Pure and Applied Mathematics Quarterly
Volume 17, Number 5, 1861–1898, 2021

# The real Fourier–Mukai transform of Cayley cycles

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**Abstract:** The real Fourier–Mukai transform sends a section of a torus fibration to a connection over the total space of the dual torus fibration. By this method, Leung, Yau and Zaslow introduced deformed Hermitian Yang–Mills (dHYM) connections for Kähler manifolds and Lee and Leung introduced deformed Donaldson–Thomas (dDT) connections for $G_2$- and Spin(7)-manifolds.

In this paper, we suggest an alternative definition of a dDT connection for a manifold with a Spin(7)-structure which seems to be more appropriate by carefully computing the real Fourier–Mukai transform again. We also post some evidences showing that the definition we suggest is compatible with dDT connections for a $G_2$-manifold and dHYM connections of a Calabi–Yau 4-manifold.

Another importance of this paper is that it motivates our study in our other papers. That is, based on the computations in this paper, we develop the theories of deformations of dDT connections for a manifold with a Spin(7)-structure and the "mirror" of the volume functional, which is called the Dirac–Born–Infeld (DBI) action in physics.

**Keywords:** Mirror symmetry, deformed Donaldson–Thomas, special holonomy, calibrated submanifold.



Received January 14, 2021.

**2010 Mathematics Subject Classification:** Primary 53C07; secondary 53D37, 53C25, 53C38.

*The first author is supported by JSPS KAKENHI Grant Number JP17K14181.

†The second author is supported by JSPS KAKENHI Grant Number JP18K13415 and Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

## 1. Introduction

In the context of mirror symmetry, in particular Kontsevich's homological mirror symmetry conjecture, one vital need is to provide a geometric functor from one side to its mirror side. Originally, the conjecture was stated for Calabi–Yau manifolds, however, the applicable scope has been extended to the other special holonomy cases, $G_2$ and Spin(7). Firstly, for the Calabi–Yau case, Leung, Yau and Zaslow [10] in 2000 found a natural and promising candidate for such a functor, which is called the real Fourier–Mukai transform nowadays.

The real Fourier–Mukai transform sends a section of a torus fibration to a connection over the total space of the dual torus fibration. In their paper [10], Leung, Yau and Zaslow proved that the real Fourier–Mukai transform of a special Lagrangian cycle is a *deformed Hermitian Yang–Mills (dHYM) connection*. This can be considered as a correspondence between supersymmetric A-cycles and B-cycles in the sense of mirror symmetry.

Even in the case where the total space is not a Calabi–Yau manifold, the real Fourier–Mukai transform can also work. Actually, Lee and Leung [9] computed the real Fourier–Mukai transform of an associative and a coassociative cycle in a $G_2$-manifold and of a Cayley cycle in a Spin(7)-manifold. In [9], they picked some properties which the real Fourier–Mukai transform satisfies and called such a connection a *deformed Donaldson–Thomas (dDT) connection*. In this paper, we suggest an alternative definition of a dDT connection for a manifold with a Spin(7)-structure which seems to be more appropriate by carefully computing the real Fourier–Mukai transform again.

As the real Fourier–Mukai transform of a submanifold written as a graph of a section of a trivial $T^4$-fibration over a flat 4-dimensional base $B$, we obtain the following.

**Theorem 1.1** (Theorem 5.1)**.** *Let $B \subset \mathbb{R}^4$ be an open set and $f : B \to T^4$ be a smooth function. Denote by $S = \{ (x, f(x)) \mid x \in B \}$ the graph of $f$, a 4-dimensional submanifold in $X = B \times T^4$. By the real Fourier–Mukai transform, $S$ corresponds to a Hermitian connection $\nabla^S$ of a trivial complex line bundle over $B \times (T^4)^* \cong X$. Denote by $F^S_\nabla \in \sqrt{-1}\Omega^2(X)$ the curvature 2-form of $\nabla^S$.*

*Then, the graph $S$ is a Cayley submanifold with an appropriate orientation if and only if*

$$\pi^2_7 \left( F^S_\nabla + \frac{1}{6} * (F^S_\nabla)^3 \right) = 0 \quad \text{and} \quad \pi^4_7 \left( (F^S_\nabla)^2 \right) = 0.$$

*Here, $\pi^k_\ell : \Omega^k \to \Omega^k_\ell$ is the projection and $\Omega^k_\ell \subset \Omega^k$ is the subspace of the space of $k$-forms corresponding to the $\ell$-dimensional irreducible representation of* Spin(7) *as in Subsection 3.4.*

We also compute the real Fourier–Mukai transform of a Cayley cycle, a Cayley submanifold with an ASD connection over it, and show the following.

**Theorem 1.2** (Theorem 5.7)**.** *Let $B \subset \mathbb{R}^4$ be an open set and $f : B \to T^4$ be a smooth function. Denote by $S = \{ (x, f(x)) \mid x \in B \}$ the graph of $f$, a 4-dimensional submanifold in $X = B \times T^4$. Let $\nabla^B$ be a Hermitian connection of a trivial complex line bundle $B \times \mathbb{C} \to B$. Denote by $F^B_\nabla \in \sqrt{-1}\Omega^2(B)$ the curvature of $\nabla^B$.*

*By the real Fourier–Mukai transform, the pair $(S, \nabla^B)$ corresponds to a Hermitian connection $\nabla$ of a trivial complex line bundle over $B \times (T^4)^* \cong X$. Denote by $F_\nabla \in \sqrt{-1}\Omega^2(X)$ the curvature 2-form of $\nabla$. Then, the following conditions are equivalent.*

1. *The graph $S$ is a Cayley submanifold with an appropriate orientation and if we identify $-\sqrt{-1}F^B_\nabla \in \Omega^2(B)$ with a 2-form on $S$, it is anti-self-dual with respect to the induced metric and the orientation which makes $S$ Cayley.*

2. *The Hermitian connection $\nabla$ satisfies*

$$\pi^2_7\left(F_\nabla + \frac{1}{6} * F^3_\nabla\right) = 0 \quad \textit{and} \quad \pi^4_7\left(F^2_\nabla\right) = 0.$$

Based on these theorems, we suggest the following definition.

**Definition 1.3.** Let $X^8$ be an 8-manifold with a Spin(7)-structure $\Phi \in \Omega^4$ and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Denote by $\Omega^k_\ell \subset \Omega^k$ the subspace of the space of $k$-forms corresponding to the $\ell$-dimensional irreducible representation of Spin(7) as in Subsection 3.4. Let $\pi^k_\ell : \Omega^k \to \Omega^k_\ell$ be the projection. A Hermitian connection $\nabla$ of $(L, h)$ satisfying

$$\pi^2_7\left(F_\nabla + \frac{1}{6} * F^3_\nabla\right) = 0 \quad \text{and} \quad \pi^4_7(F^2_\nabla) = 0 \tag{1.1}$$

is called a *deformed Donaldson–Thomas connection for a manifold with a* Spin(7)*-structure (a* Spin(7)*-dDT connection)*. Here, we regard the curvature 2-form $F_\nabla$ of $\nabla$ as a $\sqrt{-1}\mathbb{R}$-valued closed 2-form on $X$.

In this paper, we post some evidences showing that Definition 1.3 we suggest for a Spin(7)-manifold is compatible with dDT connections for a $G_2$-manifold and dHYM connections for a Calabi–Yau 4-manifold in Lemmas 7.1 and 7.2.

We also compute the real Fourier–Mukai transform of (co)associative cycles in $G_2$-manifolds. This makes us confirm the definition of deformed Donaldson–Thomas connections for a manifold with a $G_2$-structure introduced by Lee and Leung [9]. This is also useful in the computation of the real Fourier–Mukai transform of Cayley cycles. It turns out that the real Fourier–Mukai transform of an associative cycle coincides with that of a coassociative cycle as stated in [9]. Moreover, the real Fourier–Mukai transform implies identities mirror to associator and Cayley equalities. In [8], we show them and dDT connections for $G_2$- and Spin(7)-manifolds minimize a kind of the volume functional, which is called the Dirac-Born-Infeld (DBI) action in physics.

This paper is organized as follows. In Section 2, we explain the real Fourier–Mukai transform in detail. Section 3 gives basic identities and some decompositions of the spaces of differential forms in $G_2$- and Spin(7)-geometry that are used in this paper. In Section 4–6 we give computations of real Fourier–Mukai transforms and show Theorems 1.1 and 1.2. In Section 7, we show compatibilities of our Spin(7)-dDT connections with dDT connections for a $G_2$-manifold and dHYM connections for a Calabi–Yau 4-manifold. In Appendix A, we summarize the notation used in this paper.

## 2. The real Fourier–Mukai transform

In this section, we explain the real Fourier–Mukai transform. We need the following two fundamental facts. The first one is that a representation $\rho : \pi_1(M) \to GL(k, \mathbb{R})$ naturally assigns a flat connection $\tilde{\nabla}$ of $\mathbb{R}^k$-bundle $E$ over a manifold $M$ by

$$E := \tilde{M} \times_\rho \mathbb{R}^k := (\tilde{M} \times \mathbb{R}^k)/\sim,$$

where $\tilde{M}$ is the universal cover of $M$ and $(x, v) \sim (x \cdot \gamma, \rho(\gamma)^{-1} v)$ for $\gamma \in \pi_1(M)$. The flat connection $\tilde{\nabla}$ of $E$ is induced from the exterior derivative $d$ on $\tilde{M} \times \mathbb{R}^k$. The second one is that an $n$-dimensional torus $T^n$ $(= \mathbb{R}^n/(2\pi\mathbb{Z})^n)$ is canonically isomorphic to

$$\mathrm{Hom}(\pi_1((T^n)^*), U(1)) = \mathrm{Hom}(((2\pi\mathbb{Z})^n)^*, U(1)),$$

the set of all homomorphisms from the first fundamental group of its dual torus $(T^n)^*$ $(= (\mathbb{R}^n)^*/((2\pi\mathbb{Z})^n)^*)$ to $U(1)$ by

$$T^n \ni a = [\tilde{a}] \mapsto \rho_a := e^{-\sqrt{-1}\langle \cdot\,, \tilde{a}\rangle} \in \mathrm{Hom}(\pi_1((T^n)^*), U(1)),$$

where $\langle \cdot\,, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ is a dual pairing. Then, combining these two facts with $M = (T^n)^*$, we see that a point $a$ in $T^n$ assigns a flat Hermitian connection $\tilde{\nabla}^a$ of a complex line bundle $E_a := (\mathbb{R}^n)^* \times_{\rho_a} \mathbb{C}$ with the standard Hermitian metric over the dual torus $(T^n)^*$. Actually, $\pi_a : E_a \to (T^n)^*$ is isomorphic to the trivial $\mathbb{C}$-bundle $\pi_0 : \underline{\mathbb{C}} \to (T^n)^*$ since we have a nonvanishing section $s(y) := [\tilde{y}, e^{\sqrt{-1}\langle \tilde{y}, \tilde{a}\rangle} \cdot 1]$ of $E_a$, where $\tilde{y} \in (\mathbb{R}^n)^*$ representing $y \in (T^n)^* = (\mathbb{R}^n)^*/((2\pi\mathbb{Z})^n)^*$ and 1 is the trivial section of $(\mathbb{R}^n)^* \times \mathbb{C}$. The bundle isomorphism $\xi : E_a \to \underline{\mathbb{C}}$ is given, on each fiber, by

$$\pi_a^{-1}(y) \ni c \cdot s(y) \mapsto c \in \mathbb{C} \; (= \pi_0^{-1}(y)).$$

Then, a flat Hermitian connection of $\underline{\mathbb{C}}$ is induced from $\tilde{\nabla}^a$ of $E_a$ and denote it by $\nabla^a$. The connection 1-form of $\nabla^a$ with respect to the section $1 \in \Gamma((T^n)^*, \underline{\mathbb{C}})$ is represented as

$$\begin{aligned}\nabla^a 1 = \xi^{-1}(\tilde{\nabla}^a(\xi(1))) =& e^{-\sqrt{-1}\langle \cdot\,, \tilde{a}\rangle} d(e^{\sqrt{-1}\langle \cdot\,, \tilde{a}\rangle} \cdot 1)\\ =& \left(\sqrt{-1} d\langle \cdot\,, \tilde{a}\rangle\right) \otimes 1.\end{aligned}$$

In summary, a point $a = [(a^1, \cdots, a^n)] \in T^n$ assigns an equivalence class of a Hermitian complex line bundle with a flat connection over $(T^n)^*$ and one of its representatives is the trivial $\mathbb{C}$-bundle with the standard Hermitian metric and a flat Hermitian connection $\nabla^a$ defined by

$$\nabla^a := d + \sqrt{-1} \sum_{i=1}^{n} a^i dy^i,$$

where $y = (y^1, \cdots, y^n)$ are the standard coordinates on $(T^n)^*$. This correspondence $a \mapsto \nabla^a$ is also explained in [2, Section 3.2.1].

