Pure and Applied Mathematics Quarterly
Volume 17, Number 5, 1669–1710, 2021

A density theorem for asymptotically hyperbolic initial data satisfying the dominant energy condition

Mattias Dahl and Anna Sakovich

Abstract: When working with asymptotically hyperbolic initial data sets for general relativity it is convenient to assume certain simplifying properties. We prove that the subset of initial data sets with such properties is dense in the set of physically reasonable asymptotically hyperbolic initial data sets. More specifically, we show that an asymptotically hyperbolic initial data set with non-negative local energy density can be approximated by an initial data set with strictly positive local energy density and a simple structure at infinity, while changing the mass arbitrarily little. This is achieved by suitably modifying the argument used by Eichmair, Huang, Lee and Schoen in the asymptotically Euclidean case.

1. Introduction

In general relativity Einstein's equations read

$$\mathrm{Ric}^{\gamma} - \tfrac{1}{2}\mathrm{Scal}^{\gamma}\gamma = \mathrm{T}. \tag{1}$$

Here Ric^{γ} and Scal^{γ} denote respectively the Ricci tensor and the scalar curvature of a spacetime $(\mathcal{M}, \gamma)$, and the symmetric divergence-free 2-tensor T is the stress-energy tensor of the spacetime. A spacetime $(\mathcal{M}, \gamma)$ satisfying (1) is said to obey the dominant energy condition if for any future directed timelike vector ν the vector $-\mathrm{T}(\nu, \cdot)^{\sharp}$ is either future directed timelike or null. This condition means that the energy density of $(\mathcal{M}, \gamma)$ is non-negative and that the energy cannot travel faster than the speed of light.

Let (M, g) be a Riemannian submanifold of the spacetime $(\mathcal{M}, \gamma)$ satisfying the Einstein equations with unit normal denoted by η and second fundamental form denoted by K. In this case (M, g) can be viewed as a "constant time slice" of $(\mathcal{M}, \gamma)$. The dominant energy condition for $(\mathcal{M}, \gamma)$ at points of M is equivalent to the inequality $\mu \geq |J|_g$ everywhere on M. Here

Received July 30, 2020.

the energy density $\mu := \mathrm{T}(\eta, \eta)$ and the momentum density $J := \mathrm{T}(\eta, \cdot)$ can be computed from g and K using the *constraint equations*

$$-2\mu = -\mathrm{Scal}^g - (\mathrm{tr}^g K)^2 + |K|_g^2, \tag{2}$$

$$J = \mathrm{div}^g K - d(\mathrm{tr}^g K). \tag{3}$$

By an *initial data set* for the Einstein equations we will mean a triple (M, g, K) consisting of a Riemannian n-manifold (M, g) and a symmetric 2-tensor field K defined on M. We will say that (M, g, K) satisfies the *dominant energy condition* if $\mu \geq |J|_g$ holds everywhere on M, where μ and J are defined through (2)–(3). By the above discussion, this means that (M, g, K) arises as a constant time slice of a spacetime satisfying the dominant energy condition.

An initial data set (M, g, K) is said to be *asymptotically Euclidean* if outside some compact set M is diffeomorphic to the complement of a ball in Euclidean space $\mathbb{R}^n$, and if under this diffeomorphism g approaches the Euclidean metric δ and K approaches zero sufficiently fast at infinity. For asymptotically Euclidean initial data sets *asymptotic charge integrals* at infinity can be defined. They are integrals which arise as boundary terms when integrating the constraint operator

$$\Phi : (g, K) \mapsto \Big(-\mathrm{Scal}^g - (\mathrm{tr}^g K)^2 + |K|_g^2,\ \mathrm{div}^g K - d(\mathrm{tr}^g K)\Big)$$

against elements in the kernel of $D\Phi^*_{(\delta,0)}$, which correspond to Killing vectors of the Minkowski spacetime. In particular, this gives rise to the Arnowitt–Deser–Misner energy E and linear momentum P. The *positive mass conjecture* asserts that $E \geq |P|_g$ provided that the dominant energy condition holds. In particular, $E \geq 0$ is expected to hold under the same assumption, which is the statement of the *positive energy conjecture*. An excellent overview of both conjectures can be found in the recent book by D. Lee [27].

For many applications in mathematical general relativity it is an advantage to work with initial data sets which have simple asymptotics at infinity. For example, an asymptotically Euclidean initial data set (M, g, K) is said to have *harmonic asymptotics* if in asymptotically Euclidean coordinates at infinity we have

$$g = u^{\frac{4}{n-2}}\delta, \qquad \pi := K - (\mathrm{tr}^g K)g = u^{\frac{2}{n-2}}(\mathcal{L}_Y\delta - \mathrm{div}^\delta Y),$$

where $\mathcal{L}$ denotes the Lie derivative, and the positive function u and the vector field Y are such that

$$u(x) = 1 + A|x|^{2-n} + O(|x|^{1-n}), \qquad Y_j(x) = B_j|x|^{2-n} + O(|x|^{1-n}),$$

for $A \in \mathbb{R}$ and $B \in \mathbb{R}^n$. In this case the Arnowitt–Deser–Misner energy and linear momentum can be easily read off from the asymptotic expansions of u and Y at infinity, namely,

$$E = 2A, \qquad P_j = -\tfrac{n-2}{n-1} B_j.$$

Further, many arguments can be simplified by working with initial data sets with strictly positive local energy density, that is such that the *strict dominant energy condition* $\mu > |J|_g$ is satisfied. This condition is preserved under "small" perturbations of the initial data set, whereas the standard dominant energy condition $\mu \geq |J|_g$ might get violated by a perturbation. In [22, Theorem 18], Eichmair, Huang, Lee, and Schoen prove that an asymptotically Euclidean initial data set satisfying dominant energy condition can be slightly perturbed to an initial data set with harmonic asymptotics which obeys strict dominant energy condition while changing the energy E and the linear momentum P arbitrarily little. That is, the set of asymptotically Euclidean initial data sets with these preferred properties is dense in the set of asymptotically Euclidean initial data sets satisfying the dominant energy condition. This result is used in the proof of the positive mass theorem by the above authors [22, Theorem 1], and it is also required for the proof of the positive energy conjecture in dimension $n = 3$ by Schoen and Yau [38], and for its extension to dimensions $3 \leq n \leq 7$ by Eichmair [21]. Another application is the analysis of the geometry and topology of initial data sets with horizons, see [4].

The goal of the current paper is to prove the analogue of this density result for asymptotically hyperbolic initial data sets. Roughly speaking, an initial data set (M, g, K) is asymptotically hyperbolic if the Riemannian metric g approaches the hyperbolic metric b on hyperbolic space $\mathbb{H}^n$ in a chart covering everything outside a compact set. For K, there are two natural choices: either $K \to 0$ at infinity (as for spacelike totally geodesic hypersurfaces in asymptotically anti-de Sitter spacetimes) or $K \to g$ at infinity (as for "hyperboloidal" hypersurfaces in asymptotically Minkowski spacetimes)[1]. In this paper we adopt the second approach and consider "hyperboloidal" initial data, see Definition 2.2. Then similar considerations as in the asymptotically Euclidean case give rise to the notion of *mass* for asymptotically hyperbolic initial data, which is a linear functional on a certain finite dimensional vector space.

[1]We note that asymptotically hyperbolic initial data sets can also be modeled on totally umbilic hyperbolic slices of de Sitter spacetime, see for example [31].

The main result of this paper is that a given asymptotically hyperbolic initial data set satisfying the dominant energy condition can be approximated by an initial data set with *conformally hyperbolic asymptotics* in the sense of Definition 2.3 which obeys the strict dominant energy condition, while changing the value of the mass functional by an arbitrarily small amount. In fact, assuming sufficient regularity, one can construct coordinates at infinity in which the approximating initial data set has *Wang's asymptotics* in the sense of Definition 5.1. In particular, we prove the following

Theorem 1.1. *Let (M, g, K) be an asymptotically hyperbolic initial data set of type $(k+1, \alpha, \tau, \tau_0)$ for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$ satisfying the dominant energy condition $\mu \geq |J|_g$. Then for every $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(M, \bar{g}, \bar{K})$ of type (k, α, n, τ_0') for some $\tau_0' > 0$ with Wang's asymptotics satisfying the strict dominant energy condition*

$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

and such that the mass functionals $\mathcal{M}$ of (M, g, K) and $\overline{\mathcal{M}}$ of $(M, \bar{g}, \bar{K})$ satisfy

$$|\mathcal{M}(V) - \overline{\mathcal{M}}(V)| < \varepsilon$$

for any $V \in \mathcal{N}$, where $\mathcal{N}$ is the linear space spanned by restrictions of coordinate functions of Minkowski spacetime to the upper unit hyperboloid.

The applications of our results are similar to those of [22, Theorem 18]. In particular, the results are used in the proofs of the positive mass theorem for asymptotically hyperbolic manifolds by Chruściel and Delay [11] and for asymptotically hyperbolic initial data sets by the second author [37]. They can also prove useful for establishing other geometric inequalities for asymptotically hyperbolic initial data, such as those discussed in [10].

The paper is organized as follows. The definition of mass and its continuity with respect to the initial data set is discussed in Section 2. In Section 3 we show that a given asymptotically hyperbolic initial data set satisfying the dominant energy condition can be perturbed slightly to satisfy the strict dominant energy condition, while changing the mass arbitrarily little. Then in Section 4 we make a further perturbation to conformally hyperbolic asymptotics, while preserving the strict dominant energy condition. In Section 5 we prove a density result concerning asymptotically hyperbolic initial data sets that have Wang's asymptotics. We also discuss how one can switch to Wang's asymptotics given the approximating initial data set constructed in Section 4. Finally, in Section 6 we give comments on possible extensions of the results of this paper. Some supplementary results concerning differential operators on asymptotically hyperbolic manifolds are contained in Appendices A, B, and C.

2. Preliminaries

2.1. Asymptotically hyperbolic initial data

We denote the hyperbolic space of dimension n by $\mathbb{H}^n$ and the hyperbolic metric by b. We choose a point in $\mathbb{H}^n$ as the origin. In polar coordinates around this point we have $b = dr^2 + \sinh^2 r\, \sigma$ on $(0, \infty) \times S^{n-1}$, where σ is the round metric on the unit sphere S^{n-1} and r is the distance to the origin. The open ball of radius R centered at the origin is denoted by B_R, and its closure is denoted by $\overline{B}_R$.

We first give the definition of an asymptotically hyperbolic Riemannian manifold.

Definition 2.1. We say that (M, g) is a $C^{l,\beta}_\tau$*-asymptotically hyperbolic manifold* for a non-negative integer l, $0 \leq \beta < 1$ and $\tau > 0$, if there exists a compact set $K_0 \subset M$, $R_0 > 0$ and a diffeomorphism

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0},$$

such that $\Psi_* g - b \in C^{l,\beta}_{loc}(\mathbb{H}^n; S^2\mathbb{H}^n)$ and

$$\|\Psi_* g - b\|_{C^{l,\beta}_\tau(\mathbb{H}^n \setminus B_{R_0}; S^2\mathbb{H}^n)} := \sup_{x \in \mathbb{H}^n \setminus B_{R_0+1}} e^{\tau r(x)} \|\Psi_* g - b\|_{C^{l,\beta}(B_1(x); S^2\mathbb{H}^n)} < \infty.$$

The diffeomorphism Ψ introduced in this definition is called a *chart at infinity* for the asymptotically hyperbolic manifold.

Let (M, g) be a $C^{l,\beta}_\tau$-asymptotically hyperbolic manifold for l, β, τ as in Definition 2.1. Suppose that u is a locally integrable section of a geometric tensor bundle E (see [30, Chapter 3] for the definition of geometric tensor bundles) over $M \setminus K_0$. In this case we say that $u \in W^{k,p}_\delta(M \setminus K_0)$ for $0 \leq k \leq l$, and $1 < p < \infty$ if

$$\|u\|_{W^{k,p}_\delta(M \setminus K_0)} := \left\| e^{\delta r} \Psi_* u \right\|_{W^{k,p}(\mathbb{H}^n \setminus B_{R_0})} < \infty.$$

Note also the following equivalent definition of the $W^{k,p}_\delta(M \setminus K_0)$ norm,

$$\|u\|_{W^{k,p}_\delta(M \setminus K_0)} = \sum_{0 \leq j \leq k} \|e^{\delta r} \nabla^j (\Psi_* u)\|_{L^p(\mathbb{H}^n \setminus B_{R_0})}.$$

Building on these definitions, it is straightforward to define *weighted Sobolev spaces* $W^{k,p}_\delta(M)$ for any δ, $0 \leq k \leq l$, and $1 < p < \infty$. The *weighted Hölder spaces* $C^{k,\alpha}_\delta(M)$ are defined in a similar fashion. We say that $u \in C^{k,\alpha}_\delta(M \setminus K_0)$ for $0 \leq k + \alpha \leq l + \beta$, and $0 \leq \alpha < 1$ if

$$\|u\|_{C^{k,\alpha}_\delta(M\setminus K_0)} := \sup_{x\in\mathbb{H}^n\setminus B_{R_0+1}} e^{\tau r(x)} \|\Psi_* u\|_{C^{k,\alpha}(B_1(x))} < \infty.$$

The following equivalent definition of the $C^{k,\alpha}_\delta(M \setminus K_0)$ norm is often useful,

$$\|u\|_{C^{k,\alpha}_\delta(M\setminus K_0)} = \sum_{0\leq j\leq k} \sup_{\mathbb{H}^n\setminus B_{R_0}} |e^{\delta r}\nabla^j(\Psi_* u)| + \|e^{\delta r}\nabla^k(\Psi_* u)\|_{C^{0,\alpha}(\mathbb{H}^n\setminus B_{R_0})}.$$

Again, these definitions can be extended to define weighted Hölder spaces $C^{k,\alpha}_\delta(M)$.

The weighted Sobolev and Hölder spaces that we have just defined are analogues of the respective spaces defined by J. Lee in [30] on conformally compact manifolds. It is easy to check that standard facts such as embedding theorems, the Rellich lemma, and density theorems hold for these spaces and that the statements of these results repeat verbatim the respective statements in [30]. In particular Lemma 3.6 and Lemma 3.9 of [30] hold for $W^{k,p}_\delta(M)$ and $C^{k,\alpha}_\delta(M)$ as defined above and we will refer to [30] for these results throughout the text.

It is straightforward to check that classical interior elliptic regularity as formulated in [30, Lemma 4.8] holds for asymptotically hyperbolic manifolds and weighted function spaces as defined in Section 2.1. In Appendix A we show that improved elliptic regularity [30, Proposition 6.5] holds in the current setting. As a consequence, Fredholm theory for geometric elliptic operators on asymptotically hyperbolic manifolds in the sense of Definition 2.1 can be established, since the proof of [30, Theorem C] can be adapted. The reader is referred to Appendix A for details.

We can now give the definition of an asymptotically hyperbolic initial data set. Recall that the energy density μ and the momentum density J are defined via the constraint equations (2)–(3).

Definition 2.2. A triple (M, g, K) is an *asymptotically hyperbolic initial data set of class* (k, α, τ) for $k \geq 2$, $0 \leq \alpha < 1$ and $\tau > 0$ if

- (M, g) is a $C^{k,\alpha}_\tau$-asymptotically hyperbolic manifold in the sense of Definition 2.1,
- a symmetric 2-tensor K is such that $K - g \in C^{k-1,\alpha}_\tau(M; S^2M)$.

