Pure and Applied Mathematics Quarterly
Volume 17, Number 5, 1619–1668, 2021

The geometry of generalized Lamé equation, III: one-to-one of the Riemann–Hilbert correspondence

Zhijie Chen, Ting-Jung Kuo, and Chang-Shou Lin

Abstract: In this paper, the third in a series, we continue to study the generalized Lamé equation $\mathrm{H}(n_0, n_1, n_2, n_3; B)$ with the Darboux–Treibich–Verdier potential

$$y''(z) = \left[\sum_{k=0}^{3} n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau) + B\right]y(z), \quad n_k \in \mathbb{Z}_{\geq 0}$$

and a related linear ODE with additional singularities $\pm p$ from the monodromy aspect. We establish the uniqueness of these ODEs with respect to the global monodromy data. Surprisingly, our result shows that the Riemann–Hilbert correspondence from the set

$$\{\mathrm{H}(n_0, n_1, n_2, n_3; B)|B \in \mathbb{C}\} \cup \{\mathrm{H}(n_0+2, n_1, n_2, n_3; B)|B \in \mathbb{C}\}$$

to the set of group representations $\rho : \pi_1(E_\tau) \to SL(2, \mathbb{C})$ is one-to-one. We emphasize that this result is not trivial at all. There is an example that for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, there are B_1, B_2 such that the monodromy representations of $\mathrm{H}(1, 0, 0, 0; B_1)$ and $\mathrm{H}(4, 0, 0, 0; B_2)$ are **the same**, namely the Riemann–Hilbert correspondence from the set

$$\{\mathrm{H}(n_0, n_1, n_2, n_3; B)|B \in \mathbb{C}\} \cup \{\mathrm{H}(n_0+3, n_1, n_2, n_3; B)|B \in \mathbb{C}\}$$

to the set of group representations is **not** necessarily one-to-one. This example shows that our result is completely different from the classical one concerning linear ODEs defined on $\mathbb{CP}^1$ with finite singularities.

1. Introduction

Throughout the paper, we use the notations $\omega_0 = 0, \omega_1 = 1, \omega_2 = \tau, \omega_3 = 1+\tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau| \operatorname{Im}\tau > 0\}$. Define $E_\tau := \mathbb{C}/\Lambda_\tau$ to be a flat torus and $E_\tau[2] := \{\frac{\omega_k}{2}|k = 0, 1, 2, 3\} + \Lambda_\tau$ to be the set consisting of

Received November 25, 2020.

the lattice points and 2-torsion points in E_τ. For $z \in \mathbb{C}$ we denote $[z] := z \pmod{\Lambda_\tau} \in E_\tau$. For a point $[z]$ in E_τ we often write z instead of $[z]$ to simplify notations when no confusion arises.

Let $\wp(z) = \wp(z|\tau)$ be the Weierstrass elliptic function with periods Λ_τ and define $e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau)$, $k = 1, 2, 3$. Let $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ be the Weierstrass zeta function with two quasi-periods $\eta_k(\tau)$, $k = 1, 2$:

$$\eta_k(\tau) := 2\zeta(\tfrac{\omega_k}{2}|\tau) = \zeta(z + \omega_k|\tau) - \zeta(z|\tau), \quad k = 1, 2, \tag{1.1}$$

and $\sigma(z) = \sigma(z|\tau) := \exp \int^z \zeta(\xi)d\xi$ be the Weierstrass sigma function. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at Λ_τ and $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ.

This is the third in a series of papers, initiated in Part I [6], to study the generalized Lamé equation (denoted by $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$):

$$y''(z) = I_{\mathbf{n}}(z; p, A, \tau)y(z), \quad z \in \mathbb{C}, \tag{1.2}$$

where the potential $I_{\mathbf{n}}(z; p, A, \tau)$ is given by

$$I_{\mathbf{n}}(z; p, A, \tau) = \begin{bmatrix} \sum_{k=0}^3 n_k(n_k + 1)\wp(z + \frac{\omega_k}{2}|\tau) + \frac{3}{4}(\wp(z + p|\tau) + \\ \wp(z - p|\tau)) + A(\zeta(z + p|\tau) - \zeta(z - p|\tau)) + B \end{bmatrix} \tag{1.3}$$

with $\mathbf{n} = (n_0, n_1, n_2, n_3)$, $n_k \in \mathbb{Z}_{\geq 0}$ for all k, $\pm[p] \notin E_\tau[2]$ and

$$B = A^2 - \zeta(2p|\tau)A - \frac{3}{4}\wp(2p|\tau) - \sum_{k=0}^{3} n_k(n_k + 1)\wp(p + \tfrac{\omega_k}{2}|\tau). \tag{1.4}$$

The (1.4) is equivalent to that $\pm[p]$ are apparent singularities (i.e. non-logarithmic); see [4] for a proof and also [5, 8, 28] for recent studies on (1.2). Remark that all singularities of $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ are apparent and

(1.5) $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ *is independent of any representative* $\tilde{p} \in p + \Lambda_\tau$ *and* $\mathrm{GLE}(\mathbf{n}, p, A, \tau) = \mathrm{GLE}(\mathbf{n}, -p, -A, \tau)$.

For convenience, we often omit some of $\{\mathbf{n}, p, A, \tau\}$ in the notations when no confusion should arise.

Our motivation of studying GLE (1.2) is inspired by the so-called *elliptic form* of Painlevé VI equation (denoted by $\mathrm{EPVI}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$):

$$\frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2} \sum_{k=0}^{3} \alpha_k \wp' \left(p(\tau) + \frac{\omega_k}{2} \middle| \tau \right), \tag{1.6}$$

where

$$\alpha_k = \tfrac{(2n_k+1)^2}{8}, \quad n_k \in \mathbb{Z}_{\geq 0}, \quad k = 0,1,2,3. \tag{1.7}$$

In [4] we proved that GLE (1.2) with $(p, A) = (p(\tau), A(\tau))$ preserves the monodromy as τ deforms if and only if $(p(\tau), A(\tau))$ satisfies the following Hamiltonian system

$$\begin{cases} \frac{dp(\tau)}{d\tau} = \frac{\partial \mathcal{H}}{\partial A} = \frac{-i}{4\pi}(2A - \zeta(2p|\tau) + 2p\eta_1(\tau)) \\ \frac{dA(\tau)}{d\tau} = -\frac{\partial \mathcal{H}}{\partial p} = \frac{i}{4\pi}\begin{pmatrix} (2\wp(2p|\tau) + 2\eta_1(\tau))A - \frac{3}{2}\wp'(2p|\tau) \\ -\sum_{k=0}^{3} n_k(n_k+1)\wp'(p + \frac{\omega_k}{2}|\tau) \end{pmatrix}, \end{cases} \tag{1.8}$$

with

$$\begin{aligned} \mathcal{H} &= \frac{-i}{4\pi}\begin{bmatrix} A^2 + (2p\eta_1(\tau) - \zeta(2p|\tau))A - \frac{3}{4}\wp(2p|\tau) \\ -\sum_{k=0}^{3} n_k(n_k+1)\wp(p + \frac{\omega_k}{2}|\tau) \end{bmatrix} \\ &= \frac{-i}{4\pi}(B + 2p\eta_1(\tau)A), \end{aligned}$$

or equivalently $p(\tau)$ is a solution of EPVI$(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$.

Since the local exponents of GLE (1.2) at $\frac{\omega_k}{2}$ (resp. at $\pm p$) are $-n_k$, $n_k + 1$ (resp. $-\frac{1}{2}$, $\frac{3}{2}$), the local monodromy matrix at $\frac{\omega_k}{2}$ (resp. at $\pm p$) is the identity matrix I_2 (resp. is $-I_2$). Denote by L the straight segment connecting $\pm p$. Then any solution $y(z)$ of GLE (1.2) can be viewed as a single-valued meromorphic function in $\mathbb{C}\backslash(L + \Lambda_\tau)$, and in this region $y(-z)$ and $y(z + \omega_j)$ are well-defined. See [4, 28] or Section 2. Let (y_1, y_2) be any linearly independent solutions of GLE (1.2). Then there are monodromy matrices $N_1, N_2 \in SL(2, \mathbb{C})$ such that

$$\begin{pmatrix} y_1(z+\omega_j) \\ y_2(z+\omega_j) \end{pmatrix} = N_j \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}, \quad j = 1, 2, \quad \text{and} \tag{1.9}$$

$$N_1 N_2 = N_2 N_1. \tag{1.10}$$

Furthermore, the monodromy group of GLE (1.2) is generated by $-I_2, N_1, N_2$. By (1.10), clearly there are two cases (see Part I [6]):

(a) Completely reducible (i.e. all the monodromy matrices have two linearly independent common eigenfunctions). Up to a common conjugation, N_1 and N_2 can be expressed as

$$N_1 = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}, \quad N_2 = \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \tag{1.11}$$

for some $(r,s)\in\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$. In particular,

$$(\mathrm{tr}N_1,\mathrm{tr}N_2)=(2\cos 2\pi s,2\cos 2\pi r)\notin\{\pm(2,2),\pm(2,-2)\}. \tag{1.12}$$

(b) Not completely reducible (i.e. the space of common eigenfunctions is of dimension 1). Up to a common conjugation, N_1 and N_2 can be expressed as

$$N_1=\varepsilon_1\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad N_2=\varepsilon_2\begin{pmatrix}1&0\\\mathcal{C}&1\end{pmatrix}, \tag{1.13}$$

where $\varepsilon_1,\varepsilon_2\in\{\pm1\}$ and $\mathcal{C}\in\mathbb{C}\cup\{\infty\}$. In particular,

$$(\mathrm{tr}N_1,\mathrm{tr}N_2)=(2\varepsilon_1,2\varepsilon_2)\in\{\pm(2,2),\pm(2,-2)\}. \tag{1.14}$$

Remark that if $\mathcal{C}=\infty$, then (1.13) should be understood as

$$N_1=\varepsilon_1\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad N_2=\varepsilon_2\begin{pmatrix}1&0\\1&1\end{pmatrix}. \tag{1.15}$$

For later usage we will briefly review it in Section 2. In this paper, GLE (1.2) (and also the $\mathrm{H}(\mathbf{n},B,\tau)$ below) is called *completely reducible* if Case (a) occurs; *not completely reducible* if Case (b) occurs.

In [4] we proved that if $p(\tau)$ is a solution of EPVI$(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ and

$$p(\tau)\to\tfrac{\omega_k}{2}=\tfrac{\omega_k(\tau_0)}{2},\quad \text{as } \tau\to\tau_0,$$

then the potential $I_{\mathbf{n}}(z;p(\tau),A(\tau),\tau)$ converges to the well-known *Darboux–Treibich–Verdier potential* $I_{\mathbf{n}_k^{\pm}}(z;B,\tau_0)$ for some $B\in\mathbb{C}$, where the Darboux–Treibich–Verdier potential is defined as ([10, 36, 37])

$$I_{\mathbf{n}}(z;B,\tau):=\sum_{k=0}^{3}n_k(n_k+1)\wp(z+\tfrac{\omega_k}{2}|\tau)+B, \tag{1.16}$$

and $\mathbf{n}_k^{\pm}$ is defined by replacing n_k in $\mathbf{n}$ with $n_k\pm1$. That is, by considering the corresponding generalized Lamé equation (denoted by $\mathrm{H}(\mathbf{n},B,\tau)$ or simply $\mathrm{H}(\mathbf{n},B)$)

$$y''(z)=I_{\mathbf{n}}(z;B,\tau)y(z),\quad z\in\mathbb{C}, \tag{1.17}$$

we have that GLE$(\mathbf{n},p(\tau),A(\tau),\tau)$ converges to $\mathrm{H}(\mathbf{n}_k,B,\tau_0)$.

For $\mathrm{H}(\mathbf{n}, B, \tau)$ we always assume $\max_k n_k \geq 1$. $\mathrm{H}(\mathbf{n}, B, \tau)$ is the elliptic form of the well-known Heun's equation and the Darboux–Treibich–Verdier potential is known as an elliptic algebro-geometric solution of the KdV hierarchy [13, 36, 37]. See also a series of papers [29, 30, 31, 32, 33] by Takemura, where $\mathrm{H}(\mathbf{n}, B, \tau)$ was studied as the eigenvalue problem for the Hamiltonian of the BC_1 (one particle) Inozemtsev model. When $\mathbf{n} = (n, 0, 0, 0)$, the potential $n(n+1)\wp(z|\tau)$ is the well-known Lamé potential and (1.17) becomes the Lamé equation

$$y''(z) = [n(n+1)\wp(z|\tau) + B]y(z), \quad z \in \mathbb{C}. \tag{1.18}$$

Ince [17] first discovered that the Lamé potential is a finite-gap potential. See also the classic texts [14, 26, 38] and recent works [3, 9, 21, 22] for more details about (1.18).

Like $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$, the local monodromy matrix of $\mathrm{H}(\mathbf{n}, B, \tau)$ at $\frac{\omega_k}{2}$ is also I_2. Thus the monodromy representation $\rho : \pi_1(E_\tau) \to SL(2, \mathbb{C})$ is abelian, i.e. the same Cases (a) or (b) occurs.

The main purpose of this paper is to study the natural problem: *Whether $H(\mathbf{n}, B)$ or $GLE(\mathbf{n}, p, A, \tau)$ is unique with respect to the monodromy representation*, or equivalently, *whether the Riemann–Hilbert correspondence from the set $\{H(\mathbf{n}, B)|B \in \mathbb{C}\}$ or $\{GLE(\mathbf{n}, p, A, \tau)|p \notin E_\tau[2], A \in \mathbb{C}\}$ to the set of group representations $\rho : \pi_1(E_\tau) \to SL(2, \mathbb{C})$ is one-to-one (i.e. injective)*?

Remark 1.1. By letting $x = \wp(z)$, $\mathrm{H}(\mathbf{n}, B)$ can be projected to the Heun's equation on $\mathbb{CP}^1$, for which the monodromy representation is *irreducible* if and only if Case (a) occurs, and *reducible* if and only if Case (b) occurs. In other words, the monodromy of $\mathrm{H}(\mathbf{n}, B)$ is easier to compute than that of the Heun's equation on $\mathbb{CP}^1$. This is an advantage of studying $\mathrm{H}(\mathbf{n}, B)$. Most of the references in the literature are devoted to irreducible representation on $\mathbb{CP}^1$, but very few are devoted to reducible representation. In this paper we deal with the both two cases for $\mathrm{H}(\mathbf{n}, B)$.

For the completely reducible case (a), the one-to-one of the Riemann–Hilbert correspondence was proved in [21, Theorem 3.3] for the Lamé case and later in Part II [7, Lemma 2.3] for the Darboux–Treibich–Verdier case (See also [7, 21] for important applications of such results). However, the proofs in [7, 21] can *not* work for the not completely reducible case (b). In this paper, we develop a new approach, which applies the deep relation with Painlevé VI equation and seems more sophisticated but works for the not completely reducible case and also $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$.

Remark that although the monodromy matrices N_j's depend on the choice of linearly independent solutions, they are unique up to a common conjugation. In particular, $\mathrm{tr}N_j$ is *independent* of the choice of solutions, i.e. $\mathrm{tr}N_j$ is uniquely determined by $\mathrm{GLE}(\mathbf{n}, p, A)$ or $\mathrm{H}(\mathbf{n}, B)$. We say

$$(r_1, s_1) \sim (r_2, s_2) \text{ if } (r_1, s_1) \equiv \pm(r_2, s_2) \bmod \mathbb{Z}^2. \tag{1.19}$$

Then in Case (a), (r, s) is uniquely determined in $(\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2)/\sim$.

Definition 1.2. Given $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ (resp. $\mathrm{H}(\mathbf{n}, B, \tau)$), we call

$$\begin{cases} (r, s) \in (\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2)/\sim & \text{if the monodromy is completely reducible} \\ (\mathrm{tr}N_1, \mathrm{tr}N_2, \mathcal{C}) & \text{if the monodromy is not completely reducible} \end{cases}$$

to be its global monodromy data.

The main purpose of this paper is to establish the uniqueness of such ODEs with respect to the global monodromy data. For $k \in \{0, 1, 2, 3\}$ and $\mathbf{n} = (n_0, n_1, n_2, n_3)$, we define $\mathbf{n}_k$ by replacing n_k in $\mathbf{n}$ with $n_k + 2$, i.e.

$$\mathbf{n}_0 = (n_0 + 2, n_1, n_2, n_3), \quad \mathbf{n}_1 = (n_0, n_1 + 2, n_2, n_3) \tag{1.20}$$

and so on. The main result of this paper is the following uniqueness theorem.

Theorem 1.3. *Fix any* $\mathbf{n}$ *and* τ. *Then the following hold.*

(1) If $\mathrm{GLE}(\mathbf{n}, p_1, A_1)$ *and* $\mathrm{GLE}(\mathbf{n}, p_2, A_2)$ *have the same global monodromy data, then* $\mathrm{GLE}(\mathbf{n}, p_1, A_1) = \mathrm{GLE}(\mathbf{n}, p_2, A_2)$.

(2) If $H(\mathbf{n}, B_1)$ *and* $H(\mathbf{n}, B_2)$ *have the same global monodromy data, then* $H(\mathbf{n}, B_1) = H(\mathbf{n}, B_2)$.

(3) Fix any $k \in \{0, 1, 2, 3\}$. *Then the global monodromy datas of* $H(\mathbf{n}, B_1, \tau)$ *and* $H(\mathbf{n}_k, B_2, \tau)$ *can not be the same for any* $B_1, B_2 \in \mathbb{C}$.

Remark 1.4. $\mathrm{H}(\mathbf{n}, B_1, \tau)$ and $\mathrm{H}(\mathbf{n}_k, B_2, \tau)$ have *different local exponents* at the singularity $\frac{\omega_k}{2}$. Therefore, it is quite surprising to us that for fixed $\mathbf{n}$, τ and k, *the Riemann–Hilbert correspondence from the set* $\{H(\mathbf{n}, B, \tau)|B \in \mathbb{C}\} \cup \{H(\mathbf{n}_k, B, \tau)|B \in \mathbb{C}\}$ *to the set of group representations* $\rho : \pi_1(E_\tau) \to SL(2, \mathbb{C})$ *is one-to-one*. We emphasize that this result is not trivial at all. For example, we can not expect the one-to-one correspondence from $\{\mathrm{H}(\mathbf{n}, B, \tau) \mid B \in \mathbb{C}\} \cup \{\mathrm{H}((n_0 + 3, n_1, n_2, n_3), B, \tau) \mid B \in \mathbb{C}\}$ to the set of group representations. Indeed, Wang and the third author [21, Theorem 4.5] proved the existence of a pre-modular form $Z^{(n)}_{r,s}(\tau)$ such that the global monodromy data of $\mathrm{H}((n,0,0,0),B,\tau)$ for some B is given by $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$ if and only if $Z_{r,s}^{(n)}(\tau)=0$. Now for $\tau_0=\frac{1}{2}+i\frac{\sqrt{3}}{2}$, it was proved in [20, Example 2.6] that

$$Z^{(1)}_{\frac{1}{3},\frac{1}{3}}(\tau_0)=0,\quad \wp(\tfrac{1+\tau_0}{3}|\tau_0)=0.$$

Inserting these and $g_2(\tau_0)=0$ into the expression of $Z_{r,s}^{(4)}(\tau)$ (see [21, (5.8)]), we obtain $Z^{(4)}_{\frac{1}{3},\frac{1}{3}}(\tau_0)=Z^{(1)}_{\frac{1}{3},\frac{1}{3}}(\tau_0)=0$, so there are B_1,B_2 such that the global monodromy datas of $\mathrm{H}((1,0,0,0),\ B_1,\tau_0)$ and $\mathrm{H}((4,0,0,0),B_2,\tau_0)$ are both $(\frac{1}{3},\frac{1}{3})$.

Remark 1.5. For a class of linear ODEs defined on $\mathbb{CP}^1$ with finite singularities, classically there is a one-to-one correspondence of such ODEs and their monodromy datas; see e.g. [11, Proposition 2.2]. However, the set of monodromy datas for such classical result contains connection matrices at each singularities. Our Theorem 1.3 is different from the classical one because no apriori information about the connection matrices are assumed in Theorem 1.3. To the best of our knowledge, Theorem 1.3 is new.