When we consider the family of this correspondence, we get the real Fourier–Mukai transform. Precisely, let $B \subset \mathbb{R}^k$ be an open set with coordinates $x = (x^1, \cdots, x^k)$ and $f = (f^1, \cdots, f^n) : B \to T^n$ be a smooth map. Then, we get two objects: a submanifold and a connection. The $k$-dimensional submanifold in $X := B \times T^n$, denoted by $S$, is defined as the graph of $f$, that is,

$$S := \{\, (x, f(x)) \mid x \in B \,\}.$$

On the other hand, taking the family of $\nabla^{f(x)}$ for all $x \in B$, we get a Hermitian connection

$$\nabla := d + \sqrt{-1} \sum_{i=1}^{n} f^i dy^i$$

of the trivial $\mathbb{C}$-bundle over $X^* := B \times (T^n)^*$. We usually identify $B \times (T^n)^*$ with $B \times T^n$. We call $\nabla$ *the real Fourier–Mukai transform* of $S$. Basically, a property on $S$ is first interpreted as one of $f$ and second reinterpreted as one of $\nabla$. We remark that the real Fourier–Mukai transform of $(S, \nabla^B)$, the pair of a graph of $f$ and a Hermitian connection $\nabla^B = d + \sqrt{-1} \sum_{i=1}^{k} A^i dx^i$ of the trivial $\mathbb{C}$-bundle over $B \cong S$, is also defined by

$$\nabla := d + \sqrt{-1} \sum_{i=1}^{k} A^i dx^i + \sqrt{-1} \sum_{i=1}^{n} f^i dy^i$$

as a Hermitian connection of the trivial $\mathbb{C}$-bundle over $X^* = B \times (T^n)^*$.

## 3. Basics on $G_2$- and Spin(7)-geometry

In this section, we collect some basic definitions and equations on $G_2$- and Spin(7)-geometry which we need in the calculations in this paper for the reader's convenience. See for example [1, 3, 4] for references.

### 3.1. The Hodge-$*$ operator

Let $V$ be an $n$-dimensional oriented real vector space with an inner product $g$. Denote by $\langle \cdot, \cdot \rangle$ the induced inner product on $\Lambda^k V^*$ from $g$. Let $*$ be the Hodge-$*$ operator. The following identities are frequently used throughout this paper.

For $\alpha, \beta \in \Lambda^k V^*$ and $v \in V$, we have

$$*^2|_{\Lambda^k V^*} = (-1)^{k(n-k)} \mathrm{id}_{\Lambda^k V^*}, \quad \langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle,$$
$$i(v) * \alpha = (-1)^k * (v^\flat \wedge \alpha), \quad *(i(v)\alpha) = (-1)^{k+1} v^\flat \wedge *\alpha.$$

### 3.2. Basics on $G_2$-geometry

Let $V$ be an oriented 7-dimensional vector space. A $G_2$-*structure* on $V$ is a 3-form $\varphi \in \Lambda^3 V^*$ such that there is a positively oriented basis $\{ e_i \}_{i=1}^7$ of $V$ with the dual basis $\{ e^i \}_{i=1}^7$ of $V^*$ satisfying

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{3.1}$$

where $e^{i_1 \cdots i_k}$ is short for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. Setting $\mathrm{vol} := e^{1 \cdots 7}$, the 3-form $\varphi$ uniquely determines an inner product $g_\varphi$ via

$$g_\varphi(u, v) \ \mathrm{vol} = \frac{1}{6} i(u)\varphi \wedge i(v)\varphi \wedge \varphi \tag{3.2}$$

for $u, v \in V$. It follows that any oriented basis $\{ e_i \}_{i=1}^7$ for which (3.1) holds is orthonormal with respect to $g_\varphi$. Thus, the Hodge-dual of $\varphi$ with respect to $g_\varphi$ is given by

$$* \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{3.3}$$

The stabilizer of $\varphi$ is known to be the exceptional 14-dimensional simple Lie group $G_2 \subset \mathrm{GL}(V)$. The elements of $G_2$ preserve both $g_\varphi$ and vol, that is, $G_2 \subset \mathrm{SO}(V, g_\varphi)$.

We summarize important well-known facts about the decomposition of tensor products of $G_2$-modules into irreducible summands. Denote by $V_k$ the $k$-dimensional irreducible $G_2$-module if there is a unique such module. For instance, $V_7$ is the irreducible 7-dimensional $G_2$-module $V$ from above, and $V_7^* \cong V_7$. For its exterior powers, we obtain the decompositions

$$\begin{aligned} &\Lambda^0 V^* \cong \Lambda^7 V^* \cong V_1, \qquad \Lambda^2 V^* \cong \Lambda^5 V^* \cong V_7 \oplus V_{14}, \\ &\Lambda^1 V^* \cong \Lambda^6 V^* \cong V_7, \qquad \Lambda^3 V^* \cong \Lambda^4 V^* \cong V_1 \oplus V_7 \oplus V_{27}, \end{aligned} \tag{3.4}$$

where $\Lambda^k V^* \cong \Lambda^{7-k} V^*$ due to the $G_2$-invariance of the Hodge isomorphism $* : \Lambda^k V^* \to \Lambda^{7-k} V^*$. We denote by $\Lambda^k_\ell V^* \subset \Lambda^k V^*$ the subspace isomorphic to $V_\ell$. Let

$$\pi^k_\ell : \Lambda^k V^* \to \Lambda^k_\ell V^*$$

be the canonical projection. Identities for these spaces we need in this paper are as follows.

$$\begin{aligned}
\Lambda^2_7 V^* =& \{\, i(u)\varphi \mid u \in V \,\} = \{\, \alpha \in \Lambda^2 V^* \mid *(\varphi \wedge \alpha) = 2\alpha \,\}, \\
\Lambda^2_{14} V^* =& \{\, \alpha \in \Lambda^2 V^* \mid *\varphi \wedge \alpha = 0 \,\} = \{\, \alpha \in \Lambda^2 V^* \mid *(\varphi \wedge \alpha) = -\alpha \,\}, \\
\Lambda^3_1 V^* =& \mathbb{R}\varphi, \\
\Lambda^3_7 V^* =& \{\, i(u) * \varphi \in \Lambda^3 V^* \mid u \in V \,\}.
\end{aligned}$$

The following equations are well-known and useful in this paper.

**Lemma 3.1.** *For any $u \in V$, we have the following identities.*

$$\begin{aligned}
\varphi \wedge i(u) * \varphi &= -4 * u^\flat, \\
*\varphi \wedge i(u)\varphi &= 3 * u^\flat, \\
\varphi \wedge i(u)\varphi &= 2 * (i(u)\varphi) = 2u^\flat \wedge *\varphi.
\end{aligned}$$

**Definition 3.2.** Let $X$ be an oriented 7-manifold. A *$G_2$-structure* on $X$ is a 3-form $\varphi \in \Omega^3$ such that at each $p \in X$ there is a positively oriented basis $\{\, e_i \,\}_{i=1}^7$ of $T_pX$ such that $\varphi_p \in \Lambda^3 T_p^* X$ is of the form (3.1). As noted above, $\varphi$ determines a unique Riemannian metric $g = g_\varphi$ on $X$ by (3.2), and the basis $\{\, e_i \,\}_{i=1}^7$ is orthonormal with respect to $g$. A $G_2$-structure $\varphi$ is called *torsion-free* if it is parallel with respect to the Levi-Civita connection of $g = g_\varphi$. A manifold with a torsion-free $G_2$-structure is called a *$G_2$-manifold*.

A manifold $X$ admits a $G_2$-structure if and only if its frame bundle is reduced to a $G_2$-subbundle. Hence, considering its associated subbundles, we see that $\Lambda^* T^* X$ has the same decomposition as in (3.4). The algebraic identities above also hold.

### 3.3. Associative and coassociative submanifolds

On a $G_2$-manifold $(X, \varphi)$, the $G_2$-structure $\varphi$ and its Hodge dual $*\varphi$ are known to be calibrations. The corresponding calibrated submanifolds are called *associative submanifolds* and *coassociative submanifolds*, respectively. By [3, 11], we can characterize these submanifolds as follows.

**Lemma 3.3.** *An oriented 3-dimensional submanifold $A \subset X$ is associative with an appropriate orientation if and only if $*\varphi(v_1, v_2, v_3, \cdot) = 0$ for any $p \in A$ and $v_1, v_2, v_3 \in T_pS$. An oriented 4-dimensional submanifold $C \subset X$ is coassociative with an appropriate orientation if and only if the restriction of $\varphi$ to $C$ vanishes.*

### 3.4. Spin(7)-geometry

Let $W$ be an 8-dimensional oriented real vector space. A Spin(7)-*structure on* $W$ is a 4-form $\Phi \in \Lambda^4 W^*$ such that there is a positively oriented basis $\{ e_i \}_{i=0}^7$ of $W$ with dual basis $\{ e^i \}_{i=0}^7$ of $W^*$ satisfying

$$\begin{aligned} \Phi := & e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\ & + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}, \end{aligned} \tag{3.5}$$

where $e^{i_1 \cdots i_k}$ is short for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. Defining forms $\varphi$ and $*_7\varphi$ on $V := \mathrm{span}\{ e_i \}_{i=1}^7 \subset W$ as in (3.1) and (3.3), where $*_7$ stands for the Hodge star operator on $V$, we have

$$\Phi = e^0 \wedge \varphi + *_7\varphi.$$

Note that $\Phi$ is self-dual, that is, $*_8\Phi = \Phi$, where $*_8$ is the Hodge star operator on $W$. It is known that $\Phi$ uniquely determines an inner product $g_\Phi$ and a volume form and the subgroup of $\mathrm{GL}(W)$ preserving $\Phi$ is isomorphic to Spin(7). As in Definition 3.2, we can define an 8-manifold with a Spin(7)-structure and a Spin(7)-manifold.

Denote by $W_k$ the $k$-dimensional irreducible Spin(7)-module if there is a unique such module. For example, $W_8$ is the irreducible 8-dimensional Spin(7)-module from above, and $W_8^* \cong W_8$. The group Spin(7) acts irreducibly on $W_7 \cong \mathbb{R}^7$ as the double cover of SO(7). For its exterior powers, we obtain the decompositions

$$\begin{aligned} &\Lambda^0 W^* \cong \Lambda^8 W^* \cong W_1, && \Lambda^2 W^* \cong \Lambda^6 W^* \cong W_7 \oplus W_{21}, \\ &\Lambda^1 W^* \cong \Lambda^7 W^* \cong W_8, && \Lambda^3 W^* \cong \Lambda^5 W^* \cong W_8 \oplus W_{48}, \\ &\Lambda^4 W^* \cong W_1 \oplus W_7 \oplus W_{27} \oplus W_{35} \end{aligned}$$

where $\Lambda^k W^* \cong \Lambda^{8-k} W^*$ due to the Spin(7)-invariance of the Hodge isomorphism $*_8 : \Lambda^k W^* \to \Lambda^{8-k} W^*$. Again, we denote by $\Lambda^k_\ell W^* \subset \Lambda^k W^*$ the subspace isomorphic to $W_\ell$ in the above notation.

The space $\Lambda^k_7 W^*$ for $k = 2, 4, 6$ is explicitly given as follows. For the explicit descriptions of the other irreducible summands, see for example [5, (4.7)].

