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Harmonic analysis on GL_n over finite fields

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Dedicated to the memory of Bertram Kostant

Abstract: There are many formulas that express interesting properties of a finite group G in terms of sums over its characters. For estimating these sums, one of the most salient quantities to understand is the *character ratio*

$$\frac{\text{trace}(\pi(g))}{\dim(\pi)},$$

for an irreducible representation π of G and an element g of G. For example, in [12] the authors stated a formula of this type for analyzing certain random walks on G.

It turns out [22, 23] that for classical groups G over finite fields (which provide most examples of finite simple groups) there are several (compatible) invariants of representations that provide strong information on the character ratios. We call these invariants collectively *rank*.

Rank suggests a new way to organize the representations of classical groups over finite and local fields – a way in which the building blocks are the "smallest" representations. This is in contrast to Harish-Chandra's *philosophy of cusp forms* that is the main organizational principle since the 60s, and in it the building blocks are the cuspidal representations which are, in some sense, among the the "largest". The philosophy of cusp forms is well adapted to establishing the Plancherel formula for reductive groups over local fields, and led to Lusztig's classification of the irreducible representations of such groups over finite fields. However, analysis of character ratios seems to benefit from a different approach.

In this note we discuss further the notion of *tensor rank* for GL_n over a finite field $\mathbb{F}_q$ and demonstrate how to get information on representations of a given tensor rank using tools coming from the recently studied *eta correspondence*, as well as the well known philosophy of cusp forms, mentioned just above.

A significant discovery so far is that although the dimensions of the irreducible representations of a given tensor rank vary by

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quite a lot (they can differ by large powers of q), for certain group elements of interest the character ratios of these irreps are nearly equal to each other. Thus, for purposes of this aspect of harmonic analysis, representations of a fixed tensor rank form a natural family to study.

For clarity of exposition, we illustrate the developments with the aid of a specific motivational example that shows how one might apply the results to certain random walks.

0. Introduction

For a finite group G we consider the set $\widehat{G}$ of (isomorphism classes of) complex finite dimensional irreducible representations (*irreps* for short) of G, and the corresponding collection of irreducible characters of G,

$$\chi_\pi, \ \pi \in \widehat{G}, \tag{0.1}$$

given by $\chi_\pi(g) = trace(\pi(g))$, $g \in G$.

Schur's orthogonality relations [48] imply that (0.1) forms a basis for the space of class functions on G. This fact gives birth to the theory of harmonic analysis on G, namely the investigation of class functions on G via their expansion as a linear combination of irreducible characters.

Starting with the work of Frobenius [13], through the work of Diaconis-Shahshahani [12] and others (see, e.g., [39, 46, 51, 52] and references there), researchers developed explicit formulas that potentially enable one to apply the harmonic analysis technique to many class functions that express interesting properties of G.

A closer look at these formulas reveals the fact that in order to make use of them, in many cases, one needs to have a good solution for the following:

Problem (Core problem of harmonic analysis on G)**.** Estimate the character ratios

$$\frac{\chi_\pi(g)}{\dim(\pi)}, \quad \pi \in \widehat{G}, g \in G. \tag{0.2}$$

We proceed to give an example.

0.1. Hildebrand's random walk example

Consider the group $G = SL_n(\mathbb{F}_q)$ of $n \times n$ matrices with entries in a finite field $\mathbb{F}_q$ and determinant equal to one. For this example let us assume that $n \geq 3$. Inside G we look at the conjugacy class C of the transvection

$$T = \begin{pmatrix} 1 & 1 & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{0.3}$$

with $T_{ii} = 1$ for $i = 1, .., n$; $T_{12} = 1$, and $T_{ij} = 0$ elsewhere.

The following is known about C.

Fact. We have,[1,2]

- The cardinality of C is $q^{2n-2} + o(\dots)$ [1].
- Every element of G can be written as a product of no more than n elements from C [36]. Moreover,[3]

$$\frac{\#(G \smallsetminus C^{<n})}{\#(G)} = 1 - O\Big(\frac{1}{q}\Big), \tag{0.4}$$

[1] The notation $a(q) = o(b(q))$ means that $a(q)/b(q) \to 0$ as $q \to \infty$.

[2] The notation $c(q) + o(\dots)$ stands for $c(q) + o(c(q))$.

[3] We write $a(q) = O(b(q))$ if there is constant A with $a(q) \leq A \cdot b(q)$ for all sufficiently large q.

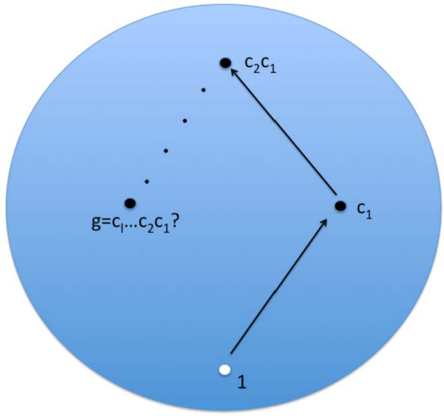

Figure 1: Random walk on G using elements from the conjugacy class C.

where $C^{<n} = \{g \in G;\ g = c_l \dots c_1 \text{ for } c_i \in C \text{ and } l < n\}$.

Formula (0.4) can be justified for example using the fact that the elements of G with all eigenvalues $\neq 1$ are outside of $C^{<n}$.

In [27] Hildebrand looked into the problem of generating random elements of G using random elements from C. The mathematical model is the following random walk on G – see Figure 1 for illustration. We start at the identity element 1 of G. Then we take element c_1 uniformly at random from C and "walk" to c_1. We can continue in this manner and walk to c_2c_1, then to $c_3c_2c_1$ etc.

Let us denote by $P_C^{*l}(g)$ the probability that in this way after l steps the product $c_l \dots c_1$ is equal to g. A very general argument [41] implies that P_C^{*l} approaches the uniform distribution U on G as $l \to \infty$.

To say more, [27] consider the distance in total variation between P_C^{*l} and U,

$$\left\| P_C^{*l} - U \right\|_{TV} = \max_{S \subset G} \left| P_C^{*l}(S) - U(S) \right|. \tag{0.5}$$

It is easy to see that $\|\cdot\|_{TV}$ is equal $\frac{1}{2}\|\cdot\|_{L_1}$, i.e., half of the L^1-norm on G [12].

The cutoff phenomenon [11] suggests that convergence to uniformity might show a sharp cutoff, namely – see Figures 2 and 3 for illustration – the distance (0.5) stays close to its maximum value (which is 1) for a while, then suddenly at some step l_M (called *mixing time*) drops to a quite small value

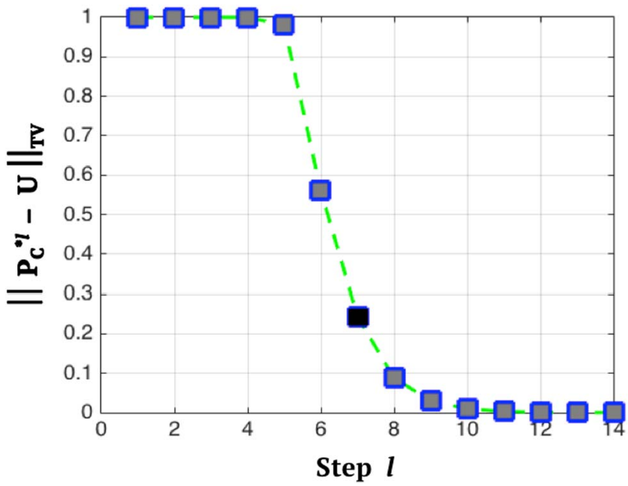

Figure 2: Numerics suggests that the mixing time for $G = SL_7(\mathbb{F}_3)$ is $l_M \approx 7$.

l	...	5	6	7	8	9	...
$\left\| P_C^{*l} - U \right\|_{TV}$	...	0.98	0.56	0.24	0.09	0.03	...

Figure 3: Numerics suggests that the mixing rate for $G = SL_7(\mathbb{F}_3)$ is $r_M \approx \frac{1}{3}$.

and then tends to zero exponentially fast with some exponent (called *mixing rate*) r_M [41].

In our case, Formula (0.4) implies that l_M can not be less than n and the numerics[4] that appears in Figure 2 illustrates, in particular, the fact that n steps are probably enough.

Theorem 0.1.1. *The random walk on $G = SL_n(\mathbb{F}_q)$, $n \geq 3$, using the collection C of transvections has, for sufficiently large q,*

(1) *Mixing time $l_M = n$.*
(2) *Mixing rate $r_M = \frac{1}{q} + O(\frac{1}{q^n})$.*

Theorem 0.1.1 was first proved in [27]. The results of this note will, among other things, provide a new proof.

[4]The numerics appearing in these notes were generated with John Cannon (Sydney) and Steve Goldstein (Madison).

0.2. Harmonic analysis of the random walk

Diaconis and Shahshahani developed in [12] formulas that, in principle, enable one to estimate the mixing time l_M and mixing rate r_M for random walks on finite groups. Here is the description that is relevant for us.

The probability distribution P_C^{*l} that we defined in Section 0.1 is a class function on G, and its expansion in terms of irreducible characters can be computed explicitly.

Proposition 0.2.1. *We have,*

$$P_C^{*l} = \frac{1}{\#(G)} \sum_{\pi \in \widehat{G}} \dim(\pi) \left(\frac{\chi_\pi(T)}{\dim(\pi)} \right)^l \chi_\pi, \tag{0.6}$$

where T is the transvection (0.3).

Indeed, Formula (0.6) can be verified using the fact that P_C^{*l} is the l-fold convolution of P_C with itself, and the standard identity for convolution of two irreducible characters.

From (0.6) we obtain:

Corollary 0.2.2. *For the random walk on G using C we have,*

(1) *The total variation distance of P_C^{*l} from uniformity satisfies*

$$\left\| P_C^{*l} - U \right\|_{TV}^2 \leq \frac{1}{4} \sum_{\mathbf{1} \neq \pi \in \widehat{G}} \dim(\pi)^2 \left| \frac{\chi_\pi(T)}{\dim(\pi)} \right|^{2l}. \tag{0.7}$$

(2) *The mixing rate satisfies* $r_M = \max_{\mathbf{1} \neq \pi \in \widehat{G}} \left| \frac{\chi_\pi(T)}{\dim(\pi)} \right|$.

Part 2 of Corollary 0.2.2 is immediate from (0.6), while for Part 1 one might in addition use the fact that the total variation norm is half of the L^1-norm, then apply Cauchy–Schwartz inequality, and finally use Schur's orthogonality of characters.

The numerics appearing in Figure 4 illustrates the possibility that a good bound on the sum at the right-hand side of (0.7) will give the desired information on the mixing time l_M.

In order to use Corollary 0.2.2 to verify Theorem 0.1.1, we want to have a method to get information on the dimensions $\dim(\pi)$, and most importantly on the character ratios $\frac{\chi_\pi(T)}{\dim(\pi)}$ of the irreps π of $G = SL_n(\mathbb{F}_q)$ at the transvection T (0.3).

l	...	5	6	7	8	9	...
$\sum_{1\neq\pi\in\widehat{G}} \dim(\pi)^2 \left\| \frac{\chi_\rho(T)}{\dim(\pi)} \right\|^{2l}$	...	204.220	5.981	0.505	0.054	0.006	...

Figure 4: The sum at the right-hand side of (0.7) for $G = SL_7(\mathbb{F}_3)$.

Recently, in [22, 23], we have discovered such a method, that seems to work nicely for all classical groups over finite fields and probably for character ratios of many other elements of interest.

0.3. Rank of a representation

Since the 1960s, Harish-Chandra's *philosophy of cusp forms* [26] is the main organizational principle in representation theory of reductive groups over finite and local fields. The central objects in his approach are the cuspidal representations. It turns out that cuspidality is a generic property, i.e., these irreps constitute a major part of all irreps, and most of them are, in some sense, among the "largest".

The philosophy of cusp forms is well adapted to establishing the Plancherel formula for reductive groups over local fields, and leads to Lusztig's classification [42] of the irreps of reductive groups over finite fields.

However, analysis of character ratios seems to require a different approach.

With this motivation in mind, we proposed in [22, 23] to turn, in some sense, things upside down, and to have an organization of the irreps of finite classical groups that is generated by the very few "smallest" representations. As a result, representations that may seem to be anomalies from the philosophy of cusp forms viewpoint play a key role here. This is interesting already in the case of $SL_2(\mathbb{F}_q)$, and this example was carried out in [24]. Although the representations of $SL_2(\mathbb{F}_q)$ have been known for a long time, we think that the perspective of rank enhances understanding of them.

Our new organization induces several (compatible) invariants of representations that provide strong information on the character ratios. We call these invariants collectively **rank**.

In this note we describe parts of the development that apply to the group $GL_n(\mathbb{F}_q)$, and deduce from it the harmonic analytic information we requested in Section 0.2 for the group $SL_n(\mathbb{F}_q)$.

In particular, for each irreducible representation ρ of $GL_n(\mathbb{F}_q)$ we attach an integer k between 0 and n, called its *tensor rank*, and show, among other things, that on the transvection T (0.3) we have,

Theorem. Fix $0 \leq k \leq n$. Then for an irrep ρ of $GL_n(\mathbb{F}_q)$ of tensor rank k, we have an estimate:

$$\frac{\chi_\rho(T)}{\dim(\rho)} = \begin{cases} \frac{1}{q^k} + o(\ldots), & \text{if} \quad k < \frac{n}{2}; \\ \frac{c_\rho}{q^k} + o(\ldots), & \text{if} \quad \frac{n}{2} \leq k \leq n-1; \\ \frac{-1}{q^{n-1}-1}, & \text{if} \quad k = n, \end{cases} \tag{0.8}$$

where c_ρ is a certain integer (independent of q) combinatorially associated with ρ.

Remark 0.3.1. *For irreps ρ of tensor rank $\frac{n}{2} \leq k \leq n-1$, the constant c_ρ in (0.8) might be equal to zero. In this case, the estimate on $\frac{\chi_\rho(T)}{\dim(\rho)}$ is simply $o(\frac{1}{q^k})$. However, it is typically non-zero, and in many cases it is 1.*

The estimates in (0.8) seem to give a significant improvement to what currently appears in the literature, and induce similar results for the irreps of $SL_n(\mathbb{F}_q)$. In particular, using some additional analytic information, Hildebrand's Theorem 0.1.1 follows.

1. Character ratios and tensor rank

We start with the problem of estimating the character[5] ratios (CRs) on the transvection T (0.3),

$$\frac{\chi_\rho(T)}{\dim(\rho)}, \quad \rho \in \widehat{GL}_n, \tag{1.1}$$

for the group $GL_n = GL_n(\mathbb{F}_q)$ of $n \times n$ invertible matrices with entries in $\mathbb{F}_q$.

1.1. Dimension

At first sight one might suspect that the size of the character ratio (1.1) is to a large extent controlled by the dimension of the representation (this is how it is usually phrased in the literature – for example see [3]) since it appears in the denominator of (1.1). This is in general **not the case** for the transvection T – see Figure 5 for illustration. In that picture, for each irreducible representation (irrep) ρ of $GL_7(\mathbb{F}_3)$ we plot[6] the (nearest integer of

[5]In this note, for clarity, we denote irreps of GL_n mostly by ρ and of SL_n mostly by π.

[6]We denote by $[x]$ the nearest integer to the real number x.

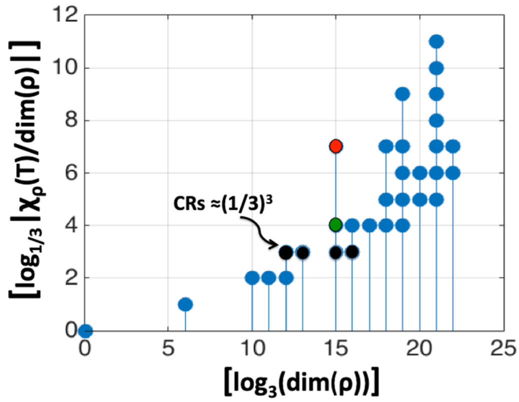

Figure 5: $\log_{1/q}$-scale of CRs vs. $\log_q$-scale of dimensions for $\rho \in \widehat{GL}_7(\mathbb{F}_q)$, $q = 3$.

the) absolute value in $\log_{1/3}$-scale of its character ratio (1.1) vs. the (nearest integer of the) $\log_3$-scale of its dimension. In particular, one learns from this numerics that there are (see the black circles in Figure 5) irreps of $GL_n(\mathbb{F}_q)$ with dimensions that differ by a multiple of large power of q, but with the same order of magnitude of CRs, and there are (see, e.g., the black-green-red circles above 15 in Figure 5) irreps of the same order of magnitude of dimension but CRs that differ by multiple of a large power of $\frac{1}{q}$.

A recent significant discovery [22, 23] is that there is an invariant, different from dimension, that seems to do a much better job in controlling the CRs (1.1) – see Figure 6 for illustration. We proceed to discuss it now.

1.2. Tensor rank

An important object attached to any finite group G is its *representation (aka Grothendieck) ring* [55]

$$R(G) = \mathbb{Z}[\widehat{G}],$$

generated from the set $\widehat{G}$ using the operations of addition and multiplication given, respectively, by direct sum $\oplus$ and tensor product $\otimes$.

It turns out [21, 22, 23, 24, 33, 34] that in the case that G is a finite classical group the ring $R(G)$ has a natural filtration that we call *tensor rank filtration*. In particular, for each irrep we get a non-negative integer that we call *tensor rank* and might be considered intuitively as its "size". Most importantly, this invariant seems to nicely control analytic properties of irreps such as character ratio.

Let us describe the development in the case of $G = GL_n$.

Consider the permutation representation[7] ω of GL_n on the space $L^2(\mathbb{F}_q^n)$ of complex valued functions on $\mathbb{F}_q^n$ given by

$$[\omega(g)(f)\,](x) = f(g^{-1}x), \tag{1.2}$$

for every $g \in GL_n$, $f \in L^2(\mathbb{F}_q^n)$, and $x \in \mathbb{F}_q^n$.

Denote by $\widehat{GL}_n(\omega^{\otimes^k})$ the set of irreps of GL_n that appear in $\omega^{\otimes^k}$- the k-fold tensor product of ω, and by $\mathbf{1}$ the trivial representation.

Proposition 1.2.1. *We have a sequence of proper containments*

$$\{\mathbf{1}\} \subsetneqq \widehat{GL}_n(\omega^{\otimes^1}) \subsetneqq \ldots \subsetneqq \widehat{GL}_n(\omega^{\otimes^n}) = \widehat{GL}_n. \tag{1.3}$$

For a proof of 1.2.1 see Appendix D.1.1.

Looking at (1.3), we see one natural way to associate a non-negative integer to an irrep, i.e.,

Definition 1.2.2 (Strict tensor rank)**.** *We say that an irrep ρ of GL_n is of <u>strict tensor rank</u> k, if in (1.3) its 1st occurrence is in $\widehat{GL}_n(\omega^{\otimes^k})$.*

We may write $\otimes$-$rank^{\star}(\rho) = k$, $r^{\star}_{\otimes}(\rho) = k$, or $rank^{\star}_{\otimes}(\rho) = k$, to indicate that an irrep ρ of GL_n is of strict tensor rank k, and denote the set of all such irreps by $(\widehat{GL}_n)^{\star}_{\otimes,k}$.

But, looking at (1.3), there is also another way to attach a non-negative integer to each irrep, taking into account the action of characters (i.e., 1-dim representations) on irreps:

Definition 1.2.3 (Tensor rank)**.** *We will say that an irrep ρ of GL_n is of <u>**tensor rank**</u> k, if it is a tensor product of a character and an irrep of strict tensor rank k, but not less.*

Again, we may use the notations $\otimes$-$rank(\rho) = k$, or $r_{\otimes}(\rho) = k$, or $rank_{\otimes}(\rho) = k$, to indicate that a representation ρ of GL_n has tensor rank k, and denote the set of all such irreps by $(\widehat{GL}_n)_{\otimes,k}$.

We extend the definition to arbitrary (not necessarily irreducible) representation of GL_n and say it is of tensor rank k if it contains irreps of tensor rank k but not of higher tensor rank.

In particular, the <u>*tensor rank filtration*</u> mentioned above is obtained by taking $F_{\otimes,k}$ to be the elements of $R(G)$ that are sums of irreps of tensor rank

[7] Up to a sign, ω is the restriction of the oscillator representation of Sp_{2n} to GL_n [18, 30, 53].

less or equal to k. Then, $F_{\otimes,(k-1)} \subset F_{\otimes,k}$, $F_{\otimes,i} \otimes F_{\otimes,j} \subset F_{\otimes,i+j}$ for every i, j, k, and $F_{\otimes,n} = R(G)$.

Sometime it is also convenient to make the following distinction and to say that a representation of GL_n is of <u>low tensor rank</u> if it is of tensor rank $k < \frac{n}{2}$.

