0}^n. \tag{20}$$

Note that while φ, φ_0 are only locally defined in terms of trivializations of L, $\varphi - \varphi_0$ is a globally defined bounded function on Z.

We have the Monge-Ampère energy functional E, defined by its variation

$$\delta E(\varphi) = \frac{1}{V}\int_Z \delta\varphi\, \omega_\varphi^n, \tag{21}$$

where V is the volume of Z with respect to ω_φ, and we define the twisted Ding functional [12] by

$$\mathcal{D}(\varphi) = -TE(\varphi) - \log\left(\int_Z e^{-T\varphi-\psi}\right). \tag{22}$$

The variation of $\mathcal{D}$ is

$$\delta\mathcal{D}(\varphi) = -TV^{-1}\int_Z \delta\varphi\, \omega_\varphi^n - \frac{\int_Z -T(\delta\varphi)e^{-T\varphi-\psi}}{\int_Z e^{-T\varphi-\psi}}, \tag{23}$$

and so the critical points satisfy

$$\omega_\varphi^n = Ce^{-T\varphi-\psi}. \tag{24}$$

Up to changing φ by addition of a constant, this is the twisted KE equation as required.

The convexity of the twisted Ding functional follows exactly Berndtsson's argument in [3] (see also [7]), and so in particular if there is a critical point, then $\mathcal{D}$ is bounded below. As in [4, 7], the key consequences of this convexity are the reductivity of the automorphism group of (Z, γ), and the vanishing of a twisted Futaki invariant.

The reductivity of the automorphism group is a generalization of Matsushima's theorem for Kähler-Einstein metrics [21] (see also [1, 2, 3, 5, 11]). Following [7], we define the Lie algebra stabilizer of (Z, γ), as a subalgebra of $\mathfrak{sl}(N+1, \mathbf{C})$ by

$$\mathfrak{g}_{Z,\gamma} = \{w \in H^0(TZ) \,:\, \iota_w \gamma = 0\}. \tag{25}$$

We then have, following [5] (see also [7, Proposition 7]).

Proposition 12. *Suppose that Z admits a twisted KE metric as above. Then $\mathfrak{g}_{Z,\gamma}$ is reductive.*

Following Chen-Donaldson-Sun [4] we also apply the convexity of the twisted Ding functional to deduce the vanishing of a twisted Futaki invariant on Z. For this we consider the variation of $\mathcal{D}$ along a 1-parameter group of automorphisms which fixes the twisting current γ. If the automorphisms are generated by a vector field v with Hamiltonian θ, then the variation of φ is θ, so we get

$$\mathrm{Fut}_{T,\gamma}(Z, v) = -TV^{-1} \int_Z \theta \omega_\varphi^n + T \frac{\int_Z \theta e^{-T\varphi - \psi}}{\int_Z e^{-T\varphi-\psi}}. \tag{26}$$

As a result we have the following.

Proposition 13. *Suppose that Z admits a twisted KE metric as above, and let $e^{-\varphi}$ be a metric on L with locally bounded potentials. Suppose that v is a holomorphic vector field on Z with a lift to L, such that the imaginary part of v acts by isometries on L, and so that $\iota_v \gamma = 0$. Let θ denote a Hamiltonian for v, i.e. $L_v \omega_\varphi = \sqrt{-1}\partial\overline{\partial}\theta$. Then $\mathrm{Fut}_{T,\gamma}(Z, v) = 0$, where $\mathrm{Fut}_{T,\gamma}(Z, v)$ is defined as in (26).*

As in [7], we need to relate this formula to the "untwisted" Donaldson-Futaki invariant. A new difficulty here is that the metric ω is not in $c_1(Z)$, and so the Donaldon-Futaki invariant can not be expressed in terms of the Ding functional. Instead we use the differential geometric formula given in Proposition 3.

Let $e^{-\varphi}$ denote the restriction of the Fubini-Study metric to L on $Z \subset \mathbf{P}^N$, and ω_φ its curvature. We can use a method similar to Ding-Tian [13] to give a more differential geometric formula for the twisted Futaki invariant. The vector field v is given by the restriction of a holomorphic vector field on $\mathbf{P}^N$, and θ is the restriction to Z of a smooth function on $\mathbf{P}^N$. It follows that we have uniform bounds $|\theta|, |\nabla\theta|, |\Delta\theta| < C$ on Z_{reg}, where we are taking the gradient and Laplacian using the metric ω_φ on Z_{reg}. In addition we have an upper bound $\mathrm{Ric}(\omega_\varphi) < C\omega_\varphi$ on Z_{reg}, and so the current $C\omega_\varphi - [\mathrm{Ric}(\omega_\varphi) - \gamma]$ is positive for a sufficiently large constant C.