**Lemma 3.4.** *Let $e^0 \in W^*$ be a unit vector. Set $V^* = (\mathbb{R}e^0)^\perp$, the orthogonal complement of $\mathbb{R}e^0$. The group* $\mathrm{Spin}(7)$ *acts irreducibly on $V^*$ as the double cover of* $\mathrm{SO}(7)$*, and hence, we have the identification $V^* \cong W_7$. Then, the following maps are* $\mathrm{Spin}(7)$*-equivariant isometries.*

$$\lambda^k : V^* \longrightarrow \Lambda^k_7 W^*, \qquad \begin{aligned} \lambda^2(\alpha) &:= \tfrac{1}{2}\left(e^0 \wedge \alpha + i(\alpha^\sharp)\varphi\right), \\ \lambda^4(\alpha) &:= \tfrac{1}{\sqrt{8}}\left(e^0 \wedge i(\alpha^\sharp) *_7 \varphi - \alpha \wedge \varphi\right), \\ \lambda^6(\alpha) &:= \tfrac{1}{3}\Phi \wedge \lambda^2(\alpha) = *_8 \lambda^2(\alpha). \end{aligned} \tag{3.6}$$

*Here, $*_8$ and $*_7$ are the Hodge star operators on $W^*$ and $V^*$, respectively.*

*Proof.* The maps above are $\mathrm{Spin}(7)$-equivariant isomorphism by [5, Lemma 4.2]. We show that these are isometries. For $\alpha \in V^*$, we compute

$$4|\lambda^2(\alpha)|^2 = \langle e^0 \wedge \alpha + i(\alpha^\sharp)\varphi, e^0 \wedge \alpha + i(\alpha^\sharp)\varphi\rangle = |\alpha|^2 + |i(\alpha^\sharp)\varphi|^2.$$

By Lemma 3.1, we have

$$|i(\alpha^\sharp)\varphi|^2 = *_7\left(i(\alpha^\sharp)\varphi \wedge *_7(i(\alpha^\sharp)\varphi)\right) = *_7\left(i(\alpha^\sharp)\varphi \wedge \alpha \wedge *_7\varphi\right) = 3|\alpha|^2. \tag{3.7}$$

Thus, we see that $\lambda^2$ is an isometry. By the definition of $\lambda^6$, this is also an isometry. We also compute

$$\begin{aligned} 8|\lambda^4(\alpha)|^2 &= \left\langle e^0 \wedge i(\alpha^\sharp) *_7 \varphi - \alpha \wedge \varphi, e^0 \wedge i(\alpha^\sharp) *_7 \varphi - \alpha \wedge \varphi \right\rangle \\ &= |i(\alpha^\sharp) *_7 \varphi|^2 + |\alpha \wedge \varphi|^2 \\ &= 2|\alpha \wedge \varphi|^2. \end{aligned}$$

The last term is computed as

$$\begin{aligned} |\alpha \wedge \varphi|^2 =& \langle \varphi, i(\alpha^\sharp)(\alpha \wedge \varphi)\rangle \\ =& \langle \varphi, |\alpha|^2\varphi - \alpha \wedge i(\alpha^\sharp)\varphi\rangle = |\alpha|^2|\varphi|^2 - |i(\alpha^\sharp)\varphi|^2 = 4|\alpha|^2, \end{aligned}$$

where we use $|\varphi|^2 = 7$ and (3.7). Hence, we see that $\lambda^4$ is an isometry. □

We give a relation between $*_8$ and $*_7$, which is useful in Section 5.

**Lemma 3.5.** *For $\alpha \in \Lambda^k V^*$, we have*

$$*_8\alpha = (-1)^k e^0 \wedge *_7\alpha, \quad *_7\alpha = *_8(e^0 \wedge \alpha). \tag{3.8}$$

*Proof.* Denote by $\mathrm{vol}_7$ the volume form on $V^*$. The volume form on $W^*$ is given by $e^0 \wedge \mathrm{vol}_7$. Then, for any $\beta \in \Lambda^k V^*$, we have

$$\beta \wedge *_8 \alpha = \langle \beta, \alpha \rangle e^0 \wedge \mathrm{vol}_7 = e^0 \wedge \beta \wedge *_7 \alpha = (-1)^k \beta \wedge e^0 \wedge *_7 \alpha,$$

which implies the first equation. The second equation follows from the first. $\square$

We give some formulas about projections onto some irreducible summands. Denote by

$$\pi^k_\ell : \Lambda^k W^* \to \Lambda^k_\ell W^* \tag{3.9}$$

the canonical projection. When $k = 2, 4, 6$ and $\ell = 7$, Lemma 3.4 implies that

$$\pi^k_\ell(\alpha^k) = \sum_{\mu=1}^{7} \langle \alpha^k, \lambda^k(e^\mu) \rangle \cdot \lambda^k(e^\mu) \tag{3.10}$$

for $\alpha^k \in \Lambda^k W^*$, where $\{ e^\mu \}_{\mu=1}^7$ is an orthonormal basis of $V^*$.

We give other descriptions of $\pi^k_\ell$ for $k = 2, 6$. Since the map $\Lambda^2 W^* \ni \alpha^2 \mapsto *_8(\Phi \wedge \alpha^2) \in \Lambda^2 W^*$ is Spin(7)-equivariant, the simple computation and Schur's lemma give the following:

$$\begin{aligned} \Lambda^2_7 W^* =& \{ \alpha^2 \in \Lambda^2 W^* \mid \Phi \wedge \alpha^2 = 3 *_8 \alpha^2 \}, \\ \Lambda^2_{21} W^* =& \{ \alpha^2 \in \Lambda^2 W^* \mid \Phi \wedge \alpha^2 = - *_8 \alpha^2 \}. \end{aligned}$$

Since $\alpha^2 = \pi^2_7(\alpha^2) + \pi^2_{21}(\alpha^2)$ and $*_8(\Phi \wedge \alpha^2) = 3\pi^2_7(\alpha^2) - \pi^2_{21}(\alpha^2)$ for a 2-form $\alpha^2 \in \Lambda^2 W^*$, it follows that

$$\pi^2_7(\alpha^2) = \frac{\alpha^2 + *_8(\Phi \wedge \alpha^2)}{4}, \quad \pi^2_{21}(\alpha^2) = \frac{3\alpha^2 - *_8(\Phi \wedge \alpha^2)}{4}. \tag{3.11}$$

Since $*_8 : \Lambda^6_\ell W^* \to \Lambda^2_\ell W^*$ is an isomorphism, we also obtain for a 6-form $\alpha^6 \in \Lambda^6 W^*$

$$\pi^6_7(\alpha^6) = \frac{\alpha^6 + \Phi \wedge *_8 \alpha^6}{4}, \quad \pi^6_{21}(\alpha^6) = \frac{3\alpha^6 - \Phi \wedge *_8 \alpha^6}{4}. \tag{3.12}$$

### 3.5. Cayley submanifolds

The 4-form $\Phi$ given by (3.5) is known to be a calibration. The corresponding calibrated submanifold is called a *Cayley submanifold*. We give a characterization of Cayley submanifolds, which is equivalent to that of [3, 11] by Lemma 3.4.

Define a Spin(7)-equivariant map $\tau : \Lambda^4 W \to \Lambda^4_7 W^*$ by

$$\tau(u_0, u_1, u_2, u_3) = \pi^4_7(u_0^\flat \wedge u_1^\flat \wedge u_2^\flat \wedge u_3^\flat). \tag{3.13}$$

If $\{ e^\mu \}_{\mu=1}^7$ is an oriented orthonormal basis of $V^*$, (3.10) implies that

$$\tau = \sum_{\mu=1}^{7} \lambda^4(e^\mu) \otimes \lambda^4(e^\mu).$$

**Lemma 3.6.** *For any $u_0, u_1, u_2, u_3 \in W$, we have*

$$|\Phi(u_0, u_1, u_2, u_3)|^2 + 8|\tau(u_0, u_1, u_2, u_3)|^2 = |u_0 \wedge u_1 \wedge u_2 \wedge u_3|^2.$$

*Proof.* We only have to show the equation when $\{ u_0, u_1, u_2, u_3 \}$ is orthonormal. Since the both sides are Spin(7)-invariant and Spin(7) acts transitively on $\mathrm{Gr}_3(W)$, the Grassmannian of 3-planes in $W$, we may assume that $u_0 = e_0, u_1 = e_1$ and $u_2 = e_2$. Since the stabilizer at $\mathrm{span}\{ e_0, e_1, e_2 \}$ in Spin(7) is the group SU(2) acting on the plane $\mathrm{span}\{ e_3, e_4, e_5, e_6, e_7 \} \cong \mathbb{R} \oplus \mathbb{C}^2$, we may assume that $u_3 = ke_3 + \ell e_4$, where $k^2 + \ell^2 = 1$. Then, (3.5) implies that

$$|\Phi(u_0, u_1, u_2, u_3)|^2 = k^2.$$

By (3.6) and (3.3), we have

$$\begin{aligned}\sqrt{8}\lambda^4(e^\mu)(u_0, u_1, u_2, u_3) &= *_7\, \varphi(e_\mu, e_1, e_2, ke_3 + \ell e_4) \\ &= - *_7\varphi(e_1, e_2, ke_3 + \ell e_4, e_\mu) \\ &= (e^{56} + e^{47})(ke_3 + \ell e_4, e_\mu) = \ell \delta_{\mu 7}.\end{aligned}$$

Then, we have

$$8|\tau(u_0, u_1, u_2, u_3)|^2 = 8 \sum_{\mu=1}^{7} |\lambda^4(e^\mu)(u_0, u_1, u_2, u_3)|^2 = \ell^2.$$

Since $|u_0 \wedge u_1 \wedge u_2 \wedge u_3|^2 = k^2 + \ell^2$, the proof is completed. □

Lemma 3.6 immediately implies the following.

**Lemma 3.7.** *An oriented 4-dimensional submanifold $C \subset W$ is Cayley with an appropriate orientation if and only if the restriction of $\tau$ to $C$ vanishes.*

## 4. The real Fourier–Mukai transform for coassociative $T^4$-fibrations

In this section, we compute the real Fourier–Mukai transform of associative cycles. This makes us confirm the definition of deformed Donaldson–Thomas connections for a manifold with a $G_2$-structure introduced by Lee and Leung [9]. This is also useful in the computation of Section 5.

Let $B \subset \mathbb{R}^3$ be an open set with coordinates $(x^1, x^2, x^3)$ and $f = (f^4, f^5, f^6, f^7) : B \to T^4$ be a smooth function with values in $T^4$. We use coordinates $(y^4, y^5, y^6, y^7)$ for $T^4$. Put

$$S := \{ (x, f(x)) \mid x \in B \}$$

the graph of $f$, a 3-dimensional submanifold in $X := B \times T^4$. Set

$$\omega_1 = dy^{45} + dy^{67}, \quad \omega_2 = dy^{46} + dy^{75}, \quad \omega_3 = -(dy^{47} + dy^{56}).$$

By (3.1) and (3.3), the standard $G_2$-structure $\varphi$ on $X$ and its Hodge dual $*\varphi$ are described as

$$\varphi = dx^{123} + \sum_{i=1}^{3} dx^i \wedge \omega_i, \tag{4.1}$$

$$*\varphi = dy^{4567} + \sum_{k \in \mathbb{Z}/3} dx^{k,k+1} \wedge \omega_{k+2}. \tag{4.2}$$

Let

$$\nabla^B = d + \sqrt{-1} \sum_{j=1}^{3} A^j dx^j$$

be a Hermitian connection of a trivial complex line bundle $B \times \mathbb{C} \to B$, where $A^j : B \to \mathbb{R}$ is a smooth function.

Next, we consider the mirror side. The real Fourier–Mukai transform of $(S, \nabla^B)$ is the connection on $X^*(\cong X)$ defined by

$$\nabla := d + \sqrt{-1} \sum_{j=1}^{3} A^j dx^j + \sqrt{-1} \sum_{a=4}^{7} f^a dy^a.$$

Then, its curvature 2-form $F_\nabla$ is given by $F_\nabla = F_\nabla^B + F_\nabla^S$, where

$$F_\nabla^B = \sqrt{-1} \sum_{i,j=1}^{3} \frac{\partial A^j}{\partial x^i} dx^i \wedge dx^j, \quad F_\nabla^S = \sqrt{-1} \sum_{i=1}^{3} \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^i} dx^i \wedge dy^a. \tag{4.3}$$

We first describe the condition for $S$ to be associative in terms of $F^S_\nabla$ in Proposition 4.1. Using this, we show that the similar statement also holds for $F_\nabla$ in Proposition 4.4.