We note that,

Remark 1.2.4. *The two notions of strict tensor rank and tensor rank differ because GL_n is not simple, and is (almost) the product of SL_n and $\mathbb{F}_q^*$. The two notions agree on restriction to SL_n.*

The following example tells us how the tensor rank one and strict tensor rank one look like, and will be vastly generalized later in Section 5.

Example 1.2.5. *The irreps of tensor rank $k = 1$ of GL_n, $n \geq 2$, are (up to twist by a character) the (non-trivial) irreducible components of ω (1.2). The group $GL_1 = \mathbb{F}_q^*$ acts on the space $L^2(\mathbb{F}_q^n)$ through its action by homotheties on $\mathbb{F}_q^n$. For every character λ of $\mathbb{F}_q^*$ we have the λ-isotypic component $\omega_\lambda = \{f : \mathbb{F}_q^n \to \mathbb{C}^*;\ f(av) = \lambda(a)f(v),\ \ a \in \mathbb{F}_q,\ v \in \mathbb{F}_q^n\}$. It is not difficult to see using direct calculations that,*

(1) *For $\lambda \neq \mathbf{1}$ the space ω_λ is irreducible as a GL_n-representation, it has dimension $\frac{q^n-1}{q-1} \approx q^{n-1}$, and its CR on T (0.3) is*

$$\frac{\chi_{\omega_\lambda}(T)}{\dim(\omega_\lambda)} = \frac{q^{n-1}-1}{q^n-1} \approx \frac{1}{q} \tag{1.4}$$

(2) *The space $\omega_{\mathbf{1}}^o = \{f \in \omega_{\mathbf{1}};\ f(0) = 0$ and $\sum_{v \in \mathbb{F}_q^n} f(v) = 0\}$ is irreducible as a GL_n-representation, it has dimension $\frac{q^n-q}{q-1} \approx q^{n-1}$, and its CR on T is*

$$\frac{\chi_{\omega_{\mathbf{1}}^o}(T)}{\dim(\omega_{\mathbf{1}}^o)} = \frac{q^{n-2}-1}{q^{n-1}-1} \approx \frac{1}{q}.$$

In particular, one deduces that there are roughly q^2 irreps of $\otimes$-rank $k = 1$.

Remark 1.2.6. *In the case of the group GL_2, using the terminology of the "philosophy of cusp forms" [26], we have,*

(1.5)
$$(\widehat{GL}_2)_{\otimes,0} = \textit{characters},\ (\widehat{GL}_2)_{\otimes,1} = \textit{principal series},\ (\widehat{GL}_2)_{\otimes,2} = \textit{cuspidals}.$$

1.3. Intrinsic characterization of strict tensor rank and tensor rank

Definitions 1.2.3 and 1.2.2 of, respectively, tensor rank and strict tensor rank, are not intrinsic as they use the representation ω (1.2). At various places of this note, it will be useful for us to use the following intrinsic characterization (given in [23]) of these notions.

For $0 \leq k \leq n$, consider the subgroup $H_k \subset GL_n$ of elements that pointwise fix the first k-coordinates subspace in $\mathbb{F}_q^n$, i.e.,

$$H_k = \left\{ \begin{pmatrix} I_k & * \\ 0 & A_{n-k} \end{pmatrix};\ A_{n-k} \in GL_{n-k} \right\}.$$

Note that $H_0 = GL_n$, $H_n = \{1\}$, and $H_k \subset H_{k-1}$, for every $k = 1, \ldots, n$.

In [23] we observed that,

Proposition 1.3.1 (Intrinsic characterisation)**.** *A representation $\rho \in \widehat{GL}_n$ is of tensor (respectively, strict tensor) rank k if and only if it admits an eigenvector (respectively, invariant vector) for H_k, but not for H_{k-1}.*

1.4. Numerics

In this note, we will think on tensor rank as a formal notion of size of a representation. But, is it going to do a good job in controlling the CRs on the transvection (0.3)?

At this stage let us present numerical data collected for the group $GL_7(\mathbb{F}_3)$ that hints toward a positive answer to the above question.

Indeed, a comparison of Figures 6 and 5 indicates that the tensor rank of a representation does a much better job than dimension in telling what should be expected for the order of magnitude of the CRs on the transvection T. Indeed, Figures 6 show something from the general truth: For tensor rank $k < \frac{n}{2}$ (i.e., the low tensor rank) irreps, although the dimensions might differ by a factor of a large power of q, all the CRs are essentially of the same size $\frac{1}{q^k}$ (compare the black circles in both figures); Moreover, for higher rank $\frac{n}{2} \leq k \leq n-1$, the CRs are of the order of magnitude of $\frac{1}{q^k}$ time a constant (independent of q), and it seems that for all tensor rank n irreps the CRs are exactly $\frac{1}{q^{n-1}}$ in absolute value; Finally, irreps of the same dimensions can have different character ratios (compare the black-green-red circles above 15 in Figure 5 with how they appear in Figure 6) which are accounted for by looking at tensor rank.

The above numerical results can be quantified precisely and proved. This is part of what we do next.

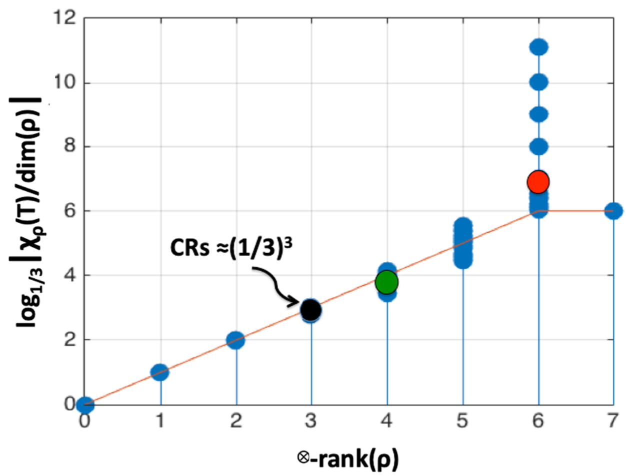

Figure 6: $\log_{1/q}$-scale of CRs vs. ⊗-rank for $\rho \in \widehat{GL}_7(\mathbb{F}_q)$, $q = 3$.

2. Analytic information on tensor rank k irreps of GL_n

In this section we present information concerning the character ratios and dimensions of the irreps of ⊗-rank k, i.e., the members of $(\widehat{GL_n})_{\otimes,k}$, including the cardinality of that set.

2.1. Character ratios on the transvection

For the CRs on the transvection T (0.3) we obtain the following, essentially sharp, estimate in term of the tensor rank.

Theorem 2.1.1. *Fix $0 \leq k \leq n$. Then, for $\rho \in (\widehat{GL_n})_{\otimes,k}$, we have an estimate:*

$$\frac{\chi_\rho(T)}{\dim(\rho)} = \begin{cases} \frac{1}{q^k} + o(\ldots), & \text{if} \quad k < \frac{n}{2}; \\ \frac{c_\rho}{q^k} + o(\ldots), & \text{if} \quad \frac{n}{2} \leq k \leq n-1; \\ \frac{-1}{q^{n-1}-1}, & \text{if} \quad k = n, \end{cases} \tag{2.1}$$

where c_ρ is a certain integer (independent of q) combinatorially associated with ρ.

Remark 2.1.2. *For irreps ρ of tensor rank $\frac{n}{2} \leq k \leq n-1$, the constant c_ρ in (2.1) might be equal to zero. In this case, the estimate on $\frac{\chi_\rho(T)}{\dim(\rho)}$ is simply $o(\frac{1}{q^k})$. However, the possibility of $c_\rho = 0$ is fairly rare, and (at least for $k \neq n-1$) we are not sure if it happens at all.*

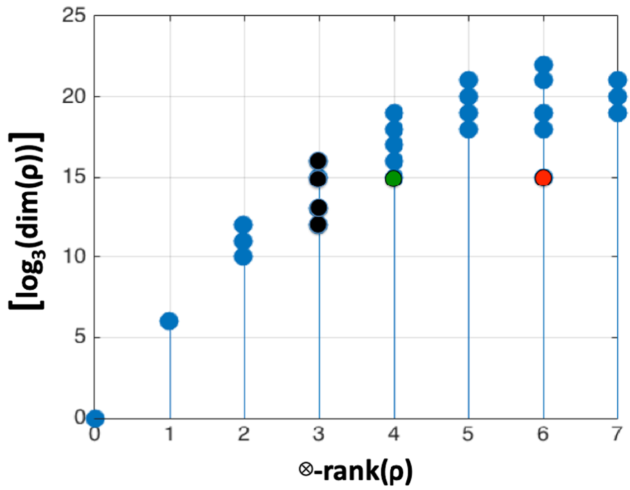

Figure 7: $\log_q$-scale of dimension vs. $\otimes$-rank for irreps ρ of $GL_7(\mathbb{F}_q)$, $q = 3$.

For a derivation of Estimates (2.1), see Section 6.1.

Note that (2.1) is a formal validation to some of the phenomena that Figure 6 illustrates.

2.2. Dimensions

We proceed to present information on the dimensions of the irreps of tensor rank k.

Figure 7 gives a numerical illustration for the distribution of the dimensions of the irreps of $GL_7(\mathbb{F}_3)$ within each given tensor rank.

In this note we obtain sharp lower and upper bounds (that formally explain Figure 7; the black-green-red dots were discussed in Section 1.1) on the dimensions of the $\otimes$-rank k irreps. Indeed, we have,

Theorem 2.2.1. *Fix* $0 \leq k \leq n$. *Then, for* $\rho \in (\widehat{GL_n})_{\otimes,k}$, *we have sharp estimate:*

$$
(2.2) \qquad q^{k(n-k)+\frac{k(k-1)}{2}} + o(\dots) \geq \dim(\rho)
$$

$$
\geq \begin{cases} q^{k(n-k)} + o(\dots), & \text{if} \quad k < \frac{n}{2}; \\ q^{(n-k)(3k-n)} + o(\dots), & \text{if} \quad \frac{n}{2} \leq k < \frac{2n}{3}; \\ q^{k(n-k)+\frac{k^2}{4}} + o(\dots), & \text{if} \quad \frac{2n}{3} \leq k \leq n, \ \textit{even}; \\ q^{k(n-k)+\frac{(k-3)^2}{4}+3(k-2)} + o(\dots), & \text{if} \quad \frac{2n}{3} \leq k \leq n, \ \textit{odd}. \end{cases}
$$

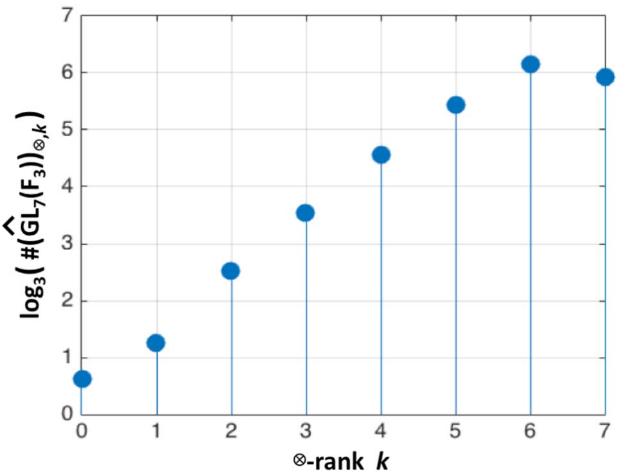

Figure 8: $\log_q$-scale of the number of ⊗-rank irreps of $GL_7(\mathbb{F}_q)$, $q = 3$.

For a proof of Theorem 2.2.1 see Section 6.2.

In [20], the authors give bounds on the dimensions of irreps of GL_n of tensor rank k. However, the estimates (2.2) are optimal for each tensor rank k, and in general stronger than those given in the cited paper.

2.3. The number of irreps of tensor rank k of GL_n

Finally, we present information concerning the cardinality of the set of irreps of ⊗-rank k – see Figure 8 for illustration.

In this aspect, we have the following sharp estimate:

Theorem 2.3.1. *Fix $0 \leq k \leq n$. Then, we have an estimate:*

$$\#((\widehat{GL_n})_{\otimes,k}) = \begin{cases} q^{k+1} + o(\ldots), & \text{if} \quad k \leq n-2; \\ c_k q^n + o(\ldots), & \text{if} \quad n-2 < k, \end{cases} \tag{2.3}$$

where $0 < c_{n-1},\, c_n < 1,\, c_{n-1} + c_n = 1.$

For a proof of Theorem 2.3.1 see Section 6.3.

2.4. Perspective

We would like to make several remarks concerning the analytic information announced just above, and to put it in some perspective to our storyline, and

Tensor rank	Number of irreps	Dimension varies by factor	Character ratio at T
$k < \frac{n}{2}$	q^{k+1}	$q^{\frac{k(k-1)}{2}}$	$\frac{1}{q^k}$
$\frac{n}{2} \leq k \leq n-1$	q^{k+1}	$q^{\frac{k^2}{4}}$ to $q^{\frac{k^2}{2}}$	$\frac{c}{q^k}$
$k = n$	q^n	$q^{\frac{k^2}{4}}$	$\frac{1}{q^{n-1}}$

Figure 9: CRs vs. variation in dimensions (in order of magnitude of power of q) for $\widehat{GL_n}$.

to what seems to be the best known estimates in the literature on character ratios at the transvection.

2.4.1. Tensor rank vs. dimension as indicator for size of character ratio Looking on the analytic information presented in the sections just above, we observe the following:

(A) For irreps in a given tensor rank.

A comparison of (2.2) and (2.1) demonstrates – see Figure 9 for a summary – what we illustrated in Sections 1.1 and 1.4: Within a given tensor rank k the dimensions may vary by a large factor (around $q^{\frac{k(k-1)}{2}}$ for rank $k < \frac{n}{2}$, and between $q^{\frac{k^2}{4}}$ to $q^{\frac{k^2}{2}}$ for $\frac{n}{2} \leq k$ – quantities are given in approximate order of magnitude of power of q) but the CRs are practically the same, of size around $\frac{1}{q^k}$ (for $\frac{n}{2} \leq k \leq n-1$ a multiple of $\frac{1}{q^k}$ by a constant independent of q).

(B) For irreps of different tensor ranks.

Looking on (2.2) we notice that:

- for $n > \frac{(k+1)(k+2)}{2}$, the upper bound for the dimension of $\otimes$-rank k irreps is (for sufficiently large q) smaller than the lower bound for rank $k+1$. But,
- when $n < \frac{(k+1)(k+2)}{2}$, the range of dimensions for $\otimes$-rank k irreps overlaps (for large enough q) the range for $k+1$, and the overlap grows with k. For k in this range, representations of the same dimension can have different character ratios, which are accounted for by looking at rank.

In conclusion, it seems that tensor rank of a representation is a better indicator than dimension for the size of its character ratio, at least on elements such as the transvection.

Tensor rank	Number of irreps	Bound on CRs at T in literature	Bound on CRs at T of this note
$k = 1$	q^2	$\frac{1}{q}$	$\frac{1}{q}$
$2 \leq k < \frac{n}{2}$	q^{k+1}	$\frac{q^{\frac{k(k-1)}{n-1}}}{q^k}$	$\frac{1}{q^k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = \frac{n}{4}$	$q^{\frac{n}{4}+1}$	$\frac{1}{q^{\frac{n}{16}}}$	$\frac{1}{q^{\frac{n}{4}}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = \frac{n}{2}$	$q^{\frac{n}{2}+1}$	$\frac{1}{q^{\frac{n}{4}}}$	$\frac{1}{q^{\frac{n}{2}}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = n$	q^n	$\frac{1}{q^{\frac{n}{4}}}$	$\frac{1}{q^{n-1}}$

Figure 10: Bounds on CRs: Current literature vs. this note (in order of magnitude).

2.4.2. Comparison with existing formulations in the literature In most of the literature on character ratios that we have seen (see, e.g., [3] or [20], and the references there), estimates on character ratios are given in terms of the dimension of representations.

Although the dimension is a standard invariant of representations, as we have seen in Parts (A) and (B) of Section 2.4.1, the dimensions of representations with a given tensor rank can vary substantially (i.e., by large powers of q), while the character ratio stays more or less constant (at least for $k < \frac{n}{2}$). Thus, using only dimension to bound character ratio will often lead to non-optimal estimates.

In particular, the estimates in this note for the character ratio on the transvection are optimal (in term of the tensor rank), and are, in general, stronger than the corresponding estimates in the papers cited above. For example, for $k < \frac{n}{2}$, rather than the bound of $\frac{1}{q^k}$, the paper [3] gives bounds of the order of magnitude of $\frac{q^{\frac{k(k-1)}{n-1}}}{q^k}$, and the exponent $\frac{k(k-1)}{n-1}$ can be fairly large when n is large and k is near $\frac{n}{2}$ (the second cited paper obtained slightly weaker bounds, on the transvection, from the first, and also formulated the result only for irreps of tensor rank $k < \sqrt{n}$). The table in Figure 10 gives some examples of the relationship between the results of this note, and of the literature cited above.

We proceed to deduce information on irreps of SL_n.

3. Analytic information on tensor rank k irreps of SL_n

In this section we describe analytic results for the irreps of SL_n, $n \geq 3$, of a given tensor rank k. In some cases these estimates can be derived as an immediate corollary of the corresponding results for GL_n, and sometime we will need more information on the irreps of GL_n and in such cases we will postpone the proofs to Section 7. The case of SL_2 is somewhat special – see Remark 3.2.8 below.

3.1. Tensor rank for representations of SL_n

First we introduce the following terminology. We assume $n \geq 3$.

Definition 3.1.1. *We will say that an irreducible representation π of SL_n has tensor rank k if it appears in the restriction of a tensor rank k (and not less) irrep of GL_n.*

As before, we denote by $(\widehat{SL_n})_{\otimes,k}$ the set of irreps of SL_n of $\otimes$-rank k.

Remark 3.1.2. *Note that the condition that π should satisfy in Definition 3.1.1 is equivalent to the requirement that (replacing GL_n by SL_n) in (1.3) it will appear in the set $\widehat{SL}_n(\omega^{\otimes k})$ but not at earlier stage. In particular, the two notions of strict tensor rank and tensor rank for irreps of GL_n, agree on restriction to SL_n.*

Our technique to get information on irreps of SL_n is through the way they appear inside irreps of GL_n. Let us start with some information on this relation.

3.2. Some properties of the restriction of irreps from GL_n to SL_n, $n \geq 3$

Take a representation ρ of GL_n and consider its restriction to SL_n. We will call the set of irreps that appear in this way the SL_n-*spectrum* of ρ. The group GL_n acts on $\widehat{SL}_n$ through its action by conjugation on SL_n. This in turn induces an action of GL_n on the SL_n-spectrum of any representation of GL_n.

Irreducibility implies that,

Claim 3.2.1. *The SL_n-spectrum of an irrep of GL_n consists of a single GL_n-orbit.*

It is helpful to know that irreps of GL_n that share the same SL_n-spectrum have the following simple relation:

Fact 3.2.2. *Two irreps of GL_n have the same SL_n-spectrum (equivalently share any representation of SL_n) iff they differ by a twist by a character of GL_n.*

The restriction can be described more precisely as follows. For each $a \in C = \mathbb{F}_q^*$ and each $\pi \in \widehat{SL}_n$, denote by π_a the representation $\pi_a(g) = \pi(s(a)gs(a)^{-1})$, $g \in SL_n$, where $s(a) \in GL_n$ is the diagonal matrix with a in the first entry and all other diagonal entries equal to 1. Then,

Fact 3.2.3. *The restriction of an irreducible representation ρ to SL_n is multiplicity free. Moreover, for any π in the SL_n-spectrum of ρ we have,*

$$\rho_{|SL_n} = \sum_{a \in C/C_\pi} \pi_a,$$

where C_π is the stabilizer of π in C.

Facts 3.2.2 and 3.2.3 are special cases of general results (see Corollary A.2.4) in Clifford-Mackey's theory [9, 44] (that we recall in Appendix A) on restriction of representations from a group to general normal subgroups (first fact), and normal subgroup with cyclic quotient (second fact).

We proceed to derive the estimates on the character ratios.

3.2.1. Character ratios on the transvection

We have the following useful Lemma:

Lemma 3.2.4. *Any element of SL_n whose centralizer in GL_n maps onto $\mathbb{F}_q^*$ under determinant will have the same character ratios on any irrep of GL_n and any irrep appearing in its restriction to SL_n.*

For a proof of Lemma 3.2.4 see Appendix D.2.1.

Since, in the case $n \geq 3$, the transvection T (0.3) meets the conditions of Lemma 3.2.4, we have, using result (2.1), the following sharp estimates:

Corollary 3.2.5. *Fix $n \geq 3$, and $0 \leq k \leq n$. Then, for $\pi \in (\widehat{SL}_n)_{\otimes,k}$, we have an estimate:*

$$\frac{\chi_\pi(T)}{\dim(\pi)} = \begin{cases} \frac{1}{q^k} + o(\ldots), & \textit{if} \quad k < \frac{n}{2}; \\ \frac{c_\pi}{q^k} + o(\ldots), & \textit{if} \quad \frac{n}{2} \leq k \leq n-1; \\ \frac{-1}{q^{n-1}-1}, & \textit{if} \quad k = n, \end{cases} \tag{3.1}$$

where c_π is a certain integer (independent of q) combinatorially associated with π.