If, in addition, $(\mu, J) \in C^{k-2,\alpha}_{n+\tau_0}$ for some $\tau_0 > 0$ then (M, g, K) is an *asymptotically hyperbolic initial data set of class* $(k, \alpha, \tau, \tau_0)$.

Abusing notation slightly we may summarize the content of this definition as follows: (M, g, K) is an asymptotically hyperbolic initial data set of class (k, α, τ) for $0 \le \alpha < 1$ and $\tau > 0$ if $(g - b, K - g) \in C^{k,\alpha}_\tau \times C^{k-1,\alpha}_\tau$. In this case $(\mu, J) \in C^{k-2,\alpha}_\tau$. The necessity for the faster decay $(\mu, J) \in C^{k-2,\alpha}_{n+\tau_0}$ will become clear in Section 2.2.

Given an asymptotically hyperbolic initial data set (M, g, K) of class (α, τ) it is convenient to set

$$\pi := (K - g) - \operatorname{tr}^g(K - g)g.$$

Note that $\pi \in C^{k-1,\alpha}_\tau$ and that π contains the same information as K, since $K = \pi + g - \frac{1}{n-1}(\operatorname{tr}^g \pi)g$. It is therefore equivalent to work with (M, g, π) as an initial data set, and in this paper we will only work with initial data sets given in this form.

In terms of (g, π) the constraint equations (2)–(3) are written as

$$\begin{aligned} -2\mu &= -(\operatorname{Scal}^g + n(n-1)) + 2\operatorname{tr}^g \pi - \frac{(\operatorname{tr}^g \pi)^2}{n-1} + |\pi|^2_g, \\ J &= \operatorname{div}^g \pi. \end{aligned}$$

By the *constraint map* we mean the map

$$\Phi : (g, \pi) \mapsto \left(-(\operatorname{Scal}^g + n(n-1)) + 2\operatorname{tr}^g \pi - \frac{(\operatorname{tr}^g \pi)^2}{n-1} + |\pi|^2_g, \operatorname{div}^g \pi \right). \tag{4}$$

Finally, we define initial data sets with conformally hyperbolic asymptotics. Recall that the *conformal Killing operator* $\mathring{\mathcal{L}}$ is defined by

$$(\mathring{\mathcal{L}}_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i - \tfrac{2}{n}(\operatorname{div}^g Y) g_{ij},$$

that is, $(\mathring{\mathcal{L}}_Y g)_{ij}$ is the trace-free part of the Lie derivative $(\mathcal{L}_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i$.

Definition 2.3. We say that an initial data set (M, g, π) has *conformally hyperbolic asymptotics* if there exists a compact set K_0, a radius $R_0 > 0$, and a diffeomorphism

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0},$$

such that

$$\Psi_* g = (1+v)^{\frac{4}{n-2}} b, \quad \Psi_* \pi = (1+v)^{\frac{2}{n-2}} \mathring{\mathcal{L}}_Y b,$$

where the function v and the components of the 1-form Y can be written in the form

$$\begin{aligned} v &= v_0 e^{-nr} + v_1, \\ Y_r &= (Y_0)_r e^{-nr} + (Y_1)_r, \\ Y_\varphi &= (Y_0)_\varphi e^{-(n-1)r} + (Y_1)_\varphi, \end{aligned} \tag{5}$$

where φ refers to a coordinate on S^{n-1}, $(v_0, Y_0) \in C^{k,\alpha}_{loc}$ is independent of r and $(v_1, Y_1) \in C^{k,\alpha}_{n+1}$ for $k \geq 2$ and $0 \leq \alpha < 1$.

2.2. The mass functional for asymptotically hyperbolic initial data

In this section we review the concept of mass in the asymptotically hyperbolic setting and discuss the continuity of mass with respect to the initial data. We first recall how the asymptotic charge integrals are defined, following Michel [33].

Let (M, g, π) be an asymptotically hyperbolic initial data set of type (k, α, τ) for $k \geq 2$, $0 \leq \alpha < 1$ and $\tau > 0$, and let Ψ be the chart at infinity as in Definition 2.1. Clearly, in this case we have $e := \Psi_* g - b \to 0$ and $\eta := \Psi_* \pi \to 0$ at infinity. Let the constraint map Φ be defined by (4). Since $\Phi(b, 0) = 0$, linearization gives us

$$\Phi(\Psi_*(g, \pi)) = D\Phi|_{(b,0)}(e, \eta) + \mathcal{Q}(e, \eta), \tag{6}$$

where $\mathcal{Q}(e, \eta)$ is a remainder term of second order. For any function V and 1-form ϖ there is a 1-form $\mathbb{U}_{(V,\varpi)}(e, \eta)$ such that

$$\langle D\Phi|_{(b,0)}(e, \eta), (V, \varpi)\rangle = \operatorname{div}^b \mathbb{U}_{(V,\varpi)}(e, \eta) + \langle (e, \eta), D\Phi^*_{(b,0)}(V, \varpi)\rangle,$$

where $D\Phi^*_{(b,0)}$ is the formal adjoint of $D\Phi|_{(b,0)}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product induced by b on geometric tensor bundles over $\mathbb{H}^n$. Contracting (6) with $(V, \varpi) \in \ker D\Phi^*_{(b,0)}$ we obtain

$$\langle \Phi(\Psi_*(g, \pi)), (V, \varpi)\rangle = \operatorname{div}^b \mathbb{U}_{(V,\varpi)}(e, \eta) + \langle \mathcal{Q}(e, \eta), (V, \varpi)\rangle. \tag{7}$$

In this way we assign to every $(V, \varpi) \in \ker D\Phi^*_{(b,0)}$ the *charge integral*

$$\mathbb{Q}_{(V,\varpi)}(g, \pi) := \lim_{R \to \infty} \int_{S_R} \mathbb{U}_{(V,\varpi)}(e, \eta)(\nu)\, d\mu^b,$$

where ν is the outer unit normal of the $(n-1)$-dimensional sphere S_R in $\mathbb{H}^n$.

The structure of the kernel of $D\Phi^*_{(b,0)}$ is well understood, see Moncrief [34]. Namely, $(V, \varpi^\sharp)$ corresponds to the normal-tangential (or lapse-shift) decomposition of the restriction along the unit hyperboloid of a Killing vector field of Minkowski spacetime. In other words, $(V, \varpi^\sharp)$ is a *Killing initial data* (or *KID*) for Minkowski spacetime given on the unit hyperboloid.

In particular, we have $(V, -dV) \in \ker D\Phi^*_{(b,0)}$ for $V \in \mathcal{N}$, where the vector space $\mathcal{N}$ is spanned by the functions

$$V_{(0)} = \cosh r, \quad V_{(1)} = x^1 \sinh r, \quad \dots, \quad V_{(n)} = x^n \sinh r$$

expressed in polar coordinates on $\mathbb{H}^n = (0,\infty) \times S^{n-1}$. Here $x^1, \dots, x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}.

For these KIDs we have the following result.

Proposition 2.4. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type $(k, \alpha, \tau, \tau_0)$ for $k \geq 2$, $0 \leq \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. Then for every $V \in \mathcal{N}$ the charge integral $\mathbb{Q}_{(V,-dV)}(g, \pi)$ is well-defined and can be computed by the formula*

(8)
$$\mathbb{Q}_{(V,-dV)}(g,\pi) = \lim_{R\to\infty} \int_{S_R} \left(V(\operatorname{div}^b e - d\operatorname{tr}^b e) + (\operatorname{tr}^b e)dV - (e + 2\eta)(\nabla^b V, \cdot) \right)(\nu)\, d\mu^b.$$

Proof. Integrating (7) over $\mathbb{H}^n \setminus B_{R_0}$ and using the divergence theorem we obtain

$$\mathbb{Q}_{(V,-dV)}(g,\pi) = \int_{\mathbb{H}^n \setminus B_{R_0}} \langle \Phi(\Psi_*(g,\pi)) - \mathcal{Q}(e,\eta), (V, -dV)\rangle\, d\mu^b + \int_{S_{R_0}} \mathbb{U}_{(V,-dV)}(e,\eta)(\nu)\, d\mu^b.$$

Estimating $\mathcal{Q}(e,\eta)$ as in [33, Equation (12)] and using our assumptions on the decay of the initial data, we see that the integral over $\mathbb{H}^n \setminus B_{R_0}$ converges, hence $\mathbb{Q}_{(V,-dV)}(g,\pi)$ is well-defined.

We refer to [33, Section IV.2.B] and references therein for the derivation of the formula (8). ∎

Definition 2.5. Let (M, g, π) be an asymptotically hyperbolic initial data set. Then the *mass* of (M, g, π) is the linear functional $\mathcal{M}_{(g,\pi)} : \mathcal{N} \to \mathbb{R}$ given by

$$\mathcal{M}_{(g,\pi)}(V) = \tfrac{1}{2(n-1)\omega_{n-1}} \mathbb{Q}_{(V,-dV)}(g,\pi),$$

where ω_{n-1} denotes the volume of the unit sphere (S^{n-1}, σ).

This is the same as the expression for the Bondi mass obtained by Chruściel, Jesierski, and Łęski in [14], under asymptotic decay conditions that however do not allow for gravitational radiation. See [15], [13], and [33] for discussions on coordinate covariance.

As Proposition 2.4 shows, the mass functional is well defined for asymptotically hyperbolic initial data sets of type $(k, \alpha, \tau, \tau_0)$ for $k \geq 2$, $0 \leq \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. It is also straightforward to check that the mass functional is trivial for asymptotically hyperbolic initial data sets of type (α, τ) with $\tau > n$. The following are two examples of the "critical" case $\tau = n$.

Example 2.6. The Anti-de Sitter Schwarzschild Riemannian metric is given by

$$g_{\mathrm{AdSS}} = \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 \sigma$$

on $[a, \infty) \times S^{n-1}$, where the inner radius a depends on m, see for example [18, Appendix A]. It can be realized as an *umbilic* (that is, $g = K$) asymptotically hyperbolic initial data set for Schwarzschild spacetime, see Brendle and Wang [8]. In this case

$$\mathcal{M}(V_{(0)}) = m, \qquad \text{and} \qquad \mathcal{M}(V_{(i)}) = 0,$$

for $i = 1, \ldots, n$, where m coincides with the mass parameter of the Schwarzschild metric.

Example 2.7. For initial data sets with conformally hyperbolic asymptotics as in Definition 2.3 it is not complicated to compute that

$$\mathcal{M}(V_{(0)}) = \tfrac{2(n+1)}{(n-2)\omega_{n-1}} \int_{S^{n-1}} v_0 \, d\mu^\sigma + \tfrac{2(n+1)}{n\omega_{n-1}} \int_{S^{n-1}} (Y_0)_r \, d\mu^\sigma,$$

and

$$\mathcal{M}(V_{(i)}) = \tfrac{2(n+1)}{(n-2)\omega_{n-1}} \int_{S^{n-1}} x^i v_0 \, d\mu^\sigma + \tfrac{2(n+1)}{n\omega_{n-1}} \int_{S^{n-1}} x^i (Y_0)_r \, d\mu^\sigma$$

for $i = 1, \ldots, n$.

Concluding this section, we confirm that the mass is continuous as a function of asymptotically hyperbolic initial data sets of type $(k, \alpha, \tau, \tau_0)$, where $k \geq 2$, $0 \leq \alpha < 1$, $\tau > \frac{n}{2}$, and $\tau_0 > 0$. For simplicity, the charts at infinity are suppressed in the statement of the result and in the proof.

Proposition 2.8. *Let (g,π) and $(\bar g,\bar\pi)$ be asymptotically hyperbolic initial data sets of type (k,α,τ,τ_0) for $k\geq 2$, $0\leq\alpha<1$, $\tau>\frac{n}{2}$, and $\tau_0>0$. Let (μ,J) and $(\bar\mu,\bar J)$ denote the respective energy and momentum densities defined via the constraint equations (2)–(3). Given $\varepsilon>0$ there exists $\delta>0$ depending only on (g,π) and ε, such that if*

$$\|g-\bar g\|_{C^2_\tau}\leq\delta,\quad \|\pi-\bar\pi\|_{C^1_\tau}\leq\delta, \tag{9}$$

and

$$\|(\mu,J)-(\bar\mu,\bar J)\|_{C^0_{n+\tau_0}}\leq\delta, \tag{10}$$

then for any $V\in\{V_{(0)},V_{(1)},\ldots,V_{(n)}\}$ we have

$$\left|\mathcal{M}_{(g,\pi)}(V)-\mathcal{M}_{(\bar g,\bar\pi)}(V)\right|\leq\varepsilon.$$

Proof. Fix $R\geq R_0$. Arguing as in the proof of Proposition 2.4 we find that

$$\begin{aligned}
&2(n-1)\omega_{n-1}\left(\mathcal{M}_{(g,\pi)}(V)-\mathcal{M}_{(\bar g,\bar\pi)}(V)\right)\\
&\quad=\int_{\mathbb{H}^n\setminus B_R}\langle\Phi(g,\pi)-\Phi(\bar g,\bar\pi),(V,-dV)\rangle\,d\mu^b\\
&\qquad-\int_{\mathbb{H}^n\setminus B_R}\langle\mathcal{Q}(e,\eta)-\mathcal{Q}(\bar e,\bar\eta),(V,-dV)\rangle\,d\mu^b\\
&\qquad+\int_{S_R}\left(\mathbb{U}_{(V,-dV)}(e,\eta)-\mathbb{U}_{(V,-dV)}(\bar e,\bar\eta)\right)(\nu)\,d\mu^b.
\end{aligned}$$

Now suppose that (g,π) is fixed and that δ and $(\bar g,\bar\pi)$ are such that (9) and (10) hold. Then by assumption (10) the absolute value of the first integral over $\mathbb{H}^n\setminus B_R$ is bounded by $C\delta$ for some $C>0$ depending only on (g,π). The same is true for the second integral over $\mathbb{H}^n\setminus B_R$ by assumption (9) combined with the fact that the remainder term $\mathcal{Q}(e,\eta)$ in (6) is at least quadratic in e and η and their derivatives of order up to 2 and 1 respectively. As for the inner boundary integral, we see that its absolute value is bounded by $C\delta e^{(n-\tau)R}$ for $C>0$ depending only on (g,π). From this it is clear that δ can be chosen so that the sum of the absolute values of these three integrals is less than ε. □

3. Perturbation to strict inequality in the dominant energy condition

This section is devoted to the following result.