**Proposition 4.1.** *The following conditions are equivalent.*

1. *The graph $S$ is an associative submanifold with an appropriate orientation.*
2. $(F^S_\nabla)^3/6 + F^S_\nabla \wedge *\varphi = 0$.
3. $(F^S_\nabla)^3/6 + F^S_\nabla \wedge *\varphi = 0$ *and* $\varphi \wedge *(F^S_\nabla)^2 = 0$.

**Remark 4.2.** A similar statement for graphical submanifolds is given by Harvey and Lawson in [3, Chapter IV, Theorem 2.4]. In terms of differential equations for $f$, they obtained two equations (2)' and (3)', which correspond to (2) and (3), respectively. Then, they stated that (1) and (2)' are equivalent in the theorem, and (3)' appeared only in the proof. They first showed that (1) and (3)' are equivalent. Using the assumption that $S$ is a graph, they showed that (2)' implies (1). Since (3)' obviously implies (2)', they obtained the equivalence. Actually, (2) and (3) are equivalent in general. See [6, Remark 3.3].

We can also consider the real Fourier–Mukai transform of a coassociative graph in associative $T^3$-fibrations. In Proposition 6.1, we show that we obtain the same equations as stated in [9].

*Proof.* Since (3) $\Rightarrow$ (2) is obvious and the converse holds by [6, Remark 3.3], (2) and (3) are equivalent. We show the equivalence of (1) and (3). By Lemma 3.3, $S$ is associative with an appropriate orientation if and only if $*\varphi(v_1, v_2, v_3, \cdot\,) = 0$ for any $p \in S$ and $v_1, v_2, v_3 \in T_pS$. Set $\partial_i := \partial/\partial x^i$ and $\partial_a := \partial/\partial y^a$ for $1 \leq i \leq 3$ and $4 \leq a \leq 7$. Then, the tangent space of $S$ is spanned by $v_1, v_2, v_3$, where

$$v_j := \partial_j + \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^j} \partial_a.$$

By (4.3), note that

$$v_j^\flat = dx^j + i(\partial_j)F$$

where we set $F = -\sqrt{-1}F^S_\nabla$. Since $*\varphi(v_1, v_2, v_3, \cdot\,) = 0$ is equivalent to $v_1^\flat \wedge v_2^\flat \wedge v_3^\flat \wedge \varphi = 0$, we have

$$0 = \Big(dx^1 + i(\partial_1)F\Big) \wedge \Big(dx^2 + i(\partial_2)F\Big) \wedge \Big(dx^3 + i(\partial_3)F\Big) \wedge \varphi. \tag{4.4}$$

Since $dx^{123} \wedge \varphi = 0$, this is equivalent to

$$0 = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{k \in \mathbb{Z}/3} dx^{k,k+1} \wedge i(\partial_{k+2}) F \wedge \varphi, \\ I_2 &= \sum_{k \in \mathbb{Z}/3} dx^{k} \wedge i(\partial_{k+1}) F \wedge i(\partial_{k+2}) F \wedge \varphi, \\ I_3 &= i(\partial_1) F \wedge i(\partial_2) F \wedge i(\partial_3) F \wedge \varphi. \end{aligned}$$

Since $I_1$ and $I_3$ are linear combinations of $dx^{123} \wedge dy^{abc}$'s and $I_2$ is a linear combination of $dx^{ij} \wedge dy^{4567}$'s, $S$ is associative with an appropriate orientation if and only if

$$I_1 + I_3 = 0, \quad I_2 = 0. \tag{4.5}$$

Now, we compute $I_1, I_2$ and $I_3$. By (4.1), we have

$$I_1 = \sum_{k \in \mathbb{Z}/3} dx^{k,k+1} \wedge i(\partial_{k+2}) F \wedge (dx^{k+2} \wedge \omega_{k+2}) = -dx^{123} \wedge \sum_{k=1}^{3} \omega_k \wedge i(\partial_k) F.$$

Since $i(\partial_k)F$ is the linear combination of $dy^a$'s and $dx^{123} \wedge \omega^k \wedge F = 0$ by (4.3), we see that

$$I_1 = -dx^{123} \wedge \sum_{k=1}^{3} i(\partial_k) \left(\omega_k \wedge F\right) = -\sum_{k=1}^{3} \left(i(\partial_k) dx^{123}\right) \wedge \omega_k \wedge F.$$

Then, by (4.2), we obtain

$$I_1 = - * \varphi \wedge F. \tag{4.6}$$

Next, we compute $I_3$. Since $i(\partial_k)F$ is the linear combination of $dy^a$'s, we see that

$$I_3 = -dx^{123} \wedge i(\partial_1) F \wedge i(\partial_2) F \wedge i(\partial_3) F$$

and

$$\begin{aligned} i(\partial_3) i(\partial_2) i(\partial_1) \left(\frac{1}{6} F^3\right) =& i(\partial_3) i(\partial_2) \left(\frac{1}{2} i(\partial_1) F \wedge F^2\right) \\ =& i(\partial_3) \left(-i(\partial_1) F \wedge i(\partial_2) F \wedge F\right) \\ =& -i(\partial_1) F \wedge i(\partial_2) F \wedge i(\partial_3) F. \end{aligned}$$

By (4.3), $F^3$ is the linear combination of $dx^{123} \wedge dy^{abc}$'s, and hence, we obtain

$$I_3 = \frac{1}{6} F^3. \tag{4.7}$$

Finally, we compute $I_2$. By (4.1), we have

$$dx^k \wedge \varphi = dx^k \wedge (dx^{k+1} \wedge \omega_{k+1} + dx^{k+2} \wedge \omega_{k+2}).$$

Since $i(\partial_k)F$ is the linear combination of $dy^a$'s, we see that

$$i(\partial_j) i(\partial_i) \left( \frac{1}{2} F^2 \right) = -i(\partial_i)F \wedge i(\partial_j)F.$$

Then, it follows that

$$\begin{aligned} I_2 &= - \sum_{k \in \mathbb{Z}/3} \left( dx^{k,k+1} \wedge \omega_{k+1} + dx^{k,k+2} \wedge \omega_{k+2} \right) \wedge i(\partial_{k+2}) i(\partial_{k+1}) \left( \frac{1}{2} F^2 \right) \\ &= -\frac{1}{2} (I_{2,1} + I_{2,2}), \end{aligned}$$

where

$$\begin{aligned} I_{2,1} &= \sum_{k \in \mathbb{Z}/3} dx^{k,k+1} \wedge \omega_{k+1} \wedge i(\partial_{k+2}) i(\partial_{k+1}) F^2 \\ &= \sum_{k \in \mathbb{Z}/3} i(\partial_{k+2}) \left( dx^{k,k+1} \wedge \omega_{k+1} \wedge i(\partial_{k+1}) F^2 \right) \end{aligned}$$

and

$$\begin{aligned} I_{2,2} &= \sum_{k \in \mathbb{Z}/3} dx^{k+1,k} \wedge \omega_k \wedge i(\partial_k) i(\partial_{k+2}) F^2 \\ &= \sum_{k \in \mathbb{Z}/3} i(\partial_{k+2}) \left( dx^{k,k+1} \wedge \omega_k \wedge i(\partial_k) F^2 \right). \end{aligned}$$

Since $dx^{k,k+1} \wedge \omega_{k+1} \wedge F^2 = 0$, which is an 8-form, it follows that

$$\begin{aligned} I_{2,1} &= \sum_{k \in \mathbb{Z}/3} i(\partial_{k+2}) \left( -i(\partial_{k+1})(dx^{k,k+1}) \wedge \omega_{k+1} \wedge F^2 \right) \\ &= \sum_{k \in \mathbb{Z}/3} i(\partial_{k+2}) \left( dx^k \wedge \omega_{k+1} \wedge F^2 \right). \end{aligned}$$

Similarly, we compute

$$I_{2,2} = \sum_{k \in \mathbb{Z}/3} i(\partial_{k+2}) \left( -dx^{k+1} \wedge \omega_k \wedge F^2 \right).$$

Then, we obtain

$$\begin{aligned}2I_2 &= \sum_{k\in\mathbb{Z}/3} i(\partial_{k+2}) \left(F^2 \wedge (-dx^k \wedge \omega_{k+1} + dx^{k+1} \wedge \omega_k)\right)\\ &= \sum_{k\in\mathbb{Z}/3} i(\partial_k) \left(F^2 \wedge (-dx^{k+1} \wedge \omega_{k+2} + dx^{k+2} \wedge \omega_{k+1})\right).\end{aligned}$$

By (4.2), we have $i(\partial_k) * \varphi = dx^{k+1} \wedge \omega_{k+2} - dx^{k+2} \wedge \omega_{k+1}$, and hence,

$$2I_2 = -\sum_{k=1}^{3} i(\partial_k) \left(F^2 \wedge i(\partial_k) * \varphi\right).$$

Since

$$\begin{aligned}-F^2 \wedge i(\partial_k) * \varphi &= -\langle F^2, *(i(\partial_k) * \varphi)\rangle \mathrm{vol}\\ &= \langle F^2, dx^k \wedge \varphi\rangle \mathrm{vol}\\ &= dx^k \wedge \varphi \wedge *(F^2) = \langle \varphi \wedge *(F^2), *dx^k\rangle \mathrm{vol},\end{aligned}$$

we see that $2I_2 = \sum_{k=1}^{3}\langle \varphi \wedge *(F^2), *dx^k\rangle * dx^k$. The equation (4.3) implies that $*(F^2)$ is the linear combination of $dx^i \wedge dy^{ab}$'s, and hence, $\varphi \wedge *(F^2)$ is the linear combination of $dx^{ij} \wedge dy^{4567}$'s. Then, we have $\langle \varphi \wedge *(F^2), *dy^a\rangle = 0$ for any $4 \leq a \leq 7$. Hence, we obtain

$$I_2 = \frac{1}{2}\varphi \wedge *(F^2). \tag{4.8}$$

Then, by (4.5), (4.6), (4.7) and (4.8), the proof is completed. □

Before going further, we rewrite the associator equality [3, Chapter IV, Theorem 1.6]. This is very useful because Lemma 4.3 implies an identity that will hold in more general settings. In [8], we show that it indeed holds generally. Using this, we see that dDT connections for $G_2$-manifolds minimize a kind of the volume functional, which is called the Dirac–Born–Infeld (DBI) action in physics, and this gives further applications. For more details, see [8].

**Lemma 4.3.** *We have*

$$\begin{aligned}&\left(1 + \frac{1}{2}\langle (F^S_\nabla)^2, *\varphi\rangle\right)^2 + \left| *\varphi \wedge F^S_\nabla + \frac{1}{6}(F^S_\nabla)^3\right|^2 + \frac{1}{4}|\varphi \wedge *(F^S_\nabla)^2|^2\\ &= \det\left(\mathrm{id}_{TX} + (-\sqrt{-1}F^S_\nabla)^\sharp\right),\end{aligned}$$

*where* $(-\sqrt{-1}F^S_\nabla)^\sharp$ *is a skew symmetric endomorphism of* $TX$ *defined by*

$$\langle(-\sqrt{-1}F^S_\nabla)^\sharp u, v\rangle = -\sqrt{-1}F^S_\nabla(u,v) \qquad \text{for } u, v \in TX.$$

*Proof.* Define $\iota : B \to X = B \times T^4$ by $\iota(x) = (x, f(x))$. Set $v_i = \iota_*(\partial_i)$ for $i = 1, 2, 3$. Then, by the associator equality [3, Chapter IV, Theorem 1.6], we have

$$|\iota^*\varphi(\partial_1, \partial_2, \partial_3)|^2 + |*\varphi(v_1, v_2, v_3, \cdot)|^2 = |v_1 \wedge v_2 \wedge v_3|^2. \tag{4.9}$$

Then, since $\iota^* dx^i = dx^i$ and $\iota^* dy^a = df^a$, (4.1) implies that

$$\begin{aligned}
&\iota^*\varphi \\
=&dx^{123} + dx^1 \wedge (df^{45} + df^{67}) + dx^2 \wedge (df^{46} + df^{75}) - dx^3 \wedge (df^{47} + df^{56}) \\
=&\Big(1 + \langle dx^{23}, df^{45} + df^{67}\rangle + \langle dx^{31}, df^{46} + df^{75}\rangle - \langle dx^{12}, df^{47} + df^{56}\rangle\Big)\, dx^{123},
\end{aligned}$$

where $df^{ab}$ is short for $df^a \wedge df^b$. On the other hand, since $F^S_\nabla = \sqrt{-1}\sum_{a=4}^7 df^a \wedge dy^a$ by (4.3), we have

$$\begin{aligned}
\langle (F^S_\nabla)^2, *\varphi\rangle &= \sum_{a,b=4}^7 \sum_{k\in\mathbb{Z}/3} \langle df^{ab} \wedge dy^{ab}, dx^{k,k+1} \wedge \omega_{k+2}\rangle \\
&= 2\Big(\langle dx^{23}, df^{45}+df^{67}\rangle + \langle dx^{31}, df^{46}+df^{75}\rangle - \langle dx^{12}, df^{47}+df^{56}\rangle\Big).
\end{aligned}$$

