

# Discrete Morse theory on digraphs

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**Abstract:** In this paper, we give a necessary and sufficient condition that discrete Morse functions on a digraph can be extended to be Morse functions on its transitive closure, from this we can extend the Morse theory to digraphs by using quasi-isomorphism between path complex and discrete Morse complex, we also prove a general sufficient condition for digraphs that the Morse functions satisfying this necessary and sufficient condition.

**Keywords:** Discrete Morse theory, quasi-isomorphism, path homology.

## 1. Introduction

Digraphs are generalizations of graphs by assigning a direction or two directions to each edge. A graph is a digraph where each edge is assigned with two directions. In 2009, J. Bang-Jensen and G.Z. Gutin [5] studied digraphs and gave applications of digraphs in quantum mechanics, finite automata, deadlocks of computer processes, etc. In 2012, A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau [11] initiated the study of path complex on digraphs and defined the path homology of digraphs. In 2015, A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau [13, 14] studied the cohomology of digraphs and graphs by using the path homology theory. In 2018, A. Grigor'yan, Y. Muranov, V. Vershinin and S.T. Yau [16] generalized the path homology theory of digraphs and constructed the path homology theory of multigraphs and quivers.

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A *digraph*  $G$  is a pair  $(V, E)$  where  $V$  is a set and  $E$  is a subset of  $V \times V$ . The elements of  $V$  are called *vertices* and  $V$  is called the *vertex set*. For any vertices  $u, v \in V$ , if  $(u, v) \in E$ , then  $(u, v)$  is called a *directed edge*, and is denoted as  $u \rightarrow v$ . For any vertex  $v \in V(G)$ , the number of directed edges starting from  $v$  is called the out-degree of  $v$ , and the number of directed edges ending at  $v$  is called the in-degree of  $v$ . The sum of in-degree and out-degree is called the degree of  $v$ , denoted as  $D(v)$ . The transitive closure  $\overline{G}$  of a digraph  $G$  is the smallest digraph containing  $G$  such that for any two directed edges  $u \rightarrow v$  and  $v \rightarrow w$  of  $\overline{G}$ , there is a directed edge  $u \rightarrow w$  of  $\overline{G}$ .

Let  $G$  be a digraph and  $V$  be the vertex set of  $G$ . For each  $n \geq 0$ , an *elementary  $n$ -path* (or  *$n$ -path* for short) on  $V$  is a sequence  $v_0v_1 \cdots v_n$  of vertices in  $V$  where  $v_0, v_1, \dots, v_n \in V$ . Here the vertices  $v_0, v_1, \dots, v_n$  are not required to be distinct. An *allowed elementary  $n$ -path* on  $G$  is a  $n$ -path  $v_0v_1 \cdots v_n$  on  $V$  such that for each  $i \geq 1$ ,  $v_{i-1} \rightarrow v_i$  is a directed edge of  $G$  and  $v_{i-1} \neq v_i$  for each  $1 \leq i \leq n$ . Let  $\Lambda_n(V)$  be the free  $R$ -module consisting of all the formal linear combinations (with coefficients in a commutative ring  $R$  with unit) of the  $n$ -paths on  $V$ . Let  $P_n(G)$  be the free  $R$ -module consisting of all the formal linear combinations of allowed elementary  $n$ -paths on  $G$ . Then  $P_n(G)$  is a sub- $R$ -module of  $\Lambda_n(V)$ .

The boundary map  $\partial_n : \Lambda_n(V) \rightarrow \Lambda_{n-1}(V)$  is defined as by letting

$$\partial_n(v_0v_1 \cdots v_n) = \sum_{i=0}^n (-1)^i d_i(v_0v_1 \cdots v_n)$$

where  $d_i$  is the face map given by

$$d_i(v_0v_1 \cdots v_n) = v_0v_1 \cdots \hat{v}_i \cdots v_n.$$

Note that  $\partial_n$  is an  $R$ -linear map from  $\Lambda_n(V)$  to  $\Lambda_{n-1}(V)$  satisfying  $\partial_n \partial_{n+1} = 0$  for each  $n \geq 0$  (cf. [11, 12, 13, 14, 15, 16, 17]). Hence  $\{\Lambda_n(V), \partial_n\}_{n \geq 0}$  is a chain complex. We define

$$\begin{aligned} \Omega_n(G) &= P_n(G) \cap (\partial_n)^{-1}P_{n-1}(G), \\ \Gamma_n(G) &= P_n(G) + \partial_{n+1}P_{n+1}(G). \end{aligned}$$

Then as graded  $R$ -modules,

$$\Omega_*(G) \subseteq P_*(G) \subseteq \Gamma_*(G) \subseteq \Lambda_*(V).$$

And as chain complexes,

$$\{\Omega_n(G), \partial_n |_{\Omega_n(G)}\}_{n \geq 0} \subseteq \{\Gamma_n(G), \partial_n |_{\Gamma_n(G)}\}_{n \geq 0} \subseteq \{\Lambda_n(V), \partial_n\}_{n \geq 0}.$$

By [6, Proposition 2.4], the canonical inclusion

$$\iota : \Omega_n(G) \longrightarrow \Gamma_n(G), \quad n \geq 0$$

of chain complexes induces an isomorphism between the homology groups

$$\iota_* : H_m(\{\Omega_n(G), \partial_n |_{\Omega_n(G)}\}_{n \geq 0}) \xrightarrow{\cong} H_m(\{\Gamma_n(G), \partial_n |_{\Gamma_n(G)}\}_{n \geq 0}), \quad m \geq 0.$$

This isomorphism gives the *path homology* of  $G$ . We denote the path homology defined in this way as  $H_n(G; R)$  ( $n \geq 0$ ), or simply  $H_n(G)$  if there is no danger of confusion.

Morse theory originated from the study of homology groups and cell structure of smooth manifolds. In the 1990s, the discrete Morse theory for cell complexes and simplicial complexes was given (cf. [7, 8, 9, 10]). In recent years, the discrete Morse theory of cell complexes and simplicial complexes has been applied to graphs, and the discrete Morse theory of graphs has been studied (cf. [1, 2, 3, 4]). Discrete Morse theory can greatly reduce the number of cells and simplices, simplify the calculation of homology groups, and can be applied to topological data analysis (cf. [18, 19, 20], etc). In these study, people use clique as flag complexes on graph which is a similarity of simplicial complexes.

In this paper, we further study the discrete Morse theory for digraphs, try to make use of the discrete Morse theory to greatly reduce the initial information which does not affect the path homology groups of digraphs, so as to simplify the calculation. Not analogue to the flag complex, the sub complex of a path complex is not necessary a path complex. So we need to extend the path complex to its transitive closure and prove that path homology is invariant under this extension.

Let  $G$  be a digraph and  $f : V(G) \longrightarrow [0, +\infty)$  a discrete Morse function on  $G$  as defined in [21] and Definition 2.1 in next section. Consider the following condition

- (\*) For each vertex  $v \in V(G)$ , there exists at most one zero point of  $f$  in all allowed elementary paths starting or ending at  $v$ .

Then  $f$  can be extended to be a Morse function  $\bar{f}$  on the transitive closure  $\bar{G}$  of  $G$  such that  $\bar{f}(v) = f(v)$  for each vertex  $v \in E(G)$  if and only if  $f$  satisfies Condition (\*). This is proved in Theorem 2.12.

For  $n \geq 0$ , define an  $R$ -linear map  $\text{grad} f : P_n(G) \rightarrow P_{n+1}(G)$  such that for an allowed elementary  $n$ -path  $\alpha$  on  $G$ , if there exists an allowed elementary  $(n + 1)$ -path  $\gamma$  satisfying  $\gamma > \alpha$  and  $f(\gamma) = f(\alpha)$ , set

$$(\text{grad} f)(\alpha) = -\langle \partial \gamma, \alpha \rangle \gamma$$

and otherwise  $(\text{grad}f)(\alpha) = 0$ . We call  $\text{grad}f$  the (algebraic) discrete gradient vector field of  $f$ , denoted as  $V_f$ . Here  $\langle, \rangle$  is the inner product in  $\Lambda_n(V)$  (with respect to which the  $n$ -paths are orthonormal).