Theorem 3.1. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type $(k, \alpha, \tau, \tau_0)$ for $k \geq 2$, $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$, and $\tau_0 > 0$. Assume that (M, g, π) satisfies the dominant energy condition, $\mu \geq |J|_g$. Then for every $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$, with the energy and momentum density denoted by $(\bar{\mu}, \bar{J})$, of type $(k, \alpha, \tau, \tau_0')$ for some $\tau_0' > 0$ such that*

$$\|g - \bar{g}\|_{C^{k,\alpha}_\tau} < \varepsilon, \qquad \|\pi - \bar{\pi}\|_{C^{k-1,\alpha}_\tau} < \varepsilon,$$

and the strict dominant energy condition

$$\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$$

holds for a constant $\gamma > 0$, and

$$\left|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)\right| < \varepsilon$$

for $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

The argument follows [22, Proof of Theorem 22]. In simple terms it can be described as follows. We would like to choose symmetric 2-tensors h and w so that the perturbed initial data $\bar{g} = g + th$ and $\bar{\pi} = \pi + tw$ satisfies $\bar{\mu} > |\bar{J}|_{\bar{g}}$ for sufficiently small $t > 0$. From the Taylor expansion $\Phi(\bar{g}, \bar{\pi}) = \Phi(g, \pi) + tD\Phi|_{(g,\pi)}(h, w) + O(t^2)$, we see that $\bar{\mu} = \mu + \frac{t}{2} f + O(t^2)$ and $\bar{J} = J + tX + O(t^2)$, where $(-f, X) = D\Phi|_{(g,\pi)}(h, w)$. Further,

$$\begin{aligned} |\bar{J}|^2_{\bar{g}} &= \bar{g}^{ij}\bar{J}_i\bar{J}_j \\ &= (g^{ij} - th^{ij} + O(t^2))(J_i + tX_i + O(t^2))(J_j + tX_j + O(t^2)) \\ &= |J|^2_g + t(2X^j - h^{ij}J_i)J_j + O(t^2), \end{aligned} \tag{11}$$

where indices are raised using the metric g. Hence if we set $X^j = \frac{1}{2}h^{ij}J_i$ then $|\bar{J}|_{\bar{g}} = |J|_g + O(t^2)$, as long as the decay of $|J|^2_g$ at infinity is not faster than that of the $O(t^2)$ term in the last line of (11). This leads to the expectation that $\bar{\mu} > |\bar{J}|_{\bar{g}}$ will be achieved if we can find a pair (h, w) such that $D\Phi|_{(g,\pi)}(h, w) = (-f, X)$, where $X^j = \frac{1}{2}h^{ij}J_i$ and $f > 0$. Indeed, in this case we have

$$\bar{\mu} - |\bar{J}|_{\bar{g}} = \mu - |J|_g + tf + O(t^2) \geq tf + O(t^2) > 0$$

provided that the $O(t^2)$ term above decays at least as fast as f at infinity.

However, $D\Phi|_{(g,\pi)}$ is not a determined elliptic operator (see for example Delay [19]), and this makes it difficult to ensure that the solutions of the equation $D\Phi|_{(g,\pi)}(h,w) = (-f, X)$ will have good asymptotic behaviour. This problem can be overcome by combining the above considerations with a certain construction introduced by Corvino and Schoen in their proof of the density result in [17, Theorem 1]. The idea is similar in spirit to the conformal method of solving the constraint equations (see for example [7, Section 4.1]) and is based on the observation that by suitably choosing a first order differential operator $\mathcal{D}$ one can ensure that the linearization at $(1,0)$ of the operator

$$(u, Y) \mapsto \Phi\left(u^{\frac{4}{n-2}} g, u^{\frac{2}{n-2}}(\pi + \mathcal{D}Y)\right), \tag{12}$$

is a second order elliptic operator with nice properties.

We begin the proof of Theorem 3.1 with some preliminaries. Set $\kappa := \frac{4}{n-2}$. For $(u-1, Y) \in C^{2,\alpha}_\tau$ we let

$$\tilde{g} = u^\kappa g, \qquad \text{and} \qquad \widetilde{\pi} = u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g), \tag{13}$$

where $\mathring{\mathcal{L}}$ is the conformal Killing operator described in Section 2.1. Our choice of the operator $\mathcal{D} = \mathring{\mathcal{L}}$ in (12) is motivated by the fact that the *vector Laplacian* $\Delta_L = \operatorname{div} \mathring{\mathcal{L}}$ is a well-known elliptic operator on asymptotically hyperbolic manifolds whose Fredholm properties (see Appendix A) fit nicely into the context of the current argument. Let $\widetilde{\mu}$ and $\widetilde{J}$ be the energy and momentum densities of $(\tilde{g}, \widetilde{\pi})$ computed via the constraint equations (2)–(3) and consider the operator

$$T(u, Y) = (-2u^\kappa \widetilde{\mu}, u^{\kappa/2}\widetilde{J}).$$

This conformal rescaling of the constraint equations is needed to ensure that the dominant energy condition scales correctly when we pass to the deformed initial data set (13), see (23) and (24) below.

It is straightforward to check that

$$\begin{aligned} -2u^\kappa \widetilde{\mu} &= \tfrac{4(n-1)}{n-2} u^{-1}\Delta^g u - \operatorname{Scal}^g - n(n-1)u^\kappa + 2u^{\kappa/2} \operatorname{tr}^g \pi - \tfrac{1}{n-1}(\operatorname{tr}^g \pi)^2 \\ &\qquad + \left(|\pi|^2_g + 2\langle \pi, \mathring{\mathcal{L}}_Y g\rangle + |\mathring{\mathcal{L}}_Y g|^2_g\right), \\ u^{\kappa/2}\widetilde{J}_j &= (\Delta_L Y + \operatorname{div}^g \pi)_j + \tfrac{2(n-1)}{n-2} u^{-1}(\pi + \mathring{\mathcal{L}}_Y g)^k_j \nabla_k u - \tfrac{2}{n-2} u^{-1}\nabla_j u \operatorname{tr}^g \pi, \end{aligned} \tag{14}$$

for $j = 1, 2, \ldots, n$. Consequently, the linearization of T at $(1, 0)$ is

$$
\begin{aligned}
DT|_{(1,0)}(v, Z) = \Big(&\tfrac{4(n-1)}{n-2}(\Delta^g v - nv) + \tfrac{4}{n-2}(\operatorname{tr}^g \pi)v + 2\langle \pi, \mathring{\mathcal{L}}_Z g\rangle, \\
&(\Delta_L Z)_j + \tfrac{2(n-1)}{n-2}\pi_j^k \nabla_k v - \tfrac{2}{n-2}(\operatorname{tr}^g \pi)\nabla_j v\Big),
\end{aligned} \tag{15}
$$

for $j = 1, 2, \ldots, n$. The following lemma concerns Fredholm properties of the operator $DT|_{(1,0)}$.

Lemma 3.2. *If (M, g, π) is an asymptotically hyperbolic initial data set of type (k, α, τ) for $k \geq 2$, $0 < \alpha < 1$ and $\tau > 0$ then $DT|_{(1,0)}$ is a Fredholm operator with index zero in the following cases:*

- *as a map $C^{l,\beta}_\delta \to C^{l-2,\beta}_\delta$ for $2 \leq l \leq k$, $0 < \beta \leq \alpha$, $-1 < \delta < n$,*
- *as a map $W^{l,p}_\delta \to W^{l-2,p}_\delta$ for $2 \leq l \leq k$, $1 < p < \infty$, $-1 < \delta + \frac{n-1}{p} < n$.*

Proof. We give the proof in the case of weighted Hölder spaces, the case of weighted Sobolev spaces is treated similarly. Write $DT|_{(1,0)} = P_0 + P_1$, where

$$
P_0 : (v, Z) \mapsto \left(\tfrac{4(n-1)}{n-2}(\Delta^g v - nv), \Delta_L Z\right),
$$

and

$$
P_1 : (v, Z) \mapsto \left(\tfrac{4}{n-2}(\operatorname{tr}^g \pi)v + 2\langle \pi, \mathring{\mathcal{L}}_Z g\rangle, \tfrac{2(n-1)}{n-2}\pi_j^k \nabla_k v - \tfrac{2}{n-2}(\operatorname{tr}^g \pi)\nabla_j v\right).
$$

Here $P_0 : C^{l,\alpha}_\delta \to C^{l-2,\alpha}_\delta$ is a Fredholm operator of index zero for $\delta \in (-1, n)$, see Proposition A.2. By [30, Lemma 3.6 (a)] the map $P_1 : C^{l,\alpha}_\delta \to C^{l-1,\alpha}_{\delta+\tau}$ is continuous, whereas by the Rellich Lemma, [30, Lemma 3.6 (d)], the inclusion $C^{l-1,\alpha}_{\delta+\tau} \hookrightarrow C^{l-2,\alpha}_\delta$ is compact. We conclude that $P_1 : C^{l,\alpha}_\delta \to C^{l-2,\alpha}_\delta$ is compact for $-1 < \delta < n$, and the claim follows. □

Recall that the constraint map Φ is defined by the formula (4). A direct computation shows that the linearization of Φ is

$$
\begin{aligned}
&D\Phi|_{(g,\pi)}(h, w) \\
&\quad = \Big(\Delta^g(\operatorname{tr}^g h) - \operatorname{div}^g \operatorname{div}^g h + \langle h, \operatorname{Ric}^g\rangle \\
&\qquad\quad + 2\left(1 - \frac{\operatorname{tr}^g \pi}{n-1}\right)(\operatorname{tr}^g w - \langle h, \pi\rangle) - 2\langle h, \pi \circ \pi\rangle + 2\langle \pi, w\rangle, \\
&\qquad\quad (\operatorname{div}^g w)_k - h^{ij}\nabla_i \pi_{jk} - (\operatorname{div} h)_j \pi^j_k + \tfrac{1}{2}\nabla_j(\operatorname{tr}^g h)\pi^j_k - \tfrac{1}{2}\pi^{ij}\nabla_k h_{ij}\Big),
\end{aligned}
$$

where $(\pi \circ \pi)_{ij} = g^{kl}\pi_{ik}\pi_{jl}$. The formal adjoint of $D\Phi$ is given by

(16)
$$\begin{aligned} D\Phi^*_{(g,\pi)}(V,X) = \Bigg(&(\Delta V)g_{ij} - \nabla_i\nabla_j V + V\mathrm{Ric}_{ij} - 2V\left(1 - \frac{\operatorname{tr}^g \pi}{n-1}\right)\pi_{ij} \\ &- 2V\pi_{ik}\pi^k_j + \tfrac{1}{2}(\pi_{jk}\nabla_i X^k + \pi_{ik}\nabla_j X^k) - \tfrac{1}{2}(\operatorname{div}\pi)_k X^k g_{ij} \\ &- \tfrac{1}{4}\langle \pi, \mathcal{L}_X g\rangle g_{ij} + \tfrac{1}{2}X^k\nabla_k \pi_{ij} + \tfrac{1}{2}(\operatorname{div} X)\pi_{ij}, \\ &\quad - \tfrac{1}{2}(\mathcal{L}_X g)_{ij} + 2V\left(1 - \frac{\operatorname{tr}^g \pi}{n-1}\right) g_{ij} + 2V\pi_{ij}\Bigg). \end{aligned}$$

The following lemma is the analogue of [22, Lemma 20] in the asymptotically hyperbolic setting. The proof of the cited lemma is similar to [17, Proposition 3.1].

Lemma 3.3. *If (M, g, π) is an asymptotically hyperbolic initial data set of type (k, α, τ) for $k \geq 2$, $0 < \alpha < 1$ and $\tau > 0$ then the linear map $A : W^{2,p}_\delta \times W^{1,p}_\delta \to W^{0,p}_\delta$ defined by*

$$A(h, w) = D\Phi|_{(g,\pi)}(h, w) - (0, \tfrac{1}{2}h^l_j J_l)$$

is surjective for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$. In particular, $D\Phi|_{(g,\pi)} : W^{2,p}_\delta \times W^{1,p}_\delta \to W^{0,p}_\delta$ is surjective for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$.

Proof. The first step is to show that A has closed range. For this we compute

$$\begin{aligned} A(vg, \mathring{\mathcal{L}}_Z g) = \Big(&(n-1)(\Delta^g v - nv) + (\mathrm{Scal}^g + n(n-1))v \\ &- 2v(\operatorname{tr}^g \pi - \tfrac{1}{n-1}(\operatorname{tr}^g \pi)^2 + |\pi|^2_g) + 2\langle \pi, \mathring{\mathcal{L}}_Z g\rangle, \\ &(\Delta_L Z)_i - v(\operatorname{div}^g \pi)_i + (\tfrac{n}{2} - 1)\pi^j_i \nabla_j v - \tfrac{1}{2}\operatorname{tr}^g \pi \nabla_i v - \tfrac{1}{2} v J_j \Big). \end{aligned}$$

Reasoning as in the proof of Lemma 3.2 we conclude that the operator

$$(v, Z) \mapsto A(vg, \mathring{\mathcal{L}}_Z g)$$

is a Fredholm operator $W^{2,p}_\delta \to W^{0,p}_\delta$ for $1 < p < \infty$ and $-1 < \delta + \frac{n-1}{p} < n$. Its range is contained in the range of the operator A. Consequently, the range of the operator A has finite codimension in $W^{0,p}_\delta$, and hence it is closed.

Next we need to show that $\ker A^*$ is trivial. Let p^* be such that $\frac{1}{p} + \frac{1}{p^*} = 1$. Then $W^{0,p^*}_{-\delta}$ is dual to $W^{0,p}_\delta$ under the standard L^2 pairing, see [30, Chapter 3].

Note that we have $-1 < -\delta + \frac{n-1}{p^*} < n$ as a consequence of $-1 < \delta + \frac{n-1}{p} < n$. It follows from (16) that $\ker A^*$ consists of $(V,X) \in W^{0,p^*}_{-\delta}$ such that

$$
\begin{aligned}
(\Delta V) g_{ij} - \nabla_i \nabla_j V + V \mathrm{Ric}_{ij} = 2V \left(1 - \frac{\operatorname{tr}^g \pi}{n-1}\right) \pi_{ij} + 2V \pi_{ik} \pi^k_j \\
- \tfrac{1}{2} (\pi_{jk} \nabla_i X^k + \pi_{ik} \nabla_j X^k) \\
+ \tfrac{1}{2} (\operatorname{div} \pi)_k X^k g_{ij} + \tfrac{1}{4} \langle \pi, \mathcal{L}_X g \rangle g_{ij} \\
- \tfrac{1}{2} X^k \nabla_k \pi_{ij} - \tfrac{1}{2} (\operatorname{div} X) \pi_{ij} \\
+ \tfrac{1}{4} (X_i J_j + X_j J_i), \\
\mathcal{L}_X g = 4V \left(1 - \frac{\operatorname{tr}^g \pi}{n-1}\right) g + 4V\pi.
\end{aligned}
\tag{17}
$$

As a consequence of the second equation we have $\mathcal{L}_X g \in W^{0,p^*}_{-\delta}$. Taking the trace of the first equation we have

$$
\Delta V - nV = V \star O^\alpha(e^{-\tau r}) + X \star O^\alpha(e^{-\tau r}) + \mathcal{L}_X g \star O^\alpha(e^{-\tau r}), \tag{18}
$$

where $O^\alpha(e^{-\tau r})$ denotes a section T of some geometric tensor bundle of appropriate type such that $T \in C^{0,\alpha}_\tau$, and $A \star B$ denotes a tensor which is obtained from $A \otimes B$ by raising and lowering indices, taking a number of contractions, and switching a number of components in the product. By standard elliptic regularity [30, Lemma 4.8 (a)] we conclude that $V \in W^{2,p^*}_{-\delta}$. As a consequence of the second equation in (17) we have

$$
\mathring{\mathcal{L}}_X g = 4V \left(\pi - \frac{\operatorname{tr}^g \pi}{n} g\right).
$$

Taking the divergence, we obtain

$$
\Delta_L X = V \star O^\alpha(e^{-\tau r}) + \nabla V \star O^\alpha(e^{-\tau r}). \tag{19}
$$

Thus $X \in W^{2,p^*}_{-\delta}$, again by standard elliptic regularity. Since $(V,X) \in W^{2,p^*}_{-\delta}$ the right hand sides of equations (18) and (19) are both in $W^{0,p^*}_{-\delta+\tau}$. Using Proposition A.2, improved elliptic regularity [30, Proposition 6.5], and the continuity of the embedding $W^{k,p^*}_{\varepsilon} \hookrightarrow W^{k,p^*}_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, we conclude that $(V,X) \in W^{2,p^*}_{\gamma}$ for any γ such that $-1 < \gamma + \frac{n-1}{p^*} < n$. Therefore we may without loss of generality assume that $1 < \gamma < n - \frac{n-1}{p^*} = 1 + \frac{n-1}{p}$.