Hence, we obtain

$$\iota^*\varphi(\partial_1, \partial_2, \partial_3) = \varphi(v_1, v_2, v_3) = 1 + \frac{1}{2}\langle (F^S_\nabla)^2, *\varphi\rangle. \tag{4.10}$$

By the proof of Proposition 4.1, we have

$$\begin{aligned}
|*\varphi(v_1, v_2, v_3, \cdot)|^2 =&|v_1^\flat \wedge v_2^\flat \wedge v_3^\flat \wedge \varphi|^2 \\
=&|I_1 + I_3|^2 + |I_2|^2 \\
=&\left|*\varphi \wedge F^S_\nabla + \frac{1}{6}(F^S_\nabla)^3\right|^2 + \frac{1}{4}|\varphi \wedge *(F^S_\nabla)^2|^2.
\end{aligned} \tag{4.11}$$

Next, we compute $|v_1 \wedge v_2 \wedge v_3|^2$. Since $v_i = \iota_*(\partial_i) = \partial_i + \partial f/\partial x^i$, we have

$$|v_1 \wedge v_2 \wedge v_3|^2 = \det\left(\mathrm{id}_3 + {}^t\! AA\right),$$

where $\mathrm{id}_3$ is the identity matrix of dimension 3, $A$ is a $4 \times 3$ matrix defined by $A = \left(\frac{\partial f^a}{\partial x^i}\right)_{4 \leq a \leq 7, 1 \leq i \leq 3}$ and ${}^tA$ is the transpose of $A$. Denote by

$$\{0, \pm\sqrt{-1}\mu_1, \pm\sqrt{-1}\mu_2, \pm\sqrt{-1}\mu_3\} \quad \text{and} \quad \{\lambda_1, \lambda_2, \lambda_3\}$$

the eigenvalues of $(-\sqrt{-1}F_\nabla^S)^\sharp$ and ${}^tAA$, respectively, where $\mu_i \in \mathbb{R}$ and $\lambda_j \geq 0$. Since

$$(-\sqrt{-1}F_\nabla^S)^\sharp = \begin{pmatrix} 0 & -{}^tA \\ A & 0 \end{pmatrix}, \qquad ((-\sqrt{-1}F_\nabla^S)^\sharp)^2 = \begin{pmatrix} -{}^tAA & 0 \\ 0 & -A{}^tA \end{pmatrix}$$

and $\{0, \lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of $A{}^tA$, we see that

$$\{0, \mu_1^2, \mu_2^2, \mu_3^2\} = \{0, \lambda_1, \lambda_2, \lambda_3\}.$$

Since $(-\sqrt{-1}F_\nabla^S)^\sharp$ and $A{}^tA$ are conjugate to

$$0 \oplus \begin{pmatrix} 0 & -\mu_1 \\ \mu_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu_2 \\ \mu_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\mu_3 \\ \mu_3 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

respectively, we obtain

$$\begin{aligned} \det\left(\mathrm{id}_{TX} + (-\sqrt{-1}F_\nabla^S)^\sharp\right) &= (1+\mu_1^2)(1+\mu_2^2)(1+\mu_3^2) \\ &= (1+\lambda_1)(1+\lambda_2)(1+\lambda_3) \\ &= \det\left(\mathrm{id}_3 + {}^tAA\right) = |v_1 \wedge v_2 \wedge v_3|^2 \end{aligned}$$

and the proof is completed. □

Using Proposition 4.1, we obtain the following.

**Proposition 4.4.** *The following conditions are equivalent.*

1. *The graph $S$ is an associative submanifold with an appropriate orientation and $\nabla^B$ is flat.*
2. $F_\nabla^3/6 + F_\nabla \wedge *\varphi = 0$.
3. $F_\nabla^3/6 + F_\nabla \wedge *\varphi = 0$ *and* $\varphi \wedge *F_\nabla^2 = 0$.

*Proof.* Since (3) $\Rightarrow$ (2) is obvious and the converse holds by [6, Remark 3.3], (2) and (3) are equivalent. We show the equivalence of (1) and (2). By (4.3), we have $(F_\nabla^B)^2 = 0$ and $F_\nabla^B \wedge (F_\nabla^S)^2 = 0$. Thus, we have $F_\nabla^3 = (F_\nabla^S)^3$ and

$$F_\nabla^3/6 + F_\nabla \wedge *\varphi = \left((F_\nabla^S)^3/6 + F_\nabla^S \wedge *\varphi\right) + F_\nabla^B \wedge *\varphi.$$

By (4.3), $(F^S_\nabla)^3/6 + F^S_\nabla \wedge *\varphi$ and $F^B_\nabla \wedge *\varphi$ are linear combinations of $dx^{123} \wedge dy^{abc}$'s and $dx^{ij} \wedge dy^{4567}$'s, respectively. Hence, (2) is equivalent to

$$(F^S_\nabla)^3/6 + F^S_\nabla \wedge *\varphi = 0 \quad \text{and} \quad F^B_\nabla \wedge *\varphi = 0.$$

The first equation is equivalent to saying that $S$ is an associative submanifold with an appropriate orientation by Proposition 4.1. By (4.2) and (4.3), we have $F^B_\nabla \wedge *\varphi = F^B_\nabla \wedge dy^{4567}$. Hence, $F^B_\nabla \wedge *\varphi = 0$ if and only if $F^B_\nabla = 0$. Then, the proof is completed. □

## 5. The real Fourier–Mukai transform for Cayley $T^4$-fibrations

In this section, we compute the real Fourier–Mukai transform of Cayley cycles and prove main theorems.

Let $B \subset \mathbb{R}^4$ be an open set with coordinates $(x^0, x^1, x^2, x^3)$ and $f = (f^4, f^5, f^6, f^7) : B \to T^4$ be a smooth function with values in $T^4$. We use coordinates $(y^4, y^5, y^6, y^7)$ for $T^4$. Put

$$S := \{\, (x, f(x)) \mid x \in B \,\}$$

the graph of $f$, a 4-dimensional submanifold in $X := B \times T^4$. The standard Spin(7)-structure $\Phi$ on $X$ is described as

$$\Phi = dx^0 \wedge \varphi + *_7\varphi,$$

where we use $\varphi$ in (4.1) and $*_7$ is the Hodge star operator on $(\{\,0\,\} \times \mathbb{R}^3) \times T^4$. Setting

$$\begin{aligned} \tau_1 &= dx^{01} + dx^{23}, \quad & \tau_2 &= dx^{02} + dx^{31}, \quad & \tau_3 &= dx^{03} + dx^{12}, \\ \omega_1 &= dy^{45} + dy^{67}, \quad & \omega_2 &= dy^{46} + dy^{75}, \quad & \omega_3 &= -(dy^{47} + dy^{56}). \end{aligned}$$

$\Phi$ is also described as

$$\Phi = dx^{0123} + dy^{4567} + \sum_{i=1}^{3} \tau_i \wedge \omega_i. \tag{5.1}$$

Note that $\Phi$ in (3.5) is also described as in (5.1). Let

$$\nabla^B = d + \sqrt{-1} \sum_{j=0}^{3} A^j dx^j$$

be a Hermitian connection of a trivial complex line bundle $B \times \mathbb{C} \to B$, where $A^j : B \to \mathbb{R}$ is a smooth function.

Next, we consider the mirror side. The real Fourier–Mukai transform of $(S, \nabla^B)$ is the connection on $X^*(\cong X)$ defined by

$$\nabla := d + \sqrt{-1} \sum_{j=0}^{3} A^j dx^j + \sqrt{-1} \sum_{a=4}^{7} f^a dy^a.$$

Then, its curvature 2-form $F_\nabla$ is described as $F_\nabla = F_\nabla^B + F_\nabla^S$, where

$$F_\nabla^B = \sqrt{-1} \sum_{i,j=0}^{3} \frac{\partial A^j}{\partial x^i} dx^i \wedge dx^j, \quad F_\nabla^S = \sqrt{-1} \sum_{i=0}^{3} \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^i} dx^i \wedge dy^a. \tag{5.2}$$

Note that the real Fourier–Mukai transform of $S$ is the connection on $X^*(\cong X)$ defined by

$$\nabla^S := d + \sqrt{-1} \sum_{a=4}^{7} f^a dy^a$$

and its curvature 2-form is given by $F_\nabla^S$. We first describe the condition for $S$ to be Cayley in terms of $F_\nabla^S$ in Theorem 5.1. Using this, we show that the similar statement also holds for $F_\nabla$ in Theorem 5.7.

**Theorem 5.1.** *Use the notation of Subsection 3.4. The graph $S$ is a Cayley submanifold with an appropriate orientation if and only if*

$$\pi_7^2 \left( F_\nabla^S + \frac{1}{6} *_8 (F_\nabla^S)^3 \right) = 0 \quad \textit{and} \quad \pi_7^4 \left( (F_\nabla^S)^2 \right) = 0.$$

**Remark 5.2.** A similar statement for graphical submanifolds is given by Harvey and Lawson in [3, Chapter IV, Theorem 2.20]. They showed that $S$ is a Cayley submanifold with an appropriate orientation if and only if two equations (1)' and (2)' are satisfied. These equations are given in terms of differential equations for $f$ and correspond to the two equations above. They also showed that if the determinant of the Jacobian of $f$ is never 1, (1)' implies (2)'. This is generalized in [7].

Thus, unlike the $G_2$ case ([6, Remark 3.3]), the first equation *does not* always imply the second. Counterexamples are provided in [3, p. 132].

*Proof.* Set

$$F := -\sqrt{-1} F_\nabla^S = dx^0 \wedge F_1 + F_2$$

where $F_1 : B \to (\mathbb{R}^7)^*$ and $F_2 : B \to \Lambda^2(\mathbb{R}^7)^*$ are given by

$$F_1 = \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^0} dy^a, \quad F_2 = \sum_{i=1}^{3} \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^i} dx^i \wedge dy^a. \tag{5.3}$$

Set $\partial_i := \partial/\partial x^i$ and $\partial_a := \partial/\partial y^a$ for $0 \le i \le 3$ and $4 \le a \le 7$. By Lemma 3.7 and equations (3.10) and (3.13), $S$ is Cayley with an appropriate orientation if and only if $\lambda^4(\alpha)(v_0, v_1, v_2, v_3) = 0$ for any $p \in S$, $v_0, v_1, v_2, v_3 \in T_pS$ and $\alpha \in \mathrm{span}\{ dx^1, \cdots, dx^3, dy^4, \cdots, dy^7 \}$. The tangent space of $S$ is spanned by $v_0, v_1, v_2, v_3$, where

$$v_j := \partial_j + \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^j} \partial_a$$

for $0 \le j \le 3$. By (5.3), note that

$$v_0 = \partial_0 + F_1^\sharp, \quad v_j = \partial_j + (i(\partial_j)F_2)^\sharp$$

for $1 \le j \le 3$. Then, we compute

$$\begin{aligned}
&\sqrt{8}\lambda^4(\alpha)(v_0, v_1, v_2, v_3) \\
&= \left( dx^0 \wedge i(\alpha^\sharp) *_7 \varphi - \alpha \wedge \varphi \right) (v_0, v_1, v_2, v_3) \\
&= *_7 \varphi(\alpha^\sharp, v_1, v_2, v_3) - \alpha(v_0)\varphi(v_1, v_2, v_3) + \sum_{k \in \mathbb{Z}/3} \alpha(v_k)\varphi(v_0, v_{k+1}, v_{k+2}) \\
&= - \langle *_7\varphi(v_1, v_2, v_3, \cdot\,), \alpha \rangle - \langle \alpha, F_1 \rangle \varphi(v_1, v_2, v_3) \\
&\quad + \sum_{k \in \mathbb{Z}/3} \alpha(v_k)\varphi(F_1^\sharp, v_{k+1}, v_{k+2}).
\end{aligned}$$