Proposition 14. *We have the equality*

$$\begin{aligned} -TV^{-1}\int_Z \theta\omega_\varphi^n + T\frac{\int_Z \theta e^{-T\varphi-\psi}}{\int_Z e^{-T\varphi-\psi}} \\ = -nV^{-1}\int_Z \theta(\mathrm{Ric}(\omega_\varphi) - T\omega_\varphi - \gamma)\wedge\omega_\varphi^{n-1}. \end{aligned} \tag{27}$$

Proof. Let us define the (twisted) Ricci potential u on Z_{reg} by

$$e^{-T\varphi-\psi-u} = \omega_\varphi^n. \tag{28}$$

Interpreting this as an equality of metrics on K^{-1} (on Z_{reg}) and taking curvatures, we have

$$T\omega_\varphi + \gamma + \sqrt{-1}\partial\overline{\partial}u = \mathrm{Ric}(\omega_\varphi). \tag{29}$$

Since the current $C\omega_\varphi - [\mathrm{Ric}(\omega_\varphi) - \gamma]$ on Z_{reg} is positive, we have $\sqrt{-1}\partial\overline{\partial}u \leq C\omega_\varphi$ on Z_{reg}. Since the singular set of Z has codimension at least 2, it follows from this that u is bounded below. Consider a resolution $\pi : \tilde{Z} \to Z$, and let η be a metric on $\tilde{Z}$. Let $\omega_\epsilon = \pi^*\omega_\varphi + \epsilon\eta$. Then ω_ϵ gives a family of smooth metrics on $\tilde{Z}$ converging to $\pi^*\omega_\varphi$ as $\epsilon \to 0$. Let us denote the pullback of u to $\tilde{Z}$ by u as well. We have $\sqrt{-1}\partial\overline{\partial}u \leq C\omega_\epsilon$ away from the exceptional set, and since u is bounded below, this inequality holds on all of $\tilde{Z}$. In particular we have $\Delta_\epsilon u \leq Cn$. Following Ding-Tian [13], we integrate the inequality

$$\int_{\tilde{Z}} \frac{\Delta_\epsilon u}{1 + (u - \inf u)}\omega_\epsilon^n \leq C \tag{30}$$

by parts to obtain

$$\int_{\tilde{Z}} \frac{|\nabla u|_\epsilon^2}{(1 + (u - \inf u))^2}\,\omega_\epsilon^n \leq C. \tag{31}$$

Letting $\epsilon \to 0$, we obtain the same estimate on Z_{reg} with the metric ω_φ. Just as in [13] we have that $u \in L^p$ for any p, and in turn this implies that we have a bound

$$\int_{Z_{reg}} |\nabla u|^p \omega_\varphi^n < C_p, \tag{32}$$

for any $p < 2$.

Differentiating the equation (28) along the vector field v we get that on Z_{reg}

$$-T\theta - v(\psi) - v(u) = \Delta\theta. \tag{33}$$

Note that we can think of $v(\psi)$ as being defined by this equation (since ψ itself is only defined in local charts), since all other terms are globally defined functions. In particular by the above estimate for u we have that $v(\psi)$ is in L^p for $p < 2$. At the same time, differentiating (29), and noting that $L_v\gamma = 0$, we get

$$\sqrt{-1}\partial\overline{\partial}[T\theta + v(u) + \Delta\theta] = 0, \tag{34}$$

and therefore we also have $\sqrt{-1}\partial\overline{\partial}v(\psi) = 0$. In particular $\Lambda = v(\psi)$ is a constant on Z, and so

$$-T\theta - \Lambda = \nabla\theta \cdot \nabla u + \Delta\theta. \tag{35}$$

Since the integral

$$\int_Z e^{-T\varphi-\psi} \tag{36}$$

is unchanged by flowing along the vector field v, we obtain

$$\int_Z (-T\theta - \Lambda)e^{-T\varphi-\psi} = 0. \tag{37}$$

Rearranging this,

$$\Lambda = -T\frac{\int \theta e^{-T\varphi-\psi}}{\int e^{-T\varphi-\psi}}. \tag{38}$$

Using this formula in (35), and integrating, we get

$$-T\int \theta\omega_\varphi^n + TV\frac{\int \theta e^{-T\varphi-\psi}}{\int e^{-T\varphi-\psi}} = \int (\nabla\theta \cdot \nabla u + \Delta u)\omega_\varphi^n, \tag{39}$$

where all integrals are on Z_{reg}. To integrate by parts, note that since the singular set of Z has real codimension at least 4, we can find cutoff functions