Remark 1.6. The uniqueness with respect to the same monodromy group does not necessarily hold. For example, our later argument shows that given $\mathbf{n}$ and $m\in\mathbb{N}_{\geq 3}$, there exist (p_j,A_j), $j=1,2$ and the same τ such that for $\mathrm{GLE}(\mathbf{n},p_1,A_1)$,

$$N_1=\begin{pmatrix} e^{-2\pi i/m} & 0\\ 0 & e^{2\pi i/m}\end{pmatrix},\ N_2=\begin{pmatrix} e^{2\pi i/m} & 0\\ 0 & e^{-2\pi i/m}\end{pmatrix},$$

i.e. $(\mathrm{tr}N_1,\mathrm{tr}N_2)=(2\cos\frac{2\pi}{m},2\cos\frac{2\pi}{m})$, and for $\mathrm{GLE}(\mathbf{n},p_2,A_2)$,

$$\tilde{N}_1=\begin{pmatrix} e^{-2\pi i/m} & 0\\ 0 & e^{2\pi i/m}\end{pmatrix},\ \tilde{N}_2=\begin{pmatrix} e^{4\pi i/m} & 0\\ 0 & e^{-4\pi i/m}\end{pmatrix},$$

i.e. $(\mathrm{tr}\tilde{N}_1,\mathrm{tr}\tilde{N}_2)=(2\cos\frac{2\pi}{m},2\cos\frac{4\pi}{m})$. Thus, these two GLEs have different global monodromy datas (or equivalently, different monodromy representations). However, they have the same monodromy group (i.e. the images of the monodromy representations are the same)

$$\langle -I_2,N_1,N_2\rangle=\left\langle -I_2,\tilde{N}_1,\tilde{N}_2\right\rangle=\langle -I_2,N_1\rangle.$$

Remark 1.7. Our proof of Theorem 1.3 is purely analytic. Recently Prof. Treibich communicated with us and he conjectured that there should be a different proof of Theorem 1.3 via algebraic geometry. This is a very interesting question and deserves further study elsewhere.

The rest of the paper is organized as follows. In Section 2, we briefly review the monodromy theory of GLE($\mathbf{n}, A, p$). Our proof of Theorem 1.3 relies on the connection between GLE($\mathbf{n}, A, p$) and Painlevé VI equation established in [4], which is briefly reviewed in Section 3. In Sections 4–5, we establish the uniqueness of solutions of certain Painlevé VI equations with respect to the global monodromy datas of GLE($\mathbf{n}, A, p$). This theory will be applied to prove Theorem 1.3 in Section 6. An application of Theorem 1.3 will be given in Section 7.

2. Preliminaries

In this section, we briefly review the basic theory about the monodromy representation of GLE($\mathbf{n}, A, p$) and H($\mathbf{n}, B$) from [6, 28], which will be applied in the proof of Theorem 1.3.

2.1. The unique even elliptic solution

Let y_1, y_2 be any two solutions of GLE($\mathbf{n}, A, p$) and set $\Phi(z) = y_1(z)y_2(z)$. Then $\Phi(z)$ satisfies the second symmetric product equation for GLE($\mathbf{n}, A, p$):

$$\Phi'''(z) - 4I(z)\Phi'(z) - 2I'(z)\Phi(z) = 0, \tag{2.1}$$

where $I(z) = I_{\mathbf{n}}(z; p, A, \tau)$. The following lemma follows from [28, Propositions 2.1 and 2.9]. For later usage, we sketch the proof of the existence here, and refer the proof of the uniqueness to [28, Proposition 2.9] or Part I [6, Proposition 2.3].

Lemma 2.1 ([28]). *Equation (2.1) has a unique (up to multiplying a nonzero constant) even elliptic solution $\Phi_e(z)$.*

Proof. Fix any base point $q_0 \in E_\tau \backslash (E_\tau[2] \cup \{\pm[p]\})$. Since the local monodromy matrice at $\frac{\omega_k}{2}$ is I_2, the monodromy representation of GLE (1.2) is reduced to $\rho : \pi_1(E_\tau \backslash \{\pm[p]\}, q_0) \to SL(2, \mathbb{C})$. Let $\gamma_\pm \in \pi_1(E_\tau \backslash \{\pm[p]\}, q_0)$ be a simple loop encircling $\pm p$ counterclockwise, and $\ell_j \in \pi_1(E_\tau \backslash \{\pm[p]\}, q_0)$, $j = 1, 2$, be two fundamental cycles of E_τ connecting q_0 with $q_0 + \omega_j$ such that ℓ_j does not intersect with $L + \Lambda_\tau$ (here L is the straight segment connecting $\pm p$) and satisfies

$$\gamma_- \gamma_+ = \ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1} \quad \text{in } \pi_1\left(E_\tau \backslash \left\{\pm[p]\right\}, q_0\right). \tag{2.2}$$

Since

$$\rho(\gamma_\pm) = -I_2, \tag{2.3}$$

we have $N_j = \rho(\ell_j)$, $N_1 N_2 = N_2 N_1$ and the monodromy group of (1.2) is generated by $\{-I_2, N_1, N_2\}$, namely is abelian. So there is a common eigenfunction (or called *eigen-solution*) $y_1(z)$ of all monodromy matrices. Let ε_i be the eigenvalue: $\ell_i^* y_1(z) = \varepsilon_i y_1(z)$, where $\ell^* y(z)$ denotes the analytic continuation of $y(z)$ along the loop ℓ. Note that $y_1(z)$ have branch points only at $\pm p + \Lambda_\tau$. By (2.3), $y_1(z)$ can be viewed as a single-valued meromorphic function in $\mathbb{C}\backslash(L+\Lambda_\tau)$, and in this region, $y_1(-z)$ is well-defined and

$$y_1(z+\omega_i) = \ell_i^* y_1(z) = \varepsilon_i y_1(z), \; i = 1, 2, \tag{2.4}$$

since the fundamental circles are chosen not to intersect with $L + \Lambda_\tau$.

Let $y_2(z) = y_1(-z)$ in $\mathbb{C}\backslash(L+\Lambda_\tau)$. Clearly $y_2(z)$ is also a solution of (1.2) and (2.4) implies

$$y_2(z+\omega_i) = \ell_i^* y_2(z) = \varepsilon_i^{-1} y_2(z), \; i = 1, 2, \tag{2.5}$$

i.e. $y_2(z)$ is also an eigenfunction with eigenvalue ε_i^{-1}. Define

$$\Phi_e(z) := y_1(z) y_2(z) = y_1(z) y_1(-z).$$

Obviously, $\pm[p]$ are no longer branch points of $\Phi_e(z)$, which implies that $\Phi_e(z)$ is single-valued meromorphic in $\mathbb{C}$. By (2.4)–(2.5), $\Phi_e(z)$ is an even elliptic function. This proves the existence part. □

Since $\Phi_e(z)$ have poles at most at $\frac{\omega_k}{2}$ with order $2n_k$ and at $\pm p$ with order 2, we have

$$\Phi_e(z) = C_0 + \sum_{k=0}^{3} \sum_{j=0}^{n_k-1} b_j^{(k)} \wp(z + \tfrac{\omega_k}{2})^{n_k-j} + \frac{d}{\wp(z) - \wp(p)},$$

where $C_0, b_j^{(k)}$ and d are constants depending on $\mathbf{n}, A, p, \tau$. By a careful computation, it was proved in [28, 29] that

Theorem 2.A ([28, 29])**.** *After a normalization of multiplying a nonzero constant depending on* $\mathbf{n}, A, p, \tau$*,*

$$\Phi_e(z) = C_0(A) + \sum_{k=0}^{3} \sum_{j=0}^{n_k-1} b_j^{(k)}(A) \wp(z + \tfrac{\omega_k}{2})^{n_k-j} + \frac{d(A)}{\wp(z) - \wp(p)}, \tag{2.6}$$

where $C_0(A) = C_0(A; p, \tau)$, $b_j^{(k)}(A) = b_j^{(k)}(A; p, \tau)$ *and* $d(A) = d(A; p, \tau)$ *are all polynomials of* A *with cofficients being rational functions of* $\wp(p)$, $\wp'(p)$, $e_k(\tau)'s$, *and they do not have common zeros, and the leading coefficient of* $C_0(A)$ *can be chosen to be* $\frac{1}{2}$. *Moreover,*

$$g := \deg_A C_0(A) > \max\left\{\deg_A b_j^{(k)}(A), \deg_A d(A)\right\}.$$

Theorem 2.A will be applied in the proof of Theorems 5.3–5.4 below.

2.2. The Hermite–Halphen ansatz

Let $N = \sum_{k=0}^{3} n_k + 1$ in this section. For any $\boldsymbol{a} = (a_1, \cdots, a_N) \in \mathbb{C}^N$, we consider the Hermite–Halphen ansatz

$$y_{\boldsymbol{a}}(z) := \frac{e^{cz} \prod_{i=1}^{N} \sigma(z - a_i)}{\sqrt{\sigma(z-p)\sigma(z+p)} \prod_{k=0}^{3} \sigma(z - \frac{\omega_k}{2})^{n_k}}, \; c \in \mathbb{C}. \tag{2.7}$$

In Part I [6] we proved that the common eigen-solution of GLE($\mathbf{n}, A, p$) must be of the form $y_{\boldsymbol{a}}(z)$.

Theorem 2.B ([6]). *Let* $y_1(z)$ *be the common eigen-solution in Lemma 2.1. Then up to a nonzero constant,*

$$y_1(z) = y_{\boldsymbol{a}}(z)$$

for some $\boldsymbol{a} = (a_1, \cdots, a_N) \in \mathbb{C}^N$ *and* $c = c(\boldsymbol{a}) \in \mathbb{C}$.

Remark 2.2. Generically $\{[a_1], \cdots, [a_N]\}$ is precisely the zero set of $y_1(z) = y_{\boldsymbol{a}}(z)$. For some special A's, the local exponent of $y_1(z)$ at p might be $\frac{3}{2}$, so there are two points in $\{[a_1], \cdots, [a_N]\}$ being $[p]$, say $[a_{N-1}] = [a_N] = [p]$ for example, and in this case the zero set of $y_1(z)$ is contained in $\{[a_1], \cdots, [a_{N-2}]\}$. Similarly, $\{[a_1], \cdots, [a_N]\}$ might contain $\frac{\omega_k}{2}$'s for special A's.

Although $y_{\boldsymbol{a}}(z)$ is a multi-valued function in $\mathbb{C}$, $y_{\boldsymbol{a}}(-z)$ can be well-defined as shown in the proof of Lemma 2.1, and $y_{\boldsymbol{a}}(-z)$ is also a common eigen-solution. By using the transformation law (let $\eta_3 = \eta_1 + \eta_2$)

$$\sigma(z + \omega_k) = -e^{\eta_k (z + \frac{\omega_k}{2})} \sigma(z), \quad k = 1, 2, 3, \tag{2.8}$$

it is easy to see that in $\mathbb{C} \backslash (L + \Lambda_\tau)$,

$$y_2(z) = y_{\boldsymbol{a}}(-z) = y_{-\boldsymbol{a}}(z) \;\text{ up to a nonzero constant,} \tag{2.9}$$

which infers

$$\Phi_e(z) = y_{\boldsymbol{a}}(z)y_{-\boldsymbol{a}}(z) \text{ up to a nonzero constant.} \tag{2.10}$$

By the uniqueness of $\Phi_e(z)$, we easily see that $\pm\boldsymbol{a} \bmod \Lambda_\tau$ is unique, i.e.

$$\pm\{[a_1], \cdots, [a_N]\} \textit{ is unique for given } GLE(\mathbf{n}, p, A, \tau), \tag{2.11}$$

and for different representatives $\boldsymbol{a}, \tilde{\boldsymbol{a}} \in \mathbb{C}^N$ of the same $\{[a_1], \cdots, [a_N]\}$,

$$y_{\boldsymbol{a}}(z) = y_{\tilde{\boldsymbol{a}}}(z) \text{ up to a nonzero constant.} \tag{2.12}$$

If $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent, then the monodromy is completely reducible by definition. The following result shows that the converse assertion also holds, and in this case the monodromy data can be easily computed.

Theorem 2.3 ([6]). *If the monodromy of $GLE(\mathbf{n}, p, A, \tau)$ is completely reducible, then $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent and there exists $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$ such that with respect to $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$,*

$$N_1 = \rho(\ell_1) = \begin{pmatrix} e^{-2\pi is} & 0 \\ 0 & e^{2\pi is} \end{pmatrix}, \; N_2 = \rho(\ell_2) = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix}, \tag{2.13}$$

and

$$\sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} = r + s\tau, \quad c(\boldsymbol{a}) = r\eta_1 + s\eta_2. \tag{2.14}$$

Furthermore, if $[a_j] \neq \pm[p]$ for all j, then (recall $\eta_3 = \eta_1 + \eta_2$)

$$c(\boldsymbol{a}) = \frac{1}{2}\sum_{i=1}^{N}(\zeta(a_i + p) + \zeta(a_i - p)) - \sum_{k=1}^{3}\frac{n_k\eta_k}{2}. \tag{2.15}$$

Proof. This result was proved in Part I [6]. Here we sketch the proof for later usage. Let $y_3(z)$ be another common eigen-solution which is linearly independent to $y_{\boldsymbol{a}}(z)$. Clearly $y_3(z)y_3(-z)$ is also an even elliptic solution of (2.1), so up to nonzero constants,

$$y_3(z)y_3(-z) = \Phi_e(z) = y_{\boldsymbol{a}}(z)y_{-\boldsymbol{a}}(z). \tag{2.16}$$

Then a zero of $y_3(z)$ must be a zero of $y_{-\boldsymbol{a}}(z)$ and vice versa, so $y_3(z) = y_{-\boldsymbol{a}}(z)$ up to a nonzero constant, namely $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent. Rewrite

$$y_{\boldsymbol{a}}(z) = \frac{e^{c(\boldsymbol{a})z}\prod_{j=1}^{N}\sigma(z-a_j)}{\sigma(z)\prod_{k=0}^{3}\sigma(z-\frac{\omega_k}{2})^{n_k}} \cdot \Psi_p(z), \tag{2.17}$$

where $\Psi_p(z)$ is defined by

$$\Psi_p(z) := \frac{\sigma(z)}{\sqrt{\sigma(z+p)\sigma(z-p)}}. \tag{2.18}$$

Since $\Psi_p(z)^2$ is even elliptic and ℓ_j is chosen to have no intersection with $L + \Lambda_\tau$, we proved in Part I [6, Lemma 2.2] that $\Psi_p(z)$ is invariant under analytic continuation along ℓ_j, i.e.

$$\ell_j^*\Psi_p(z) = \Psi_p(z), \quad j = 1, 2. \tag{2.19}$$

By applying (2.19) and the transformation law (2.8) to $y_{\boldsymbol{a}}(z)/\Psi_p(z)$, we have

$$\ell_j^* y_{\boldsymbol{a}}(z) = \exp\left(c(\boldsymbol{a})\omega_j - \eta_j\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right)\right) y_{\boldsymbol{a}}(z), \ j = 1, 2. \tag{2.20}$$

Define $(r, s) \in \mathbb{C}^2$ by

$$\begin{aligned} c(\boldsymbol{a}) - \eta_1\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right) &= -2\pi i s, \\ c(\boldsymbol{a})\tau - \eta_2\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right) &= 2\pi i r. \end{aligned} \tag{2.21}$$

Then (2.14) follows by using $\tau\eta_1 - \eta_2 = 2\pi i$. Recalling the eigenvalues $\varepsilon_1, \varepsilon_2$ in Lemma 2.1, we see from Theorem 2.B and (2.20)–(2.9) that $(\varepsilon_1, \varepsilon_2) = (e^{-2\pi i s}, e^{2\pi i r})$ and hence (2.13) holds. If both $e^{2\pi i s}$ and $e^{2\pi i r} \in \{\pm 1\}$, then $y_{\boldsymbol{a}}(z) + y_{-\boldsymbol{a}}(z)$ is also a common eigen-solution, and the same argument as (2.16) gives $y_{\boldsymbol{a}}(z) + y_{-\boldsymbol{a}}(z) = c_\pm y_{\pm\boldsymbol{a}}(z)$ for some constant $c_\pm$, a contradiction. So either $e^{2\pi i r} \notin \{\pm 1\}$ or $e^{2\pi i s} \notin \{\pm 1\}$, i.e. $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$. Finally, (2.15) follows by inserting (2.7) into GLE$(\mathbf{n}, p, A)$ and computing the leading terms at singularities $\pm p$. This completes the proof. □

Now we consider the not completely reducible case.

Theorem 2.4. *Suppose the monodromy of GLE*$(\mathbf{n}, p, A, \tau)$ *is not completely reducible. Then*

$$\{[a_1], \cdots, [a_N]\} = \{-[a_1], \cdots, -[a_N]\}, \tag{2.22}$$

and there exists $(r, s) \in \frac{1}{2}\mathbb{Z}^2$ *such that*

$$\sum_{i=1}^{N} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k}{2} = r + s\tau, \quad c(\boldsymbol{a}) = r\eta_1 + s\eta_2. \tag{2.23}$$

Furthermore, there exist linearly independent solutions such that $\rho(\ell_1)$ *and* $\rho(\ell_2)$ *can be expressed as*

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix}, \tag{2.24}$$

with $\mathcal{C} \in \mathbb{C} \cup \{\infty\}$ *and*

$$(\varepsilon_1, \varepsilon_2) = \begin{cases} (1,1), & \text{if } (r,s) \equiv (0,0) \mod \mathbb{Z}^2, \\ (1,-1), & \text{if } (r,s) \equiv (\frac{1}{2},0) \mod \mathbb{Z}^2, \\ (-1,1), & \text{if } (r,s) \equiv (0,\frac{1}{2}) \mod \mathbb{Z}^2, \\ (-1,-1), & \text{if } (r,s) \equiv (\frac{1}{2},\frac{1}{2}) \mod \mathbb{Z}^2. \end{cases} \tag{2.25}$$

Remark that if $\mathcal{C} = \infty$, *then (2.24) should be understood as*

$$\rho(\ell_1) = \varepsilon_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2.26}$$

Proof. Since the monodromy is not completely reducible and $y_{\pm\boldsymbol{a}}(z)$ are both common eigen-solutions, we have $y_{\boldsymbol{a}}(z) = y_{-\boldsymbol{a}}(z)$ up to a nonzero constant, which implies: (1) $\varepsilon_j = \varepsilon_j^{-1}$, i.e. $\varepsilon_j = \pm 1$ for $j = 1, 2$; (2) (2.22) holds by using (2.7); (3) $\Phi_e(z) = y_{\boldsymbol{a}}(z)^2$ up to a nonzero constant. Again by the same argument as (2.17)–(2.21), we easily obtain (2.23) and (2.25).

To prove (2.24), we let $y_2(z)$ be a linearly independent solution of GLE (1.2) to $y_{\boldsymbol{a}}(z)$ and define $\chi(z) := y_2(z)/y_{\boldsymbol{a}}(z)$. Then $\chi(z) \not\equiv$const has no branch points, namely $\chi(z)$ is single-valued meromorphic. Furthermore, inserting $y_2(z) = \chi(z)y_{\boldsymbol{a}}(z)$ into GLE (1.2) leads to

$$\frac{\chi''(z)}{\chi'(z)} + 2\frac{y_{\boldsymbol{a}}'(z)}{y_{\boldsymbol{a}}(z)} = 0, \text{ i.e. } \chi'(z) = \text{const} \cdot \Phi_e(z)^{-1} \text{ is even elliptic.}$$

Thus $\chi(z)$ is quasi-periodic, namely there exist two constants χ_1 and χ_2 such that

$$\chi(z+\omega_j)=\chi(z)+\chi_j, \quad j=1,2.$$

Since $y_2(z)$ is not a common eigen-solution, χ_1 and χ_2 can not vanish simultaneously. Define

$$\mathcal{C}:=\chi_2/\chi_1. \tag{2.27}$$

If $\chi_1=0$, then $\chi_2\neq 0$, $\mathcal{C}=\infty$ and a direct computation gives

$$\ell_1^*\begin{pmatrix}\chi_2 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}=\varepsilon_1\begin{pmatrix}\chi_2 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix},$$
$$\ell_2^*\begin{pmatrix}\chi_2 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}=\varepsilon_2\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\chi_2 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix},$$

which is precisely (2.26). If $\chi_1\neq 0$, then $\mathcal{C}\neq\infty$ and we easily obtain

$$\ell_1^*\begin{pmatrix}\chi_1 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}=\varepsilon_1\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\chi_1 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}, \tag{2.28}$$

$$\ell_2^*\begin{pmatrix}\chi_1 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}=\varepsilon_2\begin{pmatrix}1&0\\\mathcal{C}&1\end{pmatrix}\begin{pmatrix}\chi_1 y_{\boldsymbol{a}}(z)\\ y_2(z)\end{pmatrix}, \tag{2.29}$$

which is precisely (2.24). This completes the proof. □

Corollary 2.5. *The monodromy of $GLE(\mathbf{n},p,A,\tau)$ is completely reducible if and only if*

$$(tr\rho(\ell_1), tr\rho(\ell_2))\notin\{\pm(2,2),\pm(2,-2)\}. \tag{2.30}$$

2.3. The monodromy theory for $\mathrm{H}(\mathbf{n},B)$

Now we recall the counterpart of the above monodromy theory for $\mathrm{H}(\mathbf{n},B)$ from Part I [6], the proof of which is simpler due to the absence of singularities $\pm[p]$. In this section we denote $\tilde{N}=\sum_k n_k\geq 1$. By changing variable $z\to z+\frac{\omega_k}{2}$ if necessary, we always assume $n_0\geq 1$.