Remark 3.2.6. *For irreps π of tensor rank $\frac{n}{2} \leq k \leq n-1$, the constant c_π in (3.1) might be equal to zero. In this case, the estimate on $\frac{\chi_\pi(T)}{\dim(\pi)}$ is simply $o(\frac{1}{q^k})$.*

3.2.2. Lower and upper bounds on dimensions of tensor rank k irreps of SL_n It turns out that, most irreps of GL_n stay irreducible after restriction to SL_n, among them all the irreps that give the lower bounds and most of those that give the upper bounds on dimensions of tensor rank k irreps. As a consequence, from the corresponding results for GL_n, we obtain,

Corollary 3.2.7. *Fix $n \geq 3$, and $0 \leq k \leq n$. Then, for $\pi \in (\widehat{SL_n})_{\otimes,k}$, we have an estimate:*

$$
\begin{aligned}
(3.2) \qquad & q^{k(n-k)+\frac{k(k-1)}{2}} + o(\ldots) \geq \dim(\pi) \\
& \geq \begin{cases} q^{k(n-k)} + o(\ldots), & \text{if} \quad k < \frac{n}{2}; \\ q^{(n-k)(3k-n)} + o(\ldots), & \text{if} \quad \frac{n}{2} \leq k < \frac{2n}{3}; \\ q^{k(n-k)+\frac{k^2}{4}} + o(\ldots), & \text{if} \quad \frac{2n}{3} \leq k \leq n, \ \text{even}; \\ q^{k(n-k)+\frac{(k-3)^2}{4}+3(k-2)} + o(\ldots), & \text{if} \quad \frac{2n}{3} \leq k \leq n, \ \text{odd}. \end{cases}
\end{aligned}
$$

Moreover, the upper and lower bounds in (3.2) are attained.

The detailed derivation of Corollary 3.2.7 can be found in Section 7.1.

Remark 3.2.8. *Corollary 3.2.7 fails for $n = 2$. In that case, there are one split principal series and one cuspidal representation of GL_2 that when restricted to SL_2 decompose, respectively, into two pieces of dimension $\frac{q+1}{2}$, $\frac{q-1}{2}$. Moreover, for these representations, the character ratio is of order $\frac{1}{\sqrt{q}}$. This case, discussed in [24], arises from the "accidental" isomorphism $SL_2 \simeq Sp_2$, and these representations should be thought of as constituents of the Weil/oscillator representation for Sp_2, and to be the representations of tensor rank $k = 1$ of this group, while both the rest of the split principal series (dimension $= q + 1$) and the "discrete series" (dimension $= q - 1$) of Sp_2, should be considered to have tensor rank $k = 2$.*

3.3. The number of irreps of tensor rank k of SL_n

The fact, mentioned earlier, that most (in a quantified way) tensor rank k irreps of GL_n stay irreducible after restricting them to SL_n, implies (see estimates (2.3)) the following:

Proposition 3.3.1. *Fix $n \geq 3$, and $0 \leq k \leq n$. Then, we have an estimate:*

$$\#((\widehat{SL}_n)_{\otimes,k}) = \begin{cases} q^k + o(\ldots), & \text{if} \quad k \leq n-2; \\ c_k q^{k-1} + o(\ldots), & \text{if} \quad n-2 < k, \end{cases} \tag{3.3}$$

where $0 < c_{n-1}$, $c_n < 1$, with $c_{n-1} + c_n = 1$.

For a detailed derivation of (3.3) see Section 7.2.

4. Back to the random walk

Having the analytic information on the irreps of SL_n, $n \geq 3$, we can address the random walk problem discussed in the introduction.

4.1. Setting

Recall (see the introduction) that we consider the conjugacy class $C \subset SL_n$ of the transvection T (0.3), and use it, as a generating set, to do a random walk on SL_n. We denote by $P_C^{*l}(g)$ the probability that after l steps we arrive to a given element $g \in SL_n$.

We know that the difference of P_C^{*l} from the uniform distribution U is, in total variation,

$$\left\| P_C^{*l} - U \right\|_{TV} \approx 1, \quad \text{for} \quad l < n,$$

and want to show that the mixing time l_M is n, i.e., there is a dramatic change at the n-th step where suddenly the two distributions become close, and an exponential rate of decay – called mixing rate and denoted r_M – kicks in.

4.2. The mixing time and mixing rate

We can derive the following sharp estimates for l_M and r_M:

Theorem 4.2.1. *For the random walk on SL_n using C, as a generating set, we have, for sufficiently large q,*

1. *The mixing time $l_M = n$.*
2. *The mixing rate $r_M = \frac{1}{q} + O(\frac{1}{q^n})$.*

Part 2 of Theorem 4.2.1 follows from the fact that

$$r_M = \max_{\mathbf{1}\neq\pi\in\widehat{SL}_n} \left| \frac{\chi_\pi(T)}{\dim(\pi)} \right|,$$

and then use the estimates (1.4) and (3.1).

For a proof of Part 1 of Theorem 4.2.1, first we recall that Formula (0.4) implies that l_M can not be less than n, and then we use,

Proposition 4.2.2. *For sufficiently large q we have,*

$$\left\| P_C^{*l} - U \right\|_{TV} \leq \frac{1}{2\sqrt{q}} \left(\frac{1}{q}\right)^{l-n} + o(\ldots).$$

For a verification of Proposition 4.2.2 see Appendix D.3.1.

5. The eta correspondence and the philosophy of cusp forms

To derive the analytic results that we described in Section 2, we need to address the following:

Question. How to get information on the $\otimes$-rank k irreps of GL_n?

In this note we would like to describe a technique which leads to an answer to the above question and is based on the interplay between two methods: the *Philosophy of Cusp Forms (P-of-CF)*; and the *eta Correspondence*.

The P-of-CF was put forward in the 60s by Harish-Chandra [26]. It is one of the main organizing principles in representation theory of reductive groups over local [2] and finite fields [10, 42].

The eta correspondence was implicitly discovered in the manuscript [29]. This method can be applied in order to investigate irreps of classical groups over local and finite fields. It is based on the notion of dual pair [31] of subgroups in a finite symplectic group, and a special correspondence between certain subsets of their irreps, which is induced by restricting the oscillator (aka Weil) representation [18, 30, 53] of the relevant symplectic group to the given dual pair. In recent years we have been developing this theory much further in order to support our theory of "size" of a representation for finite classical groups [21, 22, 23, 24, 33, 34] and, in particular, in order to estimate the dimensions and character ratios of their irreps.

In this note we have refined and essentially completed the development given in [23] for the η-correspondence for the dual pair (GL_k, GL_n).

In retrospect, we note that the information we obtain on irreducible representations of tensor rank k, and strict tensor rank k, gives an essentially explicit description for the set of these representations, and for the η-correspondence. The combination of the P-of-CF with this (GL_k, GL_n)-duality provides a simple and effective way to identify the strict tensor rank and tensor rank k pieces inside the large representation $\omega^{\otimes^k}$ of GL_n that we used in Section 1.2 to define these notions.

For the rest of this section we assume $0 \leq k \leq n$.

5.1. The eta correspondence – non-explicit form

We start with a non-explicit form of the eta correspondence.

Recall (see Section 1.2) that the vector space $L^2(M_{k,n})$, of functions on the set of $k \times n$ matrices over $\mathbb{F}_q$, is a host for all (up to tensoring with characters) irreps of GL_n of strict tensor rank less or equal to k. In Section 1.2 we denoted the action of GL_n on this space by $\omega^{\otimes^k}$.

Of course we have a larger group of symmetries acting on this space, i.e., we have a pair of commuting actions

$$GL_k \overset{\omega_{kn}}{\curvearrowright} L^2(M_{k,n}) \overset{\omega_{kn}}{\curvearrowleft} GL_n, \tag{5.1}$$

given by $[\omega_{kn}(h,g)f](m) = f(h^{-1}mg)$, for every $h \in GL_k$, $m \in M_{k,n}$, $g \in GL_n$, and $f \in L^2(M_{k,n})$.

We will also refer to ω_{kn} (5.1) as the *oscillator representation* of $GL_k \times GL_n$.

For $0 < k \leq n$, the action of GL_k does not generate the full commutant of GL_n in $End(\omega_{kn})$, and vice versa (see Example 1.2.5 for $k = 1$). Let us look at the action of $GL_k \times GL_n$ on the smaller space

$$(\omega_{kn})^{\star}_{\otimes,k} < \omega_{kn}, \tag{5.2}$$

consisting of the (sums of) components of ω_{kn} that have strict tensor rank exactly k. On this space we do have,

Theorem 5.1.1. *The groups GL_k and GL_n generate each other's full commutant in $End((\omega_{kn})^{\star}_{\otimes,k})$.*

Let us write the decomposition of ω_{kn} into a direct sum of isotypic components for the irreps of GL_k as follows

$$\omega_{kn} = \sum_{\tau \in \widehat{GL}_k} \tau \otimes \Theta(\tau), \tag{5.3}$$

where the multiplicity space $\Theta(\tau)$ is a representation of GL_n.

Now, the Burnside's double commutant theorem [54] together with Theorem 5.1.1 implies that

Corollary 5.1.2 (eta correspondence – non explicit form)**.** *Each* $\Theta(\tau)$ *contains at most one irreducible component* $\eta(\tau)$ *of strict tensor rank* k *(and then, it appears with multiplicity one), in addition to irreps of lower strict tensor rank. In particular, we have a natural bijective mapping*

$$\tau \longmapsto \eta(\tau), \tag{5.4}$$

from a subcollection of $\widehat{GL}_k$ *onto the set* $(\widehat{GL}_n)^{\star}_{\otimes,k})$ *of strict tensor rank* k *irreps of* GL_n*.*

We call the mapping (5.4) the **eta correspondence** or (GL_k, GL_n)-*duality*.

Conclusion 5.1.3. *Up to twist by a character of* GL_n*,* ***all*** $\otimes$*-rank* k *irreps of* GL_n *appear in the image of the* (GL_l, GL_n)*-duality for* $l = k$*, and not before.*

Remark 5.1.4. *A proof of Theorem 5.1.1 first appeared in [29] (there, the tensor rank* k *representations were called the "new spectrum" in some relevant "Witt tower" associated with corresponding "oscillator representations" of symplectic groups), and a similar treatment was given in [23]. The outcome in both papers is the* η*-correspondence for general dual pairs in finite symplectic groups.*

In this note, we will prove Theorem 5.1.1 as a by-product of making the description of the correspondence (5.4) explicit.

5.2. The eta correspondence – explicit form

We want to get a good formula for $\eta(\tau)$ (5.4), including an explicit description of its domain in $\widehat{GL}_k$. In this section we get an approximate one (see Formula (5.10) in Theorem 5.2.2 below) showing that $\eta(\tau)$ is essentially a certain simple to write down *parabolic induction* which, in addition, can be effectively analyzed. In particular, this description will give us in Section 5.5, using the P-of-CF, an exact formula for $\eta(\tau)$ (see Equation (5.23) of Theorem 5.5.1).

We fix $0 \leq k \leq n$ and consider inside GL_n the maximal parabolic subgroup

$$P_{k,n-k} = Stab_{GL_n}(V_k), \tag{5.5}$$

stabilizing the k-dimensional subspace

$$(5.6)\qquad V_k = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix};\ x_1, \ldots, x_k \in \mathbb{F}_q \right\} \subset \mathbb{F}_q^n.$$

The group $P_{k,n-k}$ has *Levi decomposition* (see Appendix B), i.e., it can be written as a semi-direct product of subgroups

$$P_{k,n-k} = U_{k,n-k} \cdot L_{k,n-k},$$

where $U_{k,n-k}$ and $L_{k,n-k}$, called, respectively, the *unipotent radical* and the *Levi component of* $P_{k,n-k}$, are given by

$$(5.7)\qquad \begin{aligned} (1)\quad U_{k,n-k} &= \left\{ \begin{pmatrix} I_k & B \\ 0 & I_{n-k} \end{pmatrix};\ B \in M_{k,n-k} \right\}, \\ (2)\quad L_{k,n-k} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix};\ A \in GL_k,\ C \in GL_{n-k} \right\}, \end{aligned}$$

where I_k, I_{n-k}, are the corresponding identity matrices.

In particular, we have a surjective homomorphism

$$(5.8)\qquad P_{k,n-k} \overset{pr}{\twoheadrightarrow} P_{k,n-k}/U_{k,n-k} = L_{k,n-k} \simeq GL_k \times GL_{n-k}.$$

Now, take $\tau \in \widehat{GL}_k$, tensor it with the trivial representation $\mathbf{1}_{n-k}$ of GL_{n-k}, and form the *parabolic induction* (see Appendix B),

$$(5.9)\qquad I_\tau = Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k}),$$

namely, the induced representation from $P_{k,n-k}$ to GL_n of the pullback of $\tau \otimes \mathbf{1}_{n-k}$ from $GL_k \times GL_{n-k}$ via (5.8).

It turns out that,

Proposition 5.2.1 (Mutiplicity one)**.** *Consider $\tau \in \widehat{GL}_k$. Then, I_τ (5.9) is multiplicity free.*

We give a more informative version of this result in Section 5.3.2.

It is easy to see that $I_\tau < \Theta(\tau)$ for every $\tau \in \widehat{GL}_k$ (see Part (5.3.1) of Claim 5.3.1).

The representation I_τ gives the "approximate formula" (this is the meaning of Equation (5.10) below) for $\eta(\tau)$, that we mentioned at the beginning of this section. More precisely,

Theorem 5.2.2 (eta **correspondence – explicit form**). *Take $\tau \in \widehat{GL}_k$, $k \leq n$, and look at the decomposition (5.3) of ω_{kn}. We have,*

(1) ***Existence.*** *The representation $\Theta(\tau)$ contains a strict tensor rank k component if and only if τ is of strict tensor rank $\geq 2k - n$.*
Moreover, if the condition of Part (5.2.2) is satisfied, then,
(2) ***Uniqueness.*** *The representation $\Theta(\tau)$ has a unique constituent $\eta(\tau)$ of strict tensor rank k, and it appears with multiplicity one. and,*
(3) ***Formula.*** *The constituent $\eta(\tau)$ satisfies $\eta(\tau) < I_\tau < \Theta(\tau)$, and we get*

$$\boxed{I_\tau = \eta(\tau) + \sum_\rho \rho,} \tag{5.10}$$

where the sum is multiplicity free, and over certain irreps ρ which are of strict tensor rank less then k and dimension smaller than $\eta(\tau)$.
Finally, the mapping

$$\tau \longmapsto \eta(\tau), \tag{5.11}$$

gives an explicit bijective correspondence between the collection $(\widehat{GL}_k)^{\star}_{\otimes,\geq 2k-n}$ of irreps of GL_k of strict tensor rank $\geq 2k - n$, and the set $(\widehat{GL}_n)^{\star}_{\otimes,k}$ of strict tensor rank k irreps of GL_n.

Note that, indeed, Theorem 5.2.2 gives an explicit description of the eta correspondence (5.4) and hence of all members of $(\widehat{GL}_n)^{\star}_{\otimes,k}$ and (up to twist by a character) of $(\widehat{GL}_n)_{\otimes,k}$ the tensor rank k irreps. In particular, it implies Theorem 5.1.1.

The rest of this section is devoted to formulations, and proofs, of more informative versions of Proposition 5.2.1 and Theorem 5.2.2.

Remark 5.2.3. *Parts 5.2.2 and 5.2.2 were also formulated and proved, using extensive character theoretic techniques, in [20]. The techniques we use in this note, which among other things produce a proof of Theorem 5.2.2, are different. They are based on the philosophy of cusp forms and, in particular, are spectral theoretic in nature.*

5.3. Decomposing $I_\tau = Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes 1_{n-k})$ and the philosophy of cusp forms

Denote by $(M_{k,n})_k \subset M_{k,n}$, the $GL_k \times GL_n$ "open" orbit consisting of matrices of rank equal to k. We observe that

Claim 5.3.1. *The following hold:*

(1) *The strict tensor rank k part $(\omega_{kn})^{\star}_{\otimes,k}$ (5.2) of ω_{kn} is contained in $L^2((M_{k,n})_k)$.*
(2) *We have*

$$L^2((M_{k,n})_k) \simeq \sum_{\tau \in \widehat{GL}_k} \tau \otimes I_\tau,$$

as a representation of $GL_k \times GL_n$, where I_τ is given by (5.9).

For a proof of Claim 5.3.1 see Appendix D.4.2.

From Claim 5.3.1, we see that the proofs of Proposition 5.2.1 and Theorem 5.2.2 come down to learning the decomposition of I_τ. Our main tool for doing this involves the description of representations coming from the philosophy of cusp forms (P-of-CF) [26].

5.3.1. Recollection from the philosophy of cusp forms We recall some of the basics of the P-of-CF, that are relevant for us, leaving a more detailed account of this theory, including relevant references, for Appendix B.

A representation κ of GL_n is called *cuspidal* if it does not contain a non-trivial fixed vector for the unipotent radical of any parabolic subgroup stabilizing a flag in $\mathbb{F}_q^n$. Given this definition, it is easy to show that any irrep is contained in a representation induced from a representation of a parabolic subgroup P that

- is trivial on the unipotent radical U_P of P; and
- is a cuspidal representation of the quotient $L_P = P/U_P$.

Note that in the case of GL_n, the group $L_P = P/U_P$, called the *Levi component* of P, is a product of GL_m's for $m \leq n$, so that the P-of-CF provides an inductive construction of all irreps. The main ingredients needed to carry out this construction explicitly are

(1) knowledge of the cuspidal representations of the GL_m, $m \leq n$; and
(2) decomposing the representations induced from cuspidal representations.

With regard to (5.3.1), there is a very general result due to Harish-Chandra that relates different induced-from-cuspidal representations (see also Theorem B.1.2 and Corollary B.1.3):

Fact 5.3.2. *Two such representations $Ind_{P_j}^{GL_n}(\kappa_j)$, for $j = 1, 2$, are either equivalent, or they are completely disjoint – they have no irreducible constituents in common. For them to be equivalent, two conditions must be satisfied. First, the inducing parabolics must be associate, meaning that their Levi components L_{P_j} must be conjugate. Secondly, there must be an element $g \in GL_n$ that both conjugates L_{P_1} to L_{P_2}, and at the same time, conjugates the representations κ_1 and κ_2 to each other. In other words, $g(L_{P_1})g^{-1} = L_2$, and moreover, the representation $g^*\kappa_2$, which sends h in L_{P_1} to $\kappa_2(ghg^{-1})$, is equivalent to κ_1. In this way, association classes of cuspidal representations of parabolic subgroups define a partition of the unitary dual $\widehat{GL_n}$ into disjoint subsets.*

Remark 5.3.3. *Fact 5.3.2 can assist in the demonstration of the necessity statement in Part (1) of Theorem 5.2.2. See Appendix D.4.1 for a detailed proof.*

The split and spherical principal series As noted already, for GL_n any Levi component is a product of copies of groups GL_m for $m \leq n$. The collection $D_P = \{m_j\}$ of the sizes of the GL_{m_j}'s factors of a Levi component define a partition of n. Up to conjugation, we can assume that P consists of block upper triangular matrices, and, given this, we let m_j be the size of the j-th block from the upper left corner of the matrices. We will refer to this as the P-partition. Also, we will say that a cuspidal representation of GL_m has *cuspidal size m*. Thus, the cuspidal sizes of a cuspidal representation of a parabolic subgroup P also define a partition, the same as the P-partition. Up to association in GL_n, we can arrange that the block sizes m_j of P, equivalently, cuspidal sizes of a cuspidal irrep κ of P, decrease as j increases. We then also associate to the partition, and to P, a Young diagram [14], whose j-th row has length m_j.

If the block sizes are all equal to 1, then the parabolic is (conjugate to) the Borel subgroup B of upper triangular matrices. The Levi component of B is $(GL_1)^n$. Since this group is abelian, all of its irreducible representations are characters – one dimensional representations, specified by homomorphisms into $\mathbb{C}^*$. We will refer to constituents of representations induced from characters of B as the *split principal series*. Constituents of the representation induced from the trivial character $\mathbf{1}$ of B will be referred to as *spherical principal series* (or *SPS* for short). Any representation induced from a (one dimensional) character of a parabolic subgroup will have constituents all belonging to the split principal series.