Let  $\bar{V} = \text{grad}\bar{f}$  be the discrete gradient vector field on  $\bar{G}$ . By [7, Definition 6.2]), define the discrete gradient flow of  $\bar{G}$  as

$$\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial,$$

which is an  $R$ -linear map

$$\bar{\Phi} : P_n(\bar{G}) \longrightarrow P_n(\bar{G}), \quad n \geq 0.$$

Denote the stabilization map of  $\bar{\Phi}$  as  $\bar{\Phi}^\infty$ . Let  $\text{Crit}_n(G)$  be the vector space consisting of all the formal linear combinations of critical  $n$ -paths (see Definition 2.1) on  $G$ . Suppose  $\Omega_*(G)$  is  $\bar{V}$ -invariant, that is,  $\bar{V}(\Omega_*(G)) \subseteq \Omega_*(G)$ . Then in Theorem 2.17, we prove that

$$H_m(G) \cong H_m(\{\Omega_n(G) \cap \bar{\Phi}^\infty(\text{Crit}_n(\bar{G})), \partial_n\}_{n \geq 0}), \quad m \geq 0.$$

Finally, in Theorem 3.1, it is proved that Condition (\*) is not very harsh. In fact, we can define Morse functions satisfying Condition (\*) on quite general digraphs, so Theorem 2.17 can be used to simplify the calculation of their path homology groups. We give some examples to illustrate.

## 2. Discrete Morse theory for digraphs

In this section, in Theorem 2.12, we study the extendability of a Morse function  $f$  on digraph  $G$  to its transitive closure  $\bar{G}$ . We prove a quasi-isomorphism of chain complexes in Theorem 2.15 and give the discrete Morse theory for digraphs in Theorem 2.17.

### 2.1. Definitions and some properties

Let  $G$  be a digraph. For any allowed elementary paths  $\gamma$  and  $\gamma'$ , if  $\gamma'$  can be obtained from  $\gamma$  by removing some vertices, then we write  $\gamma' < \gamma$  or  $\gamma > \gamma'$ .

**Definition 2.1** (Cf. [21]). A map  $f : V(G) \longrightarrow [0, +\infty)$  is called a *discrete Morse function* on  $G$ , if for any allowed elementary path  $\alpha = v_0v_1 \cdots v_n$  on  $G$ , both of the followings hold:

- (i)  $\#\{\gamma^{(n+1)} > \alpha^{(n)} \mid f(\gamma) = f(\alpha)\} \leq 1;$
- (ii)  $\#\{\beta^{(n-1)} < \alpha^{(n)} \mid f(\beta) = f(\alpha)\} \leq 1,$

where

$$(1) \quad f(\alpha) = f(v_0v_1 \cdots v_n) = \sum_{i=0}^n f(v_i).$$

For an allowed elementary path  $\alpha$ , if in both (i) and (ii), the inequalities hold strictly, then  $\alpha$  is called *critical*.

**Definition 2.2** (Cf. [10, Definition 0.6]). A function  $f : V(G) \rightarrow [0, +\infty)$  is called a *discrete Witten–Morse function* on  $G$  if, for any allowed elementary path  $\alpha$ ,

- (i)  $f(\alpha) < \text{average}\{f(\gamma_1), f(\gamma_2)\}$  where  $\gamma_1 > \alpha$  and  $\gamma_2 > \alpha$ ;
- (ii)  $f(\alpha) > \text{average}\{f(\beta_1), f(\beta_2)\}$  where  $\beta_1 < \alpha$  and  $\beta_2 < \alpha$ .

Note that each Witten–Morse function is, in fact, a Morse function.

**Definition 2.3** (Cf. [10, Definition 0.7]). A discrete Witten–Morse function is *flat* if, for any allowed elementary path  $\alpha$

- (i)  $f(\alpha) \leq \min\{f(\gamma_1), f(\gamma_2)\}$  where  $\gamma_1 > \alpha$  and  $\gamma_2 > \alpha$ ;
- (ii)  $f(\alpha) \geq \max\{f(\beta_1), f(\beta_2)\}$  where  $\beta_1 < \alpha$  and  $\beta_2 < \alpha$ .

By (1) and Definitions 2.3, it follows that each discrete Morse function on a digraph is a discrete flat Witten–Morse function.

A directed loop on  $G$  is an allowed elementary path  $v_0v_1 \cdots v_nv_0$ ,  $n \geq 1$ .

**Lemma 2.4.** *Let  $G$  be a digraph and  $f$  a discrete Morse function on  $G$ . Let  $\alpha = v_0v_1 \cdots v_nv_0$  be a directed loop. Then for each  $0 \leq i \leq n$ ,  $f(v_i) > 0$ .*

*Proof.* Suppose to the contrary,  $f(v_i) = 0$  for some  $i$ . Let  $\alpha' = v_iv_{i+1} \cdots v_nv_0 \cdots v_{i-1}v_i$ ,  $\beta_1 = v_iv_{i+1} \cdots v_nv_0 \cdots v_{i-1}$  and  $\beta_2 = v_{i+1} \cdots v_nv_0 \cdots v_{i-1}v_i$  where  $v_{i-1} = v_n$  for  $i = 0$  and  $v_{n+1} = v_0$  for  $i = n$ . Then  $f(\alpha') = f(\beta_1) = f(\beta_2)$ . This contradicts Definition 2.1 (ii). The lemma follows.  $\square$

**Lemma 2.5.** *Let  $G$  be a digraph and  $f$  a discrete Morse function on  $G$ . Then for any allowed elementary path  $\alpha = v_0 \cdots v_n$  in  $G$ , there exists at most one index  $i$  such that  $f(v_i) = 0$ .*

*Proof.* Suppose to the contrary, there are two indices  $i$  and  $j$  such that  $f(v_i) = f(v_j) = 0$  ( $i \neq j$ ). Without loss of generality,  $i < j$ . Since  $\alpha$  is allowed,  $v_i \neq v_{i+1}$ . Let  $\alpha' = v_i \cdots v_j$ ,  $\beta_1 = v_i \cdots v_{j-1}$  and  $\beta_2 = v_{i+1} \cdots v_j$ . Then  $\beta_1 \neq \beta_2$  and  $f(\alpha') = f(\beta_1) = f(\beta_2)$ . This contradicts Definition 2.1 (ii). Hence the lemma follows.  $\square$

### 2.2. Extension of Morse functions on digraphs

**Definition 2.6** ([5, Section 2.3]). A digraph  $G$  is called *transitive*, if for any two directed edges  $u \rightarrow v$  and  $v \rightarrow w$  of  $G$ , there is a directed edge  $u \rightarrow w$  of  $G$ .

**Remark 2.7.** The digraph  $G$  is transitive if and only if  $P(G)$  is *perfect* ([11, Definition 3.4]).

The next lemma is straight-forward to verify.

**Lemma 2.8** ([5, Section 2.3]). *For any digraph  $G$ , there exists a digraph  $\overline{G}$  such that*

- (i) *each directed edge of  $G$  is a directed edge of  $\overline{G}$ ;*
- (ii)  *$\overline{G}$  is transitive;*
- (iii) *any digraph  $G'$  satisfying (i) and (ii),  $\overline{G}$  is contained in  $G'$ .*

We call  $\overline{G}$  the *transitive closure* of  $G$ . A digraph  $G$  is transitive if and only if  $\overline{G} = G$ .

**Remark 2.9.** For any directed edge  $u \rightarrow v$  in  $E(\overline{G}) \setminus E(G)$ , there exists a sequence of vertices  $w_1 w_2 \cdots w_k$  ( $k \geq 1$ ) in  $V(G)$  such that  $uw_1 \cdots w_k v$  is an allowed elementary path on  $G$ . For example,

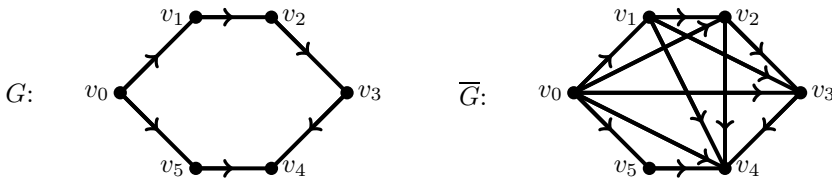


Figure 1: Remark 2.9.

$$E(\overline{G}) \setminus E(G) = \{v_0 \rightarrow v_2, v_0 \rightarrow v_3, v_0 \rightarrow v_4, v_1 \rightarrow v_3, v_1 \rightarrow v_4, v_2 \rightarrow v_4\},$$

in which there exists an allowed elementary path on  $G$  for each edge in  $E(\overline{G}) \setminus E(G)$ . Specifically,

- $v_0 \rightarrow v_2 \in E(\overline{G}) \setminus E(G)$  corresponds to  $v_0 v_1 v_2 \in P(G)$ ;
- $v_0 \rightarrow v_3 \in E(\overline{G}) \setminus E(G)$  corresponds to  $v_0 v_1 v_2 v_3 \in P(G)$ ;
- $v_0 \rightarrow v_4 \in E(\overline{G}) \setminus E(G)$  corresponds to  $v_0 v_1 v_2 v_3 v_4 \in P(G)$ ;
- $v_1 \rightarrow v_3 \in E(\overline{G}) \setminus E(G)$  corresponds to  $v_1 v_2 v_3 \in P(G)$ ;

$$v_1 \rightarrow v_4 \in E(\overline{G}) \setminus E(G) \text{ corresponds to } v_1v_2v_3v_4 \in P(G);$$

$$v_2 \rightarrow v_4 \in E(\overline{G}) \setminus E(G) \text{ corresponds to } v_2v_3v_4 \in P(G).$$

Moreover, let  $\alpha = v_0 \cdots v_4 \in P(G)$ ,  $v_i \rightarrow v_j$  ( $0 \leq i < j \leq 4$ ) are directed edges in  $E(\overline{G})$ .