In fact, we can show that $(V,X) \in C^{2,\beta}_\gamma$ for some $0 < \beta < 1$. Indeed, if $p^* < n$ then $(V,X) \in W^{2,\frac{np^*}{n-p^*}}_\gamma$ by the Sobolev embedding theorem [30, Lemma 3.6 (c)] and standard elliptic regularity applied to equations (18) and (19). Repeating this argument we obtain that $(V,X) \in W^{2,q}_\gamma$ for some $q > n$ and thus $(V,X) \in C^{1,\beta}_\gamma$ for some $0 < \beta < 1$ by Sobolev embedding [30, Lemma 3.6 (c)]. Applying standard elliptic regularity to the equations (18) and (19) we conclude that $(V,X) \in C^{2,\beta}_\gamma$.

Next we show that (V,X) vanishes to infinite order at infinity. That is, $(V,X) = O(e^{-Nr})$ for any $N > 0$. As a consequence of (17) and Definition 2.2 we see that (V,X) is a solution to the system

$$\begin{aligned}\mathrm{Hess}^b V - Vb &= V \star O(e^{-\tau r}) + \nabla V \star O(e^{-\tau r}) + X \star O(e^{-\tau r}) + \nabla X \star O(e^{-\tau r}),\\ \mathcal{L}_X b &= 4Vb + X \star O_1(e^{-\tau r}) + V \star O_1(e^{-\tau r}),\end{aligned}$$

where $O_1(e^{-\tau r})$ denotes a section T of the appropriate geometric tensor bundle such that $T \in C^1_\tau$. From the first equation and the fact that $(V,X) \in C^{2,\beta}_\gamma$ we conclude that V satisfies the ordinary differential equation

$$\partial^2_{rr} V - V = \widetilde{f}$$

along radial geodesic rays, where $\widetilde{f} = O(e^{-(\tau+\gamma)r})$. Since $\tau + \gamma > 1$, it follows that $V = O(e^{-(\tau+\gamma)r})$, see formula (46) for the explicit form of the solution. Then $V \in C^{2,\beta}_{\tau+\gamma}$ by standard elliptic regularity applied to (18). Combining this with the second equation, we see that $(\mathcal{L}_X b)_{rr} = O(e^{-(\tau+\gamma)r})$, which yields $\partial_r X_r = O(e^{-(\tau+\gamma)r})$. Integrating this relation from r to ∞, we obtain that $X_r = O(e^{-(\tau+\gamma)r})$. Note that as a consequence of this relation we also have $\partial_\mu X_r = O(e^{-(\tau+\gamma)r})$, which can be seen by first differentiating with respect to μ and then integrating from r to ∞. Here we work in polar coordinates for hyperbolic space, $(\mathbb{H}^n, b) = ((0,\infty) \times S^{n-1}, dr^2 + \sinh^2 r\, \sigma)$, the subscript r denotes the radial component and μ denotes components in a coordinate system on the sphere. It follows that

$$\partial_r X_\mu - 2 \coth r X_\mu = \bar{f},$$

where $\bar{f} = O(e^{-(\tau+\gamma-1)r})$, and hence $X_\mu = \sinh^2 r \int_r^\infty \frac{\bar{f}}{\sinh^2 s}\, ds = O(e^{-(\tau+\gamma-1)r})$. Thus $|X|_b = O(e^{-(\tau+\gamma)r})$, and hence $X \in C^{2,\beta}_{\tau+\gamma}$ by standard elliptic regularity applied to (19). We proceed by induction and deduce that $(V,X) = O(e^{-Nr})$ for any $N > 0$.

To conclude the proof, note that, as a consequence of (17), $(V,X) \in C^2$ satisfies a differential inequality

$$|\Delta(V,X)| \le C\left(|(V,X)| + |\nabla(V,X)|\right),$$

where $\Delta = \nabla^*\nabla$ is the rough Laplacian. Since (V, X) vanishes to infinite order at infinity, a standard unique continuation argument, see Appendix C, implies that (V, X) vanishes identically. □

We use the subscript c on the notation for a function space to denote the subspace of sections with compact support.

Lemma 3.4. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type (k, α, τ), where $k \geq 2$, $0 < \alpha < 1$ and $\frac{n}{2} < \tau < n$. Then for any $f \in C^{k-2,\alpha}_{\tau}$ there exist $(v, Z) \in C^{k,\alpha}_{\tau}$ and symmetric 2-tensors $(h, w) \in C^{k+1,\alpha}_{c}$ so that*

$$DT|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(h, w) = (f, \tfrac{1}{2}h^l_j J_l). \tag{20}$$

If in addition $f \in C^{k-2,\alpha}_{n+\tau_0}$ for some $\tau_0 > 0$ then $(v, Z) \in C^{k,\alpha}_{n}$.

Proof. For some $p > n$ we choose $\gamma > 0$ so that $-1 < \gamma + \frac{n-1}{p} < \tau$. In this case $C^{l,\alpha}_{\tau} \hookrightarrow W^{l,p}_{\gamma}$ for $l = 0, 1, \ldots, k$, see [30, Lemma 3.6 (c)]. Further, by Lemma 3.2 the operator $DT|_{(1,0)} : W^{2,p}_{\gamma} \to W^{0,p}_{\gamma}$ is Fredholm with index zero. Since the linear map $A : W^{2,p}_{\gamma} \times W^{1,p}_{\gamma} \to W^{0,p}_{\gamma}$ defined in Lemma 3.3 is surjective, we can find symmetric 2-tensors $(h_i, w_i) \in W^{2,p}_{\gamma} \times W^{1,p}_{\gamma}$, $i = 1, \ldots, N$, such that their images $A(h_i, w_i)$ span a subspace that complements $DT|_{(1,0)}(W^{2,p}_{\gamma})$ in $W^{0,p}_{\gamma}$. Note that by the density of compactly supported sections, [30, Lemma 3.9], together with the continuity of A we may assume that $(h_i, w_i) \in C^{k+1,\alpha}_{c}$. Consequently, since $f \in W^{0,p}_{\gamma}$ we can find $(v, Z) \in W^{2,p}_{\gamma}$ and $(h, w) \in C^{k+1,\alpha}_{c}$ such that (20) holds. By Sobolev embedding $(v, Z) \in C^{1,\alpha}_{\gamma}$. Since $\gamma > 0$ and $f \in C^{0,\alpha}_{\tau}$ it follows from (15) that $(\Delta v - nv, \Delta_L Z) \in C^{0,\alpha}_{\gamma}$. From [30, Proposition 6.5] we conclude that $(v, Z) \in C^{2,\alpha}_{\tau}$ and $(v, Z) \in C^{k,\alpha}_{\tau}$ follows by a standard bootstrap argument.

To prove the second claim note that outside a sufficiently large compact set $(v, Z) \in C^{k,\alpha}_{\tau}$ satisfies $(\Delta v - nv, \Delta_L Z) \in C^{k-2,\alpha}_{n+\varepsilon}$ for some $\varepsilon > 0$. This is an immediate consequence of (15) and the fact that $\tau > \frac{n}{2}$. The claim follows from Proposition B.2. □

Proof of Theorem 3.1. With the above lemmas at hand, the proof differs very little from that of [22, Theorem 22]. We choose a positive $C^{k+1,\alpha}$ function f such that

$$f = e^{-(n+\min\{1,\tau_0\})r}$$

near infinity, and let $(v, Z) \in C^{k,\alpha}_{n}$ and $(h, w) \in C^{k+1,\alpha}_{c}$ be a solution of the system

$$DT|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(h, w) = (-f, \tfrac{1}{2}h^l_j J_l),$$

which exists by Lemma 3.4. We will show that for a sufficiently small $t > 0$,

$$\bar{g} = (1 + tv)^{\kappa}(g + th) \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2}(\pi + t\mathring{\mathcal{L}}_Z g + tw)$$

is an initial data set whose existence is asserted in the theorem. Note that $\|g - \bar{g}\|_{C^{k,\alpha}_\tau} \leq \varepsilon$, $\|\pi - \bar{\pi}\|_{C^{k-1,\alpha}_\tau} \leq \varepsilon$ provided that t is sufficiently small.

We will verify that $\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$ for some $\gamma > 0$ depending on t. Set $u = 1 + tv$ and define

$$\Phi_1(1 + tv, tZ, th, tw) = (-2u^{\kappa}\bar{\mu}, u^{\kappa/2}\bar{J}).$$

Linearizing we have

(21)
$$\begin{aligned}\Phi_1(1 + tv, tZ, th, tw) &= \Phi_1(1, 0, 0, 0) + tD\Phi_1|_{(1,0,0,0)}(v, Z, h, w) + \mathcal{R}\\ &= (-2\mu, J) + tDT|_{(1,0)}(v, Z) + tD\Phi|_{(g,\pi)}(h, w) + \mathcal{R}\\ &= (-2\mu, J) + t(-f, \tfrac{1}{2}h_i^k J_k) + \mathcal{R},\end{aligned}$$

where the remainder term $\mathcal{R} = \mathcal{R}(t, v, Z, h, w)$ can be written as

$$\begin{aligned}&\mathcal{R}(t, v, Z, h, w)\\ &\quad= \Phi_1(1 + tv, tZ, th, tw) - \Phi_1(1, 0, 0, 0) - tD\Phi_1|_{(1,0,0,0)}(v, Z, h, w)\\ &\quad= t\int_0^1 \left[D\Phi_1|_{(1+\theta tv,\theta tZ,\theta th,\theta tw)} - D\Phi_1|_{(1,0,0,0)}\right](v, Z, h, w)\, d\theta,\end{aligned}$$

by the mean value theorem.

We first prove that

$$|\mathcal{R}| \leq Ct^2 e^{-2nr} = O(t^2 f), \tag{22}$$

where the constant $C > 0$ does not depend on t and is uniform for all points. For this it suffices to estimate $\mathcal{R}$ outside the support of (h, w) where it takes the form

$$\mathcal{R}(t, v, Z) = t\int_0^1 \left[DT|_{(1+\theta tv,\theta tZ)} - DT|_{(1,0)}\right](v, Z)\, d\theta.$$

Using (14) we compute

$$\begin{aligned}DT|_{(u,Y)}(v, Z) = \Big(&\tfrac{4(n-1)}{n-2}(-u^{-2}v\Delta^g u + u^{-1}\Delta^g v) - n(n-1)\kappa u^{\kappa-1}v\\ &+ \kappa u^{\frac{\kappa}{2}-1}v\,\mathrm{tr}^g\,\pi + 2\langle\pi, \mathring{\mathcal{L}}_Z g\rangle + 2\langle\mathring{\mathcal{L}}_Z g, \mathring{\mathcal{L}}_Y g\rangle,\end{aligned}$$

$$(\Delta_L Z)_j + \tfrac{2(n-1)}{n-2} \nabla_k (vu^{-1})(\pi + \mathring{\mathcal{L}}_Y g)^k_j \\ + \tfrac{2(n-1)}{n-2} u^{-1} \nabla_k u (\mathring{\mathcal{L}}_Z g)^k_j - \tfrac{2}{n-2} \nabla_j (vu^{-1}) \operatorname{tr}^g \pi \Big).$$

Then it is not complicated to check that

$$\Big| DT|_{(1+\theta t v, \theta t Z)}(v, Z) - DT|_{(1,0)}(v, Z) \Big| \leq \theta t \mathcal{Q}(v, Z)$$

where $\mathcal{Q}$ is a quadratic function of v, its first and second order covariant derivatives, and $\mathring{\mathcal{L}}_Z g$, which is uniformly bounded in $\theta \in [0,1]$. Hence (22) holds.

By (21) we have

$$u^\kappa \bar{\mu} = \mu + \frac{t}{2} f + O(t^2 f) \qquad \text{and} \qquad u^{\kappa/2} \bar{J}_i = J_i + \frac{t}{2} h_i^k J_k + O(t^2 f).$$

In particular, for sufficiently small $t > 0$, we have

$$u^\kappa \bar{\mu} > \mu + \frac{t}{3} f. \tag{23}$$

Recall that h is compactly supported, hence we may write

$$\bar{g}^{ij} = u^{-\kappa} (g^{ij} - t g^{ik} h_k^j + O(t^2 f)).$$

Since f is positive, we obtain

$$\begin{aligned} &(u^\kappa |\bar{J}|_{\bar{g}})^2 \\ &\quad = u^{2\kappa} \bar{g}^{ij} \bar{J}_i \bar{J}_j \\ &\quad = (g^{ij} - t g^{ik} h_k^j + O(t^2 f))(J_i + \frac{t}{2} h_i^l J_l + O(t^2 f))(J_j + \frac{t}{2} h_j^m J_m + O(t^2 f)) \\ &\quad = |J|_g^2 + O(t^2 |J|_g f + t^3 f^2) \\ &\quad = \left(|J|_g + \frac{tf}{4} \right)^2 - \frac{tf}{2} |J|_g - \frac{t^2 f^2}{16} + O\left(t \left(\frac{tf}{2} |J|_g + \frac{t^2 f^2}{16} \right) \right) \\ &\quad < \left(|J|_g + \frac{tf}{4} \right)^2 \end{aligned}$$

for $t > 0$ small enough, so we find that

$$u^\kappa |\bar{J}|_{\bar{g}} < |J|_g + \frac{t}{4} f \tag{24}$$

for such t.

Fix $t > 0$ such that (23) and (24) hold. Note that our choice of f implies that $\sup_M (|J|_g/f) < \infty$. Therefore for any $x \in M$ such that $|\bar{J}|_{\bar{g}}(x) \neq 0$ we have

$$\frac{\bar{\mu}}{|\bar{J}|_{\bar{g}}} = \frac{u^\kappa \bar{\mu}}{u^\kappa |\bar{J}|_{\bar{g}}} > \frac{\mu + tf/3}{|J|_g + tf/4} \geq \frac{|J|_g + tf/3}{|J|_g + tf/4} = 1 + \frac{t}{12(|J|_g/f) + 3t} \geq 1 + \gamma,$$

for

$$\gamma := \frac{t}{12 \sup_M (|J|_g/f) + 3t}.$$

At points where $|\bar{J}|_{\bar{g}}(x) = 0$ we have $\bar{\mu} > 0$ by (23). Consequently, we have $\bar{\mu} > (1+\gamma)|\bar{J}|_{\bar{g}}$ everywhere on M as desired.

Note also that $(\bar{\mu}, \bar{J}) \in C^{k-2,\alpha}_{n+\tau_0'}$ for $\tau_0' = \min\{1, \tau_0\}$ by (21), the asymptotics of u, and the properties of $\mathcal{R}$. In particular $\|(\mu, J) - (\bar{\mu}, \bar{J})\|_{C^0_{n+\tau_0'}}$ can be made arbitrarily small for a sufficiently small t. Thus Proposition 2.8 guarantees that $\left|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)\right| < \varepsilon$ holds. $\square$

4. Perturbation to conformally hyperbolic asymptotics

In this section we prove the following result.

Theorem 4.1. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type $(k, \alpha, \tau, \tau_0)$ for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$. Assume that the dominant energy condition $\mu \geq |J|_g$ holds. Then for every $\tau' < \tau$ and $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$, with the energy and momentum density denoted by $(\bar{\mu}, \bar{J})$, which has conformally hyperbolic asymptotics with respect to the same chart, and is such that*

$$\|g - \bar{g}\|_{C^{k,\alpha}_{\tau'}} < \varepsilon, \qquad \|\pi - \bar{\pi}\|_{C^{k-1,\alpha}_{\tau'}} < \varepsilon,$$

the strict dominant energy condition

$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

holds, and

$$|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)| < \varepsilon$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

In [18, Appendix B] a similar result was proven in the simpler case when $\pi = 0$. The proof of Theorem 4.1 is very similar to [22, Proof of Theorem 18]. Its main ingredients are Theorem 3.1 and the following lemma.