There is a second partition, that permits a more refined understanding of the split principal series, essentially reducing it to understanding the spherical principal series. A character χ of the Borel subgroup B is given by a collection χ_j, $1 \leq j \leq n$, of characters of $GL_1 = \mathbb{F}_q^*$, where χ_j is the restriction of the character χ to the j-th diagonal entry of an element of B. Up to association, these characters can be reordered as desired. Thus, we may assume that all diagonal entries j for which the χ_j are equal to a given character of GL_1 are consecutive. Given this, we can consider a block upper triangular parabolic subgroup such that, in each diagonal block, the characters χ_j are equal, and the χ_j's contained in different diagonal blocks are different. We may also assume when convenient that the sizes of these blocks decrease from top to bottom. This associates a well-defined partition to a given character χ of B.

Consider the parabolic subgroup P_χ defined by the blocks associated to the character χ of B, as in the preceding paragraph. Let the i-th block from the top of P_χ be GL_{m_i}. Let ρ be a constituent (i.e., an irreducible sub-rep) of the representation of P_χ induced from the character χ of B. Then, the philosophy of cusp forms tells us that:

(a) the representation $Ind_{P_\chi}^{GL_n}(\rho)$ will be irreducible;
(b) $\rho \simeq \otimes\rho_i$, where ρ_i is a constituent of the representation of GL_{m_i} induced from $B \cap GL_{m_i}$; and
(c) this process gives a bijection from the constituents of $Ind_B^{P_\chi}(\chi)$ to the constituents of $Ind_B^{GL_n}(\chi)$, and this last set is the product (in the natural sense) of the sets of constituents of the $Ind_{B\cap GL_{m_i}}^{GL_{m_i}}(\chi)$.

Moreover, because of the way P_χ was defined, each representation $Ind_{B\cap GL_{m_i}}^{GL_{m_i}}(\chi)$ has the form

$$(\chi_i \circ \det) \otimes Ind_{B\cap GL_{m_i}}^{GL_{m_i}}(\mathbf{1}),$$

where χ_i indicates the common character of GL_1 assigned to the diagonal entries of $B \cap GL_{m_i}$, and det is the determinant homomorphism from GL_{m_i} to GL_1. This means that the constituents of each $Ind_{B\cap GL_{m_i}}^{GL_{m_i}}(\chi)$ has the form $(\chi_i \circ \det) \otimes \rho_i$, where ρ_i is a member of the spherical principal series for GL_{m_i}. This leads us to focus on understanding the spherical principal series.

Spherical principal series The spherical principal series of GL_n have been studied extensively (see Appendices B.2.3 and C). They can be helpfully studied through the family of induced representations $Ind_P^{GL_n}(\mathbf{1})$, for all parabolic subgroups. Up to conjugation, it is enough to consider the parabolics that contain the Borel subgroup B, i.e., the block upper triangular parabolic subgroups. Also, it is standard that if P and P' are associate parabolics, then the representations $Ind_P^{GL_n}(\mathbf{1})$ and $Ind_{P'}^{GL_n}(\mathbf{1})$ are equivalent. Thus, we can select a representative from each association class of parabolics. We do this in the usual way, by requiring that the block sizes m_i of the diagonal blocks GL_{m_i} of P, listed from top to bottom, are decreasing with increasing i. This again gives us a partition (our third partition) of n, with an associated Young diagram D_P. (We note that, if all of the above discussion on the P-of-CF is referenced to a fixed original n, then the successive partitions we have been describing are partitions of parts of the preceding partition).

Notation. For the rest of this note, let us denote the set of partitions of n by $\mathcal{P}_n$, and the corresponding set of Young diagrams by $\mathcal{Y}_n$.

Let $P = P_D$ be a parabolic as above, with blocks whose sizes decrease down the diagonal, associated to the Young diagram $D \in \mathcal{Y}_n$. Consider the induced representation

$$I_D = Ind_{P_D}^{GL_n}(\mathbf{1}). \tag{5.12}$$

All the constituents of I_D are spherical principal series representations. We can be somewhat more precise (see Appendix C for a more detailed account). Recall that the set of isomorphism classes of representations of a group G form a free abelian semi-group (monoid) on the irreducible representations, and as such, has a natural order structure $\leq$ given by the notion of sub-representation (or equivalently given by dominance of all coefficients in the expression of a given representation as a sum of irreducibles). The set of partitions/Young diagrams also has a well-known order structure $\preceq$, the *dominance order* (see Definition C.1.1).

We know the following facts [25, 35] (see also Proposition C.2.1 in Appendix C):

Facts 5.3.4. *Consider the representations I_D (5.12). We have,*

(1) *The map $D \mapsto I_D$ is order preserving from the set $\mathcal{P}_n$ of partitions of n, with its reverse dominance order, to the semigroup of spherical representations (i.e., contains non-trivial B-invariant vectors) of GL_n.*
(2) *The representation I_D contains a constituent ρ_D with multiplicity one, and with the property that it is not contained in any $I_{D'}$ with $D' \precneqq D$ in the dominance order.*

Remark 5.3.5. *The representation ρ_D can also be distinguished by its dimension: it is the only constituent of I_D whose dimension, as a polynomial in q, has the same degree as the cardinality of GL_n/P_D (see Corollary C.4.1 in Appendix C.4)*

Facts 5.3.4 are parallels of similar facts for the symmetric group S_n [35]. For a given partition D of n, let S_D denote the stabilizer of D in S_n, and let Y_D denote the *Young module [8]*

$$Y_D = Ind_{S_D}^{S_n}(\mathbf{1}).$$

Also let σ_D be the irrep of S_n associated to the partition D. Then the analog of Facts 5.3.4 are valid. In particular, σ_D is contained in Y_D with multiplicity one. Moreover, for any two partitions D_1 and D_2 of n, the Bruhat decomposition for GL_n [4, 6], i.e., that

$$P_{D_1} \diagdown GL_n \diagup P_{D_2} \simeq S_{D_1} \diagdown S_n \diagup S_{D_1}, \tag{5.13}$$

implies [35] that we have an equality of intertwining numbers

$$\langle I_{D_1}, I_{D_2} \rangle = \langle Y_{D_1}, Y_{D_2} \rangle . \tag{5.14}$$

As a consequence of the facts just mentioned above, one can show (see Appendices C and B.2.3, and the reference [25] for more precise statement) that the description of the spherical principal series representations of GL_n is essentially the same as the representation theory of the symmetric group.

Split, unsplit, and a P-of-CF formula for general irreps In contrast to the split principal series irreps, we have the irreps that we call *unsplit*. These are the components of representations induced from cuspidal representations of parabolics with block sizes of 2 or larger (i.e., no blocks of size 1).

The general representation is gotten by combining unsplit representations and split principal series. More precisely, given a parabolic $P_D \subset GL_n$, let $P_{u,s}$ be the maximal parabolic subgroup, with Levi component $GL_u \times GL_s$, where GL_u contains all the blocks of P_D of size greater than 1, and GL_s contains all the blocks of P_D of size 1. (We remind the reader of the convention that P_D is block upper triangular, with the block sizes decreasing down the diagonal). Then a constituent ρ_U of a representation of GL_u induced by a cuspidal representation of $GL_u \cap P_D$ will be an unsplit representation of GL_u. On the other hand, $P_D \cap GL_s$ will be a Borel subgroup of GL_s, and a constituent ρ_S of the representation of GL_s induced from a cuspidal representation of $GL_s \cap P_D$ will be a split principal series of GL_s. Since $GL_u \times GL_s$ is a quotient of $P_{u,s}$, the tensor product $\rho_U \otimes \rho_S$ will also define a representation of $P_{u,s}$, and this representation will be irreducible. Now if we look at the induced representation

$$\rho_{U,S} = Ind_{P_{u,s}}^{GL_n}(\rho_U \otimes \rho_S), \tag{5.15}$$

then, the philosophy of cusp forms tells us that

(a) $\rho_{U,S}$ is irreducible; and

(b) the map $(\rho_U, \rho_S) \mapsto \rho_{U,S}$ is an injection from the relevant subsets of the unitary dual of $GL_u \times GL_s$ into the unitary dual of GL_n; and

(c) all irreducible representations of GL_n arise in this way (including the cuspidal representations, which are included in the situation when $P_D = GL_n$).

Remark 5.3.6 (Uniqueness and the P-of-CF formula)**.** *As discussed above in Section 5.3.1, the P-of-CF tells us that the split part ρ_S, appearing in (5.15), is induced irreducibly from a (standard, upper triangular) parabolic subgroup P_S of GL_s (corresponding to a partition $S = \{s_1 \geq \ldots \geq s_l\}$ of s) and representation of it such that, on each diagonal block of the parabolic the constituent representation has the form $\rho_{S_i} = (\chi_i \circ \det) \otimes \rho_{D_i}$, where ρ_{D_i} is a spherical principal series of the relevant GL_{s_i}-block of P_S, and the characters χ_i of GL_1 are distinct for different blocks of P_S. Moreover, the association class of P_S, and the inducing representations, are uniquely determined. Overall, Formula (5.15) can be replaced by the following more precise formula:*

$$\boxed{\rho_{U,S} = Ind_{P_{u,s_1,\ldots,s_l}}^{GL_n}\left(\rho_U \otimes \left[\bigotimes_{i=1}^{l}(\chi_i \circ \det) \otimes \rho_{D_i}\right]\right),} \tag{5.16}$$

where $P_{u,s_1,\ldots,s_l}$ is the standard upper triangular parabolic with blocks of sizes $u, s_1, \ldots, s_l$.

Let us call (5.16) the *unsplit-split P-of-CF formula* (or *parametrization*).

5.3.2. Decomposing $I_\tau = Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k})$ We are ready to describe the components of the induced representation $I_\tau = Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes 1_{n-k})$ given in (5.9), using their P-of-CF formulas. Let us start with the situation where the representation τ on the GL_k block is a SPS representation. In this case the Pieri rule produces such a description.

The Pieri rule Consider the induced representation

$$I_{\rho_D} = Ind_{P_{k,n-k}}^{GL_n}(\rho_D \otimes \mathbf{1}_{n-k}), \tag{5.17}$$

where ρ_D is the SPS of GL_k associated to the partition D of n.

The parallelism between the spherical principal series and the representations of the symmetric group implies that,

Claim 5.3.7. *Consider two partitions E of n and D of k. Denote by ρ_E and ρ_D the corresponding SPS representations of GL_n and GL_k, respectively. Also, denote by σ_E and σ_D the corresponding irreps of S_n and S_k, respectively. Then, we have an equality*

$$\langle \rho_E, I_{\rho_D} \rangle = \langle \sigma_E, I_{\sigma_D} \rangle, \tag{5.18}$$

where $\langle \bullet, \bullet \rangle = \dim(Hom(\bullet, \bullet))$ is the standard intertwining number, and I_{σ_D} denotes the induced representation

$$I_{\sigma_D} = Ind_{S_k \times S_{n-k}}^{S_n}(\sigma_D \otimes \mathbf{1}_{n-k}), \tag{5.19}$$

where the subgroup $S_k \times S_{n-k}$ is contained in the symmetric group S_n in the standard way, and $\mathbf{1}_{n-k}$ is the trivial representation of S_{n-k}.

For a proof of Claim 5.3.7, see Appendix D.4.3.

In conclusion, we can replace the spectral analysis of I_{ρ_D} (5.17) by that of I_{σ_D} (5.19). On the latter representation we have a complete understanding. To spell it out, let us recall [14] that if we have Young diagrams $\widetilde{D} \in \mathcal{Y}_n$ and $D \in \mathcal{Y}_k$ such that $\widetilde{D}$ contains D, denoted $\widetilde{D} \supset D$, then by removing from $\widetilde{D}$ all the boxes belonging to D, we obtain a configuration, denoted $\widetilde{D} - D$, called *skew-diagram*. If, in addition, each column of $\widetilde{D}$ is at most one box longer than the corresponding column of D, then we call $\widetilde{D} - D$ a *skew-row*. With this terminology, we have [8],

Theorem 5.3.8 (Pieri rule)**.** *Let $D \in \mathcal{Y}_k$. Then, the induced representation I_{σ_D} (5.19) is a multiplicity-free sum of irreps $\sigma_{\widetilde{D}}$ of S_n, where, the Young diagram $\widetilde{D} \in \mathcal{Y}_n$ satisfies:*

1. *$\widetilde{D} \supset D$; and*
2. *$\widetilde{D} - D$ is a skew-row.*

In fact, the Pieri Rule can be understood geometrically as a result about tensor products of representations of the complex general linear groups $GL_m(\mathbb{C})$ [32]. In particular, in Appendix D.4.4 we give a seemingly new proof of Theorem 5.3.8, by translating that result from the $GL_m(\mathbb{C})$ case to the S_n case, using the classical S_n-$GL_m(\mathbb{C})$ *Schur* (aka *Schur-Weyl*) *duality* [32, 49, 54]. Our approach was inspired by a remark of **Nolan Wallach**.

Finally, we would like to remark that, nowadays the Pieri rule can be understood as a particular case of the celebrated Littlewood-Richardson rule [40, 43], but was known [47] a long time before this general result.

As noted before, Theorem 5.3.8 together with Identity (5.18), implies the analogous "Pieri rule" for the decomposition of the induced representation I_{ρ_D} (5.17) for a spherical principal series ρ_D of GL_k.

The components of I_τ Next, the components of $I_\tau = Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k})$, for a general $\tau \in \widehat{GL}_k$, can be obtained using the P-of-CF Formula (5.16). Indeed, take non-negative integers $u, s_1, \ldots, s_l, d$, such that $u + s_1 + \ldots + s_l + d = k$. Applying Formula (5.16) with $S = \{s_1, \ldots, s_l, d\}$, we obtain an irreducible representation

$$\tau_{U,S} = Ind_{P_{u,s_1,\ldots,s_l,d}}^{GL_k}\left(\rho_U \otimes \left[\bigotimes_{i=1}^{l}(\chi_i \circ \det) \otimes \rho_{D_i}\right] \otimes \rho_D\right), \tag{5.20}$$

where, ρ_U is unsplit irrep of GL_u, the ρ_{D_i} and ρ_D are, respectively, SPS irreps of the corresponding GL_{s_i} and GL_d blocks. Moreover, the P-of-CF ensures that the $\tau_{U,S}$'s (5.20) produced in this way are all the irreps of GL_k. With the realization (5.20) of irreps, the Pieri rule implies that $I_{\tau_{U,S}} = Ind_{P_{k,n-k}}^{GL_n}(\tau_{U,S} \otimes \mathbf{1}_{n-k})$ has the following multiplicity free decomposition into sum of irreps

$$I_{\tau_{U,S}} \simeq \sum_{\widetilde{D}} Ind_{P_{u,s_1,\ldots,s_l,d+n-k}}^{GL_n}\left(\rho_U \otimes \left[\bigotimes_{i=1}^{l}(\chi_i \circ \det) \otimes \rho_{D_i}\right] \otimes \rho_{\widetilde{D}}\right), \tag{5.21}$$

where $\widetilde{D}$ runs over all Young diagrams in $\mathcal{Y}_{d+n-k}$ that satisfies conditions (1) and (2) of Theorem 5.3.8.

Note that, in particular, we obtained a proof of a more precise version of Proposition 5.2.1.

5.4. Computing tensor rank using the P-of-CF formula

The Pieri rule implies, as a corollary, that one can compute the strict tensor rank and tensor rank of a representation from its P-of-CF Formula (5.16), more precisely directly from its split principal series component. To state this, and similar results, it is convenient to use the notions of *tensor co-rank* and *strict tensor co-rank*, by which we mean, respectively, n minus the tensor rank and n minus the strict tensor rank.

Corollary 5.4.1. *We have,*

1. *For a partition $D = \{d_1 \geq \ldots\}$ of n, the tensor co-rank of the SPS representation ρ_D is the same as its strict tensor co-rank and is equal to d_1, that is, the longest row of the associated Young diagram.*
2. *The tensor co-rank of the representation $\rho_{U,S}$ of GL_n described by Formula (5.16), is the maximum of the tensor co-ranks of the SPS representations ρ_{D_i}, $i = 1, \ldots, l$, that appear in description of the split part*

of $\rho_{U,S}$. The strict tensor co-rank of $\rho_{U,S}$ is the strict tensor rank of the SPS representation ρ_{D_i} that is twisted in (5.16) by the trivial character.

For a proof of Corollary 5.4.1 see Appendix D.4.5.

5.5. Back to the explicit description of the eta correspondence

The above development implies a more informative version of Theorem 5.2.2.

Indeed, looking at Formula (5.21), we see that condition (2) of Theorem 5.3.8 (that the difference $\widetilde{D}-D$ be a skew row) means that the $n-k$ boxes of $\widetilde{D}-D$ all live in different columns of $\widetilde{D}$. In particular, $\widetilde{D}$ must contain at least $n-k$ columns, and the only way that it can contain only $n-k$ columns is for the boxes of $\widetilde{D}-D$ to belong to the first $n-k$ columns, consecutively, of $\widetilde{D}$. Thus, for all representations $\tau_{U,S}$ (5.20) of GL_k, of strict tensor co-rank not exceeding $n-k$ (which is equivalent to strict tensor rank exceeding $k-(n-k)=2k-n$), there is exactly one constituent – let us denote it by $\eta(\tau_{U,S})$ – of the corresponding induced representation $I_{\tau_{U,S}}$ (5.21) of co-rank not exceeding $n-k$. Moreover, for such $\tau_{U,S}$'s, by the philosophy of cusp forms, $\tau_{U,S}\mapsto\eta(\tau_{U,S})$ is one-to-one correspondence.

Concluding, we have completed the proof of Theorem 5.2.2 (see also Remark 5.3.3). Moreover, we obtained an explicit form of the eta correspondence. To write it in a pleasant way, let us express the representation $\tau_{U,S}$ (5.20) of GL_k as,

$$\tau_{U,S,D}=Ind_{P_{u,s,d}}^{GL_k}(\rho_U\otimes\rho_S\otimes\rho_D), \tag{5.22}$$

where u,s,d are non-negative integers such that $u+s+d=k$, $P_{u,s,d}$ is the corresponding standard block diagonal parabolic with blocks of sizes u,s,d, ρ_U is an unsplit irrep of GL_u, S is a partition $\{s_1\geq\ldots\geq s_l\}$ of s, ρ_S is the split representation of GL_s induced from $(\chi_i\circ\det)\otimes\rho_{D_i}$ on the GL_{s_i}-block of the parabolic P_S, $i=1,\ldots,l$ (see Formula (5.20)), where χ_i are non-trivial and distinct, and, finally, ρ_D is the SPS representation of GL_d associated with a Young diagram $D\in\mathcal{Y}_d$.

Theorem 5.5.1 (eta correspondence – explicit formula)**.** *Take $\tau_{U,S,D}\in\widehat{GL}_k$ of the form (5.22), and of strict tensor rank greater or equal to $2k-n$. Then, the unique constituent $\eta(\tau_{U,S,D})$ of strict tensor rank k of $I_{\tau_{U,S,D}}=Ind_{P_{k,n-k}}^{GL_n}(\tau_{U,S,D}\otimes\mathbf{1}_{n-k})$, satisfies,*

$$\boxed{\eta(\tau_{U,S,D})\simeq Ind_{P_{u,s,d+n-k}}^{GL_n}\left(\rho_U\otimes\rho_S\otimes\rho_{\widetilde{D}}\right),} \tag{5.23}$$

where $\rho_{\widetilde{D}}$ is the SPS representation of GL_{d+n-k} associated with the unique Young diagram $\widetilde{D} \in \mathcal{Y}_{d+n-k}$ whose first row has length $n-k$, and the rest of its rows make D. Moreover, the mapping $\tau_{U,S,D} \longmapsto \eta(\tau_{U,S,D})$ agrees with the eta correspondence (5.11).

6. Deriving the analytic information on ⊗-rank k irreps of GL_n

The concrete descriptions of the tensor rank k irreps of GL_n, given in the previous section, will be used now to derive the analytic information stated in Section 2.

6.1. Deriving the character ratios on the transvection

Our analysis of the character ratios (CRs) $\frac{\chi_\rho(T)}{\dim(\rho)}$, where $\rho \in \widehat{GL}_n$ and T is the transvection (0.3), proceeds in three steps: We start with the case when ρ is of tensor rank n, then we treat the case when it is a spherical principal series (SPS) representation, and finally we combine the first two cases to derive the CRs estimate for general tensor rank k irrep.

6.1.1. CRs for tensor rank n irreps Let $U_{n-1,1}$ be the unipotent radical of the parabolic $P_{n-1,1}$ (aka "mirabolic" [17]). This group is isomorphic to $\mathbb{F}_q^{n-1}$, and any non-identity element in this group is a transvection. Denote by $reg^\circ_{U_{n-1,1}} = reg_{U_{n-1,1}} - \mathbf{1}_{U_{n-1,1}}$ the regular representation of $U_{n-1,1}$ minus its trivial representation.

We have,

Proposition 6.1.1. *The restriction of a tensor rank n irrep ρ of GL_n to $U_{n-1,1}$ is a multiple of $reg^\circ_{U_{n-1,1}}$. In particular, the character ratio of such irrep on the transvection is equal to the CR of $reg^\circ_{U_{n-1,1}}$ on this element, namely,*

$$\frac{\chi_\rho(T)}{\dim(\rho)} = \frac{-1}{q^{n-1}-1}. \tag{6.1}$$

For a proof of Proposition 6.1.1 see Appendix D.5.1.