χ_ϵ with compact support in Z_{reg} such that $\chi_\epsilon = 1$ outside the ϵ-neighborhood of Z_{sing}, and $\|\nabla\chi_\epsilon\|_{L^4} < C$. We then have

$$\begin{aligned}\int_{Z_{reg}} \nabla\theta\cdot\nabla u\,\omega_\varphi^n &= \lim_{\epsilon\to 0}\int \chi_\epsilon\nabla\theta\cdot\nabla u\,\omega_\varphi^n \\ &= \lim_{\epsilon\to 0}\left[-\int\theta\nabla\chi_\epsilon\cdot\nabla u\,\omega_\varphi^n - \int\chi_\epsilon\theta\Delta u\,\omega_\varphi^n\right] \\ &= -\int\theta\Delta u\,\omega_\varphi^n,\end{aligned} \tag{40}$$

Here we used that $|\nabla u| \in L^{4/3}$, and so

$$\left|\int\theta\nabla\chi_\epsilon\cdot\nabla u\,\omega_\varphi^n\right| \le C\|\nabla\chi_\epsilon\|_{L^4}\left(\int_{\mathrm{supp}(\nabla\chi_\epsilon)}|\nabla u|^{4/3}\,\omega_\varphi^n\right)^{3/4} \to 0 \ \text{ as } \epsilon\to 0. \tag{41}$$

Similarly we can check that $\int\Delta u\,\omega_\varphi^n = 0$. In conclusion, from (39) we find that

$$-TV^{-1}\int\theta\omega_\varphi^n + T\frac{\int\theta e^{-T\varphi-\psi}}{\int e^{-T\varphi-\psi}} = -nV^{-1}\int_{Z_reg}\theta(\mathrm{Ric}(\omega_\varphi) - T\omega_\varphi - \gamma)\wedge\omega_\varphi^{n-1}, \tag{42}$$

as required. ☐

Suppose now that Z is the central fiber of a special degeneration for M induced by the one-parameter group $\lambda(t)$. Then using Proposition 3, we can relate the twisted Futaki invariant to the Donaldson-Futaki invariant as follows.

Corollary 15. *The twisted Futaki invariant above is given by*

$$\mathrm{Fut}_{T,\gamma}(Z,v) = DF(M,\lambda) + nV^{-1}\int_Z\theta(\gamma - c\omega_\varphi)\wedge\omega_\varphi^{n-1}, \tag{43}$$

where λ is a $\mathbf{C}^*$-action generated by the vector field v, and c is a constant so that the right hand side is unchanged when we add a constant to the Hamiltonian θ.

3.3. Completion of the proof of Theorem 1

We can now complete the proof of the main result. According to Corollary 9 it is enough to show that either we can find special degenerations for M with arbitrarily small twisted Futaki invariant, thereby contradicting the K-stability of (M, β), or $T = 1$ and the twisted KE metric that we obtained on Z is actually the twisted KE metric on M that we set out to find.

Let us denote by $Z \subset \mathbf{P}^N$ the Gromov-Hausdorff limit of (M, ω_{t_k}) along the continuity path (10). Using Proposition 10 we know that Z admits a twisted KE metric. In particular the pair (Z, γ) is in the closure of the $PGL(N+1)$-orbit of $(M, (1-T)\alpha + \beta)$, where $T = \lim t_k$, and we are identifying M with its image $F_1(M)$. We can now closely follow the method in [7] of approximating the forms α, β by currents of integration along divisors in M. Just like in [7], the twisted Futaki invariants become smaller as T increases (see [7, Equation (23)]). Because of this, and to simplify the discussion below, we will assume that $T = 1$. Note that unlike the setting in [7], here we still have a twisting term when $T = 1$, and so this case is not any easier than the case $T < 1$.

By assumption, the form β on M can be written as an integral of currents of integration, as in Equation (1). Recall also that we have the sequence $g_k \in PGL(N+1)$ such that $F_k = g_k \circ F_1$, and so $g_k(M) \to Z$. As in [7, Lemma 14], by choosing a subsequence we can ensure that each sequence $g_k(D)$ for $D \in |T|$ converges to a subvariety of $\mathbf{P}^N$ which we denote by $g_\infty(D)$. It follows that we have

$$(g_k)_*\beta \to \int_{|T|} [g_\infty(D)]\, d\mu(D), \tag{44}$$

in the weak topology. The twisting current γ on Z is obtained as the limit of $(g_k)_*\beta$ as $k \to \infty$, and so we have

$$\gamma = \int_{|T|} [g_\infty(D)]\, \mu(D). \tag{45}$$

Arguing as in [7, Lemma 15], we can find a finite set $D_1', \ldots, D_r' \in |T|$ such that the Lie algebra of the stabilizer of the tuple $(Z, g_\infty(D_1'), \ldots, g_\infty(D_r'))$ in $PGL(N+1)$ is $\mathfrak{g}_{Z,\gamma}$, and in particular it is reductive. In addition there is a subset $E \subset |T|$ of measure zero such that if $D_1, \ldots, D_K \notin E$, then the stabilizer of the extended tuple $(Z, g_\infty(D_1'), \ldots, g_\infty(D_r'), g_\infty(D_1), \ldots, g_\infty(D_K))$ is still reductive. Suppose that this tuple is not in the $PGL(N+1)$-orbit of $(M, D_1', \ldots, D_r', D_1, \ldots, D_K)$. Then we can find a $\mathbf{C}^*$-subgroup $\lambda_K \subset PGL(N+1)$ and an element $g_K \in PGL(N+1)$ such that