Since $- *_7 \varphi(v_1, v_2, v_3, \cdot\,) = -i(v_3)i(v_2)i(v_1) *_7 \varphi = - *_7 (v_3^\flat \wedge v_2^\flat \wedge v_1^\flat \wedge \varphi) = *_7(v_1^\flat \wedge v_2^\flat \wedge v_3^\flat \wedge \varphi)$, we have

$$\begin{aligned}
&\sqrt{8}\lambda^4(\alpha)(v_0, v_1, v_2, v_3) \\
&= \left\langle v_1^\flat \wedge v_2^\flat \wedge v_3^\flat \wedge \varphi - \varphi(v_1, v_2, v_3) *_7 F_1 + \sum_{k \in \mathbb{Z}/3} \varphi(F_1^\sharp, v_{k+1}, v_{k+2}) *_7 v_k^\flat, *_7\alpha \right\rangle.
\end{aligned}$$

By the proof of Proposition 4.1, we have

$$v_1^\flat \wedge v_2^\flat \wedge v_3^\flat \wedge \varphi = I_1 + I_2 + I_3,$$

where

$$I_1 = - *_7 \varphi \wedge F_2, \quad I_2 = \frac{1}{2}\varphi \wedge *_7 F_2^2, \quad I_3 = \frac{1}{6}F_2^3. \tag{5.4}$$

Here, we set

(5.5)

$$\begin{aligned} J_1 &= I_1 + I_3 - \varphi(v_1, v_2, v_3) *_7 F_1 + \sum_{k\in\mathbb{Z}/3} \varphi(F_1^\sharp, v_{k+1}, v_{k+2}) *_7 (i(\partial_k)F_2), \\ J_2 &= I_2 + \sum_{k\in\mathbb{Z}/3} \varphi(F_1^\sharp, v_{k+1}, v_{k+2}) *_7 dx^k. \end{aligned}$$

Then,

$$\sqrt{8}\lambda^4(\alpha)(v_0, v_1, v_2, v_3) = \langle J_1 + J_2, *_7\alpha\rangle \tag{5.6}$$

Since $*_7 I_1, *_7 I_3, F_1$ are linear combinations of $dy^a$'s and $*_7 I_2$ is a linear combination of $dx^i$'s, the graph $S$ is Cayley with an appropriate orientation if and only if

$$J_1 = 0 \quad \text{and} \quad J_2 = 0.$$

To simplify these equations, we show the following.

**Lemma 5.3.** *We have*

$$\begin{aligned} \varphi(v_1, v_2, v_3) &= 1 - \frac{1}{2} *_7 \left(\varphi \wedge F_2^2\right), \\ \sum_{k\in\mathbb{Z}/3} \varphi(F_1^\sharp, v_{k+1}, v_{k+2}) dx^k &= - *_7 (F_1 \wedge F_2 \wedge \varphi). \end{aligned}$$

*Proof.* The first equation follows from (4.10). We prove the second equation. Since $F_1$ is a linear combination of $dy^a$'s, the equation (4.1) implies that

$$\varphi(F_1^\sharp, v_{k+1}, v_{k+2}) = \varphi(F_1^\sharp, \partial_{k+1}, (i(\partial_{k+2})F_2)^\sharp) + \varphi(F_1^\sharp, (i(\partial_{k+1})F_2)^\sharp, \partial_{k+2}).$$

We compute

$$\begin{aligned} \varphi(F_1^\sharp, \partial_{k+1}, (i(\partial_{k+2})F_2)^\sharp) &= -\omega_{k+1}(F_1^\sharp, (i(\partial_{k+2})F_2)^\sharp) \\ &= -\langle F_1 \wedge i(\partial_{k+2})F_2, \omega_{k+1}\rangle \\ &= \langle i(\partial_{k+2})(F_1 \wedge F_2), \omega_{k+1}\rangle \\ &= \langle F_1 \wedge F_2, dx^{k+2} \wedge \omega_{k+1}\rangle. \end{aligned}$$

Then, we have

$$\begin{aligned}\varphi(F_1^\sharp, v_{k+1}, v_{k+2}) &= \langle F_1 \wedge F_2, dx^{k+2} \wedge \omega_{k+1} - dx^{k+1} \wedge \omega_{k+2}\rangle \\ &= -\langle F_1 \wedge F_2, i(\partial_k) *_7 \varphi\rangle \\ &= -\langle dx^k \wedge F_1 \wedge F_2, *_7\varphi\rangle = -\langle dx^k, *_7(F_1 \wedge F_2 \wedge \varphi)\rangle.\end{aligned}$$

By (4.1) and (5.3), $F_1 \wedge F_2 \wedge \varphi$ is a linear combination of $dx^{ij} \wedge dy^{4567}$'s, and hence, the proof is completed. □

Thus, by (5.4), (5.5) and Lemma 5.3, we see that

$$\begin{aligned}J_1 &= - *_7\varphi \wedge F_2 + \frac{1}{6}F_2^3 - \left(1 - \frac{1}{2} *_7 \left(\varphi \wedge F_2^2\right)\right) *_7 F_1 \\ &\quad + *_7(F_1 \wedge F_2 \wedge \varphi) \wedge *_7F_2, \\ J_2 &= \frac{1}{2}\varphi \wedge *_7F_2^2 - F_1 \wedge F_2 \wedge \varphi.\end{aligned} \tag{5.7}$$

Now, we describe $\pi_7^2(F - *_8F^3/6)$ and $\pi_7^4(F^2)$.

**Lemma 5.4.** *We have*

$$\begin{aligned}2\pi_7^2\left(F - \frac{1}{6} *_8 F^3\right) =&\lambda^2\left(*_7\left(*_7\varphi \wedge F_2 - \frac{1}{6}F_2^3 + \left(1 - \frac{1}{2} *_7 \left(\varphi \wedge F_2^2\right)\right) *_7 F_1\right.\right. \\ &\left.\left. - *_7\left(F_1 \wedge F_2 \wedge \varphi\right) \wedge *_7F_2\right)\right), \\ \sqrt{8}\pi_7^4(F^2) =&\lambda^4\left(*_7\left(2F_1 \wedge F_2 \wedge \varphi - \varphi \wedge *_7F_2^2\right)\right).\end{aligned}$$

*Proof.* Set

$$\begin{aligned}&\{\, e^0, \cdots, e^7 \,\} = \{\, dx^0, \cdots, dx^3, dy^4, \cdots, dy^7 \,\} \quad \text{and} \\ &\{\, e_0, \cdots, e_7 \,\} = \{\, \partial_0, \cdots, \partial_7 \,\}.\end{aligned}$$

Then, by (3.10) and (3.6), we have

$$\begin{aligned}2\pi_7^2(F) =&2\sum_{\mu=1}^{7}\langle F, \lambda^2(e^\mu)\rangle \cdot \lambda^2(e^\mu) \\ =&\sum_{\mu=1}^{7}\langle e^0 \wedge F_1 + F_2, e^0 \wedge e^\mu + i(e_\mu)\varphi\rangle \cdot \lambda^2(e^\mu) \\ =&\sum_{\mu=1}^{7}\left(\langle F_1, e^\mu\rangle + \langle F_2, i(e_\mu)\varphi\rangle\right) \cdot \lambda^2(e^\mu).\end{aligned}$$

Since $\langle F_2, i(e_\mu)\varphi\rangle = *_7(F_2 \wedge e^\mu \wedge *_7\varphi) = \langle e^\mu, *_7(F_2 \wedge *_7\varphi)\rangle$, we obtain

$$2\pi_7^2(F) = \lambda^2\left(F_1 + *_7(F_2 \wedge *_7\varphi)\right). \tag{5.8}$$

We also compute

$$\pi_7^2(*_8F^3) = \sum_{\mu=1}^{7}\langle *_8F^3, \lambda^2(e^\mu)\rangle \cdot \lambda^2(e^\mu) = \sum_{\mu=1}^{7}\langle F^3, \lambda^6(e^\mu)\rangle \cdot \lambda^2(e^\mu).$$

By (3.8), we have for $1 \le \mu \le 7$

$$2\lambda^6(e^\mu) = *_8\left(e^0 \wedge e^\mu + i(e_\mu)\varphi\right) = *_7e^\mu + e^\mu \wedge *_8\varphi = *_7e^\mu + e^0 \wedge e^\mu \wedge *_7\varphi,$$

and hence,

$$\begin{aligned}2\langle F^3, \lambda^6(e^\mu)\rangle &= \langle 3e^0 \wedge F_1 \wedge F_2^2 + F_2^3, *_7e^\mu + e^0 \wedge e^\mu \wedge *_7\varphi\rangle \\ &= 3\langle F_1 \wedge F_2^2, e^\mu \wedge *_7\varphi\rangle + \langle F_2^3, *_7e^\mu\rangle.\end{aligned}$$

The first term is computed as

$$\begin{aligned}3\langle F_1 \wedge F_2^2, e^\mu \wedge *_7\varphi\rangle =& 3\langle i(e_\mu)\left(F_1 \wedge F_2^2\right), *_7\varphi\rangle \\ =& 3 *_7\left(\left(\langle e^\mu, F_1\rangle F_2^2 - 2F_1 \wedge (i(e_\mu)F_2) \wedge F_2\right) \wedge \varphi\right) \\ =& 3\langle *_7(F_2^2 \wedge \varphi)F_1, e^\mu\rangle - 6 *_7\left(F_1 \wedge (i(e_\mu)F_2) \wedge F_2 \wedge \varphi\right).\end{aligned}$$

The second term is computed as

$$\begin{aligned}-6 *_7\left(F_1 \wedge (i(e_\mu)F_2) \wedge F_2 \wedge \varphi\right) &= 6\langle i(e_\mu)F_2, *_7\left(F_1 \wedge F_2 \wedge \varphi\right)\rangle \\ &= -6 *_7\left(*_7\left(F_1 \wedge F_2 \wedge \varphi\right) \wedge e^\mu \wedge *_7F_2\right) \\ &= 6\langle *_7\left(*_7\left(F_1 \wedge F_2 \wedge \varphi\right) \wedge *_7F_2\right), e^\mu\rangle.\end{aligned}$$

Summarizing these equations, we obtain

$$2\pi_7^2(*_8F^3) = \lambda^2\left(*_7\left(F_2^3 + 3 *_7\left(F_2^2 \wedge \varphi\right) *_7 F_1 + 6 *_7\left(F_1 \wedge F_2 \wedge \varphi\right) \wedge *_7F_2\right)\right). \tag{5.9}$$

Then, by (5.8) and (5.9), it follows that

$$\begin{aligned}
&2\pi_7^2\left(F-\frac{1}{6}*_8F^3\right)\\
=&\lambda^2\left(F_1+*_7(F_2\wedge *_7\varphi)\right.\\
&\left.-\frac{1}{6}*_7\left(F_2^3+3*_7(F_2^2\wedge\varphi)*_7F_1+6*_7(F_1\wedge F_2\wedge\varphi)\wedge *_7F_2\right)\right)\\
=&\lambda^2\left(*_7\left(*_7\varphi\wedge F_2-\frac{1}{6}F_2^3+\left(1-\frac{1}{2}*_7\left(\varphi\wedge F_2^2\right)\right)*_7F_1\right.\right.\\
&\left.\left.-*_7(F_1\wedge F_2\wedge\varphi)\wedge *_7F_2\right)\right),
\end{aligned}$$

which implies the first equation of Lemma 5.4.