6.1.2. CRs for spherical principal series irreps Consider a SPS representation ρ_D of GL_n, where $D = \{d_1 \geq \ldots \geq d_s\}$ is a partition of n. The tensor rank of ρ_D is equal to $n-d_1$ (see Corollary 5.4.1). We will show that,

$$\frac{\chi_{\rho_D}(T)}{\dim(\rho_D)} = \begin{cases} \frac{1}{q^{n-d_1}} + o(\dots), & \text{if} \quad d_1 > d_2; \\ \frac{c_D}{q^{n-d_1}} + o(\dots), & \text{otherwise,} \end{cases} \tag{6.2}$$

where c_D is a certain integer depending only on D (and not on q).

An effective description of the representation ρ_D is given by Proposition C.2.1 in Appendix C.2. In particular, it tells us that there are integers $m_{D'D,}$ independent of q, such that,

$$\rho_D = I_D + \sum_{D' \succneqq D} m_{D'D} I_{D'}, \tag{6.3}$$

where $\succ$ denotes the dominance relation on partitions/Young diagrams (see Definition C.1.1), and I_D (respectively $I_{D'}$) is the natural induced module (5.12) attached to D.

Remark 6.1.2. *The integers $m_{D'D}$ will certainly sometimes take negative values.*

We will show that the CR at T of the representation I_D on the right-hand side of (6.3), can be seen as the numerical quantity that implies estimate (6.2). To justify this assertion, first, recall (see Remark 5.3.5) that,

$$\dim(\rho_D) = \dim(I_D) + o(\dots). \tag{6.4}$$

Second, we have,

Proposition 6.1.3. *The character ratio of the induced representation I_D at the transvection satisfies,*

$$\frac{\chi_{I_D}(T)}{\dim(I_D)} = \frac{m_{d_1}}{q^{n-d_1}} + o(\dots), \tag{6.5}$$

where m_{d_1} is the number of times the quantity d_1 appears in D.

For a proof of Proposition 6.1.3 see Appendix D.5.2.

And third, we use,

Proposition 6.1.4. *Suppose $D = \{d_1 \geq \ldots \geq d_s\}$ is a partition of n which is strictly dominated by another partition D'. Then,*

$$\frac{\chi_{I_{D'}}(T)}{\dim(I_D)} = \begin{cases} o(\frac{\chi_{I_D}(T)}{\dim(I_D)}), & \textit{if} \quad d_1 > d_2; \\ c_{D,D'} \cdot \frac{\chi_{I_D}(T)}{\dim(I_D)} + o(\dots), & \textit{otherwise,} \end{cases} \tag{6.6}$$

where $c_{D,D'}$ is an explicit non-negative integer depending only on D and D' (and not on q).

For a proof of Proposition 6.1.4 see Appendix D.5.3.

Now, we can derive (6.2). Indeed, we have,

$$\frac{\chi_{\rho_D}(T)}{\dim(\rho_D)} = \frac{\chi_{I_D}(T) + \sum\limits_{D' \supsetneqq D} m_{D'D}\chi_{I_{D'}}(T)}{\dim(I_D)} + o(\dots) \tag{6.7}$$
$$= \begin{cases} \frac{1}{q^{n-d_1}} + o(\dots), & \text{if} \quad d_1 > d_2; \\ \frac{c_D}{q^{n-d_1}} + o(\dots), & \text{otherwise}; \end{cases}$$

where c_D is an integer depending only on D (and not on q), the first equality follows from Formulas (6.3) and (6.4), and the second equality is due to (6.5) and (6.6).

6.1.3. CRs for tensor rank k irreps – general case We now treat the general case.

We can reduce the estimation task to the specific cases discussed in Sections 6.1.1 and 6.1.2. Indeed, for the SPS representations, the computation (6.7) of the character ratios is reduced to the case of induced representations of the form $I_D = Ind_{P_D}^{GL_n}(\mathbf{1})$ (5.12) for a Young diagram D. In the same way, the computation of character ratio on the transvection, for general split principal series representation is reduced to the case of an induced representation from a (one-dimensional) character of a standard parabolic, i.e., $I_{D,\chi} = Ind_{P_D}^{GL_n}(\chi)$ where χ is a character of P_D. But on the transvection T the character χ is trivial, so we are back in the case of I_D. In particular, using Formula (5.16) and the standard formula [15] for computing the character of induced representations, we see that it is enough to estimate the character ratios of irreps of the form

$$\rho_{U,D} = Ind_{P_{u,d}}^{GL_n}\left(\rho_U \otimes \rho_D\right), \tag{6.8}$$

where

- ρ_U is an unsplit irrep of GL_u;
- $D \in \mathcal{Y}_d$ is a Young diagram with longest row of length $d_1 = n - k$; and
- there are m_{d_1} rows in D of that size, and ρ_D the associated SPS representation.

Denote by $G(u,n) = GL_n/P_{u,d}$ the Grassmannian of subspaces of dimension u in $V = \mathbb{F}_q^n$. Its cardinality is [1]

$$\#G(u,n) = \frac{\prod\limits_{j=u+1}^{n}(q^j-1)}{\prod\limits_{j=1}^{n-u}(q^j-1)}. \tag{6.9}$$

In particular, using (6.9), we observe that,

$$\frac{\#\left(G(u,n-1)\right)}{\#\left(G(u,n)\right)} = \frac{q^{n-u}-1}{q^n-1} = \frac{1}{q^u} + o(\ldots). \tag{6.10}$$

The group GL_n acts on the set $G(u,n)$, and the collection $(G(u,n))^T$, of elements fixed by the transvection T, decomposes into a union of two sets:

$$(G(u,n))^T = G(u,n-1) \cup G(u-1,n-1). \tag{6.11}$$

The first set in the union consists of subspaces of dimension u that live inside the kernel of $T-I$, so we can identify it with the Grassmannian $G(u,n-1)$; while the second set consists with those subspaces of dimension u containing the line $L = \mathrm{Im}(T-I)$, so we can identify it with the Grassmannian $G(u-1,n-1)$. Note that the two sets at the right-hand side of (6.11) overlap on the set $G(u-1,n-2)$ of those V_u's that contain L, and live inside $\ker(T-I)$.

For each subspace V_u of dimension u, that is fixed by T, we identify $GL(V_u) \simeq GL_u$ and $GL(V/V_u) \simeq GL_d$. In this way, we can think of the induced actions of T on V_u and on V/V_u, as elements $T_u \in GL_u$ and $T_d \in GL_d$, respectively. In conclusion, we obtain that

(6.12)

$$\begin{aligned}
&\frac{\chi_{\rho_{U,D}}(T)}{\dim(\rho_{U,D})}\\
&\quad= \frac{\#\left(G(u,n-1)\right)}{\#\left(G(u,n)\right)} \cdot \frac{\chi_{\rho_D}(T_d)}{\dim(\rho_D)} + \frac{\#(G(u-1,n-1))}{\#(G(u,n))} \cdot \frac{\chi_{\rho_U}(T_u)}{\dim(\rho_U)} + o(\ldots)\\
&\quad= \begin{cases} \frac{1}{q^k} + o(\ldots), & \text{if} \quad k < \frac{n}{2};\\ \frac{c_D}{q^k} + o(\ldots), & \text{if} \quad \frac{n}{2} \le k < n-1;\\ \frac{c_D - 1}{q^{n-1}} + o(\ldots), & \text{if} \quad k = n-1;\\ \frac{-1}{q^{n-1}} + o(\ldots), & \text{if} \quad k = n, \end{cases}
\end{aligned}$$

where: The first equality is a consequence of the standard formula for computing the character of induced reps; The little-o in the first row of Formula (6.12) comes from the overlap of the two sets on the right-hand side of (6.11); The second equality incorporates results (6.1), (6.2) – specialized to the case $n - d_1 = k$, and (6.9), (6.10); Finally, note that appearance in (6.12) of the constant c_D, an integer depending only on D (and not on q) that comes from its appearance in (6.2).

This completes the derivation of the result (2.1) on the CRs of the irreps of GL_n on the transvection T.

6.2. Deriving the estimates on dimensions

It is enough, as was in the case of the computations of the CRs just above, to compute the dimensions of the irreps of the form (6.8) where ρ_U is an unsplit representation of GL_u attached to cuspidal datum associated with Young diagram $U \in \mathcal{Y}_u$, with (in case $u \neq 0$) rows all of which are of length greater or equal 2, and ρ_D an SPS representation attached to a Young diagram $D \in \mathcal{Y}_d$.

To compute the dimension of ρ_U, we use the following [16, 19] crude approximation to the dimension of a cuspidal representation:

Proposition 6.2.1. *The dimension of a cuspidal representation of GL_u is $q^{\frac{u(u-1)}{2}} + o(\ldots)$.*

Using Proposition 6.2.1, the standard formula for dimension of induced representation, and the explicit expression for the dimension of ρ_D (see Corollary C.4.1 in Appendix C.4), we can obtain a sharp estimate on the dimension of $\rho_{U,D}$. In particular, we can compute sharp upper and lower bounds for the dimensions of the tensor rank k irreps.

6.2.1. Upper bound for the dimensions of the tensor rank k irreps

Let us start with tensor rank n irreps.

The tensor rank n case From Part (2) of Corollary 5.4.1, we learn that an irrep ρ of GL_n is of tensor rank n if and only if it is an unsplit representation of the form $\rho = \rho_U$, associated to some Young diagram $U \in \mathcal{Y}_n$ with rows all of which are of length ≥ 2, and a corresponding cuspidal datum. The philosophy of cusp forms (in particular, Part (B.2.5) of Corollary B.2.5 in Appendix B.2.2) tells us that the ρ_U's of maximal dimension are those where the cuspidal datum consists of non-isomorphic cuspidal representation on the various blocks of the corresponding Levi component. In particular, Proposition 6.2.1 implies that all these ρ_U's are of dimension $\dim(\rho_U) = q^{\frac{n(n-1)}{2}} + o(\ldots)$.

We conclude,

Proposition 6.2.2. *The largest possible dimension of a tensor rank n irrep of GL_n is $q^{\frac{n(n-1)}{2}} + o(\ldots)$.*

The tensor rank $k < n$ case Fix $0 \leq k < n$, and consider the SPS representation ρ_D where D is the partition $\{n-k, 1, \ldots, 1\}$ of n. Formula (C.6) implies that

$$\dim(\rho_D) = q^{k(n-k)+\frac{k(k-1)}{2}} + o(\ldots). \tag{6.13}$$

The dimension appearing in (6.13) is the largest possible for a tensor rank k irrep. There are several ways to justify this assertion, and we choose to proceed with a simple construction of all the tensor rank k irreps of the form $\rho_{U,D} = Ind_{P_{u,d}}^{GL_n}(\rho_U \otimes \rho_D)$ that are candidates for winning the "maximal dimension competition", and then observe that they all have dimension as in (6.13).

First, the rank constraint implies that the Young diagram D, that defines the ρ_D datum of $\rho_{U,D}$, must have longest row of size $n-k$, so we can assume that $d = n-k+l$, for some $0 \leq l \leq k$. In particular, if we want to maximize the dimension of ρ_D, under this constraint, the dominance relation on partitions tells us that, the partition D must be of the form

$$D = \{n-k, \overbrace{1, \ldots, 1}^{l \text{ times}}\}.$$

Second, to maximize the dimension of the unsplit part ρ_U of $\rho_{U,D}$, we take it to be one of the tensor rank u representation of GL_u of maximal dimension constructed in the Section just above (see 6.2.2). In particular, $\dim(\rho_U) = q^{\frac{u(u-1)}{2}} + o(\ldots)$.

Finally, with any inducing ρ_U and ρ_D such as these we just described, the dimension of the corresponding induced representation $\rho_{U,D}$ (recall that $u = k-l$) is

$$\begin{aligned} \dim(\rho_{U,D}) &= q^{(k-l)(n-k+l)} \cdot q^{\frac{(k-l)(k-l-1)}{2}} \cdot q^{l(n-k)+\frac{l(l-1)}{2}} + o(\ldots) \\ &= q^{k(n-k)+\frac{k(k-1)}{2}} + o(\ldots). \end{aligned}$$

In conclusion, we obtain,

Proposition 6.2.3. *The largest possible dimension of a tensor rank k irreducible representation of GL_n is $q^{k(n-k)+\frac{k(k-1)}{2}}+o(\dots)$.*

Overall, this completes the verification of the assertion on upper bound on dimensions, appearing in Theorem 2.2.1.

6.2.2. Lower bound for the dimensions of the tensor rank k irreps Let us start with tensor rank n irreps.

The tensor rank n case Let as assume that $n=l\lambda$ for some $\lambda\geq 1$. Consider the standard parabolic $P_{l\times\lambda}$ with Levi component an l-fold product of GL_λ's, and on each GL_λ the same cuspidal representation κ_λ. Recall (see Appendix B.2.2) that, such a cuspidal datum is called *isobaric* and the induced representation $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$ is, in general (e.g., for $l>1$), reducible. Moreover, it has a unique component $\rho_{_{l\times\lambda}}$ of smallest dimension (see Formula (B.9)) with

$$\dim(\rho_{_{l\times\lambda}})=\frac{\dim(\kappa_\lambda)^l\cdot\#(GL_n/P_{_{l\times\lambda}})}{\#(GL_l(\mathbb{F}_{q^\lambda})/B_l(\mathbb{F}_{q^\lambda}))},\tag{6.14}$$

where B_l is the standard Borel subgroup in GL_l.

We look at two cases:

- *n even:* Consider the tensor rank n representation $\rho_{\frac{n}{2}\times 2}$ of GL_n, given by the recipe described above with $\lambda=2$ and $l=\frac{n}{2}$, i.e., the constituent of $Ind_{P_{\frac{n}{2}\times 2}}^{GL_n}(\kappa_2^{\otimes^{\frac{n}{2}}})$ of smallest dimension. Then, a direct calculation, using Formula (6.14), gives

$$\dim(\rho_{\frac{n}{2}\times 2})=q^{\frac{n^2}{4}}+o(\dots).\tag{6.15}$$

- *n odd:* Consider the $\otimes$-rank n representation $\rho_{3,n-3}$ of GL_n, given by $Ind_{P_{3,n-3}}^{GL_n}(\kappa_3\otimes\rho_{\frac{n-3}{2}\times 2})$, where $P_{3,n-3}$ is the standard parabolic with Levi blocks GL_3 and GL_{n-3}, the κ_3 is a cuspidal representation of GL_3, and finally, the $\rho_{\frac{n-3}{2}\times 2}$ is the irrep of GL_{n-3} defined in the same way as $\rho_{\frac{n}{2}\times 2}$ above. Then, using (6.15) we get,

$$\dim(\rho_{3,n-3})=q^{\frac{(n-3)^2}{4}+3(n-2)}+o(\dots).\tag{6.16}$$

In fact, optimizing using the philosophy of cusp forms and Formula (6.14), we see that Examples (6.15) and (6.16) give the minimizers, in the dimension aspect, among the tensor rank n irreps.

In conclusion,

Proposition 6.2.4. *The smallest possible dimension of a tensor rank n irreducible representation of GL_n is $q^{\frac{n^2}{4}}+o(\ldots)$ if n is even, and $q^{\frac{(n-3)^2}{4}+3(n-2)}+o(\ldots)$ if n is odd.*

The tensor rank $k<n$ cases We fix $0\le k<n$, and consider an irrep $\rho_{U,D}$ of GL_n of the form (6.8), i.e., $\rho_{U,D}=Ind_{P_{u,d}}^{GL_n}(\rho_U\otimes\rho_D)$, which is in addition of tensor rank k, namely, the Young Diagram $D\in\mathcal{Y}_d$ must contains a row of length $n-k$ and this is its longest one. Optimizing to obtain the lowest possible dimension of such irreps, we just need to decide what to do with the "left over" k boxes. Moreover, because there is no interaction between the unsplit and split inducing data, we just need to decide if k goes to the diagram $U\in\mathcal{Y}_u$ or to $D\in\mathcal{Y}_d$.

We divide the discussion to several cases, depending on the size and, sometime, also the parity of k.

Case $k<\frac{n}{2}$: Here applying (replace n by k there) the numerical results (6.15) and (6.16), we see that the winner is the SPS representation ρ_D of GL_n with D corresponds to the partition $\{n-k,k\}$. The dimension is of course

$$\dim(\rho_D)=q^{k(n-k)}+o(\ldots).$$

Case $\frac{n}{2}\le k<\frac{2n}{3}$: Here, again, by a direct comparison using the numerical results (6.15) and (6.16), we see that the lowest possible dimension is of an SPS representation, this time ρ_D with D which is associated to the partition $\{n-k,n-k,\,2k-n\}$. The dimension is

$$\dim(\rho_D)=q^{(n-k)(3k-n)}+o(\ldots),$$

using the formula in Corollary C.4.1.

Case $\frac{2n}{3}\le k<n$: Here, the comparison shows that the winner is the unsplit side, i.e., the irreps of tensor rank k and of lowest dimension are of the form $\rho_{U,D}=Ind_{P_{k,n-k}}^{GL_n}(\rho_U\otimes\rho_D)$, where ρ_D is the trivial representation of GL_{n-k} and ρ_U is the tensor rank k representation of GL_k of lowest dimension. In particular,

$$\dim(\rho_{U,D})=\begin{cases}q^{k(n-k)+\frac{k^2}{4}}+o(\ldots), & \text{if } k \text{ is even;}\\ q^{k(n-k)+\frac{(k-3)^2}{4}+3(k-2)}+o(\ldots), & \text{if } k \text{ is odd;}\end{cases}$$

Overall, this completes the verification of the assertions on lower bounds on dimensions, appearing in Theorem 2.2.1.

6.3. Deriving the cardinality of the collection of tensor rank k irreps

To calculate the number of irreps of a given tensor rank, we will use informations that come from the η-correspondence and the philosophy of cusp forms.

Let us start with the largest collections.

6.3.1. The tensor rank $n-1$ and n cases The cuspidal representations of GL_n are of tensor rank n, and from their construction [16] one knows [7, 35] that there are $aq^n + o(\ldots)$ of them, for some $0 < a < 1$. In addition, from the direct construction of the generic split principal series irreps, i.e., these induced from generic characters of the Borel (or rather its Levi component – the diagonal torus) we know that there are $bq^n + o(..)$ of them for some $0 < b < 1$. But, $\#(\widehat{GL_n}) = q^n + o(\ldots)$, and the eta correspondence implies that the number of irreps of GL_n of tensor rank $\leq n-2$, is not more than $q^{n-1} + o(\ldots)$, so we deduce that there positive constants c_{n-1}, c_n, with $c_{n-1} + c_n = 1$ such that

$$\#(\widehat{GL_n})_{\otimes,n-1} = c_{n-1}q^n + o(\ldots), \text{ and } \#(\widehat{GL_n})_{\otimes,n} = c_n q^n + o(\ldots). \tag{6.17}$$

Remark 6.3.1. *We note that,*

(1) *The estimates (6.17) hold also for* $\#(\widehat{GL_n})^{\star}_{\otimes,n-1}$ *and* $\#(\widehat{GL_n})^{\star}_{\otimes,n}$
(2) *It can be shown that the constants* c_{n-1} *and* c_n *are independent of* q.

We proceed to do the counting in the lower tensor rank cases.

6.3.2. The tensor rank $k \leq n-2$ case The eta correspondence (see Theorem 5.2.2) gives a bijection between $(\widehat{GL_k})^{\star}_{\otimes,\geq 2k-n}$ – the irreps of GL_k of strict tensor rank greater or equal to $2k-n$, and $(\widehat{GL_n})^{\star}_{\otimes,k}$ – the irreps of GL_n of strict tensor rank k. For $k \leq n-2$, the collection $(\widehat{GL_k})_{\otimes,\geq 2k-n}$ includes the irreps of strict tensor rank k and $k-1$ of GL_k, so $(\widehat{GL_n})^{\star}_{\otimes,k} = q^k + o(\ldots)$. Moreover, the description (5.23) tells us that these irreps of strict tensor rank k and $k-1$ of GL_k are

- mapped by the eta correspondence to irreps of tensor rank k of GL_n; and,

- produce non-isomorphic representations upon twist by any non-trivial character of GL_n.

As a result, overall we get

$$\#(\widehat{GL}_n)_{\otimes,k} = q^{k+1} + o(..).$$

This completes the verification of Theorem 2.3.1, and the derivation of all the analytic properties stated in Section 2.

7. Deriving the analytic information on $\otimes$-rank k irreps of SL_n

Let us now derive the estimates on dimensions of tensor rank k irreps of SL_n, $n \geq 3$, and the number of such irreps. In particular, we complete the verification of the results announced in Section 3.

We start with the dimension aspect.