$$\begin{aligned} Z &= \lim_{t\to 0} \lambda_K(t) g_K \cdot M, \\ g_\infty(D_i') &= \lim_{t\to 0} \lambda_K(t) g_K \cdot D_i', \text{ for } i = 1, \ldots, r, \\ g_\infty(D_j) &= \lim_{t\to 0} \lambda_K(t) g_K \cdot D_j, \text{ for } j = 1, \ldots, K. \end{aligned} \tag{46}$$

Suppose that λ_K is generated by a vector field w_K, with Hamiltonian θ_K, and we normalize θ_K so that it has zero average on $\mathbf{P}^N$. In addition we can scale w_K so that $\|\theta_K\|_{L^2} = 1$. Note that since Z is not contained in a hyperplane, the Hamiltonian θ_K cannot be constant on Z, unless λ_K is trivial.

We can choose $D_1, \ldots, D_K \in |T| \setminus E$ so that no $d+1$ lie on a hyperplane in $|T|$. Here d is the dimension of the projective space $|T|$. From Proposition 7 we have

$$\lim_{t\to 0} \int_{\lambda_K(t)g_K\cdot M} \theta_K\, (\lambda_K(t)g_K)_*\beta \wedge \omega_{FS}^{n-1} \tag{47}$$

is equal to

$$\frac{1}{K}\sum_{i=1}^{K} \lim_{t\to 0}\int_{\lambda_K(t)g_K\cdot D_i} \theta_K\, \omega_{FS}^{n-1} + O(1/K) = \frac{1}{K}\sum_{i=1}^{K}\int_{g_\infty(D_i)} \theta_K\, \omega_{FS}^{n-1} + O(1/K), \tag{48}$$

since d is independent of K.

At the same time given any $\epsilon > 0$ we can choose K large and the D_i so that

$$\frac{1}{K}\sum_{i=1}^{K}\int_{g_\infty(D_i)} \theta_K\, \omega_{FS}^{n-1} \le \int_Z \theta_K\, \gamma\wedge\omega_{FS}^{n-1} + \epsilon. \tag{49}$$

Let us denote by $\gamma_K = \lim_{t\to 0}(\lambda_K(t)g_K)_*\beta$ the limit current on Z. Combining our inequalities, and the assumption of twisted K-stability, we have

$$\begin{aligned} 0 &\le \mathrm{Fut}_\beta(g_K\cdot M, \lambda_K) = DF(Z,\lambda_K) + nV^{-1}\int_Z \theta_K\,(\gamma_K - c\omega_{FS})\wedge\omega_{FS}^{n-1} \\ &= DF(Z,\lambda_K) + nV^{-1}\frac{1}{K}\sum_{i=1}^{K}\int_{g_\infty(D_i)}\theta_K\,\omega_{FS}^{n-1} - cnV^{-1}\int_Z \theta_K\,\omega_{FS}^{n} + O(1/K) \\ &\le DF(Z,\lambda_K) + nV^{-1}\int_Z \theta_K\,(\gamma - c\omega_{FS})\wedge\omega_{FS}^{n-1} + \epsilon + O(1/K) \\ &= \epsilon + O(1/K). \end{aligned} \tag{50}$$

Note that in the last line we used Proposition 13 and Corollary 15. Choosing ϵ small and K sufficiently large, it follows that if the tuples

$$(Z, g_\infty(D_i'), g_\infty(D_j))_{i=1,\ldots,r,j=1,\ldots,K}$$

are not in the $PGL(N+1)$-orbit of $(M, D_i', D_j)_{i=1,\ldots,r,j=1,\ldots,K}$ for infinitely many K, then we have special degenerations for (M,β) with arbitrarily small twisted Futaki invariant. Corollary 9 then implies that (M,β) is not K-stable.

Otherwise, Z is in the $PGL(N+1)$-orbit of M, and since under our assumptions M has discrete stabilizer group, it follows that the group elements g_k are uniformly bounded. As in [7], this implies that the solutions ω_{t_k} along the continuity method satisfy uniform estimates, and so we obtain a solution for $t = T$ as well, as required.

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Julius Ross
Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago IL
USA
E-mail: julius@math.uic.edu

Gábor Székelyhidi
Department of Mathematics
University of Notre Dame
Notre Dame IN
USA
E-mail: gszekely@nd.edu