Next, we compute $\pi_7^4(F^2)$. By (3.10), we have

$$\pi_7^4(F^2)=\sum_{\mu=1}^{7}\langle F^2,\lambda^4(e^\mu)\rangle\cdot\lambda^4(e^\mu).$$

For $1\le\mu\le 7$, we have by (3.6)

$$\begin{aligned}
\sqrt{8}\langle F^2,\lambda^4(e^\mu)\rangle=&\langle 2e^0\wedge F_1\wedge F_2+F_2^2, e^0\wedge i(e_\mu)*_7\varphi-e^\mu\wedge\varphi\rangle\\
=&2\langle F_1\wedge F_2, i(e_\mu)*_7\varphi\rangle-\langle F_2^2, e^\mu\wedge\varphi\rangle
\end{aligned}$$

and

$$\begin{aligned}
2\langle F_1\wedge F_2, i(e_\mu)*_7\varphi\rangle=&-2*_7(F_1\wedge F_2\wedge e^\mu\wedge\varphi)\\
=&2\langle *_7(F_1\wedge F_2\wedge\varphi), e^\mu\rangle,\\
-\langle F_2^2, e^\mu\wedge\varphi\rangle=&-*_7(e^\mu\wedge\varphi\wedge *_7F_2^2)=-\langle *_7\left(\varphi\wedge *_7F_2^2\right), e^\mu\rangle.
\end{aligned}$$

Hence, we obtain the second equation of Lemma 5.4. □

Then, by (5.7) and Lemma 5.4, we obtain

$$*_7J_1=2(\lambda^2)^{-1}\left(\pi_7^2\left(-F+\frac{1}{6}*_8F^3\right)\right), \tag{5.10}$$

$$*_7J_2=-\frac{\sqrt{8}}{2}(\lambda^4)^{-1}\pi_7^4(F^2). \tag{5.11}$$

Hence, by (5.10) and (5.11), we see that the graph $S$ is Cayley with an appropriate orientation if and only if $\pi_7^2\left(F-*_8F^3/6\right)=0$ and $\pi_7^4(F^2)=0$. □

Before going further, we rewrite the Cayley equality [3, Chapter IV, Theorem 1.28]. This is very useful because Lemma 5.5 implies an identity that will hold in more general settings as in Lemma 4.3. We show that it indeed holds generally and gives many applications. For more details, see [8].

**Lemma 5.5.** *We have*

$$\begin{aligned}&\left(1+\frac{1}{2}\langle (F^S_\nabla)^2,\Phi\rangle+\frac{*_8(F^S_\nabla)^4}{24}\right)^2+4\left|\pi^2_7\left(F^S_\nabla+\frac{1}{6}*_8(F^S_\nabla)^3\right)\right|^2\\&\quad+2\left|\pi^4_7\left((F^S_\nabla)^2\right)\right|^2\\&=\det(\mathrm{id}_{TX}+(-\sqrt{-1}F^S_\nabla)^\sharp),\end{aligned}$$

*where* $(-\sqrt{-1}F^S_\nabla)^\sharp$ *is a skew symmetric endomorphism of* $TX$ *defined by*

$$\langle(-\sqrt{-1}F^S_\nabla)^\sharp u,v\rangle=-\sqrt{-1}F^S_\nabla(u,v)\qquad \textit{for } u,v\in TX.$$

*Proof.* Define $\iota : B \to X = B \times T^4$ by $\iota(x) = (x, f(x))$. Set $v_i = \iota_*(\partial_i)$ for $i = 0, 1, 2, 3$. Then, by the Cayley equality [3, Chapter IV, Theorem 1.6], which is equivalent to Lemma 3.6, we have

$$|\iota^*\Phi(\partial_0,\partial_1,\partial_2,\partial_3)|^2+8|\tau(v_0,v_1,v_2,v_3)|^2=|v_0\wedge v_1\wedge v_2\wedge v_3|^2, \tag{5.12}$$

where $\tau$ is defined by (3.13). Then, since $\iota^*dx^i = dx^i$ and $\iota^*dy^a = df^a$, (5.1) implies that

$$\begin{aligned}&\iota^*\Phi\\=&dx^{0123}+df^{4567}+\tau_1\wedge(df^{45}+df^{67})+\tau_2\wedge(df^{46}+df^{75})-\tau_3\wedge(df^{47}+df^{56})\\=&\Big(1+*_4(df^{4567})+\langle\tau_1,df^{45}+df^{67}\rangle\\&\quad+\langle\tau_2,df^{46}+df^{75}\rangle-\langle\tau_3,df^{47}+df^{56}\rangle\Big)dx^{0123},\end{aligned}$$

where $*_4$ is the Hodge star on the space spanned by $dy^4, \cdots, dy^7$ and $df^{a_1\cdots a_k}$ is short for $df^{a_1}\wedge\cdots\wedge df^{a_k}$. On the other hand, since $F^S_\nabla = \sqrt{-1}\sum_{a=4}^7 df^a\wedge dy^a$ by (5.2), we have

$$\begin{aligned}\frac{*_8(F^S_\nabla)^4}{24}&=\sum_{a,b,c,d=4}^{7}*_8\left(\frac{df^{abcd}\wedge dy^{abcd}}{24}\right)\\&=*_8\left(df^{4567}\wedge dy^{4567}\right)=*_4\left(df^{4567}\right),\end{aligned}$$

$$\begin{aligned}\langle (F^S_\nabla)^2, \Phi\rangle &= \sum_{a,b=4}^{7}\sum_{i=1}^{3}\langle df^{ab}\wedge dy^{ab}, \tau_i\wedge\omega_i\rangle \\ &= \sum_{a,b=4}^{7}\sum_{i=1}^{3}\langle df^{ab}, \tau_i\rangle\langle dy^{ab}, \omega_i\rangle \\ &=2\left(\langle \tau_1, df^{45}+df^{67}\rangle + \langle \tau_2, df^{46}+df^{75}\rangle - \langle \tau_3, df^{47}+df^{56}\rangle\right).\end{aligned}$$

Hence, we obtain

$$|\iota^*\Phi(\partial_0,\partial_1,\partial_2,\partial_3)|^2 = \left(1+\frac{1}{2}\langle (F^S_\nabla)^2,\Phi\rangle + \frac{*_8(F^S_\nabla)^4}{24}\right)^2. \tag{5.13}$$

Next, we compute $8|\tau(v_0,v_1,v_2,v_3)|^2$. By (3.13) and (5.6), we have

$$8|\tau(v_0,v_1,v_2,v_3)|^2 = 8\sum_{\mu=1}^{7}\left(\lambda^4(e^\mu)(v_0,v_1,v_2,v_3)\right)^2 = \sum_{\mu=1}^{7}\langle *_7J_1 + *_7J_2, e^\mu\rangle^2.$$

Recall that $*_7J_1$ and $*_7J_2$ are linear combinations of $dy^a$'s and $dx^i$'s, respectively, and $\lambda^j$ is an isometry by Lemma 3.4. Then, by (5.10) and (5.11), we obtain

$$\begin{aligned}8|\tau(v_0,v_1,v_2,v_3)|^2 &= \sum_{\mu=1}^{7}\langle *_7J_1, e^\mu\rangle^2 + \langle *_7J_2, e^\mu\rangle^2 \\ &=4\left|\pi^2_7\left(F^S_\nabla + \frac{1}{6}*_8(F^S_\nabla)^3\right)\right|^2 + 2\left|\pi^4_7\left((F^S_\nabla)^2\right)\right|^2.\end{aligned} \tag{5.14}$$

By the same argument as in the proof of Lemma 4.3, we see that

$$|v_0\wedge v_1\wedge v_2\wedge v_3|^2 = \det\left(\mathrm{id}_{TX} + (-\sqrt{-1}F^S_\nabla)^\sharp\right)$$

and the proof is completed. □

Using Theorem 5.1, we obtain the following Theorem 5.7. We first prove the following lemma.

**Lemma 5.6.** *Let $U\subset\mathbb{R}^8$ be a Cayley subspace, a subspace of $\mathbb{R}^8$ which is a Cayley submanifold. Denote by $U^\perp$ the orthogonal complement of $U$. We identify $\Lambda^kU^*$ with the subspace of $\Lambda^k(\mathbb{R}^8)^*$ by*

$$\Lambda^kU^* = \{\,\alpha\in\Lambda^k(\mathbb{R}^8)^* \mid i(v)\alpha = 0 \text{ for any } v\in U^\perp\,\}.$$

*Then, $\alpha \in \Lambda^2 U^*$ is anti-self-dual with respect to the induced metric if and only if $\pi^2_7(\alpha) = 0$.*

*Proof.* Since $U$ is Cayley, there is an orthonormal basis

$$\{\, \partial/\partial x^0, \cdots, \partial/\partial x^3, \partial/\partial y^4, \cdots, \partial/\partial y^7 \,\}$$

with its dual $\{\, dx^0, \cdots, dx^3, dy^4, \cdots, dy^7 \,\}$ such that $U$ is spanned by $\partial/\partial x^0, \cdots, \partial/\partial x^3$, which is positively oriented, $U^\perp$ is spanned by $\partial/\partial y^4, \cdots, \partial/\partial y^7$ and (5.1) holds.

Denote by $*_4$ and $*_8$ the Hodge stars on $U$ and $\mathbb{R}^8$, respectively. Then, by (3.11), we have

$$\begin{aligned} 4\pi^2_7(\alpha) =& \alpha + *_8(\Phi \wedge \alpha) \\ =& \alpha + *_8 \left( dy^{4567} \wedge \alpha + \sum_{j=1}^{3} \alpha \wedge \tau_i \wedge \omega_i \right) \\ =& \alpha + *_4 \alpha + *_8 \left( \sum_{j=1}^{3} \langle \alpha, \tau_i \rangle dx^{0123} \wedge \omega_i \right) \\ =& \alpha + *_4 \alpha + \sum_{j=1}^{3} \langle \alpha, \tau_i \rangle \omega_i . \end{aligned}$$

Since $\{\partial/\partial x^0, \cdots, \partial/\partial x^3\}$ is positively oriented, $\{\, \tau_1, \tau_2, \tau_3 \,\}$ is a basis of the space of self-dual 2-forms on $U$. Hence, the proof is completed. $\square$

**Theorem 5.7.** *The following conditions are equivalent.*

1. *The graph $S$ is a Cayley submanifold with an appropriate orientation and if we identify $-\sqrt{-1}F^B_\nabla \in \Omega^2(B)$ with a 2-form on $S$, it is anti-self-dual with respect to the induced metric and the orientation which makes $S$ Cayley.*
2. $$\pi^2_7 \left( F_\nabla + \frac{1}{6} *_8 F^3_\nabla \right) = 0 \quad \textit{and} \quad \pi^4_7 \left( F^2_\nabla \right) = 0.$$

*Proof.* By (5.2), we have $(F^B_\nabla)^3 = 0$ and $(F^B_\nabla)^2 \wedge F^S_\nabla = 0$. Thus, we have $F^3_\nabla = 3F^B_\nabla \wedge (F^S_\nabla)^2 + (F^S_\nabla)^3$. Hence,

$$\begin{aligned} &\pi^2_7 \left( F_\nabla + \frac{1}{6} *_8 F^3_\nabla \right) \\ =&\pi^2_7 \left( \left( F^S_\nabla + \frac{1}{6} *_8 (F^S_\nabla)^3 \right) + \left( F^B_\nabla + \frac{1}{2} *_8 \left( F^B_\nabla \wedge (F^S_\nabla)^2 \right) \right) \right). \end{aligned}$$

Note that $F^S_\nabla + *_8 (F^S_\nabla)^3/6$, $F^B_\nabla$ and $*_8 \left(F^B_\nabla \wedge (F^S_\nabla)^2/2\right)$ are linear combinations of $dx^i \wedge dy^a$'s, $dx^{ij}$'s and $dy^{ab}$'s, respectively. Then, by (3.11) and (5.1), the first term $\pi^2_7 \left(F^S_\nabla + *_8 (F^S_\nabla)^3/6\right)$ is a linear combination of $dx^i \wedge dy^a$'s and the second term $\pi^2_7 \left(F^B_\nabla + *_8 \left(F^B_\nabla \wedge (F^S_\nabla)^2/2\right)\right)$ is that of $dx^{ij}$'s and $dy^{ab}$'s. Hence, $\pi^2_7 \left(F_\nabla + *_8 F^3_\nabla/6\right) = 0$ if and only if

$$\pi^2_7 \left(F^S_\nabla + \frac{1}{6} *_8 (F^S_\nabla)^3\right) = 0, \quad \pi^2_7 \left(F^B_\nabla + \frac{1}{2} *_8 \left(F^B_\nabla \wedge (F^S_\nabla)^2\right)\right) = 0. \tag{5.15}$$