(i) Any solution of $H(\mathbf{n},B,\tau)$ is meromorphic in $\mathbb{C}$. The corresponding second symmetric product equation

$$\Phi'''(z;B)-4I_{\mathbf{n}}(z;B,\tau)\Phi'(z;B)-2I_{\mathbf{n}}'(z;B,\tau)\Phi(z;B)=0$$

has a unique even elliptic solution $\Phi_e(z;B)$ expressed by

$$\Phi_e(z;B) = C_0(B) + \sum_{k=0}^{3}\sum_{j=0}^{n_k-1} b_j^{(k)}(B)\wp(z+\tfrac{\omega_k}{2})^{n_k-j} \tag{2.31}$$

where $C_0(B), b_j^{(k)}(B)$ are all polynomials in B with $\deg C_0 > \max_{j,k}\deg b_j^{(k)}$ and the leading coefficient of $C_0(B)$ being $\frac{1}{2}$. Moreover, $\Phi_e(z;B) = y_1(z;B)$ $y_1(-z;B)$, where $y_1(z;B)$ is a common eigenfunction of the monodromy matrices of $H(\mathbf{n}, B, \tau)$ and up to a constant, can be written as

$$y_1(z;B) = \tilde{y}_{\boldsymbol{a}}(z) := \frac{e^{c(\boldsymbol{a})z}\prod_{i=1}^{\tilde{N}}\sigma(z-a_i)}{\prod_{k=0}^{3}\sigma(z-\frac{\omega_k}{2})^{n_k}} \tag{2.32}$$

with some $\boldsymbol{a} = (a_1, \cdots, a_{\tilde{N}})$ and $c(\boldsymbol{a}) \in \mathbb{C}$. See (2.34) for the expression of $c(\boldsymbol{a})$ in the completely reducible case. By (2.32) and the transformation law (2.8), it is easy to see that $y_1(-z;B) = \tilde{y}_{-\boldsymbol{a}}(z)$ up to a sign $(-1)^{n_1+n_2+n_3}$.

(ii) Let W be the Wroskian of $y_1(z;B)$ and $y_1(-z;B)$, then $W^2 = Q_{\mathbf{n}}(B;\tau)$, where

$$Q_{\mathbf{n}}(B;\tau) := \Phi_e'(z;B)^2 - 2\Phi_e(z;B)\Phi_e''(z;B) + 4I_{\mathbf{n}}(z;B,\tau)\Phi_e(z;B)^2$$

is a monic polynomial in B with *odd degree* and independent of z.

(iii) The monodromy of $\mathrm{H}(\mathbf{n}, B, \tau)$ is completely reducible if and only if $y_1(z;B) = \tilde{y}_{\boldsymbol{a}}(z)$ and $y_1(-z;B) = \tilde{y}_{-\boldsymbol{a}}(z)$ are linearly independent, which is also equivalent to

$$\{[a_1], \cdots, [a_{\tilde{N}}]\} \cap \{-[a_1], \cdots, -[a_{\tilde{N}}]\} = \emptyset. \tag{2.33}$$

In this case, since $a_j \neq 0$ in E_τ for all j and $n_0 \neq 0$, we have

$$c(\boldsymbol{a}) = \sum_{i=1}^{\tilde{N}}\zeta(a_i) - \sum_{k=1}^{3}\frac{n_k\eta_k}{2}, \tag{2.34}$$

which follows by inserting (2.32) into $\mathrm{H}(\mathbf{n}, B, \tau)$ and computing the leading terms at the singularity 0. Besides, the (r,s) defined by

$$\begin{cases} \sum_{i=1}^{\tilde{N}} a_i - \sum_{k=1}^{3}\frac{n_k\omega_k}{2} = r + s\tau \\ \sum_{i=1}^{\tilde{N}}\zeta(a_i) - \sum_{k=1}^{3}\frac{n_k\eta_k}{2} = r\eta_1 + s\eta_2 \end{cases} \tag{2.35}$$

satisfies $(r,s)\notin \frac{1}{2}\mathbb{Z}^2$. Furthermore, with respect to $\tilde{y}_{\boldsymbol{a}}(z)$ and $\tilde{y}_{-\boldsymbol{a}}(z)$,

$$N_1=\rho(\ell_1)=\begin{pmatrix} e^{-2\pi is} & 0\\ 0 & e^{2\pi is}\end{pmatrix},\ N_2=\rho(\ell_2)=\begin{pmatrix} e^{2\pi ir} & 0\\ 0 & e^{-2\pi ir}\end{pmatrix}. \tag{2.36}$$

(iv) For the not completely reducible case, Theorem 2.4 and so Corollary 2.5 also hold for $\mathrm{H}(\mathbf{n},B,\tau)$.

3. GLE and Painlevé VI equation

In order to prove Theorem 1.3, we need to apply the deep connection [4] between GLE and Painlevé VI equation. The well-known Painlevé VI equation with four free parameters $(\alpha,\beta,\gamma,\delta)$ (denoted by PVI$(\alpha,\beta,\gamma,\delta)$) is written as

$$\begin{aligned}\frac{d^2\lambda}{dt^2}=&\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\ &+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right].\end{aligned} \tag{3.1}$$

Due to its connection with many different disciplines in mathematics and physics, PVI has been extensively studied in the past several decades. We refer the readers to the text [18] for a detailed introduction of PVI.

One of the fundamental properties for PVI is the so-called *Painlevé property*, which says that any solution $\lambda(t)$ of PVI has neither movable branch points nor movable essential singularities; in other words, for any $t_0\in\mathbb{C}\backslash\{0,1\}$, either $\lambda(t)$ is holomorphic at t_0 or $\lambda(t)$ has a pole at t_0. Therefore, it is reasonable to lift PVI to the universal covering space $\mathbb{H}=\{\tau|\operatorname{Im}\tau>0\}$ of $\mathbb{C}\backslash\{0,1\}$ by the following transformation:

$$t=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},\quad \lambda(t)=\frac{\wp(p(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}. \tag{3.2}$$

Then it is known (cf. [1, 23]) that $\lambda(t)$ solves PVI if and only if $p(\tau)$ satisfies the *elliptic form* (1.6) with parameters given by

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=\left(\alpha,-\beta,\gamma,\tfrac{1}{2}-\delta\right). \tag{3.3}$$

The Painlevé property implies that function $\wp(p(\tau)|\tau)$ is a single-valued meromorphic function in $\mathbb{H}$. This is an advantage of making the transformation (3.2).

Remark 3.1. Clearly for any $m_1, m_2 \in \mathbb{Z}$, $\pm p(\tau)+m_1+m_2\tau$ is also a solution of the elliptic form (1.6). Since they all give the same $\lambda(t)$ via (3.2), we always identify all these $\pm p(\tau)+m_1+m_2\tau$ with the same one $p(\tau)$.

Another important feature of PVI is that it is closely related to the isomonodromy theory of a second order Fuchsian ODE on $\mathbb{CP}^1$, which has five regular singular points $\{0, 1, t, \lambda(t), \infty\}$. Among them, $\lambda(t)$ (as a solution of PVI) is an apparent singularity. In fact, PVI (3.1) is equivalent to the following Hamiltonian system

$$\frac{d\lambda(t)}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu(t)}{dt} = -\frac{\partial K}{\partial \lambda}, \tag{3.4}$$

where $K = K(\lambda, \mu, t)$ is given by

$$K = \frac{1}{t(t-1)} \left\{ \begin{array}{l} \lambda(\lambda-1)(\lambda-t)\mu^2 + \theta_0(\theta_0+\theta_4)(\lambda-t) \\ - \left[\begin{array}{l} \theta_1(\lambda-1)(\lambda-t) + \theta_2\lambda(\lambda-t) \\ +(\theta_3-1)\lambda(\lambda-1) \end{array} \right] \mu \end{array} \right\}, \tag{3.5}$$

and the relation of parameters is given by

$$(\alpha, \beta, \gamma, \delta) = \left(\tfrac{1}{2}\theta_4^2,\ -\tfrac{1}{2}\theta_1^2,\ \tfrac{1}{2}\theta_2^2,\ \tfrac{1}{2}\left(1-\theta_3^2\right)\right), \tag{3.6}$$

$$2\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = 1. \tag{3.7}$$

For the Hamiltonian system (3.4), we consider a second order Fuchsian differential equation on $\mathbb{CP}^1$ as follows:

$$\frac{d^2 f}{dx^2} + p_1(x)\frac{df}{dx} + p_2(x) f = 0, \tag{3.8}$$

which has five regular singular points at $\{0, 1, t, \lambda, \infty\}$ with the Riemann scheme

$$\begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ 0 & 0 & 0 & 0 & \theta_0 \\ \theta_1 & \theta_2 & \theta_3 & 2 & \theta_0+\theta_4 \end{pmatrix}, \tag{3.9}$$

and λ is an apparent singularity. Under these conditions, we have

$$p_1(x) = \frac{1-\theta_3}{x-t} + \frac{1-\theta_1}{x} + \frac{1-\theta_2}{x-1} - \frac{1}{x-\lambda}, \tag{3.10}$$

$$p_2(x) = \frac{\theta_0\,(\theta_0+\theta_4)}{x(x-1)} - \frac{t(t-1)K}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \tag{3.11}$$

where $K = K(\lambda, \mu, t)$ is given by (3.5); see e.g. [18]. The following result was proved in [12, 24]: *Suppose that* $\theta_1, \theta_2, \theta_3, \theta_4 \notin \mathbb{Z}$ *and* λ *is an apparent singularity of (3.8). Then (3.8) is monodromy preserving as* t *deforms if and only if* $(\lambda(t), \mu(t))$ *satisfies the Hamiltonian system (3.4). In particular,* $\lambda(t)$ *is a solution of PVI (3.1).*

On the other hand, there are works studying the isomonodromic deformation on elliptic curves and its Hamiltonian structure; see e.g. [19] and references therein. Recently, we [4] developed an analogous isomonodromy theory for the elliptic form (1.6). First we proved that the elliptic form (1.6) is equivalent to the new Hamiltonian system (1.8). Then we proved that this Hamiltonian system governs the isomonodromic deformation of GLE($\mathbf{n}$, $p(\tau)$, $A(\tau), \tau$).

Theorem 3.A ([4])**.** *GLE*$(\mathbf{n}, p(\tau), A(\tau), \tau)$ *with* $p(\tau)$ *being an apparent singularity is monodromy preserving as* τ *deforms if and only if* $(p(\tau), A(\tau))$ *satisfies the Hamiltonian system (1.8). In particular,* $p(\tau)$ *is a solution of the elliptic form (1.6) with parameter (1.7).*

Remark that Theorem 3.A holds for any $n_k \in \mathbb{C}\backslash(\frac{1}{2}+\mathbb{Z})$ (i.e. non-resonant condition), but we only consider $n_k \in \mathbb{Z}_{\geq 0}$ in this paper.

Given any solution $p(\tau)$ of the elliptic form (1.6) with parameter (1.7), we define $A(\tau)$ by the first equation of (1.8). Then for any τ such that $p(\tau) \notin E_\tau[2]$, $A(\tau)$ is finite and so GLE$(\mathbf{n}, p(\tau), A(\tau), \tau)$ is well-defined, which is called *the associated GLE* of $p(\tau)$ in this paper.

In view of Theorem 3.A and the monodromy theory of GLE discussed in Section 2, we give the following definition for convenience.

Definition 3.2. A solution $p(\tau)$ of the elliptic form (1.6) with parameter (1.7) is called a completely reducible solution if the monodromy of the associated GLE($\mathbf{n}$, $p(\tau), A(\tau), \tau$) is completely reducible; otherwise, $p(\tau)$ is called a not completely reducible solution.

A natural problem is *how to classify (not) completely reducible solutions* $p(\tau)$ *in terms of the global monodromy data of the associated GLE*$(\mathbf{n}, p(\tau)$, $A(\tau), \tau)$. This is crucial for us to prove Theorem 1.3. In Sections 4–5, we answer this question for the special case $\mathbf{n} = \mathbf{0}$, i.e. $n_k = 0$ for all k and the general case $\mathbf{n} \neq \mathbf{0}$, respectively.

4. The special case $\mathbf{n} = \mathbf{0}$

Note from (1.7) that $\alpha_k = \frac{1}{8}$ for all k if $\mathbf{n} = \mathbf{0}$. This section is devoted to the classification of all solutions of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$

$$\frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{32\pi^2} \sum_{k=0}^{3} \wp'\left(p(\tau) + \frac{\omega_k}{2}\middle|\tau\right), \tag{4.1}$$

or equivalently PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$, in terms of the global monodromy data of the associated GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$. PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ was first studied by Hitchin [15] and later by Takemura [28]. Therefore, part of the results in this section do overlap with the existing literature. However, there are a number of issues which we were unable to locate satisfactory in the literature. Here we attempt to provide a self-contained account of solutions of PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ for later usage in Section 5.

First we recall Hitchin's famous formula. For any $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, let $p^{\mathbf{0}}_{r,s}(\tau)$ be defined by

$$\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) := \wp(r + s\tau|\tau) + \frac{\wp'(r + s\tau|\tau)}{2(\zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau))}. \tag{4.2}$$

In [15] Hitchin proved the following remarkable result for PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$.

Theorem 4.A ([15]). *For any $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, $p^{\mathbf{0}}_{r,s}(\tau)$ given by (4.2) is a solution to EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$; or equivalently, $\lambda^{\mathbf{0}}_{r,s}(t) := \frac{\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}$ via (4.2) is a solution to PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$.*

The following result shows that $p^{\mathbf{0}}_{r,s}(\tau)$ represents the completely reducible solutions in the sense of Definition 3.2.

Theorem 4.1. *Suppose $p^{\mathbf{0}}(\tau)$ is a solution of (4.1). Then*

(i) $p^{\mathbf{0}}(\tau)$ is completely reducible if and only if there is a complex pair $(r, s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$ such that $p^{\mathbf{0}}(\tau) = p^{\mathbf{0}}_{r,s}(\tau)$ given by (4.2). In this case, the monodromy of the associated GLE$(\mathbf{0}, p^{\mathbf{0}}(\tau), A(\tau), \tau)$ satisfies (2.13).

(ii) $\wp(p^{\mathbf{0}}_{r_1,s_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{0}}_{r_2,s_2}(\tau)|\tau) \iff (r_1, s_1) \equiv \pm(r_2, s_2) \bmod \mathbb{Z}^2$.

Proof. (i) Take $\tau_0 \in \mathbb{H}$ such that $p^{\mathbf{0}}(\tau) \notin E_\tau[2]$ in a neighborhood U of τ_0. We only need to prove $p^{\mathbf{0}}(\tau) = p^{\mathbf{0}}_{r,s}(\tau)$ in a neighborhood U for some $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$ and then the result follows by analytic continuation.

First we prove the necessary part. Since $p^{\mathbf{0}}(\tau)$ is completely reducible, the associated GLE$(\mathbf{0}, p^{\mathbf{0}}(\tau), A(\tau), \tau)$ is well-defined in U and preserves its completely reducible monodromy for $\tau\in U$. Then by Theorem 2.3 and (2.11)–(2.12), there exists $(r,s)\in\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ independent of τ such that

$$y_{a_1(\tau)}(z)=\frac{e^{c(\tau)z}\sigma(z-a_1(\tau))}{\sqrt{\sigma(z-p^{\mathbf{0}}(\tau))\sigma(z+p^{\mathbf{0}}(\tau))}} \tag{4.3}$$

is a solution to GLE$(\mathbf{0},p^{\mathbf{0}}(\tau),A(\tau),\tau)$, where

$$a_1(\tau)=r+s\tau, \tag{4.4}$$

$$\begin{aligned} c(\tau)&=r\eta_1(\tau)+s\eta_2(\tau)\\ &=\tfrac{1}{2}\left[\zeta(a_1(\tau)+p^{\mathbf{0}}(\tau))+\zeta(a_1(\tau)-p^{\mathbf{0}}(\tau))\right]. \end{aligned}\tag{4.5}$$

Here $[a_1(\tau)]\neq\pm[p^{\mathbf{0}}(\tau)]$ because the local exponents are $\frac{-1}{2},\frac{3}{2}$ at $\pm p^{\mathbf{0}}(\tau)$. Applying the addition formula

$$\zeta(u+v)+\zeta(u-v)-2\zeta(u)=\frac{\wp'(u)}{\wp(u)-\wp(v)}, \tag{4.6}$$

it is easy to see that the second equality in (4.5) is equivalent to

$$\wp\left(p^{\mathbf{0}}(\tau)|\tau\right)=\wp(r+s\tau|\tau)+\frac{\wp'(r+s\tau|\tau)}{2(\zeta\left(r+s\tau|\tau\right)-r\eta_1(\tau)-s\eta_2(\tau))}, \tag{4.7}$$

i.e. $\wp(p^{\mathbf{0}}(\tau)|\tau)=\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau)$ for $\tau\in U$. This proves $p^{\mathbf{0}}(\tau)=p^{\mathbf{0}}_{r,s}(\tau)$ by Remark 3.1.

Next we prove the sufficient part. Since $p^{\mathbf{0}}(\tau)=p^{\mathbf{0}}_{r,s}(\tau)$, the above argument shows the validity of the second equality of (4.5) by defining $a_1(\tau)=r+s\tau$. Since $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$, we may assume $a_1(\tau)\notin E_\tau[2]$ and hence $a_1(\tau)\not\equiv\pm p^{\mathbf{0}}(\tau) \bmod \Lambda_\tau$ for $\tau\in U$. Then we define $c(\tau)$ by (4.5) and $y_{a_1(\tau)}(z)$ by (4.3) in U. Consequently, a direct computation shows that $y_{a_1(\tau)}(z)$ is a solution to GLE$(\mathbf{0},\ p^{\mathbf{0}}(\tau),\ \tilde{A}(\tau),\tau)$ with

$$\tilde{A}(\tau):=\tfrac{1}{2}\left[\zeta(a_1(\tau)+p^{\mathbf{0}}(\tau))-\zeta(a_1(\tau)-p^{\mathbf{0}}(\tau))-\zeta(2p^{\mathbf{0}}(\tau))\right]. \tag{4.8}$$

Indeed, since

$$\begin{aligned} \frac{y'_{a_1}(z)}{y_{a_1}(z)}&=c(\tau)+\zeta(z-a_1)-\tfrac{1}{2}[\zeta(z+p^{\mathbf{0}})+\zeta(z-p^{\mathbf{0}})],\\ \left(\frac{y'_{a_1}(z)}{y_{a_1}(z)}\right)'&=-\wp(z-a_1)+\tfrac{1}{2}[\wp(z+p^{\mathbf{0}})+\wp(z-p^{\mathbf{0}})], \end{aligned}$$

are all elliptic functions, we have

$$\begin{aligned}\frac{y''_{a_1}(z)}{y_{a_1}(z)} &= \left(\frac{y'_{a_1}(z)}{y_{a_1}(z)}\right)' + \left(\frac{y'_{a_1}(z)}{y_{a_1}(z)}\right)^2 \\ &= \tfrac{3}{4}[\wp(z+p^{\mathbf{0}}) + \wp(z-p^{\mathbf{0}})] + \tilde{A}[\zeta(z+p^{\mathbf{0}}) - \zeta(z-p^{\mathbf{0}})] + \tilde{B},\end{aligned}$$

with some $\tilde{B} \in \mathbb{C}$ and $\tilde{A} = -c(\tau) + \zeta(p^{\mathbf{0}} + a_1) - \frac{1}{2}\zeta(2p^{\mathbf{0}})$, i.e. (4.8) holds by using the second equality of (4.5).