7.1. Deriving the estimates on dimensions

As we remarked earlier, the estimates in the dimension aspect are the same as for GL_n for a simple reason: the restrictions to SL_n of the irreps of GL_n that give the lowest and the largest dimensions in a given tensor rank, typically stay irreducible.

The main tool we use to check the irreducibility in question is the Clifford-Mackey criterion which says (see Corollary A.2.6 in Appendix A) that the restriction to SL_n of an irrep GL_n, stays irreducible if and only if it is not fixed by twist of any non-trivial character of GL_n.

As was done for GL_n, we do some case by case computations.

7.1.1. Upper bound on dimensions of tensor rank k irreps

Tensor rank n case The cuspidal representations of GL_n have a parametrization by the complex characters of the maximal anisotropic torus of GL_n modulo the action of the Galois group of the degree n extension of the finite field [16, 35]. In particular, there exist cuspidal representations (in fact most of them have this property) of GL_n which are not fixed by any twist by a character of GL_n, and so their restrictions to SL_n stay irreducible and have the dimension $q^{\frac{n(n-1)}{2}} + o(\ldots)$. This is the sharp upper bound announced in (3.2) for tensor rank n irreps.

Tensor rank $k < n$ case Consider the SPS representation ρ_D, associated to the partition D given by $\{n-k, 1, \ldots, 1\}$ of n. It gives (see Section 6.2.1) the upper bound for the dimensions of tensor rank $k < n$ irreps of GL_n, and stays (for example by the criterion stated just above) irreducible after restriction to SL_n. This shows that, indeed, in the range $k < n$ the upper bound $q^{k(n-k)+\frac{k(k-1)}{2}} + o(\ldots)$ appearing in (3.2) is sharp.

7.1.2. Lower bound on dimensions of tensor rank k irreps

Tensor rank n case Take a cuspidal representation of GL_2 which is not fixed by a twist of any character of GL_2. In addition, take a cuspidal representation of GL_3 with similar property (see Section 7.1.1 for more detailed discussion). Then, apply the construction of the tensor rank n irreps of GL_n of minimal dimension. They will be irreducible after restriction to SL_n, and have the dimension given as a sharp lower bound in (3.2) for tensor rank n irreps.

Tensor rank $k < n$ case As in the case of GL_n (see Section 6.2.2) we go over several cases.

Case $k < \frac{n}{2}$: Here, the representation of GL_n with smallest possible dimension is (up to tensoring with a character) given by the SPS representation ρ_D, where D is the partition $\{n-k, k\}$ of n. By the Clifford-Mackey criterion it stays irreducible after restriction to SL_n, This shows that, indeed, in the range $k < \frac{n}{2}$ the dimension $q^{k(n-k)} + o(\ldots)$ appearing in (3.2) is a lower bound and, indeed, a sharp one.

Case $\frac{n}{2} \leq k < \frac{2n}{3}$: In this interval, for the GL_n, the lowest possible dimension $q^{(n-k)(3k-n)} + o(\ldots)$ is (again up to tensoring by a character) of the SPS representation ρ_D with D the partition $\{n-k, n-k,\ 2k-n\}$ of n. Again, by the Clifford-Mackey criterion it stays irreducible after restriction to SL_n, confirming that also in this case what appear in (3.2) is a lower bound, and a sharp one.

Case $\frac{2n}{3} \leq k < n$: The same reasoning, using what we have for GL_n (see Section 6.2.2), implies that also for this interval the estimate in (3.2) is a sharp lower bound.

We proceed to discuss the cardinality aspect.

7.2. Deriving the number of irreps of tensor rank k of SL_n

We have sharp estimate on the number of tensor rank k irreps of GL_n – see Formula (2.3). We also know (see Fact 3.2.2) that irreps of GL_n share any constituent (hence all) after restriction to SL_n if and only if they differ by twist by a character of GL_n. Finally, we can show that most tensor rank k irreps of GL_n stay irreducible after restriction to SL_n. Indeed, we have the following quantitative result:

Proposition 7.2.1. *Consider the irreps of GL_n of tensor rank k. Then,*

(1) *For $k < \frac{n}{2}$ all of them stay irreducible after restriction to SL_n.*
(2) *For $\frac{n}{2} \leq k$ the proportion of them which stay irreducible after restriction to SL_n is $1 - o(\frac{1}{q})$.*

For a proof of Proposition 7.2.1 see Appendix D.6.1.

With the help of Proposition 7.2.1 we can get the exact estimates that stated in Formula (3.3). We go over two cases:

Case $k < \frac{n}{2}$: Here the conclusion is clear, after restriction, taking into account Fact 3.2.2 and Formula (2.3), we get that $\#((\widehat{SL}_n)_{\otimes,k}) = q^k + o(\ldots)$.

Case $\frac{n}{2} \leq k$: Here, using Part (7.2.1) of Proposition 7.2.1, we see, again using Fact 3.2.2, that $\#((\widehat{SL}_n)_{\otimes,k}) = q^k + o(\ldots)$ for $k < n-1$, and there are two positive constants c_{n-1}, c_n, with $c_{n-1} + c_n = 1$, such that $\#((\widehat{SL}_n)_{\otimes,k}) = c_k q^{n-1} + o(\ldots)$, for $k = n-1, n$.

This completes the derivation of estimates (3.3), and of all the analytic properties announced in Section 3.

Appendix A. Clifford-Mackey theory

We describe some parts from *Clifford theory/Mackey's little group method* [9, 44] that are relevant to this note.

A.1. Setting

Suppose you have a finite group G which is a semi-direct product

$$G = C \ltimes N,$$

where N is a normal subgroup, and C is cyclic.

A simple version of Clifford-Mackey theory gives the construction of the irreps of G from the irreps of N, and describes how irreps of G decompose under restriction to N.

A.2. The construction

Note that the group C acts on $\widehat{N}$, the unitary dual of N, by conjugation. We will call the members of $\widehat{N}$ that appear in, the restriction to N, of a representation $\rho \in \widehat{G}$, the N-spectrum of ρ. The irreducibility of ρ implies that,

Claim A.2.1. *The N-spectrum of $\rho \in \widehat{G}$ is a single orbit for the action of C on $\widehat{N}$.*

Let us construct all $\rho \in \widehat{G}$ sharing a given N-spectrum. Take a representation $\pi \in \widehat{N}$, and let $C_\pi \subset C$ be the stabilizer of π in C. Then for each c in C_π, there is an operator $\sigma(c)$ on the space of π such that

$$\sigma(c)\pi(n)\sigma(c)^{-1} = \pi(cnc^{-1}), \tag{A.1}$$

and this $\sigma(c)$ is determined up to a scalar multiple, by Schur's Lemma.

Claim A.2.2. *We can choose the operators $\sigma(c)$, $c \in C_\pi$, from (A.1) in such a way that they form a representation of C_π.*

Claim A.2.2 follows from the fact that C_π is cyclic.[8] Indeed, if c_0 is a generator of C_π, then we can choose $\sigma(c_0^k) = \sigma(c_0)^k$ for $0 \leq k < \#C_\pi = m$. Moreover, equation (A.1) implies that $\sigma(c_0)^m$ is a scalar multiple of the identity. We can multiply $\sigma(c_0)$ by a scalar to arrange that $\sigma(c_0)^m$ is exactly the identity. Then, with this definition of σ we get an extension $\widetilde{\pi}$ of π to $C_\pi \ltimes N$, namely the representation $\widetilde{\pi} = \sigma \ltimes \pi$ on the space of π given by

$$\widetilde{\pi}(c,n) := \sigma(c) \circ \pi(n), \quad c \in C_\pi,\ n \in N. \tag{A.2}$$

We can get other extensions by twisting this with a character of C_π.
Clifford-Mackey's theory [9, 44] then says,

Theorem A.2.3. *We have,*

(1) *The irreps of the form (A.2) are (up to twist by a character of C_π) all the possible extensions of π from N to $C_\pi \ltimes N$.*

[8] If C_π is not cyclic, then it may not happen that the $\sigma(c)$ can be chosen to form a representation. The prime example is when G is the Heisenberg group, and N is its center.

(2) *All irreps ρ of G containing π are obtained by inducing one of these extensions from $C_\pi \ltimes N$ to G.*

As a result we obtain,

Corollary A.2.4. *We have,*

(1) *Irreps of G have the same N-spectrum iff they differ by twist by a character of C.*
(2) *The restriction to N of any member of $\widehat{G}$ is multiplicity free.*

For a proof of Corollary A.2.4 see Appendix D.7.1 (Part (A.2.4) was proved in [38]).

Finally, let us rewrite Part (A.2.4) of Corollary A.2.4 in a slightly different and more quantitative way. Denote by $\widehat{G}_\pi$ the collection of all irreps of G having $\pi \in \widehat{N}$ in their N-spectrum. Theorem A.2.3 implies,

Corollary A.2.5. *The group of characters of C_π acts naturally on $\widehat{G}_\pi$ and this action is free and transitive.*

In particular,

Corollary A.2.6. *The restriction to N of an irrep ρ of G stays irreducible iff ρ is not fixed by a twist of any non-trivial character of C.*

Appendix B. Harish-Chandra's "philosophy of cusp forms"

In this section we recall several facts from Harish-Chandra's *"philosophy of cusp forms"* (P-of-CF) [26] for the description/classification of the set of irreps of GL_n. We follow closely the exposition of ideas given in [35] (where the reader can find more details, including proofs of the various statements). Other good sources are [7] and the comprehensive study done in [55].

The upshot of the P-of-CF is a process that exhausts $\widehat{GL_n}$ in three steps:

Step 1. Determining the "cuspidal" irreps of the groups GL_m, $m \leq n$.

Step 2. Dividing $\widehat{GL_n}$ into subsets parametrized by "cuspidal data".

Step 3. Parametrizing the irreps associated with each cuspidal datum.

We will give now more details on Step 2 (see Section B.1) and Step 3 (see Section B.2), leaving the classification of the irreps to be given in term of the building blocks – the cuspidal representations of Step 1, which we will not discuss explicitly in this note (see [7, 16, 35, 55]).

B.1. Cuspidal data attached to parabolic subgroups

Let us denote $V_n = \mathbb{F}_q^n$, and for each $m \leq n$ denote by $V_m \subset V_n$ subspace of V_m of vectors having their last $n-m$ coordinates equal to zero, and by V_{n-m}^o the complementary subspace consisting of vectors having their first m coordinates set to zero.

Recall that the *standard flag* associated with an increasing subsequence of integers

$$A = \{0 = a_0 < a_1 < \ldots < a_l = n\}, \tag{B.1}$$

is the nested sequence of spaces of V_n,

$$0 = V_{a_0} \subset V_{a_1} \subset \ldots \subset V_{a_l} = V_n. \tag{B.2}$$

To the flag (B.2), we associate the following triple of groups:

$$\text{(B.3)} \quad \begin{array}{llll} (1) & P_A & = & \{g \in GL_n;\ g(V_{a_i}) = V_{a_i} \text{ for every } i\}, \\ (2) & U_A & = & \{g \in P_A;\ (g-1)(V_{a_i}) \subset V_{a_{i-1}} \text{ for every } i\}, \\ (3) & L_A & = & \{g \in P_A;\ g(V_{n-a_i}^o) = V_{n-a_i}^o \text{ for every } i\}. \end{array}$$

The group P_A is called the *standard parabolic* subgroup associated with the flag (B.2), and the groups U_A, L_A, are, respectively, the *unipotent radical* and *Levi component* of P_A. We have,

$$\begin{cases} P_A = L_A U_A; \\ L_A \simeq \prod\limits_i GL_{\lambda_i}, \end{cases} \tag{B.4}$$

where $\lambda_i = a_i - a_{i-1}$, form a partition of n.

Now, we can illustrate a recursive process leading to the P-of-CF.

Take $\rho \in \widehat{GL_n}$ and consider a standard parabolic subgroup $P_A \subset GL_n$. Suppose ρ contains a vector invariant under U_A, the unipotent radical of P_A. Then Frobenius reciprocity [50] implies that there is a representation κ of P_A, trivial on U_A, such that ρ is contained in the induced representation of κ from P_A to GL_n,

$$\rho < Ind_{P_A}^{GL_n}(\kappa).$$

Since a representation of P_A, trivial on U_A, is a representation of $L_A = P_A/U_A$, and L_A is a product of GL_m's for $m < n$, the problem of determining the possibilities for κ (i.e., determining $\widehat{L}_A$) is presumably easier than that of determining $\widehat{GL}_n$. Thus, the problem of determining all $\rho \in \widehat{GL}_n$ with U_A invariant fixed vectors is reduced to the problem of determining $\widehat{L}_A$ and decomposing induced representations.

We described above an inductive procedure for determining $\widehat{GL}_n$, the building blocks of which are those representations for which no such reduction is possible, i.e., those irreps κ of GL_m, $m \leq n$, which contain no U_A-invariant vectors for any $A \neq \{0, n\}$. Harish-Chandra called these irreps *cuspidal,* a term suggested by the theory of automorphic forms.

To make the P-of-CF description of $\widehat{GL}_n$ more precise, one introduces the following key definition:

Definition B.1.1. *A cuspidal datum is a pair (P_A, κ) where $P_A \subset GL_n$ is a standard parabolic, and κ is a cuspidal irrep of its Levi subgroup L_A. Two cuspidal data (P_A, κ) and $(P_{A'}, \kappa')$ are associate if there is a $g \in GL_n$ that conjugates the Levi subgroups $L_A \subset P_A$ to $L_{A'} \subset P_{A'}$ and the corresponding cuspidal representations κ to κ'.*

A main result of the P-of-CF is

Theorem B.1.2 (Harish-Chandra)**.** *Suppose $\rho \in \widehat{GL}_n$. Up to association, there exists a unique cuspidal datum (P_A, κ) with*

$$\rho < Ind_{P_A}^{GL_n}(\kappa). \tag{B.5}$$

As a consequence we obtain

Corollary B.1.3. *The induced representations $Ind_{P_A}^{GL_n}(\kappa)$ and $Ind_{P_{A'}}^{GL_n}(\kappa')$ have components in common if and only if the cuspidal data (P_A, κ) and $(P_{A'}, \kappa')$ are associated. Moreover, in this case the induced representations are equivalent.*

Remark B.1.4. *We would like to elaborate a bit more on the structure of a cuspidal datum.*

Note that since L_A is a product of GL_{λ_i} (see (B.4)) then any irrep κ of L_A will be a tensor product

$$\kappa = \bigotimes_i \kappa_{\lambda_i}, \tag{B.6}$$

of representations κ_{λ_i} of the GL_{λ_i}. In particular, κ will be cuspidal if and only if every factor κ_{λ_i} in (B.6) is cuspidal.

Moreover, since up to association the factors GL_{λ_i} of L_A can be permuted arbitrarily, it will be useful for us to have certain "standard" organization of the cuspidal datum:

First, we will call the A (see (B.1)) for which the differences $\lambda_i = a_i - a_{i-1}$ are monotonically (weakly) decreasing the standard representative of its association class. Moreover, in this case we will call a cuspidal representation of L_A of the form (B.6) standard iff, for any cuspidal representation κ_λ of GL_λ, the set of indices i such that $\lambda_i = \lambda$ and $\kappa_{\lambda_i} \simeq \kappa_\lambda$ is a consecutive set. When these conditions hold, we will also say that (P_A, κ) is a standard cuspidal datum.

Second, we define the decomposition parabolic $P_{\widetilde{A}}$ attached to a given standard cuspidal datum with parabolic P_A. This is a parabolic $P_{\widetilde{A}}$ that contains P_A and defined by an increasing sequence $\widetilde{A} = \{\widetilde{a}_i\} \subset A$, where the $\widetilde{a}_i$ are such that, two blocks of L_A belong to the same block of $L_{\widetilde{A}}$ if and only if the cuspidal representations attached to the two blocks by the given cuspidal datum are isomorphic.

In conclusion, Theorem B.1.2 implies that the process of forming induced representations from parabolic subgroups using cuspidal representations of Levi subgroups, partitions $\widehat{GL_n}$ into disjoint subsets parametrized by association classes of standard cuspidal data.

B.2. Parametrizing the irreps associated with a cuspidal datum

The next step in the philosophy of cups forms is to parametrize the irreducible components of the induced representations $Ind_{P_A}^{GL_n}(\kappa)$.

We first observe that, for a standard cuspidal datum (P_A, κ) with decomposition parabolic $P_{\widetilde{A}}$, we have

Proposition B.2.1. *If σ is any irreducible component of $Ind_{P_A}^{P_{\widetilde{A}}}(\kappa)$ then $Ind_{P_{\widetilde{A}}}^{GL_n}(\sigma)$ is irreducible.*

Proposition B.2.1 follows from Theorem B.1.2 and Mackey irreducibility criteria [45, 50].

B.2.1. Parametrizing the irreducible components of $\boldsymbol{Ind_{P_A}^{P_{\widetilde{A}}}(\kappa)}$
Thus, by Proposition B.2.1, all the reducibility of $Ind_{P_A}^{GL_n}(\kappa)$ happens in the blocks of the Levi component $L_{\widetilde{A}}$ and analysis of reducibility reduces to analysis of cuspidal datum for which all the blocks have equivalent representations – see Figure 11 for illustration. A datum of this kind will be called an isobaric cuspidal datum.

Let us elaborate a bit more on the reduction to an isobaric case.

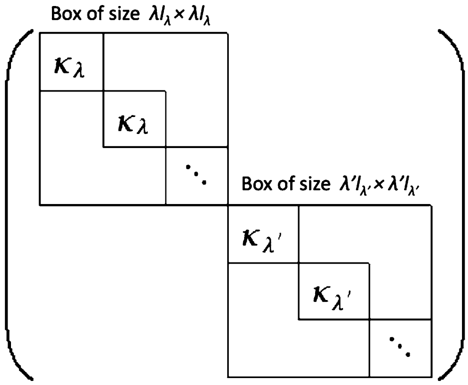

Figure 11: Cuspidal datum consisting of two blocks of an isobaric cuspidal data.

Indeed, the standard cuspidal datum (P_A, κ) is given by a representation

$$\kappa = \bigotimes_\lambda \kappa_\lambda^{\otimes l_\lambda} \quad \text{of the Levi component} \quad L_A = \prod_\lambda (GL_\lambda)^{l_\lambda},$$

and the decomposition parabolic $P_{\widetilde{A}} \supset P_A$ has Levi component

$$L_{\widetilde{A}} = \prod_\lambda GL_{\mu_\lambda}, \quad \mu_\lambda = \lambda l_\lambda.$$

Moreover, since κ is trivial on $U_A \supset U_{\widetilde{A}}$, the representation $Ind_{P_A}^{P_{\widetilde{A}}}(\kappa)$ will be effectively the representation $Ind_{L_{\widetilde{A}} \cap P_A}^{L_{\widetilde{A}}}(\kappa)$ of $L_{\widetilde{A}}$ induced from κ considered as a representation of the parabolic subgroup

$$L_{\widetilde{A}} \cap P_A = \prod_\lambda \left(GL_{\mu_\lambda} \cap P_A \right).$$

In particular, we have

$$Ind_{L_{\widetilde{A}} \cap P_A}^{L_{\widetilde{A}}}(\kappa) \simeq \bigotimes_\lambda Ind_{GL_{\mu_\lambda} \cap P_A}^{GL_{\mu_\lambda}}(\kappa_\lambda^{\otimes l_\lambda}).$$

The Levi component of $GL_{\mu_\lambda} \cap P_A$ is a product of l_λ copies of GL_λ.

The conclusion is that, indeed, in order to parametrize the irreducible components of $Ind_{P_A}^{P_{\widetilde{A}}}(\kappa)$ it suffices to analyze the case of parabolic induction attached to an isobaric cuspidal datum.

B.2.2. Parametrizing the irreps attached to isobaric cuspidal datum We take $n = l\lambda$ and consider a parabolic $P_{l\times\lambda} \subset GL_n$ with Levi component $L_{l\times\lambda} = (GL_\lambda)^l$ equipped with a representation of the form $\kappa_\lambda^{\otimes^l}$, where κ_λ is an irreducible cuspidal representation of GL_λ.

We would like to parametrize the irreducible components of $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$.

Consider the intertwining algebra $End_{GL_n}(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l}))$. From general theory (e.g., from Burnside's double commutant theorem [54]) follows that

Proposition B.2.2. *The joint action of* GL_n *and* $End_{GL_n}(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l}))$ *on* $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$ *induces a canonical bijection between the irreps of* $End_{GL_n}(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l}))$ *and the irreducible components of* $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$.