Let  $G$  be a digraph and  $f$  a discrete Morse function on  $G$ . Then  $f$  may **not** be extendable to be a discrete Morse function on  $\overline{G}$ .

**Example 2.10.** Let  $G$  be a digraph with the set of vertices  $V(G) = \{v_0, v_1, v_2, v_3\}$  and the set of directed edges  $E(G) = \{v_0 \rightarrow v_3, v_1 \rightarrow v_2, v_2 \rightarrow v_3\}$ . Then

$$P(G) = R\{v_0, v_1, v_2, v_3, v_0v_3, v_1v_2, v_2v_3, v_1v_2v_3\}$$

and the transitive closure of  $G$  is a digraph  $\overline{G}$  with  $V(\overline{G}) = V(G)$  and

$$E(\overline{G}) = E(G) \cup \{v_1 \rightarrow v_3\}.$$

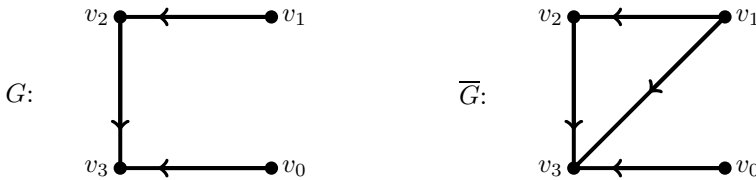


Figure 2: Example 2.10.

Let  $f$  be a function on  $V(G)$  with  $f(v_1) = f(v_0) = 0$  and  $f(v_2) > 0$ ,  $f(v_3) > 0$ . Then by Definition 2.1,  $f$  is a discrete Morse function on  $G$ . However, since  $f(v_3) = f(v_0v_3) = f(v_1v_3)$ ,  $f$  is not a discrete Morse function on  $\overline{G}$ . Hence, there does not exist any discrete Morse function  $\overline{f}$  on  $\overline{G}$  such that the restriction of  $\overline{f}$  to  $G$  equals  $f$ .

To give a condition for extendability of a Morse function  $f$  on  $G$  to its transitive closure, we study the property of the discrete Morse function on transitive digraph in the following lemma first.

**Lemma 2.11.** *Let  $G$  be a transitive digraph and  $f : V(G) \rightarrow [0, +\infty)$  a discrete Morse function on  $G$ . Then for any vertex  $v$  with  $D(v) \geq 2$ , there exists at most one vertex  $w \neq v$  such that  $f(w) = 0$  and  $v \rightarrow w$  or  $w \rightarrow v$  is a directed edge in  $E(G)$ .*

*Proof.* Suppose to the contrary, there are three cases to be considered.

CASE 1. There are two vertices  $w_1, w_2 \in V(G)$  such that  $f(w_1) = f(w_2) = 0$  and  $v \rightarrow w_1, w_2 \rightarrow v \in E(G)$ .

CASE 2. There are two vertices  $w_1, w_2 \in V(G)$  such that  $f(w_1) = f(w_2) = 0$  and  $w_1 \rightarrow v, w_2 \rightarrow v \in E(G)$ .

CASE 3. There are two vertices  $w_1, w_2 \in V(G)$  such that  $f(w_1) = f(w_2) = 0$  and  $v \rightarrow w_1, v \rightarrow w_2 \in E(G)$ .

Without loss of generality, we only give the proof of Case 1. Let  $\alpha = v$ ,  $\gamma_1 = vw_1$  and  $\gamma_2 = w_2v$ . Then  $f(\alpha) = f(\gamma_1) = f(\gamma_2)$  and  $\gamma_1 > \alpha, \gamma_2 > \alpha$ . This contradicts that  $f$  is a discrete Morse function on  $G$ . The lemma follows.  $\square$

Next, we give a necessary and sufficient condition for a discrete Morse function on a digraph to be extended to its transitive closure in the following theorem.

**Theorem 2.12.** *Let  $G$  be a digraph and  $f : V(G) \rightarrow [0, +\infty)$  a discrete Morse function on  $G$ . Then  $f$  can be extended to be a Morse function  $\bar{f}$  on  $\bar{G}$  such that  $\bar{f}(v) = f(v)$  for each vertex  $v \in E(G)$  if and only if Condition (\*) is satisfied.*

*Proof.* Suppose the discrete Morse function  $f$  on  $G$  can be extended to be a discrete Morse function on  $\bar{G}$ . Let  $\alpha = v_0 \cdots v_n$  be an allowed elementary path on  $G$  with  $v_0 = v$  or  $v_n = v$ . By Lemma 2.8, for each  $v_k$  ( $0 \leq k \leq n$ ) such that  $v_k \neq v$ ,  $v \rightarrow v_k$  or  $v_k \rightarrow v$  is a directed edge in  $E(\bar{G})$ . Then by Lemma 2.5 and Lemma 2.11,  $f$  satisfies Condition (\*) obviously.

On the other hand, suppose  $f$  satisfies Condition (\*). Let  $\alpha = v_0v_1 \cdots v_n$  be an allowed elementary path on  $\bar{G}$ . Firstly, we assert that there exists at most one index  $i$  ( $0 \leq i \leq n$ ) such that  $f(v_i) = 0$  and  $d_i(\alpha)$  is an allowed elementary  $(n-1)$ -path on  $\bar{G}$ .

CASE 1.  $\alpha \in P(G)$ . Then by Lemma 2.5, the assertion follows.

CASE 2.  $\alpha \notin P(G)$ . Then there exists a directed edge  $v_iv_{i+1} \in E(\bar{G}) \setminus E(G)$  for some  $0 \leq i \leq n-1$ . Hence, by Remark 2.9, there exists an allowed elementary path  $\alpha' = v_0 \cdots v_iw_1 \cdots w_kv_{i+1} \cdots v_n$  on  $G$  with  $k \geq 1$  and  $w_1, \dots, w_k \in V(G)$ . Hence, by Lemma 2.5, the assertion follows.

Secondly, we assert that there exists at most one allowed elementary  $(n+1)$ -path  $\alpha' = v_0 \cdots v_juv_{j+1} \cdots v_n$  on  $\bar{G}$  with  $f(u) = 0$ ,  $u \in V(G)$ . Suppose to the contrary, we consider the following two cases.

CASE 3.  $\alpha'' = v_0 \cdots v_iwv_{i+1} \cdots v_n$  is another allowed elementary  $(n+1)$ -path on  $\bar{G}$  with  $f(w) = 0$  and  $i \neq j$ . Without loss of generality,  $j > i$ . Then by Remark 2.9, there is an allowed elementary path with  $w$  as the starting point and  $u$  the ending point on  $G$ . This contradicts Lemma 2.5.



CASE 4.  $\alpha'' = v_0 \cdots v_i w v_{i+1} \cdots v_n$  is another allowed elementary  $(n + 1)$ -path on  $\overline{G}$  with  $f(w) = 0$  and  $i = j$ . Then  $u \neq w$ . Hence, by Remark 2.9, there are two allowed elementary paths on  $G$  with  $v_i$  as the starting point and  $u, w$  as the ending point respectively, or with  $u, w$  as the starting point respectively and  $v_{i+1}$  as the ending point. This contradicts Condition (\*).

Summarizing above cases,  $f$  can be extended to be a Morse function  $\overline{f}$  on  $\overline{G}$ .

Therefore, the theorem is proved. □

### 2.3. Quasi-isomorphism, discrete Morse theory for digraphs

Let  $f$  be a discrete Morse function on digraph  $G$  and  $\overline{f}$  the extension of  $f$  on the transitive closure  $\overline{G}$  of  $G$ . Let  $V_f = \text{grad} f$  and  $\overline{V} = \text{grad} \overline{f}$  be discrete gradient vector fields on  $G$  and  $\overline{G}$  respectively. Generally,  $\overline{V}|_{P(G)} \neq V_f$ .