By (4.5) and $a_1(\tau) = r + s\tau$, the same argument as Theorem 2.3 implies that (2.13) holds with respect to $y_{a_1(\tau)}(z)$ and $y_{-a_1(\tau)}(z)$, i.e. the monodromy of GLE($\mathbf{0}$, $p^{\mathbf{0}}(\tau)$, $\tilde{A}(\tau), \tau$) is completely reducible and preserves for $\tau \in U$. Then Theorem 3.A implies that $(p^{\mathbf{0}}(\tau), \tilde{A}(\tau))$ satisfies the Hamiltonian system (1.8), namely $\tilde{A}(\tau) = A(\tau)$ and so the monodromy of the associated GLE($\mathbf{0}$, $p^{\mathbf{0}}(\tau)$, $A(\tau), \tau$) of $p^{\mathbf{0}}(\tau)$ is completely reducible. This proves that $p^{\mathbf{0}}(\tau)$ is a completely reducible solution.

(ii) The sufficient part is trivial so we prove the necessary part. Suppose $\wp(p^{\mathbf{0}}_{r_1,s_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{0}}_{r_2,s_2}(\tau)|\tau)$. Take $\tau_0 \in \mathbb{H}$ such that $p^{\mathbf{0}}_{r_i,s_i}(\tau) \notin E_\tau[2]$, $i = 1, 2$, in a neighborhood U of τ_0. Then $p^{\mathbf{0}}_{r_1,s_1}(\tau) = \pm p^{\mathbf{0}}_{r_2,s_2}(\tau) + m + n\tau$ for $\tau \in U$. Let $A_i(\tau)$ be defned by the first equation of the Hamiltonian system (1.8), then $A_1(\tau) = \pm A_2(\tau)$. Together with (1.5), we conclude that these two associated GLE($\mathbf{0}$, $p^{\mathbf{0}}_{r_i,s_i}(\tau)$, $A_i(\tau), \tau$) must be the same. Consequently, it follows from the assertion (i) that

$$e^{2\pi i s_1} = e^{\pm 2\pi i s_2} \text{ and } e^{2\pi i r_1} = e^{\pm 2\pi i r_2},$$

which is precisely $(r_1, s_1) \equiv \pm (r_2, s_2) \bmod \mathbb{Z}^2$. The proof is complete. □

Next we study the not completely reducible solutions of EPVI($\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}$). Recall (3.5) that the corresponding Hamiltonian $K = K(\lambda, \mu, t)$ is given by

$$K = \frac{1}{t(t-1)}\left\{\lambda(\lambda-1)(\lambda-t)\mu^2 - \tfrac{1}{2}(\lambda^2 - 2t\lambda + t)\mu\right\}. \tag{4.9}$$

In general, solutions of PVI($\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}$) might also come from Riccati equations. It is easy to see from (4.9) that the Hamiltonian system (3.4) has four families of solutions $(\lambda(t), \mu(t))$, where $\lambda(t)$ satisfies four different Riccati equations as follows:

$$\frac{d\lambda}{dt} = -\frac{1}{2t(t-1)}(\lambda^2 - 2t\lambda + t), \quad \mu \equiv 0; \tag{4.10}$$

(4.11) $$\frac{d\lambda}{dt} = \frac{1}{2t(t-1)}(\lambda^2 - 2\lambda + t), \quad \mu \equiv \frac{1}{2\lambda};$$

(4.12) $$\frac{d\lambda}{dt} = \frac{1}{2t(t-1)}(\lambda^2 - t), \quad \mu \equiv \frac{1}{2(\lambda-1)};$$

(4.13) $$\frac{d\lambda}{dt} = \frac{1}{2t(t-1)}(\lambda^2 + 2(t-1)\lambda - t), \quad \mu \equiv \frac{1}{2(\lambda-t)}.$$

Theorem 4.2. *Suppose $p(\tau)$ is a solution of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. Then $p(\tau)$ is not completely reducible if and only if the corresponding solution $\lambda(t)$ (via (3.2)) of PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ solves one of the four Riccati equations (4.10)–(4.13).*

Proof. Let $p(\tau)$ be a solution of the elliptic form (4.1). We can take $\tau_0 \in \mathbb{H}$ such that

(4.14) $$[p(\tau)] \notin E_\tau[2] \text{ and } A(\tau) \text{ is finite in a neighborhood } U \text{ of } \tau_0,$$

namely the associated GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$ is well-defined and preserves the monodromy for $\tau \in U$. Recalling (4.8), we let $\pm a_1(\tau)$ be defined by

(4.15) $$A(\tau) = \frac{1}{2}[\zeta(a_1(\tau) + p(\tau)) - \zeta(a_1(\tau) - p(\tau)) - \zeta(2p(\tau))], \tau \in U.$$

Then (4.14) gives

(4.16) $$[a_1(\tau)] \neq \pm[p(\tau)], \;\; \tau \in U.$$

Consequently, the same argument as that in the proof of Theorem 4.1-(i) shows that

$$y_{\pm a_1(\tau)}(z) = \frac{e^{\pm c(\tau) z}\sigma(z \mp a_1(\tau))}{\sqrt{\sigma(z - p(\tau))\sigma(z + p(\tau))}}$$

with

$$c(\tau) = \frac{1}{2}\left[\zeta(a_1(\tau) + p(\tau)) + \zeta(a_1(\tau) - p(\tau))\right]$$

are both solutions of GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$. By Theorem 2.3, the monodromy is not completely reducible if and only if $y_{a_1(\tau)}(z)$ and $y_{-a_1(\tau)}(z)$ are linearly dependent, which is equivalent to $a_1(\tau) \equiv -a_1(\tau) \bmod \Lambda_\tau$, i.e.

(4.17) $$[a_1(\tau)] = [\tfrac{\omega_k}{2}] \text{ for } \tau \in U \text{ and some } k \in \{0, 1, 2, 3\}.$$

On the other hand, by the addition formula (4.6) and $\frac{\wp''(p)}{2\wp'(p)} = \zeta(2p) - 2\zeta(p)$, we can rewrite (4.15) as

$$A(\tau) = \frac{\wp'(p(\tau))}{2\left[\wp(p(\tau)) - \wp(a_1(\tau))\right]} - \frac{\wp''(p(\tau))}{4\wp'(p(\tau))}. \tag{4.18}$$

Recall that $\lambda(t)$ defined via (3.2) is a solution of PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$. Then by defining $\mu(t)$ via the first equation of the Hamiltonian system (3.4), $(\lambda(t), \mu(t))$ satisfies the Hamiltonian system (3.4). It follows from (5.20) below that the relation of $\mu(t)$ and $A(\tau)$ is given by

$$\mu(t(\tau)) = \frac{1}{8}\frac{\mathfrak{p}'(\lambda)}{\mathfrak{p}(\lambda)} + \frac{A\wp'(p)}{\left(e_2(\tau) - e_1(\tau)\right)^2 \mathfrak{p}(\lambda)}, \tag{4.19}$$

where

$$\mathfrak{p}(x) = 4x(x-1)(x-t). \tag{4.20}$$

Notice from (4.20), (3.2) and $\wp'(z)^2 = 4\prod_{k=1}^{3}(\wp(z) - e_k)$ that

$$\mathfrak{p}(\lambda(t)) = \frac{\wp'(p(\tau))^2}{(e_2(\tau) - e_1(\tau))^3}, \quad \mathfrak{p}'(\lambda(t)) = \frac{2\wp''(p(\tau))}{(e_2(\tau) - e_1(\tau))^2}.$$

Inserting these and (4.18) into (4.19), we easily obtain

$$\begin{aligned} \mu(t) &= \frac{(e_2(\tau) - e_1(\tau))\left(4A(\tau)\wp'(p(\tau)) + \wp''(p(\tau))\right)}{4\wp'(p(\tau))^2} \\ &= \frac{e_2(\tau) - e_1(\tau)}{2\left[\wp(p(\tau)) - \wp(a_1(\tau))\right]}. \end{aligned} \tag{4.21}$$

Remark that (4.21) always holds no matter with whether $p(\tau)$ is a completely reducible solution or not.

Recall that the monodromy is not completely reducible if and only if (4.17) holds. By (3.2) and (4.21), this is equivalent to

$$\mu(t) = \begin{cases} 0, \text{ if } k = 0, \\ \frac{1}{2\lambda(t)}, \text{ if } k = 1, \\ \frac{1}{2(\lambda(t)-1)}, \text{ if } k = 2, \\ \frac{1}{2(\lambda(t)-t)}, \text{ if } k = 3, \end{cases} \quad \text{in a neighborhood of } t(\tau_0), \tag{4.22}$$

namely one of (4.10)–(4.13) holds after the analytic continuation. The proof is complete. □

Now we want to find the expression of a not completely reducible solution $p(\tau)$. Assume $[a_1]=[\frac{\omega_k}{2}]\in E_\tau[2]$ by (4.17), and recall (4.15) that

$$A(\tau)=\frac{1}{2}[\zeta(\tfrac{\omega_k}{2}+p(\tau))-\zeta(\tfrac{\omega_k}{2}-p(\tau))-\zeta(2p(\tau))]. \tag{4.23}$$

By using $2\zeta(z)-\zeta(2z)=-\frac{1}{2}\frac{\wp''(z)}{\wp'(z)}$, (4.23) is equivalent to

$$A(\tau)=-\frac{1}{4}\frac{\wp''(p(\tau)-\frac{\omega_k}{2})}{\wp'(p(\tau)-\frac{\omega_k}{2})}. \tag{4.24}$$

As in Theorem 2.4, we let

$$y_1(z)=y_{a_1}(z)=\frac{e^{\frac{1}{2}[\zeta(a_1+p)+\zeta(a_1-p)]z}\sigma(z-a_1)}{\sqrt{\sigma(z-p)\sigma(z+p)}},\quad a_1=\tfrac{\omega_k}{2}, \tag{4.25}$$

and $y_2(z)=\chi(z)y_1(z)$ be linearly independent solutions of the associated GLE$(\mathbf{0},p(\tau),A(\tau),\tau)$, where

$$\chi'(z)=\text{const}\cdot y_1(z)^{-2}. \tag{4.26}$$

Define

$$(\varepsilon_{k,1},\varepsilon_{k,2})=\begin{cases}(1,1), \text{ if } k=0,\\ (1,-1), \text{ if } k=1,\\ (-1,1), \text{ if } k=2,\\ (-1,-1), \text{ if } k=3.\end{cases} \tag{4.27}$$

First we consider the case $[a_1]=[0]$. Then $y_1(z)=\frac{\sigma(z)}{\sqrt{\sigma(z-p)\sigma(z+p)}}=\Psi_p(z)$ (see Theorem 2.3 for $\Psi_p(z)$) and

$$y_1(z)^{-2}=\frac{\sigma(z+p)\sigma(z-p)}{\sigma(z)^2}=c(\wp(z)-\wp(p)),\ c\neq 0.$$

So (4.26) yields that we can take $\chi(z)=\zeta(z)+\wp(p)z$, namely for any $c(\tau)\neq 0$, $(c(\tau)y_1,y_2)$ with $y_2(z)=(\zeta(z)+\wp(p)z)y_1(z)$ is a fundamental system of solutions to GLE$(\mathbf{0},p(\tau),A(\tau),\tau)$. In particular, (2.19) implies

$$\ell_j^*\begin{pmatrix}c(\tau)y_1\\ y_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ \frac{\eta_j+\wp(p)\omega_j}{c(\tau)} & 1\end{pmatrix}\begin{pmatrix}c(\tau)y_1\\ y_2\end{pmatrix},\quad j=1,2. \tag{4.28}$$

Proposition 4.3. *The solutions of the Riccati equation (4.10) can be parameterized by $\mathcal{C} \in \mathbb{CP}^1$:*

$$\lambda^{\mathbf{0}}_{0,\mathcal{C}}(t) = \frac{\wp(p^{\mathbf{0}}_{0,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \ \wp(p^{\mathbf{0}}_{0,\mathcal{C}}(\tau)|\tau) = \frac{\eta_2(\tau) - \mathcal{C}\eta_1(\tau)}{\mathcal{C} - \tau}. \tag{4.29}$$

Moreover, the monodromy of the associated GLE satifies

$$\rho(\ell_1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix}. \tag{4.30}$$

Here when $\mathcal{C} = \infty$, it should be understand as

$$\rho(\ell_1) = I_2, \quad \rho(\ell_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{4.31}$$

Proof. In this proof, we omit $\mathbf{0}, 0$ in the notations.

Step 1. We prove that for any constant $\mathcal{C} \in \mathbb{CP}^1$, $\lambda_{\mathcal{C}}(t)$ given by (4.29) solves the Riccati equation (4.10).

Fix any $\mathcal{C} \in \mathbb{CP}^1$ and let $p(\tau) = p_{\mathcal{C}}(\tau)$, $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau))}{\wp'(p(\tau))}$ in GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$. If $\mathcal{C} = \infty$, then $\wp(p(\tau)) = -\eta_1(\tau)$. Choose $c(\tau) = \eta_2(\tau) + \wp(p(\tau))\tau$. By the Legendre relation $\tau\eta_1(\tau) - \eta_2(\tau) = 2\pi i$ we have $c(\tau) = -2\pi i$. Thus by (4.28), we obtain (4.31). That is, GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$ is monodromy preserving as τ deforms, so $p(\tau) = p_\infty(\tau)$ is a solution of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.

If $\mathcal{C} \neq \infty$, then (4.29) gives $\eta_1(\tau) + \wp(p(\tau)) \not\equiv 0$ and $\mathcal{C} = \frac{\eta_2(\tau)+\wp(p(\tau))\tau}{\eta_1(\tau)+\wp(p(\tau))}$. Choose $c(\tau) = \eta_1(\tau) + \wp(p(\tau))$. Clearly except a set of discrete points in $\mathbb{H}$, $c(\tau) \neq 0$ and so (4.28) gives (4.30). Again we conclude that $p(\tau) = p_{\mathcal{C}}(\tau)$ is a solution of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. Formula (4.29) can be found in [15, 28]. Here together with $a_1 = 0$ and (4.22), we note that $\lambda_{\mathcal{C}}(t)$ actually solves the Ricatti equation (4.10).

Step 2. Let $\lambda(t)$ be any solution of the Riccati equation (4.10). We prove the existence of $\mathcal{C} \in \mathbb{CP}^1$ such that $\lambda(t) = \lambda_{\mathcal{C}}(t)$.

Define $\pm[p(\tau)]$ by $\lambda(t)$ via (3.2) and $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau))}{\wp'(p(\tau))}$. Then $p(\tau)$ is a solution of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and the associated GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$ is monodromy preserving as τ deforms. So there exists a fundamental system of solutions $(\tilde{y}_1(z;\tau), \tilde{y}_2(z;\tau))$ such that the monodromy matrices M_1, M_2, which are defined by

$$\ell_j^* \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = M_j \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}, \ j = 1, 2,$$

are independent of τ. We may assume $\wp(p(\tau)|\tau) \not\equiv \wp(p_\infty(\tau)|\tau)$, otherwise we are done. Then $c(\tau) := \eta_1(\tau) + \wp(p(\tau)) \not\equiv 0$. For any τ such that $c(\tau) \neq 0$, $(c(\tau)y_1, y_2)$ given by (4.25)–(4.28) is also a fundamental system of solutions, so there is an invertible matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \gamma \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}$. Clearly the monodromy matrices of $(c(\tau)y_1, y_2)$ is given by (4.30), where

$$\mathcal{C} = \frac{\eta_2(\tau) + \wp(p(\tau)|\tau)\tau}{\eta_1(\tau) + \wp(p(\tau)|\tau)} \tag{4.32}$$

might depend on τ at the moment. Then

$$M_1 = \gamma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 + \frac{bd}{ad-bc} & \frac{-b^2}{ad-bc} \\ \frac{d^2}{ad-bc} & 1 - \frac{bd}{ad-bc} \end{pmatrix},$$
$$M_2 = \gamma \begin{pmatrix} 1 & 0 \\ \mathcal{C} & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 + \frac{bd}{ad-bc}\mathcal{C} & \frac{-b^2}{ad-bc}\mathcal{C} \\ \frac{d^2}{ad-bc}\mathcal{C} & 1 - \frac{bd}{ad-bc}\mathcal{C} \end{pmatrix}.$$

Since M_1, M_2 are independent of τ and $|b|^2 + |d|^2 \neq 0$, we conclude that $\mathcal{C}$ is a constant independent of τ. Consequently, (4.32) implies $\wp(p(\tau)|\tau) = \wp(p_{\mathcal{C}}(\tau)|\tau)$ and so $\lambda(t) = \lambda_{\mathcal{C}}(t)$. □

Similarly, we can prove that all solutions of the other three Riccati equations can be parameterized by $\mathbb{CP}^1$. The calculation is as follows. Fix $k \in \{1, 2, 3\}$. By (4.25) it is easy to see that

$$\chi(z) := -\frac{\wp(p) - e_k}{(e_k - e_i)(e_k - e_j)}\zeta(z - \tfrac{\omega_k}{2}) - \left(1 + e_k \frac{\wp(p) - e_k}{(e_k - e_i)(e_k - e_j)}\right) z$$

satisfies (4.26), where $\{i, j\} = \{1, 2, 3\}\backslash\{k\}$. As before, for any $c(\tau) \neq 0$, $(c(\tau)y_1(z), y_2(z))$ with $y_2(z) = \chi(z)y_1(z)$ is a fundamental system of solutions to GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$. In particular, as in Theorem 2.4 we easily obtain

$$\ell_1^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \varepsilon_{k,1} \begin{pmatrix} 1 & 0 \\ -\frac{D\eta_1 + (1+De_k)}{c(\tau)} & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix},$$
$$\ell_2^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \varepsilon_{k,2} \begin{pmatrix} 1 & 0 \\ -\frac{D\eta_2 + \tau(1+De_k)}{c(\tau)} & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}, \tag{4.33}$$

where $(\varepsilon_{k,1}, \varepsilon_{k,2})$ is given by (4.27) and

$$D := \frac{\wp(p) - e_k}{(e_k - e_i)(e_k - e_j)}. \tag{4.34}$$

Proposition 4.4. *For $k \in \{1,2,3\}$ and $\mathcal{C} \in \mathbb{CP}^1$, we let*

$$\lambda^{\mathbf{0}}_{k,\mathcal{C}}(t) = \frac{\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)},$$

where

$$\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) := \frac{e_k(\mathcal{C}\eta_1(\tau) - \eta_2(\tau)) + (\frac{g_2}{4} - 2e_k^2)(\mathcal{C} - \tau)}{\mathcal{C}\eta_1(\tau) - \eta_2(\tau) + e_k(\mathcal{C} - \tau)}. \tag{4.35}$$

Then $\lambda^{\mathbf{0}}_{k,\mathcal{C}}(t)$ satisfies the Ricatti equation (4.11) if $k = 1$, (4.12) if $k = 2$, (4.13) if $k = 3$. Conversely, such $\lambda^{\mathbf{0}}_{k,\mathcal{C}}(t)$ give all the solutions of these three Riccati equations respectively. Furthermore, the monodromy of its associated GLE satisfies

$$\rho(\ell_1) = \varepsilon_{k,1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \rho(\ell_2) = \varepsilon_{k,2}\begin{pmatrix}1 & 0\\ \mathcal{C} & 1\end{pmatrix}, \tag{4.36}$$

where as before, when $\mathcal{C} = \infty$, it should be understand as

$$\rho(\ell_1) = \varepsilon_{k,1} I_2, \quad \rho(\ell_2) = \varepsilon_{k,2}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}. \tag{4.37}$$