Lemma 4.2. *Let (g, π) be an asymptotically hyperbolic initial data set of type (k, α, τ) for $k \geq 2$, $0 < \alpha < 1$ and $\frac{n}{2} < \tau < n$ and suppose that $\frac{n}{2} < \tau' < \tau$. Then there are positive constants C_0 and δ_0 such that for any $(\bar{\mu}, \bar{J}) \in C^{k-2,\alpha}_\tau$ with $\|(\mu - \bar{\mu}, J - \bar{J})\|_{C^{k-2,\alpha}_\tau} \leq \delta \leq \delta_0$, there exists an initial data set $(\bar{g}, \bar{\pi})$ of type (k, α, τ) with the following properties:*

- *The energy and momentum densities of $(\bar{g}, \bar{\pi})$ are $\bar{\mu}$ and $\bar{J}$.*
- *Outside of a compact set $(\bar{g}, \bar{\pi})$ is of the form*

$$\bar{g} = u^\kappa b, \qquad \bar{\pi} = u^{\kappa/2} \mathring{\mathcal{L}}_Y b$$

  for $(u - 1, Y) \in C^{k,\alpha}_\tau$.
- *The initial data set $(\bar{g}, \bar{\pi})$ is close to (g, π) in the sense that*

$$\|g - \bar{g}\|_{C^{k,\alpha}_{\tau'}} \leq C_0 \delta, \qquad \|\pi - \bar{\pi}\|_{C^{k-1,\alpha}_{\tau'}} \leq C_0 \delta.$$

Proof. The proof uses the construction introduced by Corvino and Schoen in [17, Proof of Theorem 1], which is similar to the one that was used in the proof of Theorem 3.1. Given (g, π) as in the statement of the theorem and $(u - 1, Y) \in C^{k,\alpha}_\tau$, we define the map

$$T_{(g,\pi)}(u, Y) = \Phi(u^\kappa g, u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g)).$$

It follows from (14) that the components of $T_{(g,\pi)}$ are given by

$$\begin{aligned} -2\widetilde{\mu} &= \tfrac{4(n-1)}{n-2} u^{-\kappa-1} \Delta^g u - u^{-\kappa} \mathrm{Scal}^g - n(n-1) + 2u^{-\frac{\kappa}{2}} \operatorname{tr}^g \pi \\ &\qquad - \tfrac{1}{n-1} u^{-\kappa} (\operatorname{tr}^g \pi)^2 + u^{-\kappa} \left(|\pi|^2_g + 2\langle \pi, \mathring{\mathcal{L}}_Y g \rangle + |\mathring{\mathcal{L}}_Y g|^2_g \right), \\ \widetilde{J}_j &= u^{-\frac{\kappa}{2}} (\Delta_L Y + \operatorname{div}^g \pi)_j + \tfrac{2(n-1)}{n-2} u^{-\frac{\kappa}{2}-1} (\pi + \mathring{\mathcal{L}}_Y g)^k_j \nabla_k u \\ &\qquad - \tfrac{2}{n-2} u^{-\frac{\kappa}{2}-1} \operatorname{tr}^g \pi \nabla_j u, \end{aligned} \tag{25}$$

for $j = 1, 2, \ldots, n$. From this formula it is straightforward to compute the linearization

$$\begin{aligned} &DT_{(g,\pi)}|_{(1,0)}(v, Z) \\ &\quad = \Big(\tfrac{4(n-1)}{n-2} \Delta^g v + \tfrac{4}{n-2} \mathrm{Scal}^g v - \tfrac{4}{n-2} (\operatorname{tr}^g \pi) v \\ &\qquad\quad + \tfrac{\kappa}{n-1} (\operatorname{tr}^g \pi)^2 v - \tfrac{4}{n-2} |\pi|^2_g v + 2\langle \pi, \mathring{\mathcal{L}}_Z g \rangle, \\ &\qquad\quad (\Delta_L Z)_j - \tfrac{2}{n-2} (\operatorname{div}^g \pi)_j v + \tfrac{2(n-1)}{n-2} \pi^k_j \nabla_k v - \tfrac{2}{n-2} (\operatorname{tr}^g \pi) \nabla_j v \Big). \end{aligned}$$

Since $\mathrm{Scal}^g + n(n-1) \in C_\tau^{k-2,\alpha}$, we may argue as in the proof of Lemma 3.2 and show that for $2 \le l \le k$ the operator $DT_{(g,\pi)}|_{(1,0)}$ is Fredholm of index zero as an operator $C_\delta^{l,\alpha} \to C_\delta^{l-2,\alpha}$ for $-1 < \delta < n$ and as an operator $W_\gamma^{l,p} \to W_\gamma^{l-2,p}$ for $-1 < \gamma + \frac{n-1}{p} < n$. We can now choose a sufficiently large $p > n$ and γ such that $\tau' < \gamma < \tau - \frac{n-1}{p}$ in which case both $C_\tau^{l,\alpha} \hookrightarrow W_\gamma^{l,p}$ for $l = 0, 1, \ldots, k$ and the operator $DT_{(g,\pi)}|_{(1,0)} : W_\gamma^{2,p} \to W_\gamma^{0,p}$ is Fredholm of index zero. Let U be the subspace complementing the kernel of $DT_{(g,\pi)}|_{(1,0)}$ in $W_\gamma^{2,p}$. Arguing as in the proof of Lemma 3.4 we conclude that there exist finitely many pairs of compactly supported symmetric 2-tensors $(h_i, w_i) \in C_c^{k+1,\alpha}$, $i = 1, \ldots, N$, such that their images $D\Phi|_{(g,\pi)}(h_i, w_i)$ form a basis for a subspace which complements $DT_{(g,\pi)}|_{(1,0)}(W_\gamma^{2,p})$ in $W_\gamma^{0,p}$. Set $V := \operatorname{span}\{(h_i, w_i)\}_{i=1,\ldots,N}$. We define the map $\Xi_{(g,\pi)} : U \times V \to W_\gamma^{0,p}$ by

$$\Xi_{(g,\pi)} : (u, Y, h, w) \mapsto \Phi(u^\kappa g + h, u^{\kappa/2}(\pi + \mathring{\mathcal{L}}_Y g) + w). \tag{26}$$

Then the linearization $D\Xi_{(g,\pi)}|_{(1,0,0,0)} : U \times V \to W_\gamma^{0,p}$ is given by

$$D\Xi_{(g,\pi)}|_{(1,0,0,0)} : (v, Z, \eta, \omega) \mapsto DT_{(g,\pi)}|_{(1,0)}(v, Z) + D\Phi|_{(g,\pi)}(\eta, \omega)$$

and is an isomorphism by construction.

Using the chart at infinity $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus B_{R_0}$, we define the cut-off function $\chi_\lambda(x) = \chi(r(x)/\lambda)$, where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\chi(r) = 1$ for $r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. For a sufficiently large $\lambda > 0$, the cut-off initial data (g_λ, π_λ) is given by

$$g_\lambda = \chi_\lambda g + (1 - \chi_\lambda)\Psi^* b, \qquad \pi_\lambda = \chi_\lambda \pi.$$

Now for any $\tau_1 < \tau$ we have

$$\|g - g_\lambda\|_{C_{\tau_1}^{k,\alpha}} \to 0 \qquad \text{and} \qquad \|\pi - \pi_\lambda\|_{C_{\tau_1}^{k-1,\alpha}} \to 0 \tag{27}$$

as $\lambda \to \infty$. Hence we also have

$$\|\Phi(g, \pi) - \Phi(g_\lambda, \pi_\lambda)\|_{C_{\tau_1}^{k-2,\alpha}} \to 0 \tag{28}$$

as $\lambda \to \infty$.

Similarly to (26), we define the map $\Xi_{(g_\lambda,\pi_\lambda)} : U \times V \to W_\gamma^{0,p}$ by

$$\Xi_{(g_\lambda,\pi_\lambda)} : (u, Y, h, w) \mapsto \Phi(u^\kappa g_\lambda + h, u^{\kappa/2}(\pi_\lambda + \mathring{\mathcal{L}}_Y g_\lambda) + w).$$

The linearization $D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)} : U \times V \to W^{0,p}_\gamma$ is given by

$$D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)} : (v, Z, \eta, \omega) \mapsto DT_{(g_\lambda,\pi_\lambda)}|_{(1,0)}(v, Z) + D\Phi|_{(g_\lambda,\pi_\lambda)}(\eta, \omega).$$

As a consequence of (27), the operators $D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)}$ converge to the isomorphism $D\Xi_{(g,\pi)}|_{(1,0,0,0)}$ as $\lambda \to \infty$ in the uniform operator topology. It follows that there exists a positive λ_0 such that for any $\lambda \geq \lambda_0$ the linearization $D\Xi_{(g_\lambda,\pi_\lambda)}|_{(1,0,0,0)}$ is an isomorphism. Note that $\Xi_{(g_\lambda,\pi_\lambda)}(1, 0, 0, 0) = \Phi(g_\lambda, \pi_\lambda) = (-2\mu_\lambda, J_\lambda)$. Applying the Inverse Function Theorem, see for example [24, Theorem 4.2 and Remark 4.3]), it is not complicated to check that there exists $\rho_0 > 0$ depending only on (g, π) such that $\Xi_{(g_\lambda,\pi_\lambda)} : B_{\rho_0}(1, 0, 0, 0) \to \Xi_{(g_\lambda,\pi_\lambda)}(B_{\rho_0}(1, 0, 0, 0))$ is a diffeomorphism for any $\lambda \geq \lambda_0$. Furthermore, there exists a constant $C > 0$ depending only on (g, π) such that

$$C\|(u, Y, h, w)-(1, 0, 0, 0)\|_{W^{2,p}_\gamma} \leq \|\Xi_{(g_\lambda,\pi_\lambda)}(u, Y, h, w)-(-2\mu_\lambda, J_\lambda)\|_{W^{0,p}_\gamma} \tag{29}$$

holds for any $(u, Y, h, w) \in B_{\rho_0}(1, 0, 0, 0)$, and such that

$$B_{C\rho_0}(-2\mu_\lambda, J_\lambda) \subset \Xi_{(g_\lambda,\pi_\lambda)}(B_{\rho_0}(1, 0, 0, 0)).$$

Now suppose that $(\bar\mu, \bar J)$ is such that $\|(2(\mu-\bar\mu), J-\bar J)\|_{C^{0,\alpha}_\tau} \leq \delta$. By (28) we may assume that $\lambda \geq \lambda_0$ is such that $\|(2(\mu-\mu_\lambda), J-J_\lambda)\|_{C^{0,\alpha}_{\tau_1}} \leq \delta$ where $\tau_1 < \tau$. If we further assume that $\tau_1 > \gamma + \frac{n-1}{p}$ so that $C^{0,\alpha}_{\tau_1} \hookrightarrow W^{0,p}_\gamma$ it follows that there exists $\delta_0 > 0$ depending on C and ρ_0 such that $(-2\bar\mu, \bar J) \in B_{C\rho_0}(-2\mu_\lambda, J_\lambda)$ as long as $\delta \leq \delta_0$. Then it follows from the above discussion that there exists $(u, Y, h, w) \in B_{\rho_0}(1, 0, 0, 0)$ such that $\Xi_{(g_\lambda,\pi_\lambda)}(u, Y, h, w) = (-2\bar\mu, \bar J)$. As a consequence of (29)

$$\|(u-1, Y, h, w)\|_{W^{2,p}_\gamma} \leq C_0\delta$$

for a uniform constant C_0. By the Sobolev embedding we have $(u-1, Y) \in C^{1,\alpha}_\gamma$ and it follows from (25) that outside of a compact set (u, Y) satisfies

$$\begin{aligned} -2u^{\kappa+1}\bar\mu &= \tfrac{4(n-1)}{n-2}\Delta^b u + n(n-1)(u-u^{\kappa+1}) + u|\mathring{\mathcal{L}}_Y b|^2_b, \\ u^{\kappa/2}\bar J_j &= (\Delta_L Y)_j + \tfrac{2(n-1)}{n-2}u^{-1}(\mathring{\mathcal{L}}_Y b)^k_j \nabla_k u. \end{aligned} \tag{30}$$

Consequently, $v = u-1$ and Y are such that $(v, Y) \in C^{1,\alpha}_\gamma$ and recalling that $\gamma > n/2$ it follows from (30) that

$$(\Delta^b v - nv, \Delta_L Y) \in C^{0,\alpha}_\tau.$$

Hence $(v, Y) \in C^{2,\alpha}_\tau$ by the improved elliptic regularity [30, Proposition 6.5]. Further regularity follows by a standard bootstrap argument.

It is now straightforward to check that there exists a constant $C_0 > 0$ such that the initial data

$$\bar{g} = u^\kappa g_\lambda + h, \qquad \bar{\pi} = u^{\kappa/2}(\pi_\lambda + \mathring{\mathcal{L}}_Y g_\lambda) + w$$

has all the required properties. In particular, the last claim of the theorem follows using the fact that $\|(u-1, Y)\|_{C^{1,\alpha}_{\tau'}} \le C_0\delta$ (up to increasing C_0 if necessary) as a consequence of $\gamma > \tau'$ and the Sobolev embedding, and applying Schauder estimates [30, Lemma 4.8(b)] throughout. ◻

The following result is not complicated to prove.

Proposition 4.3. *Suppose that (M, g, π) is an asymptotically hyperbolic initial data set of type $(k, \alpha, \tau, \tau_0)$ for $k \ge 2$, $0 < \alpha < 1$ and $\tau_0 \ge 1$ such that*

$$g = u^\kappa b, \qquad \pi = u^{\kappa/2}\mathring{\mathcal{L}}_Y b$$

outside of a compact set for some $(u-1, Y) \in C^{k,\alpha}_\tau$. Then (g, π) has conformally hyperbolic asymptotics in the sense of Definition 2.3.

Proof. Let $v = u - 1$. It follows from (30) that outside of a compact set

$$(\Delta^b v - nv, \Delta_L Y) \in C^{k-2,\alpha}_{n+\varepsilon}$$

for some $\varepsilon > 0$. Then $(v, Y) \in C^{k,\alpha}_n$ by Proposition B.2. More specifically, from the proof of Proposition B.2 we see that (v, Y) is of the form (5), where (v_0, Y_0) does not depend on r, and $(v_1, Y_1) \in C^{2,\alpha}_{k+\varepsilon}$.