**Example 2.13.** Let  $G$  be a digraph with  $V(G) = \{v_0, v_1, v_2, v_3\}$  and  $E(G) = \{v_0 \rightarrow v_1, v_1 \rightarrow v_2, v_2 \rightarrow v_3, v_0 \rightarrow v_3\}$ . Then the transitive closure of  $G$  is a digraph  $\overline{G}$  with  $V(\overline{G}) = V(G)$  and

$$E(\overline{G}) = E(G) \cup \{v_0 \rightarrow v_2, v_1 \rightarrow v_3\}.$$

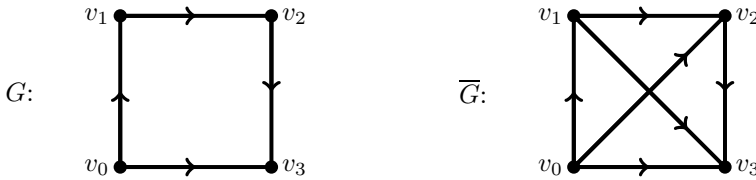


Figure 3: Example 2.13.

Let  $f$  be a function on  $V(G)$  with  $f(v_2) = 0, f(v_0) > 0, f(v_1) > 0,$  and  $f(3) > 0$ . Then  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*). By Theorem 2.12,  $f$  can be extended to be a Morse function  $\overline{f}$  on  $\overline{G}$  such that  $\overline{f}(v) = f(v)$  for each vertex  $v \in V(G)$ .

Let  $\alpha = v_0 v_3 \in P(G) \subseteq P(\overline{G})$ . Since there exists no allowed elementary path  $\gamma \in P(G)$  such that  $\gamma > \alpha$  and  $f(\gamma) = f(\alpha)$ . Then  $V_f(\alpha) = 0$ . Since  $\overline{f}(v_0 v_2 v_3) = \overline{f}(\alpha)$  and  $\partial(v_0 v_2 v_3) = v_2 v_3 - v_0 v_3 + v_0 v_2$ , then  $\overline{V}(\alpha) = v_0 v_2 v_3 \in P(\overline{G})$ . Hence the restriction of  $\overline{V}$  on  $P(G)$  may not be  $V_f$ .

Denote

$$\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial,$$

as the discrete gradient flow of  $\bar{G}$ . Similar with the proof of [7, Theorem 6.3, Theorem 6.4], the main properties of  $\bar{V}$  and  $\bar{\Phi}$  are contained in the following theorem.

**Theorem 2.14** (Cf. [7, Theorem 6.3, Theorem 6.4]).

- (i)  $\bar{V} \circ \bar{V} = 0$ ;
- (ii)  $\#\{\beta^{(n-1)} \mid \bar{V}(\beta) = \pm\alpha\} \leq 1$  for any  $\alpha \in P(\bar{G})$ ;
- (iii)  $\alpha$  is critical  $\iff \{\alpha \notin \text{Image}(\bar{V}) \text{ and } \bar{V}(\alpha) = 0\}$  for any  $\alpha \in P(\bar{G})$ ;
- (iv)  $\bar{\Phi}\partial = \partial\bar{\Phi}$ .

*Proof.* The simple proof of the theorem is as follows.

- (i). For any allowed elementary path  $\beta$  on  $\bar{G}$ , if  $\bar{V}(\beta) = \pm\alpha$ , then  $\beta < \alpha$  and  $\bar{f}(\beta) = \bar{f}(\alpha)$ . By Lemma 2.5, there is no allowed elementary path  $\gamma$  on  $\bar{G}$  such that  $\gamma > \alpha$  and  $\bar{f}(\gamma) = \bar{f}(\alpha)$ . Hence  $\bar{V} \circ \bar{V}(\beta) = 0$ . (i) is proved.
- (ii). If  $\bar{V}(\beta) = \pm\alpha$ , then  $\beta < \alpha$  and  $\bar{f}(\beta) = \bar{f}(\alpha)$ . Hence by Definition 2.1(ii), (ii) is proved.
- (iii). By Definition 2.1,

$\alpha$  is critical

$$\iff \text{for any allowed elementary path } \gamma > \alpha, \bar{f}(\gamma) > \bar{f}(\alpha) \text{ and} \\ \text{for any allowed elementary path } \beta < \alpha, \bar{f}(\beta) < \bar{f}(\alpha).$$

That is equivalent to there is no allowed elementary path  $\gamma$  on  $\bar{G}$  such that  $\gamma > \alpha$  and  $\bar{f}(\gamma) = \bar{f}(\alpha)$ , and no allowed elementary path  $\beta$  on  $\bar{G}$  such that  $\beta < \alpha$  and  $\bar{f}(\beta) = \bar{f}(\alpha)$ , which implies (iii).

(iv).

$$\partial\bar{\Phi} = \partial(\text{Id} + \partial\bar{V} + \bar{V}\partial) = \partial + \partial\bar{V}\partial,$$

$$\bar{\Phi}\partial = (\text{Id} + \partial\bar{V} + \bar{V}\partial)\partial = \partial + \partial\bar{V}\partial. \quad \square$$

Let

$$P_*^{\bar{\Phi}}(\bar{G}) = \left\{ \sum_i a_i \alpha_i \in P_*(\bar{G}) \mid \bar{\Phi}\left(\sum_i a_i \alpha_i\right) = \sum_i a_i \alpha_i, a_i \in R \right\}.$$

By Theorem 2.14 (iv), the boundary operator  $\partial$  maps  $P_n^{\bar{\Phi}}(\bar{G})$  to  $P_{n-1}^{\bar{\Phi}}(\bar{G})$ . Thus  $\{P_*^{\bar{\Phi}}(\bar{G}), \partial_*\}$  is a sub-chain complex of  $\{P_*(\bar{G}), \partial_*\}$  consisting of all  $\bar{\Phi}$ -invariant chains, called the Morse complex. Moreover, by Theorem 2.14 and [7, Theorem 7.2], we have that

$$\bar{\Phi}^N = \bar{\Phi}^{N+1} = \dots = \bar{\Phi}^\infty$$

for some sufficiently large positive integer  $N$ , where  $\bar{\Phi}^\infty = \lim_{N \rightarrow \infty} \bar{\Phi}^N$ .

To give the discrete Morse theory for digraphs, we first prove a quasi-isomorphism of chain complexes.

**Theorem 2.15.** *Suppose  $\Omega_*(G)$  is  $\bar{V}$ -invariant (that is,  $\bar{V}(\Omega_n(G)) \subseteq \Omega_{n+1}(G)$  for each  $n \geq 0$ ). There is a quasi-isomorphism*

$$\Omega_*(G) \longrightarrow \Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G}).$$

*Proof.* By the proof of [7, Theorem 7.3], we have the following chain homotopy

$$(2) \quad \begin{aligned} \bar{\Phi}^\infty : P_*(\bar{G}) &\longrightarrow P_*^{\bar{\Phi}}(\bar{G}); \\ \iota : P_*^{\bar{\Phi}}(\bar{G}) &\longrightarrow P_*(\bar{G}). \end{aligned}$$

Here  $\iota$  is the canonical inclusion. It is proved in [7, Theorem 7.3] that

$$(3) \quad \bar{\Phi}^\infty \circ \iota = \text{Id}$$

and

$$(4) \quad \iota \circ \bar{\Phi}^\infty \simeq \text{Id}.$$

Firstly, we will prove that

$$(5) \quad \bar{\Phi}^\infty|_{\Omega_*(G)} : \Omega_*(G) \longrightarrow \Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G})$$

is well-defined.

For any  $x = \sum a_i \alpha_i \in \Omega_n(G) \subseteq P_n(\bar{G})$  where  $a_i \in R$  and  $\alpha_i$  are allowed elementary  $n$ -paths on  $G$ . Since  $\Omega_*(G)$  is  $\bar{V}$ -invariant, then  $\bar{\Phi}(x) \in \Omega_n(G)$ . Hence

$$\bar{\Phi}^\infty(x) \in \Omega_n(G).$$

On the other hand, by (2),

$$\overline{\Phi}^\infty(x) \in P_*^{\overline{\Phi}}(\overline{G}).$$

Hence (5) is well-defined.

Secondly, by (3) and (4), we have

$$(\overline{\Phi}^\infty|_{\Omega_*(G)}) \circ (\iota|_{\Omega_*(G) \cap P_*^{\overline{\Phi}}(\overline{G})}) = \text{Id}.$$

It follows that

$$(\overline{\Phi}^\infty|_{\Omega_*(G)})_* \circ (\iota|_{\Omega_*(G) \cap P_*^{\overline{\Phi}}(\overline{G})})_* = \text{Id}$$

and  $(\overline{\Phi}^\infty|_{\Omega_*(G)})_*$  is surjective. Here  $(\overline{\Phi}^\infty|_{\Omega_*(G)})_*$  and  $(\iota|_{\Omega_*(G) \cap P_*^{\overline{\Phi}}(\overline{G})})_*$  are homomorphisms between homology groups  $H_*(\Omega_*(G))$  and  $H_*(\Omega_*(G) \cap P_*^{\overline{\Phi}}(\overline{G}))$  induced by morphisms between chain complexes  $\Omega_*(G)$  and  $\Omega_*(G) \cap P_*^{\overline{\Phi}}(\overline{G})$ . It leaves us to show  $(\overline{\Phi}^\infty|_{\Omega_*(G)})_*$  is injective. That is, for any element  $x \in \text{Ker} \partial|_{\Omega_n(G)}$ , if

$$(\overline{\Phi}^\infty|_{\Omega_n(G)})_*(x)$$

is a boundary in  $\Omega_{n+1}(G) \cap P_{n+1}^{\overline{\Phi}}(\overline{G})$ , then  $x$  is a boundary in  $\Omega_{n+1}(G)$ .