Proof. We sketch the proof for fixed $k \in \{1,2,3\}$ and omit $\mathbf{0}$,k in the notations. For any $\mathcal{C} \in \mathbb{CP}^1$, we let $p(\tau) = p_{\mathcal{C}}(\tau)$, $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau)-\frac{\omega_k}{2})}{\wp'(p(\tau)-\frac{\omega_k}{2})}$ in GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$. If $\mathcal{C} = \infty$, i.e. $D\eta_1 + (1 + De_k) \equiv 0$, then we choose $c(\tau) = -[D\eta_2 + \tau(1 + De_k)] = \frac{-2\pi i}{\eta_1(\tau)+e_k(\tau)} \not\equiv 0$. By (4.33) we obtain (4.37). If $\mathcal{C} \neq \infty$, then (4.35) gives $D\eta_1 + (1 + De_k) \not\equiv 0$ and $\mathcal{C} = \frac{D\eta_2+\tau(1+De_k)}{D\eta_1+(1+De_k)}$. Choose $c(\tau) = -[D\eta_1 + (1 + De_k)]$, then we immediately obtain (4.36). In both cases, GLE$(\mathbf{0}, p(\tau), A(\tau), \tau)$ is monodromy preserving, so $p(\tau) = p_{\mathcal{C}}(\tau)$ is a solution of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. Formula (4.35) was first obtained in [28]. Here by $a_1 = \frac{\omega_k}{2}$ and (4.22), we note that $\lambda_{\mathcal{C}}(t)$ actually satisfies the Ricatti equation (4.11) if $k = 1$, (4.12) if $k = 2$, (4.13) if $k = 3$. The rest of the proof is similar to that of Proposition 4.3. □

Remark that the explict expression of $\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau)$ immediately implies

$$\wp(p^{\mathbf{0}}_{k,\mathcal{C}_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{0}}_{k,\mathcal{C}_2}(\tau)|\tau) \Longleftrightarrow \mathcal{C}_1 = \mathcal{C}_2. \tag{4.38}$$

The above results completely classify all the solutions of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ in terms of the global monodromy data of the associated GLE. For a completely reducible solution $p^{\mathbf{0}}_{r,s}(\tau)$, we denote the corresponding $\mu(t)$ by $\mu^{\mathbf{0}}_{r,s}(t)$

and (4.21) gives

$$\mu^{\mathbf{0}}_{r,s}(t) = \frac{e_2(\tau) - e_1(\tau)}{2\left[\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) - \wp(r+s\tau|\tau)\right]}. \tag{4.39}$$

For a not completely reducible solution $p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)$, we denote the corresponding $\mu(t)$ by $\mu^{\mathbf{0}}_{k,\mathcal{C}}(t)$, and by (4.10)–(4.13) or (4.21),

$$\mu^{\mathbf{0}}_{0,\mathcal{C}}(t) \equiv 0, \quad \mu^{\mathbf{0}}_{k,\mathcal{C}}(t) = \frac{e_2(\tau) - e_1(\tau)}{2[\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) - e_k(\tau)]}, \quad k = 1, 2, 3.$$

We conclude this section by studying the precise relation between these two kinds of solutions.

Theorem 4.5. *For $\mathcal{C} \neq \infty$, there holds*

$$\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) = \begin{cases} \lim_{s\to 0} \wp(p^{\mathbf{0}}_{-\mathcal{C}s,s}(\tau)|\tau) & \text{if } k = 0, \\ \lim_{s\to 0} \wp(p^{\mathbf{0}}_{\frac{1}{2}-\mathcal{C}s,s}(\tau)|\tau) & \text{if } k = 1, \\ \lim_{s\to 0} \wp(p^{\mathbf{0}}_{\mathcal{C}s,\frac{1}{2}-s}(\tau)|\tau) & \text{if } k = 2, \\ \lim_{s\to 0} \wp(p^{\mathbf{0}}_{\frac{1}{2}+\mathcal{C}s,\frac{1}{2}-s}(\tau)|\tau) & \text{if } k = 3, \end{cases}$$

and the same holds for $\mu^{\mathbf{0}}_{k,\mathcal{C}}(t)$ as the limit of $\mu^{\mathbf{0}}_{r,s}(t)$ for $(r,s) = (-\mathcal{C}s, s)$ if $k = 0$, and so on.

For $\mathcal{C} = \infty$, there holds

$$\wp(p^{\mathbf{0}}_{k,\infty}(\tau)|\tau) = \begin{cases} \lim_{r\to 0} \wp(p^{\mathbf{0}}_{r,0}(\tau)|\tau) & \text{if } k = 0, \\ \lim_{r\to 0} \wp(p^{\mathbf{0}}_{\frac{1}{2}+r,0}(\tau)|\tau) & \text{if } k = 1, \\ \lim_{r\to 0} \wp(p^{\mathbf{0}}_{r,\frac{1}{2}}(\tau)|\tau) & \text{if } k = 2, \\ \lim_{r\to 0} \wp(p^{\mathbf{0}}_{\frac{1}{2}+r,\frac{1}{2}}(\tau)|\tau) & \text{if } k = 3, \end{cases}$$

and the same holds for $\mu^{\mathbf{0}}_{k,\infty}(t)$ as the limit of $\mu^{\mathbf{0}}_{r,s}(t)$ for $(r,s) = (r,0)$ if $k = 0$, and so on.

Proof. The proof is just by computations. For example, for $\mathcal{C} \neq \infty$, we denote $u = -\mathcal{C}s + s\tau = s(\tau - \mathcal{C})$ for convenience. Then $u \to 0$ as $s \to 0$, and it follows from the Laurent series of $\zeta(\cdot|\tau)$ and $\wp(\cdot|\tau)$ that

$$\zeta(-Cs + s\tau|\tau) = \frac{1}{u} - \frac{g_2}{60}u^3 + O(|u|^5),$$
$$\wp(-Cs + s\tau|\tau) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + O(|u|^4),$$

$$\wp'(-Cs + s\tau|\tau) = \frac{-2}{u^3} + \frac{g_2}{10}u + O(|u|^3),$$

hold uniformly for τ in any compact subset $K \subset \mathbb{H}$ as $s \to 0$. Inserting these into Hicthin's formula (4.2), we easily obtain that

$$\lim_{s\to 0} \wp(p^{\mathbf{0}}_{-\mathcal{C}s,s}(\tau)|\tau) = \frac{\eta_2(\tau) - \mathcal{C}\eta_1(\tau)}{\mathcal{C} - \tau} = \wp(p^{\mathbf{0}}_{0,\mathcal{C}}(\tau)|\tau)$$

holds uniformly for τ in any compact subset K. Therefore, as solutions of EPVI$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$, $\wp(p^{\mathbf{0}}_{0,\mathcal{C}}(\tau)|\tau) \to \wp(p^{\mathbf{0}}_{-\mathcal{C}s,s}(\tau)|\tau)$ as $s \to 0$. Furthermore, it follows from (4.39) that $\lim_{s\to 0} \mu^{\mathbf{0}}_{-\mathcal{C}s,s}(t) = 0 = \mu^{\mathbf{0}}_{0,\mathcal{C}}(t)$. The other formulas can be proved similarly and we omit the details here. □

In the next section, we will generalize the above results to the general case $\mathbf{n} \neq \mathbf{0}$ via the well known Bäcklund transformation.

5. General case via the Bäcklund transformation

The purpose of this section is to classify all the solutions of the elliptic form (1.6) with parameters

$$\alpha_k = \tfrac{(2n_k+1)^2}{8}, \ n_k \in \mathbb{Z}_{\geq 0} \text{ for all } k \text{ and } \mathbf{n} \neq \mathbf{0}, \tag{5.1}$$

or equivalently PVI with parameters

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) = \Big(&\tfrac{(2n_0+1)^2}{8}, \ -\tfrac{(2n_1+1)^2}{8}, \ \tfrac{(2n_2+1)^2}{8}, \\ &\tfrac{1}{2} - \tfrac{(2n_3+1)^2}{8}\Big), \ n_k \in \mathbb{Z}_{\geq 0} \text{ for all } k \text{ and } \mathbf{n} \neq \mathbf{0},\end{aligned} \tag{5.2}$$

in terms of the global monodromy data of the associated GLE. The idea is to apply the Bäcklund transformations.

It is known that solutions of PVI with parameter (5.2) could be obtained from solutions of PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ (i.e. $n_k = 0$ for all k) via the Bäcklund transformations ([25]). By (3.6)–(3.7), it is convenient to consider the parameter space of PVI (equivalently the Hamiltonian system (3.4)–(3.5)) as an affine space

$$\mathcal{K} = \left\{\theta = (\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{C}^5 \ : \ 2\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = 1\right\}.$$

Definition 5.1 ([25])**.** An Bäcklund transformation κ is an invertible mapping which maps solutions $(\lambda(t), \mu(t), t)$ of the Hamiltonian system (3.4) with parameter θ to solutions $(\kappa(\lambda)(t), \kappa(\mu)(t), t)$ of (3.4) with new parameter $\kappa(\theta) \in K$ where both $\kappa(\lambda)(t)$ and $\kappa(\mu)(t)$ are rational functions of λ, μ, t. In particular, $\kappa(\lambda)(t)$ is a solution to PVI (3.1) with new parameter $\kappa(\theta) \in \mathcal{K}$.

The list of the Bäcklund transformations $\kappa_j (0 \le j \le 4)$ is given in the Table 1 (cf. [35]).

Table 1: Bäcklund transformations

	θ_0	θ_1	θ_2	θ_3	θ_4	t	λ	μ
κ_0	$-\theta_0$	$\theta_1+\theta_0$	$\theta_2+\theta_0$	$\theta_3+\theta_0$	$\theta_4+\theta_0$	t	$\lambda+\frac{\theta_0}{\mu}$	μ
κ_1	$\theta_0+\theta_1$	$-\theta_1$	θ_2	θ_3	θ_4	t	λ	$\mu-\frac{\theta_1}{\lambda}$
κ_2	$\theta_0+\theta_2$	θ_1	$-\theta_2$	θ_3	θ_4	t	λ	$\mu-\frac{\theta_2}{\lambda-1}$
κ_3	$\theta_0+\theta_3$	θ_1	θ_2	$-\theta_3$	θ_4	t	λ	$\mu-\frac{\theta_3}{\lambda-t}$
κ_4	$\theta_0+\theta_4$	θ_1	θ_2	θ_3	$-\theta_4$	t	λ	μ

Among them κ_0 is due to Okamoto [25] while the others are classically known. These transformations κ_j $(0 \le j \le 4)$, which satisfy $\kappa_j \circ \kappa_j = Id$ (i.e. $\kappa_j^{-1} = \kappa_j$), generate the affine Weyl group of type $D_4^{(1)}$:

$$W(D_4^{(1)}) = \langle \kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \rangle . \tag{5.3}$$

Denote $\theta^{\mathbf{0}} := (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which corresponds to PVI$(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$. By Table 1 there exists $\kappa^{\mathbf{n}} \in W(D_4^{(1)})$ such that

$$\theta^{\mathbf{n}} := \left(-\frac{1+\sum n_k}{2}, n_1+\tfrac{1}{2}, n_2+\tfrac{1}{2}, n_3+\tfrac{1}{2}, n_0+\tfrac{1}{2}\right) = \kappa^{\mathbf{n}}(\theta^{\mathbf{0}}). \tag{5.4}$$

Note that

$$(\kappa^{\mathbf{n}})^{-1} \in W(D_4^{(1)}) \quad \text{and} \quad \theta^{\mathbf{0}} = (\kappa^{\mathbf{n}})^{-1}(\theta^{\mathbf{n}}). \tag{5.5}$$

Consequently, there exist two *rational functions* $R^{\mathbf{n}}(\cdot,\cdot,\cdot)$ and $\tilde{R}^{\mathbf{n}}(\cdot,\cdot,\cdot)$ of three independent variables with coefficients in $\mathbb{Q}$ such that for any solution $(\lambda^{\mathbf{0}}(t), \mu^{\mathbf{0}}(t))$ of the Hamiltonian system (3.4) with parameter $\theta^{\mathbf{0}}$, $(\lambda^{\mathbf{n}}(t), \mu^{\mathbf{n}}(t))$ given by

$$\lambda^{\mathbf{n}}(t) := \kappa(\lambda^{\mathbf{0}})(t) = R^{\mathbf{n}}(\lambda^{\mathbf{0}}(t), \mu^{\mathbf{0}}(t), t), \tag{5.6}$$

$$\mu^{\mathbf{n}}(t) := \kappa(\mu^{\mathbf{0}})(t) = \tilde{R}^{\mathbf{n}}(\lambda^{\mathbf{0}}(t), \mu^{\mathbf{0}}(t), t), \tag{5.7}$$

is a solution of the Hamiltonian system (3.4) with parameter $\theta^{\mathbf{n}}$, or equivalently, $\lambda^{\mathbf{n}}(t)$ is a solution of PVI with parameter (5.2).

Remark that by (5.5), there are also two *rational functions* $\mathcal{R}^{\mathbf{n}}(\cdot,\cdot,\cdot)$ and $\tilde{\mathcal{R}}^{\mathbf{n}}(\cdot,\cdot,\cdot)$ of three independent variables with coefficients in $\mathbb{Q}$ such that the rational map (5.6)–(5.7) is invertible in the following sense

$$\lambda^{\mathbf{0}}(t) = \mathcal{R}^{\mathbf{n}}(\lambda^{\mathbf{n}}(t), \mu^{\mathbf{n}}(t), t), \quad \mu^{\mathbf{0}}(t) = \tilde{\mathcal{R}}^{\mathbf{n}}(\lambda^{\mathbf{n}}(t), \mu^{\mathbf{n}}(t), t). \tag{5.8}$$

In the literature, there are also references treating the Bäcklund transformations as biholomorphic transformations on the space of initial conditions for solutions of Painlevé equations; see e.g. [27, 34]. In this paper, (5.6)–(5.8) are enough for our following arguments and so we do not need to discuss the space of initial conditions.

Notation: Let $p^{\mathbf{n}}(\tau)$ be a solution of the elliptic form (1.6) with parameter (5.1). We denote it by $p^{\mathbf{n}}_{r,s}(\tau)$ (resp. $p^{\mathbf{n}}_{k,\mathcal{C}}(\tau)$) if it comes from the solution $p^{\mathbf{0}}_{r,s}(\tau)$ (resp. $p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)$) of EPVI$(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$ via (5.6), i.e.

$$\frac{\wp(p^{\mathbf{n}}_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} = R^{\mathbf{n}}\left(\frac{\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \mu^{\mathbf{0}}_{r,s}(t), t\right), \tag{5.9}$$

$$\frac{\wp(p^{\mathbf{n}}_{k,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} = R^{\mathbf{n}}\left(\frac{\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \mu^{\mathbf{0}}_{k,\mathcal{C}}(t), t\right). \tag{5.10}$$

We use similar notations $\mu^{\mathbf{n}}_{r,s}(t)$ and $\mu^{\mathbf{n}}_{k,\mathcal{C}}(t)$ via (5.7). Consequently, it follows from (5.8) that

$$\frac{\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} = \mathcal{R}^{\mathbf{n}}\left(\frac{\wp(p^{\mathbf{n}}_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \mu^{\mathbf{n}}_{r,s}(t), t\right), \tag{5.11}$$

$$\frac{\wp(p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} = \mathcal{R}^{\mathbf{n}}\left(\frac{\wp(p^{\mathbf{n}}_{k,\mathcal{C}}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \mu^{\mathbf{n}}_{k,\mathcal{C}}(t), t\right). \tag{5.12}$$

Remark 5.2. Given $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, we write $Z = Z_{r,s}(\tau)$, $\wp = \wp(r+s\tau|\tau)$ and $\wp' = \wp'(r+s\tau|\tau)$ for convenience. Then Hitchin's formula (4.2) gives

$$\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) = \wp + \frac{\wp'}{2Z}.$$

Consequently, we see from (4.21) that

$$\mu^{\mathbf{0}}_{r,s}(t) = \frac{e_2(\tau) - e_1(\tau)}{2[\wp(p^{\mathbf{0}}_{r,s}(\tau)|\tau) - \wp]} = \frac{(e_2(\tau) - e_1(\tau))Z}{\wp'}.$$

Inserting these and $t = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ into (5.9), we conclude that

$$\wp(p^{\mathbf{n}}_{r,s}(\tau)|\tau) = \Xi_{\mathbf{n}}(Z, \wp, \wp', e_1(\tau), e_2(\tau), e_3(\tau)),$$

where $\Xi_{\mathbf{n}}$ is a rational function of six independent variables with coefficients in $\mathbb{Q}$.

Our main results of this section are as follows, which indicate that the Bäcklund transformation preserves the global monodromy data (or equivalently the monodromy representation) in both completely reducible and not completely reducible cases.

Theorem 5.3 (Completely reducible solutions)**.**

(1) $p^{\mathbf{n}}(\tau)$ *is a completely reducible solution if and only if there exists* $(r, s) \in \mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ *such that* $p^{\mathbf{n}}(\tau) = p^{\mathbf{n}}_{r,s}(\tau)$*. In this case, for any* τ *satisfying* $p^{\mathbf{n}}(\tau) \notin E_\tau[2]$*, the monodromy of the associated GLE*$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$ *satisfies (2.13), i.e. the global monodromy data is precisely this* (r, s)*.*

(2) $\wp(p^{\mathbf{n}}_{r_1,s_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{n}}_{r_2,s_2}(\tau)|\tau) \Longleftrightarrow (r_1, s_1) \equiv \pm(r_2, s_2) \mod \mathbb{Z}^2$*.*

Theorem 5.4 (Not completely reducible solutions)**.**

(1) $p^{\mathbf{n}}(\tau)$ *is a not completely reducible solution if and only if there exist* $k \in \{0, 1, 2, 3\}$ *and* $\mathcal{C} \in \mathbb{C} \cup \{\infty\}$ *such that* $p^{\mathbf{n}}(\tau) = p^{\mathbf{n}}_{k,\mathcal{C}}(\tau)$*. In this case, for any* τ *satisfying* $p^{\mathbf{n}}(\tau) \notin E_\tau[2]$*, the monodromy of the associated GLE*$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$ *satisfies (4.36)–(4.37), i.e. the global monodromy data is precisely* $(2\varepsilon_{k,1}, 2\varepsilon_{k,2}, \mathcal{C})$*.*

(2) $\wp(p^{\mathbf{n}}_{k,\mathcal{C}_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{n}}_{k,\mathcal{C}_2}(\tau)|\tau)$ *if and only if* $\mathcal{C}_1 = \mathcal{C}_2$*.*

The rest of this section is devoted to the proofs of these theorems. First we note that by applying the gauge transformation

$$f(x) = \phi(x)F(x) \quad \text{with} \quad \phi(x) = (x-\lambda)x^{\frac{\theta_1}{2}}(x-1)^{\frac{\theta_2}{2}}(x-t)^{\frac{\theta_3}{2}}, \tag{5.13}$$

equation (3.8) is normalized into a new Fuchsian ODE

$$\frac{d^2F}{dx^2} + P_1(x)\frac{dF}{dx} + P_2(x)F = 0, \tag{5.14}$$

where

$$P_1 = p_1 + 2\frac{\phi'}{\phi}, \quad P_2 = p_2 + \frac{\phi'}{\phi}p_1 + \frac{\phi''}{\phi}.$$

Clearly the Riemann scheme of (5.14) is

$$(5.15)\qquad \begin{pmatrix} 0 & 1 & t & \lambda & \infty \\ -\frac{\theta_1}{2} & -\frac{\theta_2}{2} & -\frac{\theta_3}{2} & -1 & \frac{3-\theta_4}{2} \\ \frac{\theta_1}{2} & \frac{\theta_2}{2} & \frac{\theta_3}{2} & 1 & \frac{3+\theta_4}{2} \end{pmatrix},$$

and λ is still an apparent singularity of (5.14). As in [16], equation (5.14) is called the *normal form* of (3.8). By (5.15) it is easy to see that the normal form (5.14) has its monodromy group contained in $SL(2,\mathbb{C})$, which is an important advantage comparing to (3.8).