Inserting $u = v+1$ and Y in (30) we conclude that $(v_1, Y_1) \in C^{k,\alpha}_{n+\varepsilon}$ satisfies

$$(\Delta^b v_1 - nv_1, \Delta_L Y_1) \in C^{k-2,\alpha}_{n+1}.$$

Thus $(v_1, Y_1) \in C^{k,\alpha}_{n+1}$ by Proposition B.2. ◻

Proof of Theorem 4.1. By Theorem 3.1 we may without loss of generality assume that $\mu > (1+\gamma)|J|_g$ for some $\gamma > 0$. Let ξ be a smooth function such that $\xi(x) = e^{-r(x)}$ outside a compact set. For χ_λ as in the proof of Lemma 4.2 set $\xi_\lambda := \chi_\lambda + (1-\chi_\lambda)\xi$. Then $(\xi_\lambda\mu, \xi_\lambda J) \in C^{k-2,\alpha}_{n+1+\tau_0}$ and

$$\|(\mu, J) - (\xi_\lambda\mu, \xi_\lambda J)\|_{C^{k-2,\alpha}_{n+\tau_0'}} \to 0 \tag{31}$$

as $\lambda \to \infty$ for any $\tau_0' < \tau_0$. By Lemma 4.2 and Proposition 4.3 we may construct initial data sets (g_λ, π_λ) with conformally hyperbolic asymptotics, whose energy and momentum densities are $(\xi_\lambda \mu, \xi_\lambda J)$, and such that

$$\|g - g_\lambda\|_{C^{k,\alpha}_{\tau'}} \to 0, \qquad \|\pi - \pi_\lambda\|_{C^{k-1,\alpha}_{\tau'}} \to 0 \tag{32}$$

for any $\tau' < \tau$ as $\lambda \to \infty$. In particular, $\|g - g_\lambda\|_{C^0} \to 0$ as $\lambda \to \infty$ thus $|J|^2_{g_\lambda} = |J|^2_g(1 + o(1))$ as $\lambda \to \infty$. It follows that

$$\xi_\lambda \mu > \xi_\lambda (1+\gamma)|J|_g \geq \left(1 + \frac{\gamma}{2}\right) |\xi_\lambda J|_{g_\lambda}$$

for λ sufficiently large, hence (g_λ, π_λ) satisfies the strict dominant energy condition. Further, since (31) and (32) hold for any $0 < \tau_0' < \tau_0$ and $\frac{n}{2} < \tau' < \tau$, it follows by Proposition 2.8 that

$$\mathcal{M}_{(g_\lambda, \pi_\lambda)}(V) \to \mathcal{M}_{(g,\pi)}(V)$$

for all $V \in \{V_{(0)}, V_{(1)}, \dots, V_{(n)}\}$ as $\lambda \to \infty$. $\square$

5. Initial data sets with Wang's asymptotics

In this section we refine the results of Section 3 for another important class of asymptotically hyperbolic initial data.

Definition 5.1. Let (M, g, π) be an asymptotically hyperbolic initial data set of type (k, α, n, τ_0) for $k \geq 2$, $0 \leq \alpha < 1$ and $\tau_0 > 0$. We say that (M, g, π) has *Wang's asymptotics* with respect to the chart at infinity

$$\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0}$$

if the following holds:

1. The pushforward of the metric g under Ψ satisfies

$$\Psi_* g = dr^2 + \sinh^2 r g_r, \tag{33}$$

where

$$g_r = \sigma + m e^{-nr} + O^{k,\alpha}(e^{-(n+1)r})$$

is an r-dependent family of symmetric 2-tensors on S^{n-1}, $m \in C^{k,\alpha}$ is a symmetric 2-tensor on S^{n-1}, and the expression $O^{k,\alpha}(e^{-(n+1)r})$ stands for a tensor in the weighted Hölder space $C^{k,\alpha}_{n+1}(\mathbb{H}^n)$.

2. The pushforward of the 2-tensor π under Ψ satisfies

$$(34)\qquad \begin{aligned}(\Psi_*\pi)_{rr} &= p_{rr}e^{-nr} + q_{rr},\\ (\Psi_*\pi)_{r\mu} &= p_{r\mu}e^{-(n-1)r} + q_{r\mu},\\ (\Psi_*\pi)_{\mu\nu} &= p_{\mu\nu}e^{-(n-2)r} + q_{\mu\nu},\end{aligned}$$

where $p \in C^{k-1,\alpha}_{loc}$ does not depend on r, $q \in C^{k-1,\alpha}_{n+1}$, and μ, ν denote components in a coordinate system on the sphere.

Asymptotically hyperbolic metrics g with asymptotics (33) were considered by Wang in [39] and have been studied in various contexts, see for example [1], [5], [35], and [36].

Note that any sufficiently regular conformally compactifiable metric with the round sphere as the boundary at infinity and deviating from the hyperbolic metric at the "critical" order $|g - b|_b = O(e^{-nr})$ can be written in the form (33) in appropriate coordinates. See, for example, [5, Section IV], [1, Section 3], or [16]. We will make use of this fact in the proofs of Theorem 5.2 and Theorem 5.3 below.

In the case when (M, g, π) is an initial data set with Wang's asymptotics a direct computation shows that the mass functional is given by

$$\mathcal{M}(V_{(0)}) = \tfrac{1}{2(n-1)\omega_{n-1}} \int_{S^{n-1}} (n \operatorname{tr}_\sigma m - 2p_{rr})\, d\mu^\sigma,$$

and

$$\mathcal{M}(V_{(i)}) = \tfrac{1}{2(n-1)\omega_{n-1}} \int_{S^{n-1}} x^i (n \operatorname{tr}_\sigma m - 2p_{rr})\, d\mu^\sigma,$$

for $i = 1, \ldots, n$, where m and p_{rr} are as in Definition 5.1. Furthermore, we have the following result.

Theorem 5.2. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type $(k+1, \alpha, n, \tau_0)$ for $k \geq 2$, $0 < \alpha < 1$ and $\tau_0 > 0$ such that it has Wang's asymptotics with respect to the chart at infinity $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0}$ and satisfies the dominant energy condition $\mu \geq |J|_g$. Then, for any $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$, with the energy and momentum density denoted by $(\bar{\mu}, \bar{J})$, of type $(k+1, \alpha, n, \tau_0')$ for some $\tau_0' > 0$ such that*

$$(35)\qquad \|g - \bar{g}\|_{C^{k+1,\alpha}_n} < \varepsilon, \qquad \textit{and} \qquad \|\pi - \bar{\pi}\|_{C^{k,\alpha}_n} < \varepsilon,$$

the strict dominant energy condition

$$(36)\qquad \bar{\mu} > |\bar{J}|_{\bar{g}}$$

holds, and

$$|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)| < \varepsilon \tag{37}$$

for any $V \in \{V_{(0)}, V_{(1)}, \dots, V_{(n)}\}$. *Furthermore, there is a coordinate chart at infinity* $\Phi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0}$ *such that* $(M, \bar{g}, \bar{\pi})$ *is an asymptotically hyperbolic initial data set of type* (k, α, n, τ_0') *with Wang's asymptotics with respect to this chart.*

Proof. Let $\varepsilon > 0$ be fixed. Since $C^{k,\alpha}_n \hookrightarrow C^{k,\alpha}_\tau$ for $n > \tau$ and $k = 0, 1, \dots$, we may view (M, g, π) as initial data of type $(k+1, \alpha, \tau, \tau_0)$ for $k \geq 2$, $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$. Arguing as in the proof of Theorem 3.1 one shows that there exist $(v, Z) \in C^{k,\alpha}_n$, and $(h, w) \in C^{k+1,\alpha}_c$ such that for some sufficiently small $t > 0$ the perturbed initial data set

$$\bar{g} = (1 + tv)^\kappa (g + th) \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2}(\pi + t\mathring{\mathcal{L}}_Z g + tw)$$

satisfies (35) and (36). Moreover, if the positive function f used in this construction is chosen so $f = O(e^{-(n+1)r})$ then we have $(v, Z) = (v_0, Z_0)e^{-nr} + (v_1, Z_1)$ for $(v_0, Z_0) \in C^{k,\alpha}_{loc}$ independent of r and $(v_1, Z_1) \in C^{k,\alpha}_{n+1}$ as a consequence of (15) and the fact that (M, g, π) is initial data of type $(k+1, \alpha, n, \tau_0)$ (compare the proof of Proposition 4.3). Since in this case f might decay faster than $J = O(e^{-(n+\tau_0)r})$, it is not clear that there is a $\gamma > 0$ such that $\bar{\mu} > (1 + \gamma)|\bar{J}|_{\bar{g}}$. However, this is not important in the current setting since we do not intend to make a further perturbation of $(\bar{g}, \bar{\pi})$.

Next we estimate the difference between the masses of the initial data sets (g, π) and $(\bar{g}, \bar{\pi})$. Outside a compact set we have

$$\bar{g} = (1 + tv)^\kappa g \qquad \text{and} \qquad \bar{\pi} = (1 + tv)^{\kappa/2}(\pi + t\mathring{\mathcal{L}}_Z g). \tag{38}$$

Set $U = (1 + tv)^\kappa - 1$, then $\bar{g} - g = Ug$, where $U = O_1(tv) = O_1(e^{-nr})$. We also have

$$\bar{\pi} - \pi = \Big((1 + tv)^{\kappa/2} - 1\Big)\pi + t(1 + tv)^{\kappa/2}\mathring{\mathcal{L}}_Z g = t\mathring{\mathcal{L}}_Z b + O(e^{-2nr}).$$

Then a straightforward computation shows that for any $V \in \{V_{(0)}, V_{(1)}, \dots, V_{(n)}\}$ we have

$$\begin{aligned}
&\mathcal{M}_{(\bar{g},\bar{\pi})}(V) - \mathcal{M}_{(g,\pi)}(V) \\
&\quad = \tfrac{1}{2(n-1)\omega_{n-1}} \lim_{R\to\infty} \int_{S_R} \Big((n-1)(UdV - VdU)(\nu) - 2t\mathring{\mathcal{L}}_Z b(\nabla^b V, \nu)\Big)\, d\mu^b.
\end{aligned}$$

Consequently, we have

$$|\mathcal{M}_{(\bar{g},\bar{\pi})}(V) - \mathcal{M}_{(g,\pi)}(V)| \leq Ct(|v|_{C^1_n} + |Z|_{C^1_n})$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ and (37) follows, after decreasing t if necessary.

Now recall that g has asymptotic expansion (33) with respect to the chart Ψ at infinity. With respect to this chart, $\bar{g}$ has the expansion

$$\bar{g} = (1+tv)^\kappa dr^2 + \sinh^2 r \left(\sigma + (m + t\kappa v_0 \sigma)e^{-nr} + O^{k+1,\alpha}(e^{-(n+1)r})\right),$$

where the term $O^{k+1,\alpha}(e^{-(n+1)r})$ is an r-dependent tensor on S^{n-1} as described in Definition 5.1. While this expansion is not of the form (33), there are standard techniques to find coordinates near infinity in which $\bar{g}$ has the desired form (see for example [1, Section 3.2.1], [5, Section IV], [12], or [16, Section 2.2]). For completeness, we provide the details here.

First, using the substitution

$$r = \operatorname{arcsinh}\left(\sinh^{-1}\rho\right)$$

we bring $\bar{g}$ to the conformally compact form

$$\bar{g} = \sinh^{-2}\rho \left((1+tv)^\kappa d\rho^2 + \sigma + 2^{-n}(m + t\kappa v_0\sigma)\rho^n + \eta\right).$$

The expression in the brackets is a $C^{k+1,\alpha}$-metric on $\{0 \leq \rho \leq \rho_0\}$ for some $\rho_0 > 0$, and $\eta = \eta_{\mu\nu}dy^\mu dy^\nu$ is a ρ-dependent tensor on S^{n-1} with components $\eta_{\mu\nu} = O^{k+1,\alpha}(\rho^{n+1})$ which is to be understood in the following sense:

$$\begin{aligned} &f = O^{K,\alpha}(\rho^N) \Leftrightarrow \\ &\rho^{l-N}\partial^l_\rho \partial^{(m)}_{y^i} f \in C^{K-l-|m|,\alpha}(\{0 \leq \tau \leq \tau_0\}) \text{ for } 0 \leq l + |m| \leq K. \end{aligned} \tag{39}$$

Note that ρ is a smooth defining function in the sense of [28] and the manifold has the round sphere (S^{n-1}, σ) as conformal infinity.

Next, we eliminate the term $2^{-n}t\kappa v_0\rho^n$ in the coefficient

$$(1+tv)^\kappa = 1 + 2^{-n}t\kappa v_0\rho^n + O^{k+1,\alpha}(\rho^{n+1})$$

by changing the defining function according to

$$\rho = \tau - \frac{t\kappa}{n2^{n+1}}v_0\,\tau^{n+1}.$$

This gives us

$$\begin{aligned}\bar{g} = \sinh^{-2}\tau \Big\{ & \left(1 + O^{k+1,\alpha}(\tau^{n+1})\right) d\tau^2 - \frac{t\kappa}{n2^n}\tau^{n+1}\partial_\mu v_0\, d\tau dy^\mu \\ & + \Big[\sigma_{\mu\nu} + 2^{-n}\left(m_{\mu\nu} + \tfrac{t\kappa(n+1)}{n} v_0\sigma_{\mu\nu}\right)\tau^n \\ & + \frac{t^2\kappa^2}{n^2 4^{n+1}}\partial_\mu v_0 \partial_\nu v_0 \tau^{2n+2} + O^{k+1,\alpha}(\tau^{n+1})\Big] dy^\mu dy^\nu \Big\}\end{aligned} \tag{40}$$

where the notation $O^{K,\alpha}(\tau^N)$ is understood as in (39) with ρ replaced by τ.

We will now perform the change of conformal gauge as described in [1, Section 3.2.1]. The idea is to write

$$\bar{g} := (\sinh\tau)^{-2}\tilde{g} = (\theta\sinh\tau)^{-2}(\theta^2\tilde{g})$$

and then choose the function θ so that $\theta\sinh\tau = \sinh\chi$, where χ is the geodesic distance to the boundary $\{\tau = 0\}$ with respect to the metric $\theta^2\tilde{g}$. For this θ is required to satisfy the equation

$$\phi\tilde{g}(d\theta, d\theta) + 2\theta\tilde{g}(d\theta, d\phi) = \theta^4\phi + \theta^2\phi^{-1}(1 - \tilde{g}(d\phi, d\phi)), \tag{41}$$

where $\phi = \sinh\tau$, see [1, Section 3.2.1]. This is a first order PDE with characteristics transversal to the boundary $\{\tau = 0\}$ so the solution θ satisfying the boundary condition $\theta = 1$ exists in $\{0 \le \tau \le \tau_0\}$ for some $\tau_0 > 0$. Due to the regularity of the coefficients of (41) we conclude that $\theta \in C^{k,\alpha}(\{0 \le \tau \le \tau_0\})$[2], that is at this step there is a loss of regularity by one derivative. Similar to [1, Section 3.2.1] we conclude that $\theta = 1 + O^{k,\alpha}(\tau^{n+1})$ and $\chi = \tau(1 + O^{k,\alpha}(\tau^{n+1}))$. All in all, we obtain

$$\bar{g} = \sinh^{-2}\chi\left(d\chi^2 + \sigma + 2^{-n}\left(m + \tfrac{t\kappa(n+1)}{n}v_0\sigma\right)\chi^n + \bar{\eta}\right),$$

where $\bar{\eta} = \bar{\eta}_{\mu\nu}dy^\mu dy^\nu$ is a χ-dependent tensor on S^{n-1} with components $\eta_{\mu\nu} = O^{k,\alpha}(\chi^{n+1})$ in the sense of (39) with ρ replaced by χ.