Suppose there exists an element  $y \in \Omega_{n+1}(G) \cap P_{n+1}^{\overline{\Phi}}(\overline{G})$  such that  $\partial y = \overline{\Phi}^\infty(x)$ . Since

$$\begin{aligned} &\overline{\Phi}^\infty(x) \\ &= \overline{\Phi}^N(x) \\ &= (\text{Id} + \partial\overline{V})^N(x) \\ &= \left( C_N^0(\text{Id})^N + C_N^1\text{Id}^{(N-1)}\partial\overline{V} + C_N^2\text{Id}^{(N-2)}(\partial\overline{V})^2 + \dots + C_N^N(\partial\overline{V})^N \right) (x) \\ &= x + \left( C_N^1\partial\overline{V} + C_N^2(\partial\overline{V})^2 + \dots + C_N^N(\partial\overline{V})^N \right) (x) \\ &= x + \partial \left( C_N^1\overline{V} + C_N^2(\overline{V}\partial\overline{V}) + \dots + C_N^N(\overline{V}\partial\overline{V} \dots \partial\overline{V}) \right) (x) \end{aligned}$$

and  $\Omega_*(G)$  is  $\overline{V}$ -invariant, then  $L(x) \in \Omega_{n+1}(G)$  where

$$L = C_N^1\overline{V} + C_N^2(\overline{V}\partial\overline{V}) + \dots + C_N^N(\overline{V}\partial\overline{V} \dots \partial\overline{V}).$$

Hence

$$\partial y - \partial L(x) = x$$

which implies  $x = \partial(y - L(x))$  and  $x$  is a boundary in  $\Omega_{n+1}(G)$ .

The theorem is proved. □

Denote

$$\Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G}) = \Omega_*^{\bar{\Phi}}(G).$$

By Theorem 2.15, we have the discrete Morse theory for digraphs as follows.

**Corollary 2.16.** *Let  $G$  be a digraph and  $f$  a discrete Morse function on  $G$  satisfying Condition (\*). Let  $\bar{f}$  be the extension of  $f$  on  $\bar{G}$  and  $\bar{V} = \text{grad}\bar{f}$  the discrete gradient vector field on  $\bar{G}$ . Suppose  $\Omega_*(G)$  is  $\bar{V}$ -invariant. Then*

$$H_m(G) \cong H_m(\Omega_*^{\bar{\Phi}}(G)), m \geq 0.$$

Furthermore, for each  $n \geq 0$ , let  $\text{Crit}_n(G)$  be the free  $R$ -module consisting of all the formal linear combinations of critical  $n$ -paths on  $G$ . Then  $\text{Crit}_n(G)$  is a sub- $R$ -module of  $P_n(G)$ . By [7, Theorem 8.2], there is an isomorphism of graded  $R$ -modules

$$\bar{\Phi}^\infty|_{\text{Crit}_*(\bar{G})}: \text{Crit}_*(\bar{G}) \longrightarrow P_*^{\bar{\Phi}}(\bar{G}).$$

Hence, by Corollary 2.16, we have that

**Theorem 2.17.** *Let  $G$  be a digraph and  $f$  a discrete Morse function on  $G$  satisfying Condition (\*). Let  $\bar{f}$  be the extension of  $f$  on  $\bar{G}$  and  $\bar{V} = \text{grad}\bar{f}$  the discrete gradient vector field on  $\bar{G}$ . Suppose  $\Omega_*(G)$  is  $\bar{V}$ -invariant. Then*

$$(6) \quad H_m(G) \cong H_m(\{\Omega_n(G) \cap \bar{\Phi}^\infty(\text{Crit}_n(\bar{G})), \partial_n\}_{n \geq 0}).$$

**Example 2.18.** Consider the following digraph  $G$  and its transitive closure  $\bar{G}$ . Then

$$\Omega_*(G) = R(v_0, v_1, v_2, v_3, v_0v_1, v_1v_2, v_2v_3, v_0v_3)$$

Let  $f : V(G) \longrightarrow [0, +\infty)$  be a function on  $G$  with  $f(v_0) = 0$  and  $f(v_i) > 0, 0 < i \leq 3$ . Then  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*). By Theorem 2.12,  $f$  can be extended to be a Morse function  $\bar{f}$  on  $\bar{G}$  such that  $\bar{f}(v) = f(v)$  for all vertices  $v \in V(\bar{G})$ .

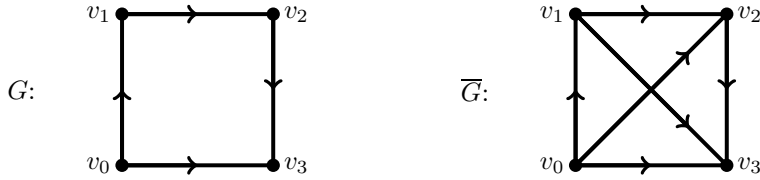


Figure 4: Example 2.18.

Since  $\bar{f}(v_0) = 0$  and  $v_0 \rightarrow v_i$  ( $i \neq 0$ ) are directed edges on  $\bar{G}$ , all allowed elementary paths on  $\bar{G}$  except for the 0-path  $\{v_0\}$  are not critical. Hence,

$$\text{Crit}_*(\bar{G}) = R(v_0).$$

Let  $\bar{V} = \text{grad}\bar{f}$  be the discrete gradient vector field on  $\bar{G}$ . Then

$$(7) \quad \begin{aligned} \bar{V}(v_1) &= -v_0v_1, & \bar{V}(v_2) &= -v_0v_2, & \bar{V}(v_3) &= -v_0v_3, \\ \bar{V}(v_1v_3) &= -v_0v_1v_3, & \bar{V}(v_2v_3) &= -v_0v_2v_3, & \bar{V}(v_1v_2) &= -v_0v_1v_2 \\ \bar{V}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}, \end{aligned}$$

in which (7) implies that  $\Omega_*(G)$  is not  $\bar{V}$ -invariant.

Let  $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$  be the discrete gradient flow of  $\bar{G}$ . Then

$$\begin{aligned} \bar{\Phi}(v_0) &= v_0, & \bar{\Phi}(v_1) &= v_0, \\ \bar{\Phi}(v_2) &= v_0, & \bar{\Phi}(v_3) &= v_0, \\ \bar{\Phi}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}. \end{aligned}$$

By calculate directly, we have that  $\bar{\Phi}^\infty = \bar{\Phi}$ . Then

$$\begin{aligned} \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0), \\ \Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0). \end{aligned}$$

Hence,

$$\begin{aligned} H_0(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &\cong R, \\ H_m(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0 \text{ for } m > 0. \end{aligned}$$

By [11, Proposition 4.7],  $H_1(G) \cong R$  and by [11, Theorem 4.6],  $H_1(\bar{G}) = 0$ .

$$\begin{aligned} H_1(G) &\neq H_1(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))), \\ H_1(G) &\neq H_1(\bar{G}). \end{aligned}$$

**Remark 2.19.** If the condition that  $\Omega_*(G)$  is  $\overline{V}$  – invariant in Theorem 2.17 does not hold for some digraphs, there may be no isomorphism of homology groups given in (6).

**Remark 2.20.** In general, the homology groups of a digraph  $G$  and its transitive closure  $\overline{G}$  are different (in the sense of isomorphism).

### 3. Digraphs satisfying condition (\*)

In this section, we will show that there are quite general digraphs on which Morse functions satisfying Condition (\*) can be defined, and give some examples to illustrate Theorem 2.17.

Inspired by Lemma 2.4, we give the following theorem.

**Theorem 3.1.** *Let  $G$  be a digraph and  $f : V(G) \rightarrow [0, +\infty)$  a function on  $G$  given by*

$$f(v) = \begin{cases} 0, & \text{if } v = v', \\ \neq 0, & \text{if } v \neq v'. \end{cases}$$

*Here  $v' \in V(G)$  does not belong to any directed loop on  $G$ . Then  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*).*

*Proof.* Let  $\alpha = v_0 \cdots v_n$  be an arbitrary allowed elementary path on  $G$ . There are two cases.