Example B.2.3. *The standard parabolic attached to the set* $\{0, 1, \ldots, n\}$ *is the Borel subgroup* $B \subset GL_n$ *of upper triangular matrices. The irreducible representations that appear in* $Ind_B^{GL_n}(\mathbf{1})$ *are called* *<u>spherical principal series (SPS)</u> representations. Consider the algebra (under convolution)* $H(GL_n//B)$ *of functions on* GL_n *which are bi-invariant with respect to* B*. This algebra is called the* *<u>spherical Hecke algebra</u>. Realizing* $Ind_B^{GL_n}(\mathbf{1})$ *on the space of functions on* G/B *we obtain an identification*

$$End_{GL_n}(Ind_B^{GL_n}(\mathbf{1})) = \mathcal{H}(GL_n//B). \tag{B.7}$$

Identity (B.7) has an important generalization as follows. Consider the spherical Hecke algebra $\mathcal{H}(GL_l(\mathbb{F}_{q^\lambda})//B(\mathbb{F}_{q^\lambda}))$. It turns out that, $End_{GL_n}(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l}))$ has a presentation in term of generators and relations that is identical to the presentation using standard generators and relations for $\mathcal{H}(GL_l(\mathbb{F}_{q^\lambda})//B(\mathbb{F}_{q^\lambda}))$ [28, 35, 42]. In particular,

Theorem B.2.4. *There is an explicit isomorphism*

$$End_{GL_n}(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})) \simeq \mathcal{H}(GL_l(\mathbb{F}_{q^\lambda})//B(\mathbb{F}_{q^\lambda})),$$

that preserves the natural L^2*-structures on the two algebras up to multiples.*

As a corollary we get

Corollary B.2.5. *There is a canonical bijection*

$$\beta : \widehat{GL}_n(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})) \longleftrightarrow \widehat{GL}_l(Ind_{B(\mathbb{F}_{q^\lambda})}^{GL_l(\mathbb{F}_{q^\lambda})}(\mathbf{1})), \tag{B.8}$$

with the following properties:

(1) *The multiplicity of* ρ *in* $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$ *is equal the multiplicity of* $\beta(\rho)$ *in* $Ind_{B(\mathbb{F}_{q^\lambda})}^{GL_l(\mathbb{F}_{q^\lambda})}(\mathbf{1})$.

(2) *For every* $\rho\in\widehat{GL}_n(Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l}))$ *we have*

$$\frac{\dim(\rho)}{\dim(\beta(\rho))}=\dim(\kappa_\lambda^{\otimes^l})\frac{\#((GL_n(\mathbb{F}_q)/P_{l\times\lambda}(\mathbb{F}_q))}{\#(GL_l(\mathbb{F}_{q^\lambda})/B(\mathbb{F}_{q^\lambda}))}. \tag{B.9}$$

According to (B.8), in order to parametrize irreps attached to isobaric cuspidal datum, it is enough to decompose the space $Ind_B^{GL_l}(\mathbf{1})$.

B.2.3. Parametrizing the spherical principal series representations We want to parametrize the SPS representations, i.e., the irreps that appear in $Ind_B^{GL_l}(\mathbf{1})$.

Let us denote by W the standard Weyl group (i.e., the permutation matrices) in GL_l. In addition, for a standard parabolic $P_A\subset GL_l$, we consider the induced representation $I_A=Ind_{P_A}^{GL_l}(\mathbf{1})$, the subgroup $W_A=W\cap P_A<W$, and the induced representation $Y_A=Ind_{W_A}^{W}(\mathbf{1}))$.

The following theorem gives an effective parametrization of the SPS representations:

Theorem B.2.6. *There is a unique bijection*

$$\alpha:\widehat{GL}_l(Ind_B^{GL_l}(\mathbf{1}))\longleftrightarrow\widehat{W}, \tag{B.10}$$

such that for every standard parabolic subgroup P_A *we have* $\rho\in\widehat{GL}_l(I_A)$ *if and only if* $\alpha(\rho)\in\widehat{W}(Y_A)$. *Moreover, in that case we have*

$$\dim(Hom_{GL_l}(\rho,I_A))=\dim(Hom_W(\alpha(\rho),Y_A)). \tag{B.11}$$

The standard justification for Theorem B.2.6 that we are aware of (see [5, 35, 37]), goes by the name “Tits's deformation argument”. However, due to its fundamental rule in the representation theory of the finite general linear groups, it might be worthwhile to give other derivations of Theorem B.2.6. In particular, in Appendix C we sketch a modified approach to the proof of Theorem B.2.6, which seems to be more elementary than the approach currently used in the literature, and might give additional valuable information on representations of GL_l.

In conclusion, we obtain a classification of the irreducible components that appear in $Ind_B^{GL_l}(\mathbf{1})$. The parametrization is given in term of partitions of l as is the case for the irreps of $W=S_l$ [8, 15, 54]. As a consequence, using (B.8) and (B.10) we get a parametrization of the irreps that appear inside $Ind_{P_{l\times\lambda}}^{GL_n}(\kappa_\lambda^{\otimes^l})$ in terms of partitions of l.

B.3. Summary

We have learned that, according to the philosophy of cusp forms, an irrep of GL_n is specified by its cuspidal datum, augmented by a collection of partitions. Precisely, for each cuspidal representation κ_λ that appears in the datum, if l is the number of times that κ_λ appears, then we augment the datum with a partition of l attached to κ_λ.

Appendix C. Representations of S_l and the spherical principal series for GL_l

We sketch (for a more comprehensive treatment, including proofs of the main statements, see [25]) a seemingly not so well known organization of the representation theories of – on the one hand the symmetric group S_l, and on the other hand the spherical principal series (SPS) representations of GL_l.

The modified perspective, starts by putting at the forefront two naturally arising structures – the *symmetric* and *spherical* monoids. Then, as a logical outcome of their intrinsic qualities, one is able to reproduce, in an elegant way, first the classifications – in terms of partitions – of the irreps of the symmetric group, and of the SPS representations of the finite general linear group; and second to recast the bijection stated in Theorem B.2.6, in terms of an isomorphism – the only one possible – between these two aforementioned monoids.

As a by-product we get additional valuable information on the SPS representations of GL_l from those of S_l. For example, this is how we obtained the Pieri rule for GL_l in Section 5.3.2.

C.1. The symmetric monoid and the classification of the irreps of S_l

Consider the set $M(S_l)$ of representations of the symmetric group S_l up to equivalence. The direct sum operation $\oplus$ on representations induces, in a natural way, a structure of a monoid on $M(S_l)$ with identity element given by the 0 representation. We will call it the symmetric monoid. It is well known [50] that the symmetric monoid (and the analogous structure for any finite group) is a free abelian semigroup on the irreducible representations.

The symmetric monoid is equipped naturally with

- an "inner product", given by the non-degenerate symmetric bilinear form,
$$\langle \sigma, \sigma' \rangle = \dim(Hom(\sigma, \sigma')), \quad \sigma, \sigma' \in M(S_l);$$

and,

- a partial order, given by

$$\sigma < \sigma' \text{ iff } \sigma \text{ is a sub-representation of } \sigma'.$$

The monoid $M(S_l)$ has an easily defined and much-studied collection of elements, called *Young modules*, parametrized by partitions of l [8]. Indeed, for a partition $L = \{l_1 \geq l_2 \geq \ldots \geq l_s\}$ of l, we define the *Young module* associated with L, to be the induced module

$$Y_L = Ind_{S_{l_1}\times\ldots\times S_{l_s}}^{S_l}(\mathbf{1}), \tag{C.1}$$

where the subgroup $S_{l_1} \times \ldots \times S_{l_s}$ is contained in S_l in the standard way.

We would like to point out two properties of the collection (C.1) of Young modules. Both involve the *dominance relation* on the set $\mathcal{P}_l$ of partitions of l. Recall that,

Definition C.1.1. *If, in addition to L as above, we have another partition $L' = \{l'_1 \geq l'_2 \geq \ldots \geq l'_r\}$ of l, then we say that L' dominates L, and write $L \prec L'$, if $r \leq s$ and*

$$\sum_{i=1}^{j} l_i \leq \sum_{i=1}^{j} l'_i, \qquad \text{for } j = 1, \ldots, r.$$

With this terminology we have,

Proposition C.1.2. *Suppose L is a partition of l. Then,*

1. *For any partition L' of l, we have $Y_{L'} \lneqq Y_L$ if and only if $L \precneqq L'$.*
2. *There is a unique irrep*

$$\sigma_L < Y_L, \tag{C.2}$$

which is not contained in $Y_{L'}$, for any partition L' that strictly dominates L. Moreover, the multiplicity of σ_L in Y_L is one.

An elementary proof of Proposition C.1.2 can be found in [25, 35].

As a corollary of Proposition C.1.2 we reproduce the well known classification of irreps of the symmetric group:

Corollary C.1.3 (Classification)**.** *The irreps σ_L, $L \in \mathcal{P}_l$, are pairwise non-isomorphic, and exhaust $\widehat{S}_l$. In particular, the irreps of S_l are naturally parametrized by partitions of l.*

C.2. The spherical monoid and the classification of the constituents of $\boldsymbol{Ind_B^{GL_l}(1)}$

Consider the set $M_B(GL_l)$ of representations (up to equivalence) of the group GL_l for which all subreps have a B-invariant vector. *Mutatis mutandem,* as in the case of the symmetric monoid, the set $M_B(GL_l)$ with the direct sum operation $\oplus$ is a monoid. It is the free abelian semigroup on the irreducible representations of GL_l that appear in $Ind_B^{GL_l}(\mathbf{1})$, that is, in the permutation action of GL_l on the variety of complete flags in $\mathbb{F}_q^l$. It inherits a partial order structure $<$ and inner product $\langle\cdot,\cdot\rangle$ from the monoid of all representations of GL_l. We will call it the spherical monoid.

As in the case of the symmetric monoid, also the spherical monoid has a easily defined and much-studied collection of elements parametrized by partitions. In this case, starting with a partition $L = \{l_1 \geq l_2 \geq \ldots \geq l_s\}$ of l, we consider the (parabolically) induced module

$$I_L = I_{A(L)} = Ind_{P_{A(L)}}^{GL_l}(\mathbf{1}) < \ Ind_B^{GL_l}(\mathbf{1}), \tag{C.3}$$

where $P_{A(L)}$ is the standard parabolic (see Section B.1) associated with the set $A(L) = \{0, l_1, \ldots, l_1 + \ldots + l_s = l\}$.

The collection of induced modules I_L (C.3) satisfies the properties:

Proposition C.2.1. *Suppose L is a partition of l. Then,*

(1) *For any partition L' of l, we have $I_{L'} \lneqq I_L$ if and only if $L \not\gneqq L'$.*
(2) *There is a unique irrep*

$$\rho_L < I_L, \tag{C.4}$$

which is not contained in $I_{L'}$, for any partition L' that strictly dominates L. Moreover, the multiplicity of ρ_L in I_L is one.

An elementary proof of Proposition C.2.1 can be found in [25, 35].

As a corollary of Proposition C.2.1 we reproduce the well known classification of the spherical principal series representations of GL_l:

Corollary C.2.2 (Classification)**.** *The irreps ρ_L, $L \in \mathcal{P}_l$, are pairwise non-isomorphic, and exhaust $\widehat{GL}_l(Ind_B^{GL_l}(\mathbf{1}))$. In particular, the SPS representations of GL_l are naturally parametrized by partitions of l.*

C.3. Correspondence between irreps of S_l and the spherical principal series of GL_l

Consider the symmetric and spherical monoids, $M(S_l)$ and $M_B(GL_l)$, respectively. We have,

Theorem C.3.1. *The assignment*

$$I_L \longmapsto Y_L, \quad L \in \mathcal{P}_l,$$

extends uniquely to an isomorphism of monoids

$$\alpha : M_B(GL_l) \xrightarrow{\sim} M(S_l), \tag{C.5}$$

that satisfies the following (equivalent) conditions:

C1 *α preserves the partial orders $<$ on both monoids.*
C2 *α preserves the inner products $\langle , \rangle$ on both monoids.*

Moreover, the aforementioned extension α satisfies $\alpha(\rho_L) = \sigma_L$, for every $L \in \mathcal{P}_l$ (see (C.4) and (C.2)).

The uniqueness part of Theorem C.3.1 is immediate, while the existence part is a direct consequence of the Bruhat decomposition [4, 6] (for more details see [25, 35]).

Finally, we note that Theorem B.2.6 follows from Theorem C.3.1.

C.4. Estimating the dimensions of the spherical principal series representations of GL_l

Proposition C.2.1 has the following consequences for the dimensions of the SPS representations:

Corollary C.4.1 (Dimension)**.** *We have,*

(1) ***Formula.*** *The dimension of the SPS representation ρ_L (C.4) attached to a partition $L = \{l_1 \geq l_2 \geq \ldots \geq l_s\}$of l, satisfies* $\dim(\rho_L) = \dim(I_L) + o(\ldots)$*, as $q \to \infty$, and, in particular,*

$$\dim(\rho_L) = q^{d_L} + o(\ldots), \tag{C.6}$$

where $d_L = \sum_{1\leq i<j\leq s} l_i l_j$.

(2) ***Monotonicity***. *Suppose L and L' are two partitions of l, with $L \precneqq L'$. Then, $d_{L'} < d_L$.*

For a proof of Corollary C.4.1 see Appendix D.8.1.

Appendix D. Proofs

D.1. Proofs for Section 1

D.1.1. Proof of Proposition 1.2.1

Proof. Note that $(L^2(\mathbb{F}_q^n))^{\otimes^k} = L^2(M_{k,n})$, where $M_{k,n}$ denotes the space of matrices of size $k \times n$ over $\mathbb{F}_q$. In particular, the space $L^2(M_{k,n})$ contains the regular representation if and only if $k = n$, and the existence of cuspidal representations, for example, tells us that the filtration does not stabilize before that stage. This completes the proof of the Proposition. □

D.2. Proofs for Section 3

D.2.1. Proof of Lemma 3.2.4

Proof. Consider the section s for the determinant morphism $GL_n \xrightarrow{\det} \mathbb{F}_q^*$, sending $a \in \mathbb{F}_q^*$ to the diagonal matrix with diagonal $(a, 1, \ldots, 1)$.

Suppose $\pi \in \widehat{SL_n}$ appears in the restriction of $\rho \in \widehat{GL_n}$ to SL_n. Then, by Fact 3.2.3 we have,

(D.1)
$$\rho_{|SL_n} = \sum_{a \in C/C_\pi} \pi_a,$$

where $C = \mathbb{F}_q^*$, $\pi_a \in \widehat{SL_n}$ for $a \in C$, is given by $\pi_a(g) = \pi(s(a)gs(a)^{-1})$, and C_π is the stabilizer of π in C.

In particular, for an element $g \in SL_n$ with centralizer in GL_n satisfying our assumption, we have $\chi_{\pi_a}(g) = \chi_\pi(g)$ for every $a \in \mathbb{F}_q^*$. It follows that,

$$\frac{\chi_\rho(g)}{\dim(\rho)} = \frac{\chi_\pi(g)}{\dim(\pi)},$$

as claimed. □

D.3. Proofs for Section 4

D.3.1. Proof of Proposition 4.2.2

Proof. We remark that the tensor rank $k = 1$ irreps that appear in example 1.2.5 stay irreps after restriction to SL_n, $n \geq 3$ (using the argument given in Appendix D.6.1).

Next, we compute,

$$\begin{aligned}
4\left\|P_C^{*l}-U\right\|_{TV}^2 &\leq \sum_{\mathbf{1}\neq\rho\in\widehat{SL}_n} \dim(\rho)^2\left|\frac{\chi_\rho(T)}{\dim(\rho)}\right|^{2l} \\
&= \sum_{k=1}^{n}\sum_{\rho\in(\widehat{SL}_n)_{\otimes,k}} \dim(\rho)^2\left|\frac{\chi_\rho(T)}{\dim(\rho)}\right|^{2l} \\
&\leq (q-1)\left(\frac{q^n-1}{q-1}\right)^2\left(\frac{q^{n-1}-1}{q^n-1}\right)^{2l}+o(\ldots) \\
&\leq \frac{1}{q}\left(\frac{1}{q^2}\right)^{l-n}+o(\ldots),
\end{aligned}$$

where the first inequality is (0.7); the second inequality incorporates Example 1.2.5 for tensor rank $k=1$, and Formulas (3.1), (3.2), and (3.3) for higher ranks. This completes the verification of the proposition. □

D.4. Proofs for Section 5

D.4.1. Proof of the necessity statement in Part (1) of Theorem 5.2.2

Proof. We make use of characterization of strict tensor rank given by Proposition 1.3.1.

If τ has strict tensor rank $2k-n-a$, for some $k\geq a>0$, then τ has a fixed vector for H'_{2k-n-a}, the stabilizer of the first $2k-n-a$ coordinates subspace in $\mathbb{F}_q^k$. So, τ is contained in $Ind_{H_{2k-n-a}}^{GL_k}(\mathbf{1})$.

Consider the parabolic $P_{2k-n-a,\,2(n-k)+a}\subset GL_n$, its unipotent radical $U_{2k-n-a,\,2(n-k)+a}$, and Levi component $L_{2k-n-a,\,2(n-k)+a}\simeq GL_{2k-n-a}\times GL_{2(n-k)+a}$. In particular, inside this parabolic we have the group $\mathcal{G}_{2k-n-a,2(n-k)+a}=U_{2k-n-a,\,2(n-k)+a}\cdot GL_{2(n-k)+a}$.

Next, consider the parabolic $P_{n-k+a,n-k}\subset GL_{2(n-k)+a}$, with Levi $L_{n-k+a,n-k}=GL_{n-k+a}\times GL_{n-k}$ and unipotent radical $U_{n-k+a,n-k}\simeq M_{n-k+a,n-k}$, namely,

$$P_{n-k+a,n-k}$$
$$=\left\{\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}\in GL_{2(n-k)+a}; A\in GL_{n-k+a}, B\in GL_{n-k}, C\in M_{n-k+a,n-k}\right\}.$$

Denote by $\widetilde{\rho}_{n-k+a,n-k}$ the pullback to $\mathcal{G}_{2k-n-a,2(n-k)+a}$ of

$$\rho_{n-k+a,n-k}=Ind_{P_{n-k+a,n-k}}^{GL_{2(n-k)+a}}(\mathbf{1}),$$

and observe that,

$$Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k}) < Ind_{\mathcal{G}_{2k-n-a,2(n-k)+a}}^{GL_n}(\widetilde{\rho}_{n-k+a,n-k}).$$

Note that the parabolics $P_{n-k+a,n-k}$ and $P_{n-k,n-k+a}$ are associate. This implies, using Theorem B.1.2, that the induced representations $\rho_{n-k+a,n-k}$ and (the similarly defined) $\rho_{n-k,n-k+a}$ are equivalent. In particular, the pullback $\widetilde{\rho}_{n-k,n-k+a}$ of $\rho_{n-k,n-k+a}$ to $\mathcal{G}_{2k-n-a,2(n-k)+a}$ satisfies

$$Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k}) < Ind_{\mathcal{G}_{2k-n-a,2(n-k)+a}}^{GL_n}(\widetilde{\rho}_{n-k,n-k+a}).$$

But the group $H_{k-a} \subset GL_n$, that fixes the first $k-a$ coordinates subspace in $\mathbb{F}_q^n$, acts trivially on $Ind_{\mathcal{G}_{2k-n-a,2(n-k)+a}}^{GL_n}(\widetilde{\rho}_{n-k,n-k+a})$, so, using Proposition 1.3.1, we conclude that $Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k})$ contains irreps of strict tensor rank at most $k-a$.

This completes the proof of the necessity statement. □

D.4.2. Proof of Claim 5.3.1

Proof. Consider the natural $GL_k \times GL_n$-action on the set of matrices $M_{k,n}$. An orbit for this action is described by the rank of its elements. The rank r can vary from 0 to k (we assume $k \leq n$), and we denote the corresponding orbit by $(M_{k,n})_r$. In particular, we have a decomposition into direct sum of $GL_k \times GL_n$-representations,

$$L^2(M_{k,n}) = \sum_{r=0}^{k} L^2((M_{k,n})_r).$$

Note that, if $0 \leq r < l \leq k$, then as GL_n-representation $L^2((M_{k,n})_r) < L^2(M_{r,n})$, because each GL_n-orbit of matrices of lower rank is equivalent to the open orbit in the matrices of that rank. So, we see that, representations supported on matrices of lower rank are of lower strict tensor rank, and, in particular, the strict tensor rank k part of $L^2(M_{k,n})$ is contained in $L^2((M_{k,n})_k)$. This proves Part (5.3.1) of Claim 5.3.1.