We conclude by performing the coordinate change $\bar{r} = \operatorname{arcsinh}(\sinh^{-1}\chi)$ and obtain

$$\bar{g} = d\bar{r}^2 + \sinh^2\bar{r}\left(\sigma + (m + \tfrac{t\kappa(n+1)}{n}v_0\sigma)e^{-n\bar{r}} + O^{k,\alpha}(e^{-(n+1)\bar{r}})\right),$$

[2]In particular, we only have $\phi^{-1}(1 - \tilde{g}(d\phi, d\phi)) \in C^{k,\alpha}(\{0 \le \tau \le \tau_0\})$, see also (40). To establish the regularity of the solution, one may argue as in the proof of [29, Theorem 22.39] while keeping track of regularity as it is done in the proof of [28, Lemma 5.1]. The details are left to the reader.

with the $O^{k,\alpha}(e^{-(n+1)\bar{r}})$ as in Definition 5.1, which is in the form (33). Finally, we note that the coordinate change $r \to \bar{r} = r - \frac{t\kappa}{2n} v_0 e^{-nr} + O^{k,\alpha}(e^{-(n+1)r})$ does not change the mass of the initial data set $(\bar{g}, \bar{\pi})$ which is readily checked by a direct computation. In fact, the effect of this change can be described in rough terms as moving the mass content of the metric $\bar{g}$ from the radial part $\bar{g}_{rr}$ to the tangential part $\bar{g}_{\mu\nu}$, while preserving the mass. □

The change of the radial coordinate performed in the proof of Theorem 5.2 gives us a mean to modify conformally hyperbolic asymptotics to Wang's asymptotics. In particular, it is straightforward to obtain the following consequence of Theorem 4.1.

Theorem 5.3. *Let (M, g, π) be an asymptotically hyperbolic initial data set of type $(k+1, \alpha, \tau, \tau_0)$ for $0 < \alpha < 1$, $\frac{n}{2} < \tau < n$ and $\tau_0 > 0$. Assume that the dominant energy condition $\mu \geq |J|_g$ holds. Then for every $\varepsilon > 0$ there exists an asymptotically hyperbolic initial data set $(\bar{g}, \bar{\pi})$ of type (k, α, n, τ_0') for some $\tau_0' > 0$ with Wang's asymptotics (possibly with respect to a different chart at infinity) satisfying the strict dominant energy condition*

$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

and such that

$$|\mathcal{M}_{(g,\pi)}(V) - \mathcal{M}_{(\bar{g},\bar{\pi})}(V)| < \varepsilon$$

for any $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$.

Proof. By Theorem 4.1, we can approximate (g, π) by an initial data set $(\bar{g}, \bar{\pi})$ of the same regularity and with conformally hyperbolic asymptotics. Noting that outside a compact set $(\bar{g}, \bar{\pi})$ is of the form (38) with $(g, \pi) = (b, 0)$ the result follows by performing the coordinate change $r \to \bar{r}$ as in the proof of Theorem 5.2. □

6. Concluding remarks

Remark 6.1. In this paper we have focused on the charge integrals $\mathbb{Q}_{(V,-dV)}$, where $V \in \{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$. These charge integrals are associated (as described in Section 2.2) to the Killing vectors $\partial_t, \partial_{x^1}, \ldots, \partial_{x^n}$ of Minkowski spacetime which generate infinitesimal translations in time and space. One may ask if the analogue of Theorem 4.1 can be proven for the remaining charges associated with the Killing vectors $x^i \partial_t + t\partial_{x^i}$, $1 \leq i \leq n$, and $x^i \partial_{x^j} - x^j \partial_{x^i}$, $1 \leq i < j \leq n$, which generate respectively infinitesimal boosts and rotations. In fact, using the general theory by Michel [33, Section IV.B] it is straightforward to check that these charges are well-defined and continuous under the assumptions of Proposition 2.4 and Proposition 2.8. As a consequence, we expect that the perturbation results of this paper can be extended to apply to these charges as well. Note that this is quite different from the situation in the asymptotically Euclidean setting, where the charges associated with boosts and rotations are determined by terms of lower order in the asymptotic expansion of the initial data set than the charges associated with translations in time and space. For this reason a given asymptotically Euclidean initial data set can be perturbed slightly to achieve any value of angular momentum and center of mass in such a way that the mass and linear momentum do not change, see Huang, Schoen, and Wang [25]. (In particular, this shows that the mass and angular momentum inequality will in general not hold for asymptotically Euclidean initial data sets without the assumption of axial symmetry.) One does not expect such a result to hold in the asymptotically hyperbolic setting.

Remark 6.2. Using the appropriate notion of mass (see for example [33, Section 4.2] or [9, Section 4]) it is straightforward to extend our results to the case of asymptotically hyperbolic initial data representing slices of asymptotically anti-de Sitter spacetimes.

Remark 6.3. Regarding the extension of the results to the case of weighted Sobolev spaces, note that in this case it is not possible to rely on the beautiful work of J. Lee [30]. Instead, methods for operators *asymptotic to geometric operators on hyperbolic space* (compare Bartnik [6, Definition 1.5]) can be used, see for example the proof of Lemma 3.2. Some results in this direction have been obtained in [20].

Appendix A. Fredholm operators on asymptotically hyperbolic manifolds: chart-dependent approach

Theorem C in J. Lee's monograph [30] proves the Fredholm property of geometric elliptic operators acting on weighted Sobolev and Hölder spaces on conformally compact manifolds. In this appendix we will show that the same result holds for asymptotically hyperbolic manifolds in the sense of Definition 2.1. We use the same definition of geometric tensor bundles and geometric elliptic partial differential operators as in the cited monograph.

Let (M, g) be a $C^{l,\beta}_\tau$-asymptotically hyperbolic n-manifold in the sense of Definition 2.1 for $n \geq 2$, $l \geq 2$, $0 \leq \beta < 1$, and $\tau > 0$. Let $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus \overline{B}_{R_0}$ be the chart at infinity. Given a geometric elliptic partial differential operator $P : C^\infty(M; E) \to C^\infty(M; E)$ of order $m \le l$ we define the *indicial map* $I_s(P) : E|_{S^{n-1}} \to E|_{S^{n-1}}$ by setting

$$I_s(P)\overline{u} := \lim_{r\to\infty} e^{sr} P(e^{-sr}\overline{u}).$$

Following [30, Section 4], we call $s \in \mathbb{C}$ a *characteristic exponent* at $p \in S^{n-1}$ if $I_s(P)$ is singular at p. Using the fact that $|\Psi_* g - b|_b = O(e^{-\tau r})$ it is not complicated to check that the characteristic exponents of P are constant on S^{n-1}. Further, if P is formally self-adjoint one may verify that the set of characteristic exponents is symmetric about the line $\operatorname{Re} s = \frac{n-1}{2} - k$, where $k = k_1 - k_2$ is the rank of the geometric tensor bundle $E \subset T^{k_1}_{k_2} M$, see [30, Proposition 4.4]. Similarly to the conformally compact case, we define the *indicial radius* of P as the smallest non-negative number R such that P has a characteristic exponent whose real part is $\frac{n-1}{2} - k + R$.

Theorem A.1. *Let (M, g) be a connected asymptotically hyperbolic n-manifold of class $C^{l,\beta}_\tau$, with $n \ge 2$, $l \ge 2$, $0 \le \beta < 1$, and $\tau > 0$ and let $E \to M$ be a geometric tensor bundle over M. Suppose that $P : C^\infty(M; E) \to C^\infty(M; E)$ is an elliptic, formally self-adjoint, geometric partial differential operator of order m, $0 < m \le l$, and assume that there exists a compact set $K \subset M$ and a positive constant C such that*

$$\|u\|_{L^2} \le C \|Pu\|_{L^2} \tag{42}$$

for all $u \in C^\infty_c(M \setminus K; E)$. Let R be the indicial radius of P.

- *If $1 < p < \infty$ and $m \le k \le l$ then the natural extension*
$$P : W^{k,p}_\delta(M; E) \to W^{k-m,p}_\delta(M; E)$$
is Fredholm for $|\delta + \frac{n-1}{p} - \frac{n-1}{2}| < R$. In that case, its index is zero, and its kernel is equal to the L^2 kernel of P.
- *If $0 < \alpha < 1$ and $m < k + \alpha \le l + \beta$ then the natural extension*
$$P : C^{k,\alpha}_\delta(M; E) \to C^{k-m,\alpha}_\delta(M; E)$$
is Fredholm for $|\delta - \frac{n-1}{2}| < R$. In that case, its index is zero, and its kernel is equal to the L^2 kernel of P.

Proof. The proof goes as in [30, Chapter 6], except for the steps which explicitly use coordinates at infinity. We verify that these steps can be carried out in our setting, that is for asymptotically hyperbolic manifolds as in Definition 2.1. Specifically, we need to adapt the construction of a parametrix given in Proposition 6.2 and Corollary 6.3 of [30]. In fact, the adaptation turns out to be rather straightforward since we have a single chart at infinity naturally replacing (finitely many) boundary Möbius charts of [30, Chapter 6].

Let $\Psi : M \setminus K_0 \to \mathbb{H}^n \setminus B_{R_0}$ be a chart at infinity as in Definition 2.1. As in [23, Appendix A] we use this chart to construct a bundle $\breve{E} \to \mathbb{H}^n$ which is defined using the same $O(n)$-representation as the one which defines E, and an isomorphism $\Upsilon : \breve{E}|_{\mathbb{H}^n \setminus B_{R_0}} \to E|_{M\setminus K_0}$. The isomorphism Υ, its inverse and their first l derivatives all have uniformly bounded norms on $\mathbb{H}^n \setminus B_{R_0}$, respectively $M \setminus K_0$. Let $\breve{P} : C^\infty(\mathbb{H}^n; \breve{E}) \to C^\infty(\mathbb{H}^n; \breve{E})$ be the operator on hyperbolic space with the same local coordinate expression as P. We define $P' : C^\infty(\mathbb{H}^n \setminus B_{R_0}; \breve{E}) \to C^\infty(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$ by

$$P'u = \Upsilon^{-1} P \Upsilon u.$$

Let $R_1 \geq R_0$. Since P is a geometric operator and g is $C^{l,\beta}_\tau$-asymptotically hyperbolic, we conclude that for each $\delta \in \mathbb{R}$, $0 < \alpha < 1$, $1 < p < \infty$, and k such that $m \leq k \leq l$ and $m < k + \alpha \leq l + \beta$ there exists a positive constant C independent of R_1 such that

$$\|P'u - \breve{P}u\|_{C^{k-m,\alpha}_\delta(\mathbb{H}^n\setminus B_{R_1};\breve{E})} < Ce^{-\tau R_1}\|u\|_{C^{k,\alpha}_\delta(\mathbb{H}^n\setminus B_{R_1};\breve{E})} \tag{43}$$

holds for all $u \in C^{k,\alpha}_\delta(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$, and

$$\|P'u - \breve{P}u\|_{W^{k-m,p}_\delta(\mathbb{H}^n\setminus B_{R_1};\breve{E})} < Ce^{-\tau R_1}\|u\|_{W^{k,p}_\delta(\mathbb{H}^n\setminus B_{R_1};\breve{E})} \tag{44}$$

holds for all $u \in W^{k,p}_\delta(\mathbb{H}^n \setminus B_{R_0}; \breve{E})$.

Now suppose that P satisfies (42). Then, by the properties of Υ, the operator P' also satisfies (42) (perhaps with a larger constant). Consequently, if R_1 is sufficiently large, it follows by (44) and standard elliptic regularity that $\breve{P}$ satisfies $\|u\|_{L^2} \leq C\|\breve{P}u\|_{L^2}$ for all $u \in C^\infty_c(\mathbb{H}^n \setminus B_{R_1}; \breve{E})$, possibly with a larger value of C than in (42). By [30, Theorems 5.7 and 5.9] we conclude that $\breve{P}$ is invertible as an operator $W^{k,p}_\delta(\mathbb{H}^n; \breve{E}) \to W^{k-m}_\delta(\mathbb{H}^n; \breve{E})$ for $|\delta + \frac{n-1}{p} - \frac{n-1}{2}| < R$ and as an operator $C^{k,\alpha}_\delta(\mathbb{H}^n; \breve{E}) \to C^{k-m,\alpha}_\delta(\mathbb{H}^n; \breve{E})$ for $|\delta - \frac{n-1}{2}| < R$.

Now assume that $R_1 \geq R_0$ is sufficiently large and let K_N be such that $M \setminus K_N = \Psi^{-1}(\mathbb{H}^n \setminus B_{NR_1})$ for $N = 1, 2, \ldots$. We define two smooth bump functions: ψ_0 equal to 1 on K_2 and supported on K_4 and ψ_1 equal to 1 on $M \setminus K_2$ and supported on $M \setminus K_1$. Let

$$\phi = \frac{\psi_1}{\sqrt{\psi_0^2 + \psi_1^2}},$$

so that $\{1 - \phi^2, \phi^2\}$ is a partition of unity subordinate to the cover $\{K_4, M \setminus K_1\}$. Clearly, $\phi \in C^{l,\beta}(M)$.

We proceed by defining the operators $Q, S : C_c^\infty(M, E) \to C_c^\infty(M, E)$ by

$$\begin{aligned} Qu &= \phi \Upsilon \breve{P}^{-1} \Upsilon^{-1}(\phi u), \\ Su &= \phi \Upsilon \breve{P}^{-1}(P' - \breve{P}) \Upsilon^{-1}(\phi u), \\ Tu &= \phi \Upsilon \breve{P}^{-1} \Upsilon^{-1}([\phi, P]u). \end{aligned}$$

A straightforward computation as in [30, Proof of Proposition 6.2] shows that

$$QPu = u + Su + Tu.$$

Furthermore, it follows from the above discussion that Q, S, and T extend to bounded maps

$$\begin{aligned} Q &: W^{0,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E), \\ S &: W^{m,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E), \\ T &: W^{m-1,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E) \end{aligned}$$

for $|\delta + \frac{n-1}{p} - \frac{n-1}{2}| < R$, and to bounded maps

$$\begin{aligned} Q &: C^{0,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E), \\ S &: C^{m,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E), \\ T &: C^{m-1,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E) \end{aligned}$$

for $|\delta - \frac{n-1}{2}| < R$. In particular, as a consequence of (43) and (44), we see that if u is supported in $M \setminus K_1$, then

$$\|Su\|_{W^{m,p}_\delta} \le Ce^{-\tau R_1} \|u\|_{W^{m,p}_\delta}, \quad \|Su\|_{C^{m,\alpha}_\delta} \le Ce^{-\tau R_1} \|u\|_{C^{m,\alpha}_\delta}$$

holds for some constant C independent of R_1 and u. Without loss of generality we may assume that $Ce^{-\tau R_1} < \frac{1}{2}$, and it follows that the operators

$$\begin{aligned} \mathrm{Id} + S &: W^{m,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E), \\ \mathrm{Id} + S &: C^{m,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E) \end{aligned}$$

have bounded inverses. This implies that, whenever u has support in $M \setminus K_1$, we have

$$\widetilde{Q}Pu = u + \widetilde{T}u,$$

where $\widetilde{Q} = (\mathrm{Id} + S)^{-1} \circ Q$ is bounded as an operator

$$\begin{aligned}
\widetilde{Q} &: W^{0,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E), \\
\widetilde{Q} &: C^{0,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E)
\end{aligned}$$

and $\widetilde{T} = (\mathrm{Id} + S)^{-1} \circ T$ is bounded as an operator

$$\begin{aligned}
\widetilde{T} &: W^{m-1,p}_\delta(M \setminus K_1; E) \to W^{m,p}_\delta(M \setminus K_1; E), \\
\widetilde{T} &: C^{m-1,\alpha}_\delta(M \setminus K_1; E) \to C^{m,\alpha}_\delta(M \setminus K_1; E)
\end{aligned}$$

where δ is in the same range as above. As a consequence of this parametrix construction, improved elliptic regularity results [30, Proposition 6.5] hold for asymptotically hyperbolic manifolds as in Definition 2.1.