CASE 1.  $v_i \neq v'$  for any  $0 \leq i \leq n$ . We assert that there exists only one index  $k$  with  $-1 \leq k \leq n$  such that  $\gamma' = v_0 \cdots v_k v' v_{k+1} \cdots v_n$  (for  $k = -1$ ,  $\gamma' = v' v_0 \cdots v_n$ ) is an allowed elementary path on  $G$ . Suppose to the contrary,  $\gamma'' = v_0 \cdots v_j v' v_{j+1} \cdots v_n$  is another allowed elementary path on  $G$ ,  $j \neq k$ . Without loss of generality,  $k < j$ . Then  $\tilde{\gamma} = v' v_{k+1} \cdots v_j v'$  is a directed loop on  $G$  which contradicts that  $v'$  does not belong to any directed loop. Hence for any allowed elementary path  $\gamma > \alpha$ ,

$$\#\{\gamma > \alpha \mid f(\gamma) = f(\alpha)\} \leq 1$$

and for any allowed elementary path  $\beta < \alpha$ ,  $f(\beta) < f(\alpha)$ .

CASE 2.  $v_i = v'$  for some  $0 \leq i \leq n$ . We assert that there is no any other vertex  $v_j = v'$  for  $0 \leq j \neq i \leq n$ . Suppose to the contrary,  $v_i = v_j = v'$ ,  $i \neq j$ . Without loss of generality,  $i < j$ . Since  $\alpha$  is an allowed elementary path on  $G$ ,  $v_{k-1} \neq v_k$  for each  $1 \leq k \leq n$ . Then  $j \neq i + 1$ . Let  $\alpha' = v_i \cdots v_j$ . Then  $\alpha'$  is a

directed loop on  $G$  which contradicts that  $v'$  does not belong to any directed loop. Hence, for any allowed elementary path  $\beta < \alpha$ ,

$$\#\{\beta < \alpha \mid f(\beta) = f(\alpha)\} \leq 1$$

and for any allowed elementary path  $\gamma > \alpha$ ,  $f(\gamma) > f(\alpha)$ .

Combining Case 1 and Case 2, by Definition 2.1,  $f$  is a discrete Morse function on  $G$ . Moreover, since there is only one vertex  $v' \in V(G)$  such that  $f(v') = 0$ ,  $f$  satisfies Condition (\*). The theorem follows.  $\square$

Finally, for illustrating the application of discrete Morse theory in the calculation of simplified homology groups of digraphs, we give the following examples.

**Example 3.2.** Let  $G$  be a square. Then the transitive closure of  $G$  is a digraph  $\overline{G}$  with  $V(\overline{G}) = V(G)$  and

$$E(\overline{G}) = E(G) \cup \{v_0 \rightarrow v_3\}.$$

Let  $f$  be a function on  $G$  such that

$$f(v_0) = 1, f(v_1) = 0, f(v_2) = 2, f(v_3) = 3.$$

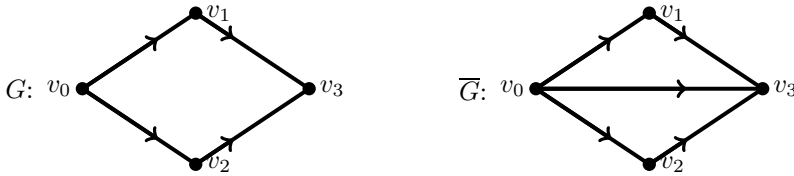


Figure 5: Example 3.2.

By Theorem 3.1,  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*) and by Theorem 2.12,  $f$  can be extended to be a Morse function  $\overline{f}$  on  $\overline{G}$  such that  $\overline{f}(v) = f(v)$  for all vertices  $v \in V(\overline{G})$ . Then

$$\begin{aligned} \text{Crit}_*(G) &= R(v_1, v_2, v_0v_2, v_2v_3, v_0v_1v_3, v_0v_2v_3), \\ \text{Crit}_*(\overline{G}) &= R(v_1, v_2, v_0v_2, v_2v_3, v_0v_2v_3), \\ \Omega_*(G) &= R(v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_1v_3, v_2v_3, v_0v_1v_3 - v_0v_2v_3). \end{aligned}$$

Note that  $\text{Crit}_*(\overline{G}) \cap P(G) \neq \text{Crit}_*(G)$ .



Let  $\bar{V} = \text{grad}\bar{f}$  be the discrete gradient vector field on  $\bar{G}$ . Then

$$\begin{aligned} \bar{V}(v_0) &= v_0v_1, & \bar{V}(v_3) &= -v_1v_3, \\ \bar{V}(v_0v_3) &= v_0v_1v_3, \\ \bar{V}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}, \\ \bar{V}(\Omega_n(G)) &\subseteq \Omega_{n+1}(G) \text{ for any } n \geq 0. \text{ That is, } \Omega_*(G) \text{ is } \bar{V} - \text{invariant.} \end{aligned}$$

Let  $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$  be the discrete gradient flow of  $\bar{G}$ . Then

$$\begin{aligned} \bar{\Phi}(v_0) &= v_1, & \bar{\Phi}(v_1) &= v_1, \\ \bar{\Phi}(v_2) &= v_2, & \bar{\Phi}(v_3) &= v_1, \\ \bar{\Phi}(v_0v_1) &= 0, & \bar{\Phi}(v_0v_2) &= v_0v_2 - v_0v_1, \\ \bar{\Phi}(v_1v_3) &= 0, & \bar{\Phi}(v_2v_3) &= v_2v_3 - v_1v_3, \\ \bar{\Phi}(v_0v_3) &= 0, & \bar{\Phi}(v_0v_1v_3) &= 0, \\ \bar{\Phi}(v_0v_2v_3) &= v_0v_2v_3 - v_0v_1v_3. \end{aligned}$$

By calculate directly, we have that  $\bar{\Phi}^\infty = \bar{\Phi}$ . Then

$$\begin{aligned} \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_1, v_2, v_0v_2 - v_0v_1, v_2v_3 - v_1v_3, v_0v_2v_3 - v_0v_1v_3), \\ \Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_1, v_2, v_0v_2 - v_0v_1, v_2v_3 - v_1v_3, v_0v_2v_3 - v_0v_1v_3). \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_1(v_0v_2 - v_0v_1) &= v_2 - v_1, & \partial_1(v_2v_3 - v_1v_3) &= v_1 - v_2 \\ \partial_2(v_0v_2v_3 - v_0v_1v_3) &= (v_0v_2 - v_0v_1) + (v_2v_3 - v_1v_3) \end{aligned}$$

and

$$\begin{aligned} H_0(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= R, \\ H_1(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0, \\ H_m(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0 \text{ for } m \geq 2, \end{aligned}$$

which are consistent with the path homology groups of  $G$  given in [11, Proposition 4.7].

**Example 3.3.** Consider the digraph  $G$  given in Example 3.2. Let  $f : V(G) \rightarrow [0, +\infty)$  be a function on  $G$  with  $f(v_0) = 0$  and  $f(v_i) > 0, 0 < i \leq 3$ , which is different from the function given in Example 3.2. By Theorem 3.1,  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*) and by Theorem 2.12,

$f$  can be extended to be a Morse function  $\bar{f}$  on  $\bar{G}$  such that  $\bar{f}(v) = f(v)$  for all vertices  $v \in V(\bar{G})$ . Obviously,

$$\Omega_*(G) = R(v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_1v_3, v_2v_3, v_0v_1v_3 - v_0v_2v_3)$$

Since  $\bar{f}(v_0) = 0$  and  $v_0 \rightarrow v_i$  ( $i \neq 0$ ) are directed edges on  $\bar{G}$ , it follows that all allowed elementary paths on  $\bar{G}$  except for the 0-path  $\{v_0\}$  are not critical. Hence  $\text{Crit}_*(\bar{G}) = R(v_0)$ .

Let  $\bar{V} = \text{grad}\bar{f}$  be the discrete gradient vector field on  $\bar{G}$ . Then

$$(8) \quad \begin{aligned} \bar{V}(v_1) &= -v_0v_1, & \bar{V}(v_2) &= -v_0v_2, & \bar{V}(v_3) &= -v_0v_3, \\ \bar{V}(v_1v_3) &= -v_0v_1v_3, & \bar{V}(v_2v_3) &= -v_0v_2v_3, \\ \bar{V}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}, \end{aligned}$$

in which (8) implies that  $\Omega_*(G)$  is not  $\bar{V}$ -invariant.

Let  $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$  be the discrete gradient flow of  $\bar{G}$ . Then

$$\begin{aligned} \bar{\Phi}(v_0) &= v_0, & \bar{\Phi}(v_1) &= v_0, \\ \bar{\Phi}(v_2) &= v_0, & \bar{\Phi}(v_3) &= v_0, \\ \bar{\Phi}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}. \end{aligned}$$

By calculate directly, we have that  $\bar{\Phi}^\infty = \bar{\Phi}$ . Then

$$\begin{aligned} \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0), \\ \Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0). \end{aligned}$$

Therefore,

$$\begin{aligned} H_0(G) &= H_0(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) \cong R, \\ H_m(G) &= H_m(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) = 0 \text{ for all } m \geq 1. \end{aligned}$$

**Remark 3.4.** By Example 3.2 and Example 3.3, we know that in the discrete Morse theory for digraphs, the selection of zero points of discrete Morse functions is very important to simplify the calculation of homology groups. Generally speaking, we can choose the vertex with larger degree in the transitive closure of a digraph as the zero point.