We proceed to the monodromy representation. Take the base point $x_0 = \frac{\wp(q_0)-e_1}{e_2-e_1} \notin \{0,1,t,\infty\}$ and let $\gamma_j \in \pi_1(\mathbb{C}\backslash\{0,1,t\},x_0)$ be a simple loop encircling the singular point 0 for $j=1$, 1 for $j=2$, t for $j=3$ respectively in the counterclockwise direction, and γ_4 be a simple loop around ∞ clockwise such that

$$\gamma_1\gamma_2\gamma_3 = \gamma_4^{-1} \quad \text{in } \pi_1(\mathbb{C}\backslash\{0,1,t\},x_0).$$

Of course we require that all these loops do not intersect except at the base point x_0. Let M_j be the monodromy matrix along the loop γ_j with respect to any fixed fundamental system of solutions $(F_1(x),F_2(x))$ of (5.14). Then $\det M_j = 1$, namely $M_j \in SL(2,\mathbb{C})$ for all j. Define

$$(5.16)\qquad \varkappa_1 := \mathrm{tr}(M_2M_3),\quad \varkappa_2 := \mathrm{tr}(M_1M_3),\quad \varkappa_3 := \mathrm{tr}(M_1M_2).$$

Then $\varkappa = (\varkappa_1,\varkappa_2,\varkappa_3) \in \mathbb{C}^3$ is *independent* of the choice of solutions, and is referred to as *global monodromy data* of (3.8) (or (5.14)) in [16]. Clearly $\varkappa_j = \varkappa_j(\theta,\lambda,\mu,t)$ is uniquely determined by equation (3.8) itself and so is a function of (θ,λ,μ,t) for all j. Then each Bäcklund transformation $\kappa \in W(D_4^{(1)})$ induces a transformation (still denoted by κ) from $\mathbb{C}^3$ to $\mathbb{C}^3$:

$$(5.17)\qquad \kappa(\varkappa_j) := \varkappa_j(\kappa(\theta),\kappa(\lambda),\kappa(\mu),t),\quad j=1,2,3.$$

We recall an important result from [16]; see also [2] for a different proof.

Theorem 5.A ([16, 2])**.** *The global monodromy data $\varkappa = (\varkappa_1,\varkappa_2,\varkappa_3)$ is invariant under the Bäcklund transformations $W(D_4^{(1)})$. Namely for any Bäcklund transformation $\kappa \in W(D_4^{(1)})$, $\kappa(\varkappa_j) = \varkappa_j$ for $j=1,2,3$.*

Theorem 5.A can be also applied to GLE$(\mathbf{n},p,A,\tau)$. Consider transformations as in [4]

$$(5.18)\qquad x = \frac{\wp(z)-e_1}{e_2-e_1},\quad t = \frac{e_3-e_1}{e_2-e_1},\quad \lambda = \frac{\wp(p)-e_1}{e_2-e_1},$$

and

$$(x-\lambda)^{-\frac{1}{2}}x^{-\frac{n_1}{2}}(x-1)^{-\frac{n_2}{2}}(x-t)^{-\frac{n_3}{2}}f(x)=y(z). \tag{5.19}$$

Then $y(z)$ solves GLE$(\mathbf{n},p,A,\tau)$ if and only if $f(x)$ satisfies the Fuchsian ODE (3.8) on $\mathbb{CP}^1$ with parameter $\theta=\theta^{\mathbf{n}}$, where μ in (3.11) is given by

$$\mu=\frac{1}{8}\frac{\mathfrak{p}'(\lambda)}{\mathfrak{p}(\lambda)}+\frac{A\wp'(p)}{(e_2-e_1)^2\mathfrak{p}(\lambda)}+\frac{n_1}{2\lambda}+\frac{n_2}{2(\lambda-1)}+\frac{n_3}{2(\lambda-t)}, \tag{5.20}$$

$$\text{where} \quad \mathfrak{p}(\lambda)=4\lambda(\lambda-1)(\lambda-t), \tag{5.21}$$

and $K=K(\lambda,\mu,t)$ is given by (3.5). Note that $\pm p\notin E_\tau[2]$ are apparent singularities of GLE$(\mathbf{n},p,A,\tau)$ is equivalent to that $\lambda\notin\{0,1,t,\infty\}$ is an apparent singularity of (3.8). See [4, Theorem 4.1] for the proof.

By (5.4), (5.13) and (5.19), we let

$$y(z)=\psi(x)F(x) \;\text{ with }\; \psi(x)=(x-\lambda)^{\frac{1}{2}}x^{\frac{1}{4}}(x-1)^{\frac{1}{4}}(x-t)^{\frac{1}{4}}. \tag{5.22}$$

Then the above argument shows that $y(z)$ is a solution to GLE$(\mathbf{n},p,A,\tau)$ if and only if $F(x)$ satisfies the normal form (5.14).

Remark 5.5. Recall the definition of $\gamma_j\in\pi_1(\mathbb{C}\backslash\{0,1,t\},x_0)$. Under the transformation (5.18), it is easy to see that the fundamental cycle ℓ_1 (resp. ℓ_2) of E_τ is mapped to a simple loop in $\pi_1(\mathbb{C}\backslash\{0,1,t\},x_0)$ which separates $\{1,t\}$ from $\{0,\infty\}$ (resp. separates $\{0,t\}$ from $\{1,\infty\}$), so (ℓ_1,ℓ_2) must be mapped to one of

$$(\gamma_2^{-1}\gamma_3^{-1},\gamma_1\gamma_3),(\gamma_3\gamma_2,\gamma_3^{-1}\gamma_1^{-1}),(\gamma_2\gamma_3,\gamma_3\gamma_1),(\gamma_3^{-1}\gamma_2^{-1},\gamma_1^{-1}\gamma_3^{-1}).$$

In this paper, by letting the base point q_0 lie inside the parallelogram with vertices $\{0,\frac{-\omega_1}{2},\frac{-\omega_2}{2},\frac{-\omega_3}{2}\}$, we can always assume that (ℓ_1,ℓ_2) is mapped to $(\gamma_2^{-1}\gamma_3^{-1},\,\gamma_1\gamma_3)$.

Recalling the global monodromy data $\varkappa=(\varkappa_1,\varkappa_2,\varkappa_3)$ of the normal form (5.14), we have the following important result.

Lemma 5.6.

$$\begin{aligned} tr\rho(\ell_1)&=-tr(M_2M_3)=-\varkappa_1,\\ tr\rho(\ell_2)&=-tr(M_1M_3)=-\varkappa_2,\\ tr(\rho(\ell_1)^{-1}\rho(\ell_2))&=-tr(M_1M_2)=-\varkappa_3. \end{aligned}$$

Proof. Let $(y_1(z), y_2(z))$ be any fundamental system of solutions to GLE($\mathbf{n}$, p, A, τ). Define a fundamental system of solutions $(F_1(x), F_2(x))$ of (5.14) via $(y_1(z), y_2(z))$ and (5.22). Recall the notation $N_j = \rho(\ell_j)$. Under the transformation (5.18), it follows from Remark 5.5 that (ℓ_1, ℓ_2) is mapped to $(\gamma_2^{-1}\gamma_3^{-1}, \gamma_1\gamma_3)$. Then

$$N_1 \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \ell_1^* \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \left(\gamma_2^{-1}\gamma_3^{-1}\right)^* \psi(x) \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$$
$$= -\psi(x) M_2^{-1} M_3^{-1} \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = -M_2^{-1} M_3^{-1} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},$$

and similarly,

$$N_2 \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = \ell_2^* \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = -M_1 M_3 \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix},$$

where the minus sign comes from the analytic continuation of $\psi(x)$. Therefore, $N_1 = -M_2^{-1}M_3^{-1}$ and $N_2 = -M_1M_3$. Since $M_j \in SL(2, \mathbb{C})$, we have

$$\operatorname{tr}(M_2^{-1}M_3^{-1}) = \operatorname{tr}((M_2M_3)^{-1}) = \operatorname{tr}(M_2M_3) = \varkappa_1,$$

which proves $\operatorname{tr}N_1 = -\varkappa_1$ and similarly $\operatorname{tr}N_2 = -\operatorname{tr}(M_1M_3) = -\varkappa_2$.

On the other hand, recall (5.4) that $\theta_j = n_j + \frac{1}{2}$ with $n_j \in \mathbb{Z}_{\geq 0}$ for $j = 1, 2, 3$, so (5.15) implies the existence of inverse matrices P_j such that

$$M_j = P_j^{-1} \begin{pmatrix} e^{-\pi i\theta_j} & 0 \\ 0 & e^{\pi i\theta_j} \end{pmatrix} P_j = (-1)^{n_j} P_j^{-1} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} P_j,$$

which infers $M_j^2 = -I_2$. Therefore,

$$\operatorname{tr}(N_1^{-1}N_2) = \operatorname{tr}(M_3M_2M_1M_3) = \operatorname{tr}(M_3^2M_2M_1)$$
$$= -\operatorname{tr}(M_2M_1) = -\operatorname{tr}(M_1M_2) = -\varkappa_3.$$

The proof is complete. □

We are in the position to prove Theorems 5.3–5.4.

Proof of Theorems 5.3–5.4. First, the assertions (2) of these two theorems follow directly from Theorem 4.1-(ii), (4.38) and (5.9)–(5.12) (i.e. the invertibility of the Bäcklund transformation implies the invertibility of the associated rational map).

Suppose $p^{\mathbf{n}}(\tau)$ is a solution of the elliptic form (1.6) with parameter (5.1), and $p^{\mathbf{0}}(\tau)$ is the corresponding solution of the elliptic form (4.1) such that under the Bäcklund transformation $\kappa^{\mathbf{n}}$, $p^{\mathbf{0}}(\tau)$ is transformed to $p^{\mathbf{n}}(\tau)$. By Theorem 5.A and Lemma 5.6, the associated GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$ and GLE$(\mathbf{0}, p^{\mathbf{0}}(\tau), A^{\mathbf{0}}(\tau), \tau)$ have the same $(\mathrm{tr}\rho(\ell_1), \mathrm{tr}\rho(\ell_2))$. Together with Corollary 2.5, we conclude that $p^{\mathbf{n}}(\tau)$ is a completely reducible solution (resp. not completely reducible) if and only if $p^{\mathbf{0}}(\tau)$ is a completely reducible solution (resp. not completely reducible).

Now we prove Theorem 5.3-(1). Let $p^{\mathbf{n}}(\tau)$ be a completely reducible solution, then so does $p^{\mathbf{0}}(\tau)$. Applying Theorem 4.1, there exists $(r,s) \in \mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ such that $p^{\mathbf{0}}(\tau) = p^{\mathbf{0}}_{r,s}(\tau)$ and the monodromy of GLE$(\mathbf{0}, p^{\mathbf{0}}(\tau),\ A^{\mathbf{0}}(\tau), \tau)$ satisfies (2.13), which implies

$$(\mathrm{tr}\rho(\ell_1),\ \mathrm{tr}\rho(\ell_2),\ \mathrm{tr}(\rho(\ell_1)^{-1}\rho(\ell_2))) \\ = (2\cos 2\pi s,\ 2\cos 2\pi r,\ 2\cos 2\pi(r+s)). \tag{5.23}$$

Thus, $p^{\mathbf{n}}(\tau) = p^{\mathbf{n}}_{r,s}(\tau)$ and Lemma 5.6 implies that (5.23) holds for GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$. Consequently, it follows from Theorem 2.3 and (2.11)–(2.12) that the monodromy of GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$ satisfies (2.13). This proves Theorem 5.3-(1).

Finally, we prove Theorem 5.4-(1). Let $p^{\mathbf{n}}(\tau)$ be a not completely reducible solution, then so does $p^{\mathbf{0}}(\tau)$. By Theorem 4.2 and Propositions 4.3–4.4, there exist $k \in \{0,1,2,3\}$ and $\mathcal{C} \in \mathbb{C}\cup\{\infty\}$ such that $p^{\mathbf{0}}(\tau) = p^{\mathbf{0}}_{k,\mathcal{C}}(\tau)$ and

$$(\mathrm{tr}\rho(\ell_1),\ \mathrm{tr}\rho(\ell_2)) = (2\varepsilon_{k,1},\ 2\varepsilon_{k,2}), \tag{5.24}$$

for GLE$(\mathbf{0}, p^{\mathbf{0}}(\tau), A^{\mathbf{0}}(\tau), \tau)$. Thus $p^{\mathbf{n}}(\tau) = p^{\mathbf{n}}_{k,\mathcal{C}}(\tau)$ and (5.24) holds for GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$.

It remains to prove that the monodromy of GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A^{\mathbf{n}}(\tau), \tau)$ satisfies (4.36)–(4.37), i.e. the global monodromy data is precisely $(2\varepsilon_{k,1}, 2\varepsilon_{k,2}, \mathcal{C})$. Note that we only need to prove this assertion for *some* τ because of the isomonodromic deformation. We take $k=1$ and $\mathcal{C} \neq \infty$ for example, and all the other cases can be proved in the same way. By Theorem 4.5 and (5.6)–(5.10), we easily obtain

$$\wp(p^{\mathbf{n}}_{1,\mathcal{C}}(\tau)|\tau) = \lim_{s\to 0}\wp(p^{\mathbf{n}}_{\frac{1}{2}-\mathcal{C}s,s}(\tau)|\tau),$$
$$\mu^{\mathbf{n}}_{1,\mathcal{C}}(t) = \lim_{s\to 0}\mu^{\mathbf{n}}_{\frac{1}{2}-\mathcal{C}s,s}(t).$$

Fix any τ such that $p^{\mathbf{n}}_{1,\mathcal{C}}(\tau) \notin E_\tau[2]$. By Remark 3.1 we may assume

$$p^{\mathbf{n}}_{1,\mathcal{C}}(\tau) = \lim_{s\to 0} p^{\mathbf{n}}_{\frac{1}{2}-\mathcal{C}s,s}(\tau)$$

and then it follows from (5.20) that the corresponding

$$A^{\mathbf{n}}_{1,\mathcal{C}}(\tau) = \lim_{s\to 0} A^{\mathbf{n}}_{\frac{1}{2}-\mathcal{C}s,s}(\tau).$$

In the rest of the proof, we omit $\mathbf{n}$, τ in the notations for convenience. Thus the associated GLE$(\mathbf{n}, p_{1,\mathcal{C}}, A_{1,\mathcal{C}})$ is a limit of GLE$(\mathbf{n}, p_{\frac{1}{2}-\mathcal{C}s,s}, A_{\frac{1}{2}-\mathcal{C}s,s})$. Denote by $\Phi_e(z)$ and $\Phi_{e,s}(z)$ respectively, to be their corresponding unique even elliptic solution stated in Theorem 2.A. Then

$$\Phi_e(z) = \lim_{s\to 0} \Phi_{e,s}(z). \tag{5.25}$$

Recall Theorem 2.4 that

$$\chi_j := \int_z^{z+\omega_j} \frac{1}{\Phi_e(\xi)} d\xi \neq \infty, \quad j = 1, 2 \tag{5.26}$$

are well-defined and independent of z. We claim that

$$\chi_2/\chi_1 = \mathcal{C}. \tag{5.27}$$

Once (5.27) is proved, then Theorem 2.4 and (5.24) imply that the monodromy of GLE$(\mathbf{n}, p_{1,\mathcal{C}}, A_{1,\mathcal{C}})$ satisfies (4.36)–(4.37) with $k = 1$, hence completes the proof of Theorem 5.4-(1).

To prove (5.27), we apply Theorem 2.3 and Theorem 4.1-(i) to GLE$(\mathbf{n}, p_{\frac{1}{2}-\mathcal{C}s,s}, A_{\frac{1}{2}-\mathcal{C}s,s})$ and denote the corresponding $y_{\pm\boldsymbol{a}}(z)$ by $y_{\pm\boldsymbol{a}(s)}(z)$, which gives

$$\ell_1^* \begin{pmatrix} y_{\boldsymbol{a}(s)}(z) \\ y_{-\boldsymbol{a}(s)}(z) \end{pmatrix} = \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix} \begin{pmatrix} y_{\boldsymbol{a}(s)}(z) \\ y_{-\boldsymbol{a}(s)}(z) \end{pmatrix},$$
$$\ell_2^* \begin{pmatrix} y_{\boldsymbol{a}(s)}(z) \\ y_{-\boldsymbol{a}(s)}(z) \end{pmatrix} = \begin{pmatrix} e^{2\pi i(\frac{1}{2}-\mathcal{C}s)} & 0 \\ 0 & e^{-2\pi i(\frac{1}{2}-\mathcal{C}s)} \end{pmatrix} \begin{pmatrix} y_{\boldsymbol{a}(s)}(z) \\ y_{-\boldsymbol{a}(s)}(z) \end{pmatrix}.$$

By (2.10) there exists a nonzero constant $c(s)$ such that

$$\Phi_{e,s}(z) = c(s) y_{\boldsymbol{a}(s)}(z) y_{-\boldsymbol{a}(s)}(z).$$

It follows from (5.25) that up to a subsequence, $\lim_{s\to 0} c(s) = c_0 \notin \{0, \infty\}$. Let

$$W(s) := y'_{\boldsymbol{a}(s)}(z) y_{-\boldsymbol{a}(s)}(z) - y_{\boldsymbol{a}(s)}(z) y'_{-\boldsymbol{a}(s)}(z)$$

be the Wronskian, which is a nonzero constant independent of z. Since GLE($\mathbf{n}$, $p_{\frac{1}{2}-\mathcal{C}s,s}, A_{\frac{1}{2}-\mathcal{C}s,s}$) converges to GLE($\mathbf{n}, p_{1,\mathcal{C}}, A_{1,\mathcal{C}}$) whose monodromy is not completely reducible, we have

$$\lim_{s\to 0} W(s) = 0. \tag{5.28}$$

Define

$$f_s(z) := \frac{y_{\boldsymbol{a}(s)}(z)}{y_{-\boldsymbol{a}(s)}(z)}.$$

Then $f_s(z)$ has no branch points and hence single-valued in $\mathbb{C}$, which satisfies

$$f_s(z+1) = e^{-4\pi i s} f_s(z), \quad f_s(z+\tau) = e^{4\pi i(\frac{1}{2}-\mathcal{C}s)} f_s(z).$$

Furthermore, a direct computation gives

$$\frac{d}{dz}\ln f_s(z) = \frac{c(s)W(s)}{\Phi_{e,s}(z)},$$

and so

$$e^{-4\pi i s} = \frac{f_s(z+1)}{f_s(z)} = \exp\left(c(s)W(s)\int_z^{z+1}\frac{1}{\Phi_{e,s}(\xi)}d\xi\right),$$
$$e^{4\pi i(\frac{1}{2}-\mathcal{C}s)} = \frac{f_s(z+\tau)}{f_s(z)} = \exp\left(c(s)W(s)\int_z^{z+\tau}\frac{1}{\Phi_{e,s}(\xi)}d\xi\right).$$

Therefore, there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$\int_z^{z+1}\frac{1}{\Phi_{e,s}(\xi)}d\xi = \frac{-4\pi i s + 2\pi i m_1}{c(s)W(s)},$$
$$\int_z^{z+\tau}\frac{1}{\Phi_{e,s}(\xi)}d\xi = \frac{4\pi i(\frac{1}{2}-\mathcal{C}s) + 2\pi i m_2}{c(s)W(s)}.$$

Together with (5.25)–(5.26), we have

$$\lim_{s\to 0}\frac{-4\pi i s + 2\pi i m_1}{c(s)W(s)} = \chi_1, \quad \lim_{s\to 0}\frac{4\pi i(\frac{1}{2}-\mathcal{C}s) + 2\pi i m_2}{c(s)W(s)} = \chi_2.$$

This, together with $\lim_{s\to 0} c(s) = c_0 \notin \{0,\infty\}$ and (5.28), yields $(m_1, m_2) = (0,-1)$ and so

$$\frac{\chi_2}{\chi_1} = \lim_{s\to 0}\frac{4\pi i(\frac{1}{2}-\mathcal{C}s) - 2\pi i}{-4\pi i s} = \mathcal{C}.$$

This proves (5.27). The proof is complete. □

6. Proofs of Theorem 1.3

This section is devoted to proving Theorem 1.3. In this section, we denote $N = \sum_k n_k + 1$. First we prove the uniqueness of $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ with respect to the monodromy data.