Next, we consider $\pi^4_7 \left(F^2_\nabla\right) = \pi^4_7 \left((F^B_\nabla)^2 + 2F^B_\nabla \wedge F^S_\nabla + (F^S_\nabla)^2\right) = 0$. By the definition of $\lambda^4$ in (3.6), (4.1) and (4.2), $\lambda^4(dx^i)$ is a linear combination of $dx^{jk} \wedge dy^{bc}$'s for each $1 \le i \le 3$, and $\lambda^4(dy^a)$ is a linear combination of $dx^j \wedge dy^{bcd}$'s and $dx^{jk\ell} \wedge dy^b$'s for each $4 \le a \le 7$. By (5.2), $(F^B_\nabla)^2$, $F^B_\nabla \wedge F^S_\nabla$ and $(F^S_\nabla)^2$ are linear combinations of $dx^{0123}$, $dx^{ijk} \wedge dy^a$'s and $dx^{ij} \wedge dy^{ab}$'s, respectively. Hence, we have $\pi^4_7 \left((F^B_\nabla)^2\right) = 0$ and $\pi^4_7 \left(F^2_\nabla\right) = 0$ if and only if

$$\pi^4_7 \left((F^S_\nabla)^2\right) = 0, \quad \pi^4_7 \left(F^B_\nabla \wedge F^S_\nabla\right) = 0. \tag{5.16}$$

The first equations of (5.15) and (5.16) are equivalent to saying that $S$ is a Cayley submanifold with an appropriate orientation by Theorem 5.1. Thus, assuming that $S$ is a Cayley submanifold, we may show that $\pi^2_7 \left(F^B_\nabla + *_8 \left(F^B_\nabla \wedge (F^S_\nabla)^2/2\right)\right) = 0$ and $\pi^4_7 \left(F^B_\nabla \wedge F^S_\nabla\right) = 0$ if and only if $-\sqrt{-1}F^B_\nabla$ is anti-self-dual with respect to the induced metric and the orientation which makes $S$ Cayley. For simplicity, set

$$(F^S)^\sharp = (-\sqrt{-1}F^S_\nabla)^\sharp,$$

where $(-\sqrt{-1}F^S_\nabla)^\sharp$ is defined in Lemma 5.5. Then, assuming that $S$ is a Cayley submanifold, $\pi^2_7 \left(F^B_\nabla + *_8 \left(F^B_\nabla \wedge (F^S_\nabla)^2/2\right)\right) = 0$ and $\pi^4_7 \left(F^B_\nabla \wedge F^S_\nabla\right) = 0$ if and only if

$$\pi^2_7 \left(\left((\mathrm{id}_{TX} + (F^S)^\sharp)^{-1}\right)^* F^B_\nabla\right) = 0 \tag{5.17}$$

by [7, Theorem A.8 (2)].

Now, we observe (5.17) pointwisely. Fix $x \in B$ and regard $(F^S)^\sharp = (F^S)^\sharp_{(x,f(x))} \in \mathrm{End}(T_{(x,f(x))}X) \cong \mathrm{End}(\mathbb{R}^8)$. By the definition of $(F^S)^\sharp$, we see that

$$(\mathrm{id}_{TX} + (F^S)^\sharp)_* \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \sum_{a=4}^{7} \frac{\partial f^a}{\partial x^i}(x) \frac{\partial}{\partial y^a},$$

$$(\mathrm{id}_{TX} + (F^S)^\sharp)_* \left(\frac{\partial}{\partial y^a}\right) = \frac{\partial}{\partial y^a} - \sum_{i=0}^{3} \frac{\partial f^a}{\partial x^i}(x) \frac{\partial}{\partial x^i}.$$

Then, we can regard $(\mathrm{id}_{TX} + (F^S)^\sharp)_* (\partial/\partial x^i)$ as an element of $T_{(x,f(x))}S$ and $(\mathrm{id}_{TX} + (F^S)^\sharp)_* (\partial/\partial y^a)$ as an element of $T^\perp_{(x,f(x))}S$. Moreover, $(\mathrm{id}_{TX} + (F^S)^\sharp)_*|_{W_0} : W_0 \to T_{(x,f(x))}S$ and $(\mathrm{id}_{TX} + (F^S)^\sharp)_*|_{V_0} : V_0 \to T^\perp_{(x,f(x))}S$ are isomorphisms, where $W_0$ and $V_0$ are subspaces of $\mathbb{R}^8$ spanned by $\partial/\partial x^0, \cdots, \partial/\partial x^3$ and $\partial/\partial y^4, \cdots, \partial/\partial y^7$, respectively. Since $F^B_\nabla$ is a linear combination of $dx^{ij}$'s, we see that

$$((\mathrm{id}_{TX} + (F^S)^\sharp)^{-1})^*(-\sqrt{-1}F^B_\nabla) \in \Lambda^2 T^*_{(x,f(x))}S$$

in the sense of Lemma 5.6. Then, by Lemma 5.6, (5.17) holds if and only if $((\mathrm{id}_{TX} + (F^S)^\sharp)^{-1})^*(-\sqrt{-1}F^B_\nabla) \in \Lambda^2 T^*_{(x,f(x))}S$ is anti-self-dual with respect to the induced metric and the orientation which makes $S$ Cayley.

Since the identification between $B$ and $S$ is given by $\kappa : B \ni x \mapsto (x, f(x)) \in S$ and $(d\kappa)_x = (\mathrm{id}_{TX} + (F^S)^\sharp)_*|_{W_0}$, where we identify $T_xB$ with $W_0$, we obtain the desired statement. $\square$

## 6. The real Fourier–Mukai transform for associative $T^3$-fibrations

In this section, we compute the real Fourier–Mukai transform of coassociative cycles using Theorems 5.1 and 5.7. It turns out that the real Fourier–Mukai transform of an associative cycle coincides with that of a coassociative cycle as stated in [9].

Let $B \subset \mathbb{R}^4$ be an open set with coordinates $(y^4, y^5, y^6, y^7)$ and $f = (f^1, f^2, f^3) : B \to T^3$ be a smooth function with values in $T^3$, where we use coordinates $(x^1, x^2, x^3)$ for $T^3$. Put

$$S := \{ (y, f(y)) \mid y \in B \}$$

the graph of $f$, a 4-dimensional submanifold in $X := B \times T^3$. The manifold $X$ admits a $G_2$-structure $\varphi$ with its Hodge dual $*\varphi = *_7\varphi$ as in (4.1) and (4.2). Let

$$\nabla^B = d + \sqrt{-1}\sum_{a=4}^{7} A^a dy^a$$

be a Hermitian connection of a trivial complex line bundle $B \times \mathbb{C} \to B$, where $A^j : B \to \mathbb{R}$ is a smooth function.

Next, we consider the mirror side. The real Fourier–Mukai transform of $(S, \nabla^B)$ is the connection on $X^*(\cong X)$ defined by

$$\nabla := d + \sqrt{-1}\sum_{a=4}^{7} A^a dy^a + \sqrt{-1}\sum_{j=1}^{3} f^j dx^j.$$

Then, its curvature 2-form $F_\nabla$ is given by $F_\nabla = F_\nabla^B + F_\nabla^S$, where

$$(6.1) \qquad F_\nabla^B = \sqrt{-1} \sum_{a,b=4}^{7} \frac{\partial A^b}{\partial y^a} dy^a \wedge dy^b, \quad F_\nabla^S = \sqrt{-1} \sum_{j=1}^{3} \sum_{a=4}^{7} \frac{\partial f^j}{\partial y^a} dy^a \wedge dx^j.$$

We first describe the condition for $S$ to be coassociative in terms of $F_\nabla^S$.

**Proposition 6.1.** *The following conditions are equivalent.*

1. *The graph $S$ is a coassociative submanifold with an appropriate orientation.*
2. $(F_\nabla^S)^3/6 + F_\nabla^S \wedge *\varphi = 0.$
3. $(F_\nabla^S)^3/6 + F_\nabla^S \wedge *\varphi = 0$ *and* $\varphi \wedge *(F_\nabla^S)^2 = 0.$

Thus, we obtain the same equations as in Proposition 4.1.

*Proof.* Since (3) obviously implies (2) and the converse holds by [6, Remark 3.3], (2) and (3) are equivalent. We show the equivalence of (1) and (3). Fixing $* \in S^1$, we have an embedding

$$\iota : B \times T^3 \cong B \times \{\, * \,\} \times T^3 \hookrightarrow B \times T^4.$$

Let $(x^0, x^1, x^2, x^3)$ be coordinates for $T^4$. We canonically identify $F_\nabla^S$ on $B \times (T^3)^*$ with a 2-form on $B \times (T^4)^*$ such that $i(\partial/\partial x^0) F_\nabla^S = 0$.

The manifold $B \times T^4$ admits a Spin(7)-structure $\Phi$ given by

$$\Phi = dx^0 \wedge \varphi + *_7 \varphi,$$

and the graph $S$ is coassociative if and only if $\iota(S)$ is Cayley. Then, Theorem 5.1 implies that $S$ is a coassociative submanifold with an appropriate orientation if and only if

$$\pi^6_7 \left( *_8 F_\nabla^S + (F_\nabla^S)^3/6 \right) = 0 \quad \text{and} \quad \pi^4_7 \left( (F_\nabla^S)^2 \right) = 0.$$

We describe these equations in terms of the $G_2$-structure $\varphi$ on $B \times T^3$. By (3.12) and (3.8), we have

$$\begin{aligned} 4\pi^6_7 \left( *_8 F_\nabla^S \right) &= *_8 F_\nabla^S + \Phi \wedge F_\nabla^S \\ &= dx^0 \wedge *_7 F_\nabla^S + (dx^0 \wedge \varphi + *_7 \varphi) \wedge F_\nabla^S \\ &= dx^0 \wedge (*_7 F_\nabla^S + \varphi \wedge F_\nabla^S) + *_7 \varphi \wedge F_\nabla^S, \end{aligned}$$

$$\begin{aligned} 4\pi_7^6\left((F_\nabla^S)^3\right) &= (F_\nabla^S)^3 + \Phi \wedge *_8(F_\nabla^S)^3 \\ &= (F_\nabla^S)^3 + (dx^0 \wedge \varphi + *_7\varphi) \wedge dx^0 \wedge *_7(F_\nabla^S)^3 \\ &= dx^0 \wedge *_7 F_\nabla^3 \wedge *_7\varphi + (F_\nabla^S)^3. \end{aligned}$$

Hence, $\pi_7^6\left(*_8 F_\nabla^S + (F_\nabla^S)^3/6\right) = 0$ is equivalent to

$$*_7 F_\nabla^S + \varphi \wedge F_\nabla^S + \frac{1}{6} *_7 (F_\nabla^S)^3 \wedge *_7\varphi = 0, \quad *_7\varphi \wedge F_\nabla^S + \frac{1}{6}(F_\nabla^S)^3 = 0.$$

Since these two equations are equivalent by [6, Lemma 3.2], $\pi_7^6( *_8 F_\nabla^S + (F_\nabla^S)^3/6) = 0$ is equivalent to $*_7\varphi \wedge F_\nabla^S + (F_\nabla^S)^3/6 = 0$.

Next, we consider $\pi_7^4\left((F_\nabla^S)^2\right) = 0$. By Lemma 3.4, $\pi_7^4\left((F_\nabla^S)^2\right) = 0$ if and only if

$$\langle dx^0 \wedge i(\alpha^\sharp) *_7 \varphi - \alpha \wedge \varphi, (F_\nabla^S)^2 \rangle = 0$$

for any $\alpha \in \Omega^1(B \times T^3)$. Since $i(\partial/\partial x^0) F_\nabla^S = 0$, this is equivalent to

$$0 = \langle \alpha \wedge \varphi, (F_\nabla^S)^2 \rangle = *_7(\alpha \wedge \varphi \wedge *_7(F_\nabla^S)^2) = \langle \alpha, *_7(\varphi \wedge *_7(F_\nabla^S)^2)\rangle.$$

Hence, the proof is completed by [6, Remark 3.3]. □

Similarly, we obtain the following Proposition 6.2 from Theorem 5.7.

**Proposition 6.2.** *The following conditions are equivalent.*

1. *The graph $S$ is a coassociative submanifold with an appropriate orientation and if we identify $-\sqrt{-1}F_\nabla^B \in \Omega^2(B)$ with a 2-form on $S$, it is anti-self-dual with respect to the induced metric and the orientation which makes $S$ coassociative.*
2. $F_\nabla^3/6 + F_\nabla \wedge *\varphi = 0.$
3. $F_\nabla^3/6 + F_\nabla \wedge *\varphi = 0$ *and* $\varphi \wedge *F_\nabla^2 = 0.$

Since the lemma corresponding to Lemma 5.6 would be interesting in itself, we write it down here.

**Lemma 6