The rest of the proof does not use coordinates at infinity, and the reader is referred to [30] for details. $\square$

Proposition A.2. *The operator $\Delta - n$ and the vector Laplacian Δ_L satisfy the conditions of Theorem A.1 with $R = \frac{n+1}{2}$.*

Proof. It is not complicated to check that the L^2-estimate at infinity (42) holds for $\Delta - n$. For Δ_L the L^2-estimate at infinity can be proven by standard methods, see for example [30, Section 7] or Appendix B in [23]. The critical exponents of both operators can be computed using the explicit expressions for their components, see the proof of Proposition B.2 below. $\square$

Appendix B. Solutions of critical order

Suppose that $P : C^\infty(M; E) \to C^\infty(M; E)$ is a formally self-adjoint geometric elliptic operator of order m satisfying the conditions of Theorem A.1 and let $\delta_- < \delta_+$ be its critical exponents. Roughly speaking, if $u = O(e^{-\delta r})$ for some $\delta \in (\delta_-, \delta_+)$ then $Pu = O(e^{-\kappa r})$ for some $\kappa \in (\delta_-, \delta_+)$ implies that $u = O(e^{-\kappa r})$, see [30, Proposition 6.5]. At the same time, $Pu = O(e^{-\delta_+ r})$ does not necessarily imply $u = O(e^{-\delta_+ r})$. An extensive study of the asymptotic behaviour of solutions outside of the Fredholm interval can be found in [3] and [2, Chapter 4] in the case of conformally compact metrics. Analogous results can be proven for asymptotically hyperbolic manifolds as in Definition 2.1 using the following simple lemma.

Lemma B.1. *Consider the ordinary differential equation*

$$u'' + Au' + Bu = f. \tag{45}$$

Assume that $A^2 - 4B > 0$ *so that the characteristic equation* $\lambda^2 - A\lambda + B = 0$ *has two distinct real roots* $\delta_- < \delta_+$. *Suppose that* (45) *holds for* $u = u(r) = O(e^{-\delta r})$ *and* $f = f(r) = O(e^{-\kappa r})$ *for some* $\kappa > \delta_+$. *Then*

- $\delta > \delta_-$ *implies that* $u = O(e^{-\delta_+ r})$, *and*
- $\delta > \delta_+$ *implies that* $u = O(e^{-\kappa r})$.

Note that we use a possibly non-standard characteristic equation which results from substituting $u = e^{-\lambda r}$ rather than $u = e^{\lambda r}$ into (45).

Proof. This is a consequence of the explicit formula

$$\begin{aligned} u &= \Lambda_- e^{-\delta_- r} + \Lambda_+ e^{-\delta_+ r} \\ &\quad - \frac{1}{\delta_+ - \delta_-}\left(e^{-\delta_- r}\int_r^\infty e^{\delta_- s} f(s)\, ds - e^{-\delta_+ r}\int_r^\infty e^{\delta_+ s} f(s)\, ds \right) \end{aligned} \tag{46}$$

for the solutions of (45). Note that Λ_- and Λ_+ do not depend on r. ☐

In this paper we use the following result.

Proposition B.2. *Let* (M, g) *be a connected asymptotically hyperbolic* n-*manifold of class* $C^{l,\beta}_\tau$, *with* $n \geq 2$, $l \geq 2$, $0 \leq \beta < 1$, *and* $\tau > 0$.

- *Assume that* $v \in C^{0,0}_\delta$ *is such that* $\Delta v - nv \in C^{k-2,\alpha}_{n+\varepsilon}$ *for* $\varepsilon > 0$, $0 < \alpha < 1$, *and* $k + \alpha \leq l + \beta$. *If* $\delta > -1$ *then* $v \in C^{k,\alpha}_n$. *If* $\delta > n$, *then* $v \in C^{k,\alpha}_{n+\varepsilon}$.
- *Assume that* $Z \in C^{0,0}_\delta$ *is such that* $\Delta_L Z \in C^{k-2,\alpha}_{n+\varepsilon}$ *for* $\varepsilon > 0$, $0 < \alpha < 1$, *and* $k + \alpha \leq l + \beta$. *If* $\delta > -1$ *then* $Z \in C^{k,\alpha}_n$. *If* $\delta > n$, *then* $Z \in C^{k,\alpha}_{n+\varepsilon}$.

Proof. We first prove the second claim. A straightforward computation shows that if $Z \in C^{2,\alpha}_{\delta'}$ then

$$\begin{aligned} (\Delta_L Z)_r &= \tfrac{2(n-1)}{n}(\partial^2_{rr} Z_r + (n-1)\partial_r Z_r - nZ_r) + O(e^{-(\delta'+\gamma)r}), \\ (\Delta_L Z)_\psi &= \partial^2_{rr} Z_\psi + (n-3)\partial_r Z_\psi - 2(n-1)Z_\psi + O(e^{-(\delta'+\gamma-1)r}) \end{aligned}$$

for $\gamma = \min\{1, \tau\}$. Note that in our case $Z \in C^{k,\alpha}_{\delta'}$ for any $\delta' \in (-1, n)$ as a consequence of improved elliptic regularity [30, Proposition 6.5] so we may assume that $\delta' + \gamma > n$. Hence the components of Z satisfy

$$\partial^2_{rr} Z_r + (n-1)\partial_r Z_r - nZ_r = O(e^{-\kappa r}),$$
$$\partial^2_{rr} Z_\psi + (n-3)\partial_r Z_\psi - 2(n-1)Z_\psi = O(e^{-(\kappa-1)r})$$

for $\kappa = \min\{\delta' + \gamma, n + \varepsilon\}$. From Lemma B.1 it follows that $Z_r = O(e^{-nr})$, and $Z_\psi = O(e^{-(n-1)r})$, hence $Z \in C^{k,\alpha}_n$ by standard elliptic regularity [30, Lemma 4.8]. Similarly, if $\delta > n$ it follows that $Z \in C^{k,\alpha}_{n+\varepsilon}$, possibly after repeating this argument finitely many times in order to ensure that $\kappa = n+\varepsilon$.

The first claim is proven similarly using Lemma B.1 and the fact that

$$\Delta v - nv = \partial^2_{rr} v + (n-1)\partial_r v - nv + O(e^{-(\delta'+\gamma)r})$$

for $\gamma = \min\{1, \tau\}$ when $v \in C^{2,\alpha}_{\delta'}$. □

Appendix C. On the unique continuation property

The following result is a straightforward consequence of the unique continuation results by Mazzeo [32, Theorem 7] and Kazdan [26, Theorem 1.8].

Proposition C.1. *Let (M, g) be a $C^{2,\beta}_\tau$-asymptotically hyperbolic manifold for $\tau > 0$ and $0 \le \beta < 1$, and let E be a geometric tensor bundle over M. Suppose that $u \in C^2(M; E)$ satisfies the differential inequality*

$$|\Delta u| \le C(|u| + |\nabla u|), \tag{47}$$

where $\Delta = -\nabla^*\nabla$ is the rough Laplacian. If u vanishes to infinite order at infinity, that is $|u| = O(e^{-Nr})$ for any $N > 0$, then $u = 0$ on M.

Proof. The hyperbolic metric $b = dr^2 + \sinh^2 r\,\sigma$ clearly satisfies the conditions (4)–(6) in [32]. We may therefore combine Theorem 7 in this reference with the fact that $g - b \in C^{2,\beta}_\tau$ to conclude that for any $z \in C^2(M; E)$ vanishing on $\{r \le r_0\}$ and to infinite order at infinity we have

$$t^3 \int_M e^{2tr}|z|^2\, d\mu^g + t\int_M e^{2tr}|\nabla z|^2\, d\mu^g \le C_0 \int_M e^{2tr}|\Delta z|^2\, d\mu^g. \tag{48}$$

Here it is assumed that t and r_0 are sufficiently large, and that C_0 does not depend on t. We now argue as in [32, Corollary 11] and set $z = \phi u$ where ϕ vanishes on $\{r \le r_0\}$, and is equal to 1 on $\{r \ge r_0 + 1\}$. As a consequence of (48) combined with (47) we obtain

$$(t^3 - 2C_0C^2)\int_{r_0+1}^\infty e^{2tr}|u|^2\, d\mu^g \le C_0 \int_{r_0}^{r_0+1} e^{2tr}|\Delta z|^2\, d\mu^g.$$

When $t \to \infty$ the left hand side is at least of order $O(t^3 e^{2(r_0+1)t})$, whereas the right hand side has order $O(e^{2(r_0+1)t})$. Hence $u = 0$ on $\{r \geq r_0 + 1\}$. To conclude the proof, it suffices to note that u satisfies the conditions of the strong unique continuation theorem [26, Theorem 1.8], thus $u = 0$ on M. $\square$

Acknowledgements

We want to thank Romain Gicquaud and Lan-Hsuan Huang for helpful discussions and comments related to the work in this paper. Further, we want to thank two anonymous referees for their careful reading and many insightful remarks helping to improve the paper.

References

[1] L. Andersson, M. Cai, and G. J. Galloway. Rigidity and positivity of mass for asymptotically hyperbolic manifolds. *Ann. Henri Poincaré*, **9**(1):1–33, 2008. MR2389888

[2] L. Andersson and P. T. Chruściel. Solutions of the constraint equations in general relativity satisfying “hyperboloidal boundary conditions”. *Dissertationes Math. (Rozprawy Mat.)*, **355**:100, 1996. MR1405962

[3] L. Andersson, P. T. Chruściel, and H. Friedrich. On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations. *Comm. Math. Phys.*, **149**(3):587–612, 1992. MR1186044

[4] L. Andersson, M. Dahl, G. J. Galloway, and D. Pollack. On the geometry and topology of initial data sets with horizons. *Asian J. Math.*, **22**(5):863–881, 2018. MR3878142

[5] T. Balehowsky and E. Woolgar. The Ricci flow of asymptotically hyperbolic mass and applications. *J. Math. Phys.*, **53**(7):072501, 15, 2012. MR2985224

[6] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, **39**(5):661–693, 1986. MR0849427

[7] R. Bartnik and J. Isenberg. The constraint equations. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 1–38. Birkhäuser, Basel, 2004. MR2098912

[8] S. Brendle and M.-T. Wang. A Gibbons–Penrose inequality for surfaces in Schwarzschild spacetime. *Comm. Math. Phys.*, **330**(1):33–43, 2014. MR3215576

[9] C. Cederbaum, J. Cortier, and A. Sakovich. On the center of mass of asymptotically hyperbolic initial data sets. *Ann. Henri Poincaré*, **17**(6):1505–1528, 2016. MR3500223

[10] Y. S. Cha, M. Khuri, and A. Sakovich. Reduction arguments for geometric inequalities associated with asymptotically hyperboloidal slices. *Classical Quantum Gravity*, **33**(3):035009, 33, 2016. MR3529561

[11] P. T. Chruściel and E. Delay. The hyperbolic positive energy theorem. http://arxiv.org/abs/1901.05263.

[12] P. T. Chruściel, G. J. Galloway, L. Nguyen, and T.-T. Paetz. On the mass aspect function and positive energy theorems for asymptotically hyperbolic manifolds. *Classical Quantum Gravity*, **35**(11):115015, 38, 2018. MR3801943

[13] P. T. Chruściel and M. Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific J. Math.*, **212**(2):231–264, 2003. MR2038048

[14] P. T. Chruściel, J. Jezierski, and S. Łęski. The Trautman–Bondi mass of hyperboloidal initial data sets. *Adv. Theor. Math. Phys.*, **8**(1):83–139, 2004. MR2086675

[15] P. T. Chruściel and G. Nagy. The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times. *Adv. Theor. Math. Phys.*, **5**(4):697–754, 2001. MR1926293

[16] J. Cortier, M. Dahl, and R. Gicquaud. Mass-like invariants for asymptotically hyperbolic metrics. https://arxiv.org/abs/1603.07952.

[17] J. Corvino and R. M. Schoen. On the asymptotics for the vacuum Einstein constraint equations. *J. Differential Geom.*, **73**(2):185–217, 2006. MR2225517

[18] M. Dahl, R. Gicquaud, and A. Sakovich. Asymptotically hyperbolic manifolds with small mass. *Comm. Math. Phys.*, **325**(2):757–801, 2014. MR3148101

[19] E. Delay. Smooth compactly supported solutions of some underdetermined elliptic PDE, with gluing applications. *Comm. Partial Differential Equations*, **37**(10):1689–1716, 2012. MR2971203

[20] E. Delay and J. Fougeirol. Hilbert manifold structure for asymptotically hyperbolic relativistic initial data. https://arxiv.org/abs/1607.05616. MR4257590

[21] M. Eichmair. The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight. *Comm. Math. Phys.*, **319**(3):575–593, 2013. MR3040369

[22] M. Eichmair, L.-H. Huang, D. Lee, and R. Schoen. The spacetime positive mass theorem in dimensions less than eight. *J. Eur. Math. Soc. (JEMS)*, **18**(1):83–121, 2016. MR3438380

[23] R. Gicquaud and A. Sakovich. A large class of non-constant mean curvature solutions of the Einstein constraint equations on an asymptotically hyperbolic manifold. *Comm. Math. Phys.*, **310**(3):705–763, 2012. MR2891872

[24] R. Howard. The inverse function theorem for Lipschitz maps. Available at http://people.math.sc.edu/howard/.

[25] L.-H. Huang, R. Schoen, and M.-T. Wang. Specifying angular momentum and center of mass for vacuum initial data sets. *Comm. Math. Phys.*, **306**(3):785–803, 2011. MR2825509

[26] J. L. Kazdan. Unique continuation in geometry. *Comm. Pure Appl. Math.*, **41**(5):667–681, 1988. MR0948075

[27] D. Lee. *Geometric relativity*, volume 201 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019. MR3970261

[28] J. M. Lee. The spectrum of an asymptotically hyperbolic Einstein manifold. *Comm. Anal. Geom.*, **3**(1–2):253–271, 1995. MR1362652

[29] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. MR1930091

[30] J. M. Lee. Fredholm operators and Einstein metrics on conformally compact manifolds. *Mem. Amer. Math. Soc.*, **183**(864):vi+83, 2006. MR2252687

[31] Z. Liang and X. Zhang. Spacelike hypersurfaces with negative total energy in de Sitter spacetime. *J. Math. Phys.*, **53**(2):022502, 10, 2012. MR2920460

[32] R. Mazzeo. Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds. *Amer. J. Math.*, **113**(1):25–45, 1991. MR1087800

[33] B. Michel. Geometric invariance of mass-like asymptotic invariants. *J. Math. Phys.*, **52**(5):052504, 14, 2011. MR2839077

[34] V. Moncrief. Spacetime symmetries and linearization stability of the Einstein equations. I. *J. Math. Phys.*, **16**:493–498, 1975. MR0363398

[35] A. Neves. Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds. *J. Differential Geom.*, **84**(1):191–229, 2010. MR2629514

[36] A. Neves and G. Tian. Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds. II. *J. Reine Angew. Math.*, **641**:69–93, 2010. MR2643925

[37] A. Sakovich. The Jang equation and the positive mass theorem in the asymptotically hyperbolic setting. *Comm. Math. Phys.*, **386**(2):903–973, 2021. MR4294283

[38] R. Schoen and S. T. Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, **79**(2):231–260, 1981. MR0612249

[39] X. Wang. The mass of asymptotically hyperbolic manifolds. *J. Differential Geom.*, **57**(2):273–299, 2001. MR1879228

Mattias Dahl
Institutionen för Matematik
Kungliga Tekniska Högskolan
100 44 Stockholm
Sweden
E-mail: dahl@kth.se

Anna Sakovich
Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)
Am Mühlenberg 1, 14476 Potsdam
Germany
Matematiska Institutionen
Uppsala University
Box 480, 751 06 Uppsala
Sweden
E-mail: anna.sakovich@math.uu.se