Proof of Theorem 1.3-(1). Fix $\mathbf{n}$ and τ_0. Suppose $\mathrm{GLE}(\mathbf{n}, p_j, A_j, \tau_0)$, $j = 1, 2$, have the same global monodromy data. Let $(p_j^{\mathbf{n}}(\tau), A_j^{\mathbf{n}}(\tau))$ be the solution of the Hamiltonian system (1.8) with initial data $(p_j^{\mathbf{n}}(\tau_0), A_j^{\mathbf{n}}(\tau_0)) = (p_j, A_j)$, $j = 1, 2$. Then $p_j^{\mathbf{n}}(\tau)$ are solutions of the elliptic form (1.6) with parameter (5.1). There are two cases.

Case 1. The monodromies of $\mathrm{GLE}(\mathbf{n}, p_j, A_j, \tau_0)$ are completely reducible with the same global monodromy data $(r_j, s_j) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$ with $(r_1, s_1) \sim (r_2, s_2)$. Then Theorem 5.3 implies $p_j^{\mathbf{n}}(\tau) = p_{r_j,s_j}^{\mathbf{n}}(\tau)$ and hence $\wp(p_1^{\mathbf{n}}(\tau)|\tau) \equiv \wp(p_2^{\mathbf{n}}(\tau)|\tau)$. In a small neighborhood U of τ_0 we may assume $p_1^{\mathbf{n}}(\tau) = \pm p_2^{\mathbf{n}}(\tau) + m_1 + m_2\tau$ for some $m_j \in \mathbb{Z}$. Then it follows from the first equation of the Hamiltonian system (1.8) that $A_1^{\mathbf{n}}(\tau) = \pm A_2^{\mathbf{n}}(\tau)$ for $\tau \in U$. In particular, these hold for τ_0 and we conclude from (1.5) that $\mathrm{GLE}(\mathbf{n}, p_1, A_1, \tau_0) = \mathrm{GLE}(\mathbf{n}, p_2, A_2, \tau_0)$.

Case 2. The monodromies of $\mathrm{GLE}(\mathbf{n}, p_j, A_j, \tau_0)$ are not completely reducible with the same global monodromy data $(2\varepsilon_{k,1}, 2\varepsilon_{k,2}, \mathcal{C})$. Thanks to Theorem 5.4, the same argument as Case 1 implies $\mathrm{GLE}(\mathbf{n}, p_1, A_1, \tau_0) = \mathrm{GLE}(\mathbf{n}, p_2, A_2, \tau_0)$. □

To prove Theorem 1.3 for $\mathrm{H}(\mathbf{n}, B, \tau)$, we need to apply the relation between $\mathrm{H}(\mathbf{n}, B, \tau)$ and $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ studied in [4].

Fix any $\tau_0 \in \mathbb{H}$ and $c_0^2 \in \{\pm i\frac{2n_0+1}{2\pi}\}$. Then for any $h \in \mathbb{C}$, it was proved in [4] that there exists a solution $p_h^{\mathbf{n}}(\tau)$ of the elliptic form (1.6) with parameters (5.1) satisfying the following asymptotic behavior

$$p_h^{\mathbf{n}}(\tau) = c_0(\tau - \tau_0)^{\frac{1}{2}}(1 + h(\tau - \tau_0) + O(\tau - \tau_0)^2) \text{ as } \tau \to \tau_0. \tag{6.1}$$

Recall Remark 3.1 that we identify the solutions $p_h^{\mathbf{n}}(\tau)$ and $-p_h^{\mathbf{n}}(\tau)$, so (6.1) gives two 1-parameter families (one family is given by $c_0^2 = i\frac{2n_0+1}{2\pi}$ and the other by $c_0^2 = -i\frac{2n_0+1}{2\pi}$) of solutions of the elliptic form (1.6) satisfying $p_h^{\mathbf{n}}(\tau) \to 0$ as $\tau \to \tau_0$. Moreover, these two 1-parameter families of solutions give all solutions $p^{\mathbf{n}}(\tau)$ of the elliptic form (1.6) such that $p^{\mathbf{n}}(\tau_0) = 0$. See [4, Section 3] for the proof.

By using (6.1), we proved that the associated $\mathrm{GLE}(\mathbf{n}, p_h^{\mathbf{n}}(\tau), A(\tau), \tau)$ converges to either $\mathrm{H}(\mathbf{n}^+, B_0, \tau_0)$ or $\mathrm{H}(\mathbf{n}^-, B_0, \tau_0)$ for some $B_0 \in \mathbb{C}$ as $\tau \to \tau_0$ where $\mathbf{n}^\pm = (n_0 \pm 1, n_1, n_2, n_3)$. More precisely, we have

Theorem 6.1 ([4]). *Let $\tau_0 \in \mathbb{H}$ and $p^{\mathbf{n}}(\tau)$ be a solution of the elliptic form (1.6) with parameters (5.1) such that $p^{\mathbf{n}}(\tau_0) = 0$. Then $p^{\mathbf{n}}(\tau) = \pm p^{\mathbf{n}}_h(\tau)$ for some $h \in \mathbb{C}$. Furthermore, the associated GLE$(\mathbf{n}, p^{\mathbf{n}}(\tau), A(\tau), \tau)$ converges to either $H(\mathbf{n}^+, B_0, \tau_0)$ if $c_0^2 = -i\frac{2n_0+1}{2\pi}$ or $H(\mathbf{n}^-, B_0, \tau_0)$ if $c_0^2 = i\frac{2n_0+1}{2\pi}$. Here*

$$B_0 = 2\pi i c_0^2 (4\pi i h - \eta_1(\tau_0)) - \sum_{k=1}^{3} n_k(n_k+1)e_k(\tau_0). \tag{6.2}$$

Proof of Theorem 1.3-(2). Fix $\mathbf{n}$ and τ_0. Suppose $\mathrm{H}(\mathbf{n}, B_j, \tau_0)$, $j = 1, 2$, have the same global monodromy data. Our goal is to prove $B_1 = B_2$.

Let $\mathbf{n}^+ = (n_0+1, n_1, n_2, n_3)$ and $c_0^2 = i\frac{2(n_0+1)+1}{2\pi}$. Define h_j, $j = 1, 2$, by (6.2) by replacing B_0 with B_j and consider the solutions $p^{\mathbf{n}^+}_{h_j}(\tau)$. By Theorem 6.1, the associated $\mathrm{GLE}(\mathbf{n}^+, p^{\mathbf{n}^+}_{h_j}(\tau), A^{\mathbf{n}^+}_{h_j}(\tau), \tau)$ converges to $\mathrm{H}(\mathbf{n}, B_j, \tau_0)$ as $\tau \to \tau_0$. The key step is to show that

(6.3) *the global monodromy data of GLE$(\mathbf{n}^+, p^{\mathbf{n}^+}_{h_j}(\tau), A^{\mathbf{n}^+}_{h_j}(\tau), \tau)$ and $H(\mathbf{n}, B_j, \tau_0)$ are the same.*

Once (6.3) is proved, then $\mathrm{GLE}(\mathbf{n}^+, p^{\mathbf{n}^+}_{h_j}(\tau), A^{\mathbf{n}^+}_{h_j}(\tau), \tau)$, $j = 1, 2$, have the same global monodromy data and so Theorem 1.3-(1) yields that these two GLEs coincide, i.e. $\wp(p^{\mathbf{n}^+}_{h_1}(\tau)|\tau) \equiv \wp(p^{\mathbf{n}^+}_{h_2}(\tau)|\tau)$. From here and $p^{\mathbf{n}^+}_{h_j}(\tau_0) = 0$ for $j = 1, 2$, we obtain $p^{\mathbf{n}^+}_{h_1}(\tau) = \pm p^{\mathbf{n}^+}_{h_2}(\tau)$ near τ_0. This implies $h_1 = h_2$ and so $B_1 = B_2$.

We only need to prove (6.3) for $j = 1$ and in the following proof we write $(p^{\mathbf{n}^+}_{h_1}(\tau), A^{\mathbf{n}^+}_{h_1}(\tau)) = (p(\tau), A(\tau))$ for convenience.

Case 1. $p(\tau) = p^{\mathbf{n}^+}_{r,s}(\tau)$ for some $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ is a completely reducible solution, i.e. the global monodromy data of $\mathrm{GLE}(\mathbf{n}^+, p(\tau), A(\tau), \tau)$ is (r, s).

Denote $\hat{N} = \sum n_k + 2$. Then by Theorem 2.3 and (2.11)–(2.12), there exists $\boldsymbol{a}(\tau) = (a_1(\tau), \cdots, a_{\hat{N}}(\tau))$ satisfying

$$\sum_{i=1}^{\hat{N}} a_i(\tau) - \sum_{k=1}^{3} \frac{n_k\omega_k}{2} = r + s\tau \tag{6.4}$$

such that

$$y_{\boldsymbol{a}(\tau)}(z) = \frac{e^{(r\eta_1(\tau)+s\eta_2(\tau))z} \prod_{i=1}^{\hat{N}} \sigma(z - a_i(\tau)|\tau)}{\sigma(z|\tau)^{n_0+2} \prod_{k=1}^{3} \sigma(z - \frac{\omega_k}{2}|\tau)^{n_k}} \times \frac{\sigma(z|\tau)}{\sqrt{\sigma(z - p(\tau)|\tau)\sigma(z + p(\tau)|\tau)}} \tag{6.5}$$

is a solution of GLE($\mathbf{n}^+, p(\tau), A(\tau), \tau$). By passing a subsequence, we may assume

$$\lim_{\tau\to\tau_0} \boldsymbol{a}(\tau) = \boldsymbol{a} = (a_1, \cdots, a_{\hat{N}}) \in E_\tau^{\hat{N}}. \tag{6.6}$$

Then

$$\sum_{i=1}^{\hat{N}} a_i - \sum_{k=1}^{3} \frac{n_k \omega_k(\tau_0)}{2} = r + s\tau_0, \tag{6.7}$$

and $p(\tau) \to p(\tau_0) = 0$ implies that

$$y_{\boldsymbol{a}}(z) := \frac{e^{(r\eta_1(\tau_0)+s\eta_2(\tau_0))z} \prod_{i=1}^{\hat{N}} \sigma(z-a_i|\tau_0)}{\sigma(z|\tau_0)^{n_0+2} \prod_{k=1}^{3} \sigma(z-\frac{\omega_k}{2}|\tau_0)^{n_k}} \tag{6.8}$$

is a solution of H($\mathbf{n}, B_1, \tau$). Note that two of $a_1, \cdots, a_{\hat{N}}$ must be 0 since the local exponents of H($\mathbf{n}, B_1, \tau$) at 0 are $-n_0, n_0+1$. By (6.7)–(6.8) and the transformation law (2.8), we immediately obtain that with respect to $y_{\mathbf{a}}(z)$ and $y_{-\mathbf{a}}(z)$, the monodromy matrices $\rho(\ell_j), j = 1, 2$, are exactly (2.13). This proves that the global monodromy data of H($\mathbf{n}, B_1, \tau$) is also the same (r, s) as that of GLE($\mathbf{n}^+, p(\tau), A(\tau), \tau$).

Case 2. $p(\tau) = p_{k,\mathcal{C}}^{\mathbf{n}^+}(\tau)$ for some $k \in \{0,1,2,3\}$ and $\mathcal{C} \in \mathbb{C} \cup \{\infty\}$ is a not completely reducible solution, i.e. the global monodromy data of GLE($\mathbf{n}^+, p(\tau), A(\tau), \tau$) is $(2\varepsilon_{k,1}, 2\varepsilon_{k,2}, \mathcal{C})$.

Recalling Theorem 2.4 and (4.27), there exists $\boldsymbol{a}(\tau) = (a_1(\tau), \cdots, a_{\hat{N}}(\tau))$ satisfying (6.4) and

$$(r,s) \equiv \begin{cases} (0,0) \mod \mathbb{Z}^2 & \text{if } k=0, \\ (\frac{1}{2},0) \mod \mathbb{Z}^2 & \text{if } k=1, \\ (0,\frac{1}{2}) \mod \mathbb{Z}^2 & \text{if } k=2, \\ (\frac{1}{2},\frac{1}{2}) \mod \mathbb{Z}^2 & \text{if } k=3, \end{cases} \tag{6.9}$$

such that $y_{\boldsymbol{a}(\tau)}(z)$ given by (6.5) is a solution of GLE($\mathbf{n}^+, p(\tau), A(\tau), \tau$). As in Case 1, we may assume (6.6), then $y_{\boldsymbol{a}}(z)$ given by (6.8) is a solution of H($\mathbf{n}, B_1, \tau$). By (6.7), (6.9) and (2.8), we easily obtain

$$y_{\boldsymbol{a}}(z+\omega_j) = \varepsilon_{k,j} y_{\boldsymbol{a}}(z), \quad j = 1, 2.$$

Since the proof of Theorem 2.4 gives $\mathcal{C}=\frac{\int_z^{z+\omega_2} y_{\boldsymbol{a}(\tau)}(\xi)^{-2}d\xi}{\int_z^{z+\omega_1} y_{\boldsymbol{a}(\tau)}(\xi)^{-2}d\xi}$, it follows from $y_{\boldsymbol{a}(\tau)}(z)^{-2}\to y_{\boldsymbol{a}}(z)^{-2}$ that

$$\frac{\int_z^{z+\omega_2} y_{\boldsymbol{a}}(\xi)^{-2}d\xi}{\int_z^{z+\omega_1} y_{\boldsymbol{a}}(\xi)^{-2}d\xi}=\mathcal{C}.$$

Therefore, the global monodromy data of $\mathrm{H}(\mathbf{n},B_1,\tau_0)$ is $(2\varepsilon_{k,1},2\varepsilon_{k,2},\mathcal{C})$, again the same as that of $\mathrm{GLE}(\mathbf{n}^+,p(\tau),A(\tau),\tau)$.

The proof is complete. □

Proof of Theorem 1.3-(3). Fix any $\mathbf{n}$, τ_0 and $k\in\{0,1,2,3\}$. Suppose that the global monodromy datas of $\mathrm{H}(\mathbf{n},B_1,\tau_0)$ and $\mathrm{H}(\mathbf{n}_k,B_2,\tau_0)$ are the same for some $B_1,B_2\in\mathbb{C}$. By changing variable $z\to z+\frac{\omega_k}{2}$, we only need to consider the case $k=0$. Then (1.20) implies

$$\mathbf{n}_0^-=(n_0+1,n_1,n_2,n_3)=\mathbf{n}^+,\quad \text{i.e. } (\mathbf{n}^+)^+=\mathbf{n}_0. \tag{6.10}$$

Define h_1 by (let $c_0^2=i\frac{2n_0+3}{2\pi}$ and $B_0=B_1$ in (6.2))

$$B_1=-(2n_0+3)\left(4\pi i h_1-\eta_1(\tau_0)\right)-\sum_{k=1}^3 n_k(n_k+1)e_k(\tau_0),$$

and h_2 by (let $c_0^2=-i\frac{2n_0+3}{2\pi}$ and $B_0=B_2$ in (6.2))

$$B_2=(2n_0+3)\left(4\pi i h_2-\eta_1(\tau_0)\right)-\sum_{k=1}^3 n_k(n_k+1)e_k(\tau_0).$$

Then it follows from (6.1) that there exist solutions $p_{h_j}^{\mathbf{n}^+}$, $j=1,2$, satisfying

$$p_{h_1}^{\mathbf{n}^+}(\tau)=c_1(\tau-\tau_0)^{\frac12}(1+h_1(\tau-\tau_0)+O(\tau-\tau_0)^2)\text{ as }\tau\to\tau_0, \tag{6.11}$$

$$p_{h_2}^{\mathbf{n}^+}(\tau)=c_2(\tau-\tau_0)^{\frac12}(1+h_2(\tau-\tau_0)+O(\tau-\tau_0)^2)\text{ as }\tau\to\tau_0, \tag{6.12}$$

with $c_1^2=i\frac{2n_0+3}{2\pi}=-c_2^2$. In particular,

$$\wp(p_{h_1}^{\mathbf{n}^+}(\tau)|\tau)\neq\wp(p_{h_2}^{\mathbf{n}^+}(\tau)|\tau)\quad\text{for }\tau\to\tau_0. \tag{6.13}$$

On the other hand, it follows from (6.10)–(6.12) and Theorem 6.1 that the associated $\mathrm{GLE}(\mathbf{n}^+,p_{h_1}^{\mathbf{n}^+}(\tau),A_{h_1}^{\mathbf{n}^+}(\tau),\tau)$ converges to $\mathrm{H}(\mathbf{n},B_1,\tau_0)$ and $\mathrm{GLE}(\mathbf{n}^+,$ $p_{h_2}^{\mathbf{n}^+}(\tau),A_{h_2}^{\mathbf{n}^+}(\tau),\tau)$ converges to $\mathrm{H}(\mathbf{n}_0,B_2,\tau_0)$ as $\tau\to\tau_0$. Then the same proof as Theorem 1.3-(2) shows that $\mathrm{GLE}(\mathbf{n}^+, p_{h_1}^{\mathbf{n}^+}(\tau), A_{h_1}^{\mathbf{n}^+}(\tau), \tau)$ has the same global monodromy data as $\mathrm{H}(\mathbf{n}, B_1, \tau_0)$ and so do for $\mathrm{GLE}(\mathbf{n}^+, p_{h_2}^{\mathbf{n}^+}(\tau), A_{h_2}^{\mathbf{n}^+}(\tau), \tau)$, $\mathrm{H}(\mathbf{n}_0, B_2, \tau_0)$. Together with our assumption, we conclude that $\mathrm{GLE}(\mathbf{n}^+, p_{h_j}^{\mathbf{n}^+}(\tau), A_{h_j}^{\mathbf{n}^+}(\tau), \tau)$ has the same global monodromy data for $j = 1, 2$. Then it follows from Theorem 1.3-(1) that these two GLEs coincide, i.e. $\wp(p_{h_1}^{\mathbf{n}^+}(\tau)|\tau) \equiv \wp(p_{h_2}^{\mathbf{n}^+}(\tau)|\tau)$, a contradiction with (6.13).

The proof is complete. □

We want to emphasize that the same proof as (6.3) improves Theorem 6.1 as follows.

Theorem 6.2. *Under the same notations and assumptions as Theorem 6.1, GLE($\mathbf{n}$, $p^{\mathbf{n}}(\tau), A(\tau), \tau$) has the same global monodromy data with its limiting equation H($\mathbf{n}^+, B_0, \tau_0$) for $c_0^2 = -i\frac{2n_0+1}{2\pi}$ (resp. H($\mathbf{n}^-, B_0, \tau_0$) for $c_0^2 = i\frac{2n_0+1}{2\pi}$).*

7. Applications

In this section, we give an application of Theorem 1.3 to $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$. First we recall the basic theory of $\mathrm{GLE}(\mathbf{n}, p, A, \tau)$ from its hyperelliptic aspect in Part I [6].