Next, we want to compute the isotypic components for the action of GL_k on $L^2((M_{k,n})_k)$. Since GL_k acts freely on $(M_{k,n})_k$, the space $L^2((M_{k,n})_k)$ contains a copy of the regular representation of GL_k. Let us denote by $\mathcal{H}_\tau$ a space on which $\tau \in \widehat{GL_k}$ is represented, and calculate the multiplicity space

$$Hom_{GL_k}(L^2((M_{k,n})_k), \mathcal{H}_\tau) =? \tag{D.2}$$

Considering the matrix $I_{k,n} \in (M_{k,n})_k$ whose first k diagonal elements are 1, and all other entries are 0, we can identify $(M_{k,n})_k = H_k \diagdown GL_n$, where $H_k = Stab_{GL_n}(I_{k,n})$. Now, to an intertwiner ι in (D.2) we can associate the function $f_\iota : GL_n \to \mathcal{H}_\tau$, given by $f_\iota(g) = \iota(\delta_{H_k g})$, where $\delta_{H_k g}$ is the delta function at the coset $H_k g$. Note that since $GL_k \cdot H_k = P_{k,n-k}$, and GL_k normalizes H_k, the assignment $\iota \mapsto f_\iota$, gives a morphism

$$\text{(D.3)} \quad Hom_{GL_k}(L^2(H_k \diagdown GL_n), \mathcal{H}_\tau) \longrightarrow \{f : GL_n \to \mathcal{H}_\tau;\ f(pg) = \widetilde{\tau}(p)f(g),\ p \in P_{k,n-k},\ g \in GL_n\},$$

where $\widetilde{\tau}$ is the composition of τ with the projection $P_{k,n-k} \twoheadrightarrow GL_k$.

The right-hand side of (D.3) is $Ind_{P_{k,n-k}}^{GL_n}(\tau \otimes \mathbf{1}_{n-k})$, and, moreover, the mapping (D.3) is an isomorphism. This proves Part (5.3.1) of Claim 5.3.1. □

D.4.3. Proof of Claim 5.3.7

Proof. We use the development described in Appendix C.

Consider the isomorphism α (C.5) given in Theorem C.3.1, between the spherical monoid $M_B(GL_n)$ and the symmetric monoid $M(S_n)$. It preserves the inner product structures, defined by the intertwining number pairings, on both sides. But, by the way the SPS representations of the general linear groups, and the irreps of the symmetric groups, are assigned to partitions (see Appendices C.2 and C.2, respectively), we know that α sends I_{ρ_D} (5.17) to I_{σ_D} (5.19). Concluding, for every partition E of n and D of k, the identity $\langle \rho_E, I_{\rho_D} \rangle = \langle \sigma_E, I_{\sigma_D} \rangle$ (5.18) holds. This completes the proof of Claim 5.3.7. □

D.4.4. Proof of Theorem 5.3.8

Proof. Recall *Schur duality* [49]: the groups $GL_m(\mathbb{C})$ and S_n both act in an obvious way on the n-fold tensor power $(\mathbb{C}^m)^{\otimes^n}$ of $\mathbb{C}^m$. The actions of $GL_m(\mathbb{C})$ and S_n commute with each other, and moreover, they generate mutual commutants in the endomorphisms of $(\mathbb{C}^m)^{\otimes^n}$. The resulting correspondence of representations is compatible with the parametrizations of the representations of S_n and of $GL_m(\mathbb{C})$ with Young diagrams [32].

If we look at the action of $S_k \times GL_m(\mathbb{C})$ on $(\mathbb{C}^m)^{\otimes^k}$, then the isotypic subspace for the representation σ_D of S_k will be isomorphic to $\sigma_D \otimes \rho^D$, as a representation of $S_k \times GL_m(\mathbb{C})$. Here ρ^D is the representation of $GL_m(\mathbb{C})$ parametrized by the Young diagram D. Similarly, if we look at the action of $S_{n-k} \times GL_m(\mathbb{C})$ on the S_{n-k} invariants, then the action of $S_{n-k} \times GL_m(\mathbb{C})$ is $\mathbf{1}_{n-k} \otimes S^{n-k}(\mathbb{C}^m)$. Here $S^a(\mathbb{C}^m)$ indicates the a-th symmetric power of $\mathbb{C}^m$, and is the representation of $GL_m(\mathbb{C})$ corresponding to the diagram with a single row of length a.

Taking the tensor product, we conclude that the isotypic component of $(\mathbb{C}^m)^{\otimes^n}$ for the representation $\sigma_D \otimes \mathbf{1}_{n-k}$ of $S_k \times S_{n-k}$, is the $S_k \times S_{n-k} \times GL_m(\mathbb{C})$-module

$$(\sigma_D \otimes \mathbf{1}_{n-k}) \otimes (\rho^D \otimes S^{n-k}(\mathbb{C}^m)).$$

The Pieri rule for the complex general linear group [32] tells us that the tensor product $\rho^D \otimes S^{n-k}(\mathbb{C}^m)$ of $GL_m(\mathbb{C})$-modules decomposes in a multiplicity-free sum of representations $\rho^{\widetilde{D}}$, where $\widetilde{D}$ is as described in the statement of the theorem: $\widetilde{D}$ has n boxes, contains D, and $\widetilde{D} - D$ is a skew row.

Now consider the $S_n \times GL_m(\mathbb{C})$-module generated by $\rho^D \otimes S^{n-k}(\mathbb{C}^m)$. From Schur duality, we know that it is the sum $\sigma_{\widetilde{D}} \otimes \rho^{\widetilde{D}}$, where $\widetilde{D}$ runs through the set of diagrams of the previous paragraph, *mutatis mutandem*, of the statement of the theorem. On the other hand, this is the $S_n \times GL_m(\mathbb{C})$-module generated by the $S_k \times S_{n-k}$-module $\sigma_D \otimes \mathbf{1}_{n-k}$. It follows that the restriction of a representation σ_E of S_n to $S_k \times S_{n-k}$ contains $\sigma_D \otimes \mathbf{1}_{n-k}$ if and only if $E = \widetilde{D}$, as described above, and then the multiplicity of $\sigma_D \otimes \mathbf{1}_{n-k}$ in $\sigma_{\widetilde{D}}$ is 1. The theorem now follows by Frobenius reciprocity. □

D.4.5. Proof of Corollary 5.4.1

Proof. It is not difficult to see (e.g., using the intrinsic characterization given by Proposition 1.3.1) that for SPS representation strict tensor rank and tensor rank agree. Take a Young diagram $D \in \mathcal{Y}_n$ with longest row of size d_1, and consider the corresponding SPS representation ρ_D. According to the Pieri rule (see Theorem 5.3.8), $k = n - d_1$ is the first such that ρ_D appears inside an induced representation of the form $I_{\rho_E} = Ind_{P_{k,n-k}}^{GL_n}(\rho_E \otimes \mathbf{1}_{n-k})$ where ρ_E is a SPS representation of the GL_k-block of the parabolic $P_{k,n-k}$. In fact, $E \in \mathcal{Y}_k$ is the Young diagram obtained from D by deleting its first row. So the strict co-tensor rank of ρ_D is d_1. This proves Part (5.4.1).

Part (5.4.1), i.e., the case of general irrep, is proved in a similar manner. The Pieri rule implies Formula (5.21), and using it and all of its twists by characters, we deduce the statement applying the same argument as in the SPS case.

This completes the proof of Corollary 5.4.1. □

D.5. Proofs for Section 6

D.5.1. Proof of Proposition 6.1.1

Proof. Recall that the unipotent radical $U_{n-1,1}$ of the parabolic $P_{n-1,1}$ is isomorphic to $\mathbb{F}_q^{n-1}$, and that any non-identity element in that group is a transvection. So all non-identity elements are conjugate, and dually, all non-identity characters are conjugate. So the restriction of any representation of GL_n to $U_{n-1,1}$ is a sum of some copies of the trivial representation $\mathbf{1}_{U_{n-1,1}}$, and some copies of $reg^{\circ}_{U_{n-1,1}} = reg_{U_{n-1,1}} - \mathbf{1}_{U_{n-1,1}}$ the regular representation minus the trivial representation.

Let ρ be an irreducible representation of GL_n. If the restriction of ρ to $U_{n-1,1}$ contains $\mathbf{1}_{U_{n-1,1}}$, then by Frobenius reciprocity, ρ must be contained in a representation induces from the parabolic $P_{n-1,1}$. This means that its realization in terms of the philosophy of cusp forms must be induction from a parabolic whose Levi component contains some GL_1 factors, which in turn means, by Corollary 5.4.1, that ρ has tensor rank at most $n-1$.

We conclude that tensor rank n irreps restricted to $U_{n-1,1}$ contain only multiple of $reg^{\circ}_{U_{n-1,1}}$, and the character ratio of such a representation on the transvection will be the character ratio, on that element, of $reg^{\circ}_{U_{n-1,1}}$, which is $\frac{-1}{q^{n-1}-1}$. This completes the proof of Proposition 6.1.1. ☐

D.5.2. Proof of Proposition 6.1.3

Proof. The proof is by a direct computation of the ratio between the cardinalities of an appropriate set of flags of vector spaces and its subset of *transvection invariant flags*.

The representation I_D, where $D = \{d_1 \geq \ldots \geq d_r\}$ is a partition of n, can be realized on the space of functions on the set X_F of flags of vector spaces in $V = \mathbb{F}_q^n$ of the form

$$F: \quad 0 = V_{a_0} \subset V_{a_1} \subset \ldots \subset V_{a_{r-1}} \subset V_{a_r} = V, \tag{D.4}$$

where $\dim(V_{a_j}) = a_j$, and $\dim(V_{a_j}/V_{a_{j-1}}) = d_j$, for $j = 1, \ldots, r$.

Let T be a transvection on V, i.e., $T - I$ (here, I stands for the identity operator) has rank one, and $(T-I)^2 = 0$.

We are interested in knowing what restrictions the flag F (D.4) must satisfy in order to be invariant under T. We treat two extreme cases, and then the general case.

Case 1. *Suppose the line* $L = \operatorname{Im}(T - I)$ *is not contained in* $V_{a_{r-1}}$.

In this case for F to be invariant under T, all the V_{a_j}, and in particular, $V_{a_{r-1}}$, must be contained in $\ker(T-I)$, which is a hyperplane – a subspace of V of co-dimension 1. In other words, F is actually a flag in $(n-1)$-space rather than n-space.

The collection of flags with the given subspace dimensions in $(n-1)$-space rather than n-space has relative cardinality $\frac{1}{q^{\dim(V_{a_{r-1}})}}+o(\ldots)$ with respect to the collection of all such flags in n-space, as one sees by comparing opposite unipotent radicals in the two situations.

Case 2. *Suppose, on another hand, that the line L is contained already in V_{a_1}.*

In this case F is guaranteed to be invariant under T. How many flags can satisfy this condition? If L is contained in V_{a_1}, then the flag F will push down to define a flag in the $(n-1)$-dimensional space V/L. Again comparing opposite unipotent radicals, we see that the relative cardinality of this collection of T-invariant F with respect to the collection of all possible F is $\frac{1}{q^{n-\dim(V_{a_1})}}+o(\ldots)$.

Case 3. *Now consider the general situation: suppose $V_{a_j} \not\supseteq L \subset V_{a_{j+1}}$.*

Here, by looking only at the sub-flag

$$F_{a_j}: \quad V_{a_1} \subset \ldots \subset V_{a_j} \subset V,$$

we conclude that F_{a_j} is part of a collection of flags of relative cardinality $\frac{1}{q^{\dim(V_{a_j})}}$ with respect to the set of all flags with the same dimension set as F_{a_j}.

On the other hand, consider the flag

$$_{a_j}F: \quad V_{a_{j+1}}/V_{a_j} \subset \ldots \subset V_{a_{r-1}}/V_{a_j} \subset V/V_{a_j}.$$

It satisfies the second simplified condition of Case 2. This implies that $_{a_j}F$ is part of in a collection of flags of relative cardinality $\frac{1}{q^{n-\dim(V_{a_{j+1}})}}+o(\ldots)$ with respect to the collection of all flags with the dimension set of $_{a_j}F$.

The mapping $F \mapsto F_{a_j}$ defines a surjective map from the set X_F of flags with dimension set the same as F, to the collection of flags with the dimension set of F_{a_j}. This map is a fibration, with fiber equal to the collection of flags with dimension set equal to that of $_{a_j}F$. Looking at the inverse image of the T-invariant set in the fiber over each point of the T-invariant set in F_{a_j} satisfying the condition $\mathrm{Im}(T-I) \subset V_{a_{j+1}}$, we conclude that the set of flags with the dimension set of F and such that $\mathrm{Im}(T-I)$ is contained in $V_{a_{j+1}}$ but not in V_{a_j} has relative cardinality

$$\frac{1}{q^{n-\dim(V_{a_{j+1}})+\dim(V_{a_j})}}+o(\ldots)=\frac{1}{q^{n-\dim(V_{a_{j+1}}/V_{a_j})}}+o(\ldots),$$

with respect to the set X_F.

Taking the minimum of the numbers

$$n-\max_j \dim(V_{a_{j+1}}/V_{a_j}),$$

which in our case is $n-d_1$, and denote by m_{d_1} the number of times the quantity d_1 appears in the partition D, we conclude that the relative cardinality of the set X_F^T, of T-invariant flags with the dimension set of F, with respect to the set X_F is

$$\frac{\#X_F^T}{\#X_F}=\frac{m_{d_1}}{q^{n-d_1}}+o(\ldots). \tag{D.5}$$

Of course (D.5) is equal to $\frac{\chi_{I_D}(T)}{\dim(I_D)}$. This completes the proof of Proposition 6.1.3. □

D.5.3. Proof of Proposition 6.1.4

Proof. Fix an algebraic closure $\mathbf{k}$ of $\mathbb{F}_q$. Consider the flag variety $\mathbf{X}_D$ of flags in $\mathbf{V}=\mathbf{k}^n$ defined by a partition $D=\{d_1\geq\ldots\geq d_r\}$ of n [14]. It is irreducible of dimension

$$\Delta=\dim(\mathbf{X}_D)=\sum_{1\leq i<j\leq r} d_i d_j.$$

Consider a transvection T acting on $\mathbf{X}_D$. It has fixed points (this was explained in the proof just above) that form a Zariski open subset of a union of flag varieties These flag varieties have dimensions

$$\Delta_i=\Delta-n+d_i.$$

Consider a partition D' that dominates D. Then D' can be reached from D by a sequence of transformations of the type $d_a\mapsto d_a+1$; and $d_b\mapsto d_b-1$, for $a<b$, and leaving the other d_i's unchanged.

For this D', we have

$$\begin{aligned}\Delta' &= \dim(\mathbf{X}_{D'}) = \Delta + \sum_{i\neq a} d_i - \sum_{i\neq b} d_i - 1 \\ &= \Delta - d_a + d_b - 1.\end{aligned}$$

Then the subvarieties of fixed points for the transvection on this $X_{D'}$ have dimensions

$$\Delta'_i = \Delta' - n + d'_i.$$

Since we have $d'_i = d_i$ except for $i = a, b$, and since $d'_b = d_b - 1$, and since $\Delta' < \Delta$, these dimensions are all less than for the corresponding subvarieties for D, except possibly for $\Delta'_a = \Delta' - n + d_a + 1$. For this to be equal to the largest dimension for D, we would need that $\Delta' = \Delta - 1$, and $d_a = d_1$.

The condition $\Delta' = \Delta - 1$ in turn implies that $d_b = d_a$. Thus, the trace of the transvection on $I_{D'} = L^2(X_{D'})$ (where $X_{D'}$ denotes the set of $\mathbb{F}_q$-rational points $X_{D'} = \mathbf{X}_{D'}(\mathbb{F}_q)$) will be of smaller order of magnitude than the trace on $I_D = L^2(X_D)$, except when $d_a = d_b$, in which case, it will be of the same order of magnitude.

This completes the proof of the Proposition. □

D.6. Proofs for Section 7

D.6.1. Proof of Proposition 7.2.1

Proof. According to Corollary (A.2.6) the restriction to SL_n of an irrep ρ of GL_n is irreducible iff ρ is not fixed by a twist of any non-trivial character of GL_n. Moreover, the eta correspondence (see Theorem 5.5.1), and the description (see Formula (5.23)) of its image, tells us that, a twist by a character of a tensor rank k representation will produce isomorphic one only if the corresponding representation of GL_k has tensor rank

$$r = 2k - n. \tag{D.6}$$

Now, let us go over various cases:

The tensor rank $k < \frac{n}{2}$ irreps of GL_n: In this domain, no irreps of GL_k has tensor rank (D.6). So every irrep of GL_n, in this range, stays irreducible after restriction to SL_n. This completes the justification of Part 1 of Proposition 7.2.1.

The tensor rank $\frac{n}{2} \leq k \leq n-2$ irreps of GL_n: In this interval $r = 2k - n \leq k - 2$, so using the counting coming from the eta correspondence our knowledge on the cardinality of the tensor rank r irreps of GL_k (see Theorem 2.3.1) we get that at most $q^{k-1} + o(\ldots)$ irreps of tensor rank k of GL_n might be reducible after restriction to SL_n. This justify Part 2 of Proposition 7.2.1, for the range under discussion.

The tensor ranks $n-1$ and n irreps of GL_n: For the irreps of tensor rank $n-1$ of GL_n, the "generic" split principal series representation, induced from characters of the standard Borel subgroup which are not fixed by twist of any character of GL_n, stays irreducible after restriction to SL_n (in fact these contribute the $c_{n-1}q^n + o(\ldots)$ to the cardinality of irreps of tensor rank $k = n-1$ given in Formula (2.3)).

A similar argument applies for the irreps of tensor rank n of GL_n. Here we use the cuspidal irreps of GL_n parametrized by generic characters of the torus of GL_n defined by the multiplicative group of the field extension $\mathbb{F}_{q^n}$ of degree n of $\mathbb{F}_q$. (See Section 7.1.1 for more information on this parametrization). This provides $\frac{1}{n}q^n + o(\ldots)$ cuspidal irreps, which all have tensor rank n, and stays irreducible after restriction to SL_n. Similar arguments can be made for any maximal torus in GL_n. Generic characters of a given torus will parametrize representations of rank n if all the irreducible factors of the torus are multiplicative groups of proper extensions of $\mathbb{F}_q$. If one or more factors of the torus is $\mathbb{F}_q^*$, then the corresponding representations will have rank $n-1$.

Overall, the above counting gives the $c_n q^n + o(\ldots)$ irreps of tensor rank $k = n$ given in Formula (2.3).

This completes the justification of Part 2 of Proposition 7.2.1. □

D.7. Proofs for Appendix A

D.7.1. Proof of Corollary A.2.4 We use the notations and definitions given in Section A.

Proof. We start with the proof of Part (A.2.4).

Of course two irreps of G that differ by a twist by a character of C have the same N-spectrum.

For the other direction. Suppose π is an irrep of N. Since C is cyclic, all possible characters of the stabilizer C_π of π arise by restriction from some character of C. If we take one irreducible ρ of G containing π, it will be induced from some representation $\widetilde{\pi}$ of $C_\pi \ltimes N$, as described in Section A (see Part (A.2.3) of Theorem A.2.3). If we twist ρ with a character of C it will be induced from the representation $\widetilde{\pi}$, twisted with this character restricted to C_π. But we know (see Part (A.2.3) of Theorem A.2.3) this will give all possible extensions of π to $C_\pi \ltimes N$, and so will give all possible representations of $\widehat{G}$ containing π when restricted to N. This completes the proof of Part (A.2.4).

Part (A.2.4), i.e., the multiplicity-freeness also follows from the description of the irreducibles containing π. They are all induced from the extension of π (which is exactly one copy of π on N) to the stabilizer of π under action of C by conjugation. So the induced representation restricted to N consists of one copy of each Ad^*C transform of π, one for each coset in $G/(C_\pi \ltimes N)$. ☐

D.8. Proofs for Appendix C

D.8.1. Proof of Corollary C.4.1

Proof. Suppose $L = \{l_1 \geq l_2 \geq \ldots \geq l_s\}$ is a partition of l. We have $\dim(I_L) = \#(GL_l/P_L) = \#(U_L)$, where U_L is the unipotent radical of P_L. An easy direct computation gives $\#U_L = q^{d_L} + o(\ldots)$, where $d_L = \sum_{1\leq i<j\leq s} l_i l_j$. It follows that

$$\dim(I_L) = q^{d_L} + o(\ldots). \tag{D.7}$$

We want to show that

$$d_{L'} < d_L, \text{ if } L' \succ L. \tag{D.8}$$

Indeed, suppose that for some $j_0 < j_1$, we have a partition of l given by $L' = \{l_1 \geq \ldots \geq l_{j_0-1} \geq l_{j_0} + 1 \geq l_{j_0+1} \geq \ldots \geq l_{j_1-1} \geq l_{j_1} - 1 \geq l_{j_1+1} \geq \ldots \geq l_r\}$. Then $L' \succ L$, and, in fact, the dominance order on partitions is generated by such inequalities [8]. In particular, it is enough to show that $d_{L'} < d_L$ for such L, L'. A direct computation implies that $d_L > d_{L'}$, iff $l_{j_0} \cdot l_{j_1} > (l_{j_0}+1)\cdot(l_{j_1}-1)$, and the latter inequality holds true since $l_{j_0} \geq l_{j_1}$.

Now, combining Formulas (D.7) and (D.8), with Proposition C.2.1, we get that $\dim(\rho_L) = \dim(I_L) + o(\ldots) = q^{d_L} + o(\ldots)$.

This completes the proof of both parts of Corollary C.4.1. ☐

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