**Example 3.5.** Consider the following digraph  $G$  and its transitive closure  $\bar{G}$ . Let  $f : V(G) \rightarrow [0, +\infty)$  be a function on  $G$  with  $f(v_0) = 0$  and  $f(v_i) > 0$ ,  $0 < i \leq 5$ .

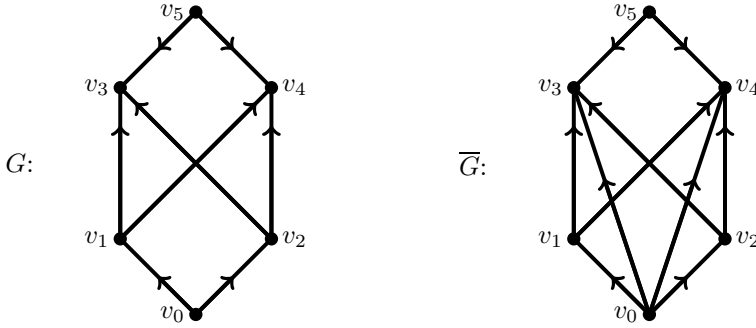


Figure 6: Example 3.5.

By Theorem 3.1,  $f$  is a discrete Morse function on  $G$  satisfying Condition (\*). By Theorem 2.12,  $f$  can be extended to be a Morse function  $\bar{f}$  on  $\bar{G}$  such that  $\bar{f}(v) = f(v)$  for all vertices  $v \in V(\bar{G})$ . Then

$$\text{Crit}_*(\bar{G}) = R(v_0, v_5, v_5v_3, v_5v_4),$$

$$\Omega_*(G) = R(v_0, v_1, v_2, v_3, v_4, v_5, v_0v_1, v_0v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_5v_3, v_5v_4, v_0v_1v_3 - v_0v_2v_3, v_0v_1v_4 - v_0v_2v_4).$$

Let  $\bar{V} = \text{grad}\bar{f}$  be the discrete gradient vector field on  $\bar{G}$ . Then

$$\begin{aligned} \bar{V}(v_1) &= -v_0v_1, & \bar{V}(v_2) &= -v_0v_2, \\ \bar{V}(v_3) &= -v_0v_3, & \bar{V}(v_4) &= -v_0v_4, \\ (9) \quad \bar{V}(v_1v_3) &= -v_0v_1v_3, & \bar{V}(v_1v_4) &= -v_0v_1v_4, \\ (10) \quad \bar{V}(v_2v_3) &= -v_0v_2v_3, & \bar{V}(v_2v_4) &= -v_0v_2v_4, \\ \bar{V}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}. \end{aligned}$$

By (9) and (10),  $\bar{V}(\Omega_1(G)) \not\subseteq \Omega_2(G)$ . This implies that  $\Omega_*(G)$  is not  $\bar{V}$ -invariant.

Let  $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$  be the discrete gradient flow of  $\bar{G}$ . Then

$$\begin{aligned} \bar{\Phi}(v_0) &= v_0, & \bar{\Phi}(v_1) &= v_0, \\ \bar{\Phi}(v_2) &= v_0, & \bar{\Phi}(v_3) &= v_0, \\ \bar{\Phi}(v_4) &= v_0, & \bar{\Phi}(v_5) &= v_5, \\ \bar{\Phi}(v_0v_1) &= 0, & \bar{\Phi}(v_0v_2) &= 0, \\ \bar{\Phi}(v_0v_3) &= 0, & \bar{\Phi}(v_0v_4) &= 0, \end{aligned}$$

$$\begin{aligned}
 \bar{\Phi}(v_1v_3) &= 0, & \bar{\Phi}(v_1v_4) &= 0, \\
 \bar{\Phi}(v_2v_3) &= 0, & \bar{\Phi}(v_2v_4) &= 0, \\
 \bar{\Phi}(v_5v_3) &= v_5v_3 - v_0v_3, & \bar{\Phi}(v_5v_4) &= v_5v_4 - v_0v_4, \\
 \bar{\Phi}(v_0v_1v_4) &= 0, & \bar{\Phi}(v_0v_1v_3) &= 0, \\
 \bar{\Phi}(v_0v_2v_4) &= 0, & \bar{\Phi}(v_0v_2v_3) &= 0.
 \end{aligned}$$

By calculate directly, we have that  $\bar{\Phi}^\infty = \bar{\Phi}$ . Then

$$\begin{aligned}
 \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0, v_5, v_5v_3 - v_0v_3, v_5v_4 - v_0v_4), \\
 \Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G})) &= R(v_0, v_5, v_5v_3 - v_0v_3, v_5v_4 - v_0v_4).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \partial_1(v_5v_3 - v_0v_3) &= v_0 - v_5 \\
 \partial_1(v_5v_4 - v_0v_4) &= v_0 - v_5
 \end{aligned}$$

and

$$\begin{aligned}
 H_0(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= R \\
 H_1(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= R \\
 H_m(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0 \text{ for } m \geq 2,
 \end{aligned}$$

which are consistent with the path homology groups of  $G$ .

**Remark 3.6.** By Example 3.3 and Example 3.5, the condition that  $\Omega_*(G)$  is  $\bar{V}$  – invariant in Theorem 2.17 is sufficient but not necessary. That is, even if the digraph does not satisfy this condition, we may have an isomorphism of homology groups given in (6).

### 4. Further discussion

In this section, we will study the matrix representation of Theorem 2.17, which will be helpful for us to find efficient algorithms computing the homology (persistent homology) groups of digraphs applying our discrete Morse theory for digraphs in the future.

Let  $G$  be a digraph and  $\bar{G}$  the transitive closure of  $G$ . Choose all allowed elementary  $n$ -paths as a basis of  $P(\bar{G})$ , denoted as  $B_n$ . Let  $M(\bullet)$  be the matrix corresponding to the operator  $\bullet$  and  $E_n$  the identity matrix of order  $n$ . Let

$$\begin{aligned}
 \bar{V}_n &: P_n(\bar{G}) \longrightarrow P_{n+1}(\bar{G}), \\
 \partial_n &: P_n(\bar{G}) \longrightarrow P_{n-1}(\bar{G}),
 \end{aligned}$$

$$\begin{aligned} \bar{\Phi}_n & : P_n(\bar{G}) \longrightarrow P_n(\bar{G}), \\ \bar{\Phi}_n^\infty|_{\text{Crit}_n(\bar{G})} & : \text{Crit}_n(\bar{G}) \longrightarrow P_n^{\bar{\Phi}}(\bar{G}), \end{aligned}$$

for each  $n \geq 0$ .

We illustrate the calculation process of homology groups in Example 3.2 with the matrix representation of operators. Since

$$\begin{aligned} P_0(\bar{G}) & = R(v_0, v_1, v_2, v_3); \\ P_1(\bar{G}) & = R(v_0v_1, v_0v_2, v_0v_3, v_1v_3, v_2v_3); \\ P_2(\bar{G}) & = R(v_0v_1v_3, v_0v_2v_3); \\ P_3(\bar{G}) & = 0. \end{aligned}$$

Then

$$(11) \quad \bar{V}_1 \begin{bmatrix} v_0v_1 \\ v_0v_2 \\ v_0v_3 \\ v_1v_3 \\ v_2v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_0v_1v_3 \\ v_0v_2v_3 \end{bmatrix},$$

$$(12) \quad \bar{V}_0 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_0v_1 \\ v_0v_2 \\ v_0v_3 \\ v_1v_3 \\ v_2v_3 \end{bmatrix},$$

$$(13) \quad \partial_2 \begin{bmatrix} v_0v_1v_3 \\ v_0v_2v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_0v_1 \\ v_0v_2 \\ v_0v_3 \\ v_1v_3 \\ v_2v_3 \end{bmatrix},$$

and

$$(14) \quad \partial_1 \begin{bmatrix} v_0v_1 \\ v_0v_2 \\ v_0v_3 \\ v_1v_3 \\ v_2v_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

By (11)–(14), we have that

$$M(\bar{V}_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M(\bar{V}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$M(\partial_2) = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix},$$

and

$$M(\partial_1) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Hence,

$$M(\bar{\Phi}_1) = E_1 + M(\partial_1)M(\bar{V}_0) + M(\bar{V}_1)M(\partial_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