Recall $\Phi_e(z)$ in Theorem 2.A. It follows from (2.1) that

$$Q_{\mathbf{n},p}(A) := \Phi_e'(z)^2 - 2\Phi_e''(z)\Phi_2(z) + 4I_{\mathbf{n}}(z; p, A, \tau)\Phi_e(z)^2$$

is a monic polynomial in A of degree $2g+2$ and independent of z. Since $\Phi_e(z) = y_1(z)y_2(z)$ (recall $y_2(z) = y_1(-z)$), it is known (cf. Part I [6, Theorem 2.7]) that the Wronskian W of $y_1(z)$ and $y_2(z)$ satisfies $W^2 = Q_{\mathbf{n},p}(A)$. Define the hyperelliptic curve $\Gamma_{\mathbf{n},p} = \Gamma_{\mathbf{n},p}(\tau)$ by

$$\Gamma_{\mathbf{n},p}(\tau) := \{(A, W)|W^2 = Q_{\mathbf{n},p}(A; \tau)\}. \tag{7.1}$$

Since $\deg_A Q_{\mathbf{n},p}(A;\tau)$ is even, the curve $\Gamma_{\mathbf{n},p}(\tau)$ has two points at infinity denoted by $\infty_\pm$, i.e. $\overline{\Gamma_{\mathbf{n},p}(\tau)} = \Gamma_{\mathbf{n},p}(\tau) \cup \{\infty_\pm\}$. Clearly $y_1(z)$ can be uniquely determined by the pair $(A, W) \in \Gamma_{\mathbf{n},p}(\tau)$ by considering the correspondence (note that $-W$ is the Wronskian of $y_2(z)$ and $y_1(z) = y_2(-z)$)

$$(y_1(z), y_2(z)) \leftrightarrow (A, W), \quad (y_2(z), y_1(z)) \leftrightarrow (A, -W).$$

Denote $N = \sum_{k=0}^3 n_k + 1$ in the sequel. Recall Section 2.2 that there is $\boldsymbol{a} = \{a_1, \cdots, a_N\}$ (unique mod Λ_τ) such that $y_1(z) = y_{\boldsymbol{a}}(z)$. Then we can define a map $i_{\mathbf{n},p} : \Gamma_{\mathbf{n},p} \to \mathrm{Sym}^N E_\tau$ by

$$i_{\mathbf{n},p}(A, W) := \{[a_1], \cdots, [a_N]\} \in \mathrm{Sym}^N E_\tau, \tag{7.2}$$

where $[a_i] := a_i \pmod{\Lambda_\tau} \in E_\tau$. Clearly this $i_{\mathbf{n},p}$ is well-defined. Furthermore, if $W \neq 0$, then we see from $y_2(z) = y_{-\mathbf{a}}(z)$ that

$$i_{\mathbf{n},p}(A, -W) = \{-[a_1], \cdots, -[a_N]\}. \tag{7.3}$$

We proved in Part I [6] that $i_{\mathbf{n},p}$ is an embedding from $\Gamma_{\mathbf{n},p}$ into $\mathrm{Sym}^N E_\tau$. Let $Y_{\mathbf{n},p}(\tau)$ be the image of $\Gamma_{\mathbf{n},p}(\tau)$ in $\mathrm{Sym}^N E_\tau$ under $i_{\mathbf{n},p}$, i.e.

$$Y_{\mathbf{n},p}(\tau) = \left\{ \begin{array}{c} [\boldsymbol{a}] = \{[a_1], \cdots, [a_N]\} \ \in \mathrm{Sym}^N E_\tau |\ y_{\boldsymbol{a}}(z) \text{ defined in} \\ (2.7) \text{ is a solution of } \mathrm{GLE}(\mathbf{n}, p, A, \tau) \text{ for some } A \end{array} \right\}, \tag{7.4}$$

and define the addition map $\sigma_{\mathbf{n},p} : Y_{\mathbf{n},p}(\tau) \to E_\tau$ by

$$\sigma_{\mathbf{n},p}([\boldsymbol{a}]) := \sum_{i=1}^{N} [a_i] - \sum_{k=1}^{3} [\tfrac{n_k \omega_k}{2}]. \tag{7.5}$$

Clearly

$$\sigma_{\mathbf{n},p}([-\boldsymbol{a}]) = -\sum_{i=1}^{N} [a_i] - \sum_{k=1}^{3} [\tfrac{n_k \omega_k}{2}] = -\sigma_{\mathbf{n},p}([\boldsymbol{a}]).$$

Furthermore, the degree $\deg \sigma_{\mathbf{n},p} = \#\sigma_{\mathbf{n},p}^{-1}(z), z \in E_\tau$, is well-defined and

$$\deg \sigma_{\mathbf{n},p} = \sum_{k=0}^{3} n_k(n_k+1) + 1.$$

Besides,

$$\overline{Y_{\mathbf{n},p}(\tau)} = Y_{\mathbf{n},p}(\tau) \cup \{\infty_+(p), \infty_-(p)\},$$

where

$$\infty_\pm(p) := \Big(\overbrace{0, \cdots, 0}^{n_0}, \overbrace{\tfrac{\omega_1}{2}, \cdots, \tfrac{\omega_1}{2}}^{n_1}, \overbrace{\tfrac{\omega_2}{2}, \cdots, \tfrac{\omega_2}{2}}^{n_2}, \overbrace{\tfrac{\omega_3}{2}, \cdots, \tfrac{\omega_3}{2}}^{n_3}, \pm p \Big).$$

The above theories can be found in Part I [6].

Let $K(E_\tau)$ and $K(\overline{Y_{\mathbf{n},p}(\tau)})$ be the field of rational functions of E_τ and $\overline{Y_{\mathbf{n},p}(\tau)}$, respectively. Then $K(\overline{Y_{\mathbf{n},p}(\tau)})$ is a finite extension over $K(E_\tau)$ and

$$\Big[K(\overline{Y_{\mathbf{n},p}(\tau)}) : K(E_\tau) \Big] = \deg \sigma_{\mathbf{n},p} = \sum_{k=0}^{3} n_k(n_k+1) + 1. \tag{7.6}$$

In this section, we consider the basic question *what a primitive generator of this field extension is.* Motivated by (2.14)–(2.15), we define

$$\mathbf{z}_{\mathbf{n},p}(a_1,\cdots,a_N) :=\zeta\left(\sum_{i=1}^{N} a_i - \sum_{k=1}^{3}\frac{n_k\omega_k}{2}\right) - \frac{1}{2}\sum_{i=1}^{N}(\zeta(a_i+p)+\zeta(a_i-p)) + \sum_{k=1}^{3}\frac{n_k\eta_k}{2}, \tag{7.7}$$

which is meromorphic and periodic in each a_i and hence defines a rational function on E_τ^N. By symmetry, it descends to a rational function on $\mathrm{Sym}^N E_\tau$. We denote the restriction $\mathbf{z}_{\mathbf{n},p}|_{\overline{Y_{\mathbf{n},p}(\tau)}}$ also by $\mathbf{z}_{\mathbf{n},p}$, which is a rational function on $\overline{Y_{\mathbf{n},p}(\tau)}$. Here as an application of Theorem 1.3, we can prove that $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a})$ is a primitive generator. The same statement as the following result was proved in [21] for the Lamé equation and later generalized to $\mathrm{H}(\mathbf{n},B,\tau)$ in Part II [7].

Theorem 7.1. *$\mathbf{z}_{\mathbf{n},p}$ is a primitive generator of the finite extension of rational function field $K(\overline{Y_{\mathbf{n},p}(\tau)})$ over $K(E_\tau)$, i.e. the minimal polynomial $W_{\mathbf{n},p}(\mathbf{z}) \in K(E_\tau)[\mathbf{z}]$ of $\mathbf{z}_{\mathbf{n},p}$ satisfies $\deg W_{\mathbf{n},p} = \deg \sigma_{\mathbf{n},p}$.*

Proof. Since $\mathbf{z}_{\mathbf{n},p} \in K(\overline{Y_{\mathbf{n},p}(\tau)})$, its minimal polynomial $W_{\mathbf{n},p}(\mathbf{z}) \in K(E_\tau)[\mathbf{z}] = \mathbb{C}(\wp(\sigma),\wp'(\sigma))[\mathbf{z}]$ exists with degree $d_{\mathbf{n},p} := \deg W_{\mathbf{n},p} | \deg \sigma_{\mathbf{n},p}$ by (7.6).

Note that if $\boldsymbol{a} = -\boldsymbol{a}$, then $\sigma_{\mathbf{n},p}(\boldsymbol{a}) \in E_\tau[2]$. To prove $d_{\mathbf{n},p} = \deg \sigma_{\mathbf{n},p}$, i.e. $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a})$ is a primitive generator, we take $\sigma_0 \in E_\tau \backslash E_\tau[2]$ outside the branch loci of $\sigma_{\mathbf{n},p} : \overline{Y_{\mathbf{n},p}(\tau)} \to E_\tau$ such that there are precisely $\deg \sigma_{\mathbf{n},p}$ different points $\boldsymbol{a}^k \in Y_{\mathbf{n},p}(\tau)$ satisfying $\sigma_{\mathbf{n},p}(\boldsymbol{a}^k) = \sigma_0$ and $\pm[p] \notin \boldsymbol{a}^k$ for $1 \le k \le \deg \sigma_{\mathbf{n},p}$. We claim that

$$\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^{k_1}) \neq \mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^{k_2}), \quad \forall k_1 \neq k_2. \tag{7.8}$$

Suppose for some $k_1 \neq k_2$ we have $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^{k_1}) = \mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^{k_2})$. Then we can take $(a_1,\cdots,a_N)$, $(b_1,\cdots,b_N) \in \mathbb{C}^N$ to be representatives of $\boldsymbol{a}^{k_1}, \boldsymbol{a}^{k_2}$ such that

$$\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i,\ \sum_{i=1}^{N}(\zeta(a_i+p)+\zeta(a_i-p)) = \sum_{i=1}^{N}(\zeta(b_i+p)+\zeta(b_i-p)).$$

By (7.4), there exist A_1, A_2 such that $y_{\boldsymbol{a}^{k_1}}(z)$ (resp. $y_{\boldsymbol{a}^{k_2}}(z)$) is a solution of $\mathrm{GLE}(\mathbf{n},p,A_1,\tau)$ (resp. $\mathrm{GLE}(\mathbf{n},p,A_2,\tau)$). Then (2.13)–(2.15) imply that $\mathrm{GLE}(\mathbf{n},p,A_1,\tau)$ and $\mathrm{GLE}(\mathbf{n},p,A_1,\tau)$ have the same global monodromy data

$(r,s)\notin\frac{1}{2}\mathbb{Z}^2$, namely $y_{\boldsymbol{a}^{k_1}}(z)$ and $y_{\boldsymbol{a}^{k_2}}(z)$ satisfy the same transformation law:

$$\ell_1^* y(z)=e^{-2\pi is}y(z),\quad \ell_2^* y(z)=e^{2\pi ir}y(z). \tag{7.9}$$

Consequently, Theorem 1.3 implies $\mathrm{GLE}(\mathbf{n},p,A_1,\tau)=\mathrm{GLE}(\mathbf{n},p,A_2,\tau)$, i.e. $y_{\boldsymbol{a}^{k_1}}(z)$ and $y_{\boldsymbol{a}^{k_2}}(z)$ are solutions of the same $\mathrm{GLE}(\mathbf{n},p,A_1,\tau)$ and satisfies the same transformation law (7.9). It follows from $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$ and (2.13) that $y_{\boldsymbol{a}^{k_1}}(z)=y_{\boldsymbol{a}^{k_2}}(z)$, so $\boldsymbol{a}^{k_1}=\boldsymbol{a}^{k_2}$, a contradiction.

This proves (7.8), which infers that these $\deg\sigma_{\mathbf{n},p}$ different points $\boldsymbol{a}^k$'s give $\deg\sigma_{\mathbf{n},p}$ different values $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^k)$'s. That is for $\sigma=\sigma_0$, the polynomial $W_{\mathbf{n},p}(\mathbf{z})\in\mathbb{C}(\wp(\sigma),\wp'(\sigma))[\mathbf{z}]$ of degree $d_{\mathbf{n},p}|\deg\sigma_{\mathbf{n},p}$ has $\deg\sigma_{\mathbf{n},p}$ distinct zeros $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a}^k)$'s, which implies $d_{\mathbf{n},p}=\deg\sigma_{\mathbf{n},p}$. The proof is complete. □

Remark 7.2. For $(r,s)\in\mathbb{C}^2\setminus\frac{1}{2}\mathbb{Z}^2$, as in [7, 21] we define

$$Z_{r,s}(\tau):=\zeta(r+s\tau|\tau)-r\eta_1(\tau)-s\eta_2(\tau).$$

Then it follows from (7.7) and (2.13)–(2.15) that $\mathbf{z}_{\mathbf{n},p}(\boldsymbol{a})=Z_{r,s}(\tau)$ with $\sigma_{\mathbf{n},p}(\boldsymbol{a})=r+s\tau$. Therefore, like the Lamé case proved in [21] and the general Darboux–Treibich–Verdier case proved in Part II [7], the monodromy data (r,s) of $\mathrm{GLE}(\mathbf{n},p,A,\tau)$ in (2.13)–(2.15) can be characterized by

$$W_{\mathbf{n},p}(Z_{r,s}(\tau))=0 \quad\text{with}\quad \sigma=r+s\tau. \tag{7.10}$$

Let us consider the special case $\mathbf{n}=\mathbf{0}$ for example. Then

$$\mathbf{z}_{\mathbf{0},p}(a)=\zeta(a)-\tfrac{1}{2}(\zeta(a+p)+\zeta(a-p))=\frac{\wp'(a)}{2(\wp(p)-\wp(a))}\in K(E_\tau),$$

i.e. its minimal polynomial $W_{\mathbf{0},p}(\mathbf{z})=\mathbf{z}-\mathbf{z}_{\mathbf{0},p}(a)$. So (7.10) is just

$$Z_{r,s}(\tau)-\frac{\wp'(r+s\tau)}{2(\wp(p)-\wp(r+s\tau))}=0,$$

which recovers Hitchin's formula

$$\wp(p|\tau)=\wp(r+s\tau|\tau)+\frac{\wp'(r+s\tau|\tau)}{2Z_{r,s}(\tau)}.$$

Therefore, (7.10) should be closely related to the formula of solutions of Painlevé VI equation with parameter (5.1)–(5.2) for general $\mathbf{n}$, which will be studied elsewhere.

Acknowledgements

We thank Prof. Treibich for his interest and valuable comments to this work. The research of Z. Chen was supported by NSFC (No. 12071240, 11871123). The research of T-J. Kuo was supported by MOST (MOST 107-2628-M-003-002-MY4).

References

[1] M. V. Babich and L. A. Bordag; *Projective differential geometrical structure of the Painlevé equations.* J. Differ. Equ. **157** (1999), 452–485. MR1713268

[2] P. Boalch; *From Klein to Painlevé via Fourier, Laplace and Jimbo.* Proc. Lond. Math. Soc. **90** (2005), 167–208. MR2107041

[3] C. L. Chai, C. S. Lin and C. L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: I.* Camb. J. Math. **3** (2015), 127–274. MR3356357

[4] Z. Chen, T. J. Kuo and C. S. Lin; *Hamiltonian system for the elliptic form of Painlevé VI equation.* J. Math. Pures Appl. **106** (2016), 546–581. MR3520447

[5] Z. Chen, T. J. Kuo and C. S. Lin; *Unitary monodromy implies the smoothness along the real axis for some Painlevé VI equation, I.* J. Geom. Phys. **116** (2017), 52–63. MR3623645

[6] Z. Chen, T. J. Kuo and C. S. Lin; *The geometry of generalized Lamé equation, I.* J. Math. Pures Appl. **127** (2019), 89–120. MR3960139

[7] Z. Chen, T. J. Kuo and C. S. Lin; *The geometry of generalized Lamé equation, II: Existence of pre-modular forms and application.* J. Math. Pures Appl. **132** (2019), 251–272. MR4030254

[8] Z. Chen, T. J. Kuo, C. S. Lin and C. L. Wang; *Green function, Painlevé VI equation, and Eisenstein series of weight one.* J. Differ. Geom. **108** (2018), 185–241. MR3763067

[9] S. Dahmen; *Counting integral Lamé equations by means of dessins d'enfants.* Trans. Amer. Math. Soc. **359** (2007), 909–922. MR2255201

[10] G. Darboux; *Sur une équation lineare.* C. R. Acad. Sci. Paris, t. **XCIV**(25), (1882), 1645–1648.

[11] A. S. Fokas, A. R. Its, A. A.Kapaev and V. Yu.Novokshenov; *Painlevé transcendents. The Riemann–Hilbert approach.* Mathematical Surveys and Monographs, Vol. 128, Amer. Math. Soc., Providence, RI, 2006. MR2264522

[12] R. Fuchs; *Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen.* Math. Ann. **63** (1907), 301–321. MR1511408

[13] F. Gesztesy and R. Weikard; *Treibich–Verdier potentials of the stationary (m)KdV hierarchy.* Math. Z. **219** (1995), 451–476. MR1339715

[14] G. H. Halphen; *Traité des Fonctions Elliptiques et de leurs Applications II*, Gauthier-Villars et Fils, Paris, 1888.

[15] N. J. Hitchin; *Twistor spaces, Einstein metrics and isomonodromic deformations.* J. Differ. Geom. **42**(1) (1995), 30–112. MR1350695

[16] M. Inaba, K. Iwasaki and M. Saito; *Bäcklund transformations of the sixth Painlevé equation in terms of Riemann–Hilbert correspodence.* Inter. Math. Res. Not. **1** (2004), 1–30. MR2036953

[17] E. Ince; *Further investigations into the periodic Lamé functions.* Proc. Roy. Soc. Edinburgh **60** (1940), 83–99. MR0002400

[18] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida; *From Gauss to Painlevé: A Modern Theory of Special Functions.* Springer, vol. E16, 1991. MR1118604

[19] S. Kawai; *Isomonodromic deformation of Fuchsian projective connections on elliptic curves.* Nagoya Math. J. **171** (2003), 127–161. MR2002016

[20] C. S. Lin and C. L. Wang; *Elliptic functions, Green functions and the mean field equations on tori.* Ann. Math. **172**(2) (2010), 911–954. MR2680484

[21] C. S. Lin and C. L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: II.* J. Éc. polytech. Math. **4** (2017), 557–593. MR3665608

[22] R. Maier; *Lamé polynomail, hyperelliptic reductions and Lamé band structure.* Philos. Trans. R. Soc. A **366** (2008), 1115–1153. MR2377687

[23] Y. Manin; *Sixth Painlevé quation, universal elliptic curve, and mirror of* $\mathbb{P}^2$. Amer. Math. Soc. Transl. (2), **186** (1998), 131–151. MR1732409

[24] K. Okamoto; *Isomonodromic deformation and Painlevé equations, and the Garnier system*. J. Fac. Sci. Univ. Tokyo Sec. IA Math. **33** (1986), 575–618. MR0866050

[25] K. Okamoto; *Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}*. Ann. Mat. Pura Appl. **146** (1986), 337–381. MR0916698

[26] E. Poole; *Introduction to the theory of linear differential equations*. Oxford University Press, 1936.

[27] T. Shioda and K. Takano; *On some Hamiltonian structures of Painleve systems, I*. Funkc. Ekvac. **40** (1997), 271–291. MR1480279

[28] K. Takemura; *The Hermite–Krichever Ansatz for Fuchsian equations with applications to the sixth Painlevé equation and to finite gap potentials*. Math. Z. **263** (2009), 149–194. MR2529492

[29] K. Takemura; *The Heun equation and the Calogero–Moser–Sutherland system I: the Bethe Ansatz method*. Comm. Math. Phys. **235** (2003), 467–494. MR1974511

[30] K. Takemura; *The Heun equation and the Calogero–Moser–Sutherland system II: perturbation and algebraic solution*. Elec. J. Differ. Equ. **2004**(15) (2004), 1–30. MR2036199

[31] K. Takemura; *The Heun equation and the Calogero–Moser–Sutherland system III: the finite-gap property and the monodromy*. J. Nonl. Math. Phys. **11** (2004), 21–46. MR2031210

[32] K. Takemura; *The Heun equation and the Calogero–Moser–Sutherland system IV: the Hermite–Krichever Ansatz*. Comm. Math. Phys. **258** (2005), 367–403. MR2171700

[33] K. Takemura; *The Heun equation and the Calogero–Moser–Sutherland system V: generalized Darboux transformations*. J. Nonl. Math. Phys. **13** (2006), 584–611. MR2263090

[34] K. Takemura; *Middle convolution and Heun's equation*. SIGMA **5** (2009), 040. MR2506172

[35] T. Tsuda, K. Okamoto and H. Sakai; *Folding transformations of the Painlevé equations*. Math. Ann. **331** (2005), 713–738. MR2148794

[36] A. Treibich; *New elliptic potentials*. Acta Appl. Math. **36** (1994), 27–48. MR1303854

[37] A. Treibich and J. L. Verdier; *Revetements exceptionnels et sommes de 4 nombres triangulaires*. Duke Math. J. **68** (1992), 217–236. MR1191559

[38] E. Whittaker and G. Watson; *A course of modern analysis*. Cambridge University Press, 1996. MR1424469

Zhijie Chen
Department of Mathematical Sciences, Yau Mathematical Sciences Center
Tsinghua University
Beijing 100084
China
E-mail: zjchen2016@tsinghua.edu.cn

Ting-Jung Kuo
Department of Mathematics
Taiwan Normal University
Taipei 11677
Taiwan
E-mail: tjkuo1215@ntnu.edu.tw

Chang-Shou Lin
Department of Mathematics
Taiwan University
Taipei 10617
Taiwan
E-mail: cslin@math.ntu.edu.tw