By calculate directly, we have that  $(M(\bar{\Phi}_1))^\infty = M(\bar{\Phi}_1) \cdot M(\bar{\Phi}_1) \cdot \dots = M(\bar{\Phi}_1) = M(\bar{\Phi}_1^\infty)$ . Since  $\text{Crit}_1(\bar{G}) = R(v_0v_2, v_2v_3)$ . Then  $M(\bar{\Phi}_1^\infty |_{\text{Crit}_1(\bar{G})})$  is the matrix composed of the second and the fifth rows of  $M(\bar{\Phi}_1)$ . That is,

$$M(\bar{\Phi}_1^\infty |_{\text{Crit}_1(\bar{G})}) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Since  $\Omega_1(G) = R(v_0v_1, v_0v_2, v_1v_3, v_2v_3)$ , it follows that

$$\Omega_1(G) \cap \bar{\Phi}^\infty(\text{Crit}_1(\bar{G})) = R(v_0v_2 - v_0v_1, v_2v_3 - v_1v_3).$$

By (14), we have that

$$(15) \quad M(\partial_1 |_{\Omega_1(G) \cap \bar{\Phi}^\infty(\text{Crit}_1(\bar{G}))}) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Hence, we can obtain that

$$\text{Ker}(\partial_1 |_{\Omega_1(G) \cap \bar{\Phi}^\infty(\text{Crit}_1(\bar{G}))}) = R(v_0v_2 - v_0v_1 + v_2v_3 - v_1v_3)$$

and

$$\text{Im}(\partial_1 |_{\Omega_1(G) \cap \bar{\Phi}^\infty(\text{Crit}_1(\bar{G}))}) = R(v_1 - v_2)$$

from (15).

Similarly, the matrix of  $\bar{\Phi}_0 : P_0(\bar{G}) \rightarrow P_0(\bar{G})$  is

$$M(\bar{\Phi}_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the matrix of  $\bar{\Phi}_2 : P_2(\bar{G}) \rightarrow P_2(\bar{G})$  is

$$M(\bar{\Phi}_2) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

By calculate directly,

$$(M(\bar{\Phi}_0))^\infty = M(\bar{\Phi}_0) = M(\bar{\Phi}_0^\infty)$$

and

$$(M(\bar{\Phi}_2))^\infty = M(\bar{\Phi}_2) = M(\bar{\Phi}_2^\infty).$$

Then

$$\begin{aligned} \bar{\Phi}_0^\infty(\text{Crit}_0(\bar{G})) &= \bar{\Phi}_0(\text{Crit}_0(\bar{G})) \\ &= \text{Crit}_0(\bar{G}). \end{aligned}$$

Hence,

$$\begin{aligned} \Omega_0(G) \cap \bar{\Phi}^\infty(\text{Crit}_0(\bar{G})) &= \text{Crit}_0(\bar{G}) \\ &= R(v_1, v_2) \end{aligned}$$

and

$$\text{Ker}(\partial_0 |_{\Omega_0(G) \cap \bar{\Phi}^\infty(\text{Crit}_0(\bar{G}))}) = R(v_1, v_2).$$

Furthermore, since  $\Omega_2(G) = R(v_0v_1v_3 - v_0v_2v_3)$  and  $\text{Crit}_2(\bar{G}) = R(v_0v_2v_3)$ , it follows that

$$\Omega_2(G) \cap \bar{\Phi}^\infty(\text{Crit}_2(\bar{G})) = R(v_0v_1v_3 - v_0v_2v_3).$$

By (13), we have that

$$(16) \quad M(\partial_2 |_{\Omega_2(G) \cap \bar{\Phi}^\infty(\text{Crit}_2(\bar{G}))}) = \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \end{bmatrix}.$$

Hence, we can obtain that

$$\text{Ker}(\partial_2 |_{\Omega_2(G) \cap \bar{\Phi}^\infty(\text{Crit}_2(\bar{G}))}) = 0$$

and

$$\text{Im}(\partial_2 |_{\Omega_2(G) \cap \bar{\Phi}^\infty(\text{Crit}_2(\bar{G}))}) = R(v_0v_1 - v_0v_2 + v_1v_3 - v_2v_3)$$

from (16).

Therefore,

$$\begin{aligned} H_0(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= R, \\ H_1(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0, \\ H_m(\Omega_*(G) \cap \bar{\Phi}^\infty(\text{Crit}_*(\bar{G}))) &= 0 \text{ for } m \geq 2, \end{aligned}$$

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## References

- [1] R. AYALA, L.M. FERNÁNDEZ AND J.A. VILCHES, Discrete Morse inequalities on infinite graphs. *Electron. J. Combin.* **16** (1) (2009), R38. [MR2491640](#)



- [2] R. AYALA, L.M. FERNÁNDEZ AND J.A. VILCHES, Morse inequalities on certain infinite 2-complexes. *Glasg. Math. J.* **49** (2) (2007), 155–165. [MR2347251](#)
- [3] R. AYALA, L.M. FERNÁNDEZ, D. FERNÁNDEZ-TERNERO AND J.A. VILCHES, Discrete Morse theory on graphs. *Topol. Appl.* **156** (2009), 3091–3100. [MR2556069](#)
- [4] R. AYALA, L.M. FERNÁNDEZ, A. QUINTERO AND J.A. VILCHES, A note on the pure Morse complex of a graph. *Topol. Appl.* **155** (2008), 2084–2089. [MR2457993](#)
- [5] J. BANG-JENSEN AND G.Z. GUTIN, *Digraphs: Theory, Algorithms and Applications. 2-nd Edition*. Springer Monographs in Mathematics, Springer, 2009. [MR2472389](#)
- [6] S. BRESSAN, J. LI, S. REN AND J. WU, The embedded homology of hypergraphs and applications. *Asian J. Math.* **23** (3) (2019), 479–500. [MR3983756](#)
- [7] R. FORMAN, Morse theory for cell complexes. *Adv. Math.* **134** (1998), 90–145. [MR1612391](#)
- [8] R. FORMAN, A user’s guide to discrete Morse theory. *Sém. Lothar. Combin.* **48** (2002), 35pp. [MR1939695](#)
- [9] R. FORMAN, Discrete Morse theory and the cohomology ring. *Trans. Amer. Math. Soc.* **354** (12) (2002), 5063–5085. [MR1926850](#)
- [10] R. FORMAN, Witten–Morse theory for cell complexes. *Topology* **37** (5) (1998), 945–979. [MR1650414](#)
- [11] A. GRIGOR’YAN, Y. LIN, Y. MURANOV AND S.T. YAU, Homologies of path complexes and digraphs. *Preprint arXiv:1207.2834v4* (2013).
- [12] A. GRIGOR’YAN, Y. LIN, Y. MURANOV AND S.T. YAU, Homotopy theory for digraphs. *Pure Appl. Math. Q.* **10** (4) (2014), 619–674. [MR3324763](#)
- [13] A. GRIGOR’YAN, Y. LIN, Y. MURANOV AND S.T. YAU, Cohomology of digraphs and (undirected) graphs. *Asian J. Math.* **19** (5) (2015), 887–932. [MR3431683](#)
- [14] A. GRIGOR’YAN, Y. LIN, Y. MURANOV AND S.T. YAU, Path complexes and their homologies. *Preprint* (2015), <https://www.math.uni-bielefeld.de/grigor/dnote.pdf>. To appear in *Int. J. Math.*

- [15] A. GRIGOR'YAN, Y. MURANOV AND S.T. YAU, Homologies of digraphs and Künneth formulas. *Commun. Anal. Geom.* **25** (5) (2017), 969–1018. [MR3733797](#)
- [16] A. GRIGOR'YAN, Y. MURANOV, V. VERSHININ AND S.T. YAU, Path homology theory of multigraphs and quivers. *Forum Math.* **30** (5) (2018), 1319–1337. [MR3849650](#)
- [17] A. GRIGOR'YAN, R. JIMENEZ, Y. MURANOV AND S.T. YAU, Homology of path complexes and hypergraphs. *Topol. Appl.* **267** (2019), 106877. [MR4002637](#)
- [18] T. LEWINER, H. LOPES AND G. TAVARES, Applications of Forman's discrete Morse theory to topology visualization and mesh compression. *Transactions on visualization and computer graphics.* **10** (5) (2004), 499–508, IEEE.
- [19] K. MISCHAIKOW AND V. NANDA, Morse theory for filtrations and efficient computation of persistent homology. *Discrete Comput. Geom.* **50** (2013), 330–353. [MR3090522](#)
- [20] H. KANNAN, E. SAUCAN, I. ROY AND A. SAMAL, Persistent homology of unweighted complex networks via discrete Morse theory. *Scientific Reports* **9** (2019), article 13817.
- [21] C. WANG AND S. REN, A Discrete Morse Theory for Digraphs. *arXiv:2007.13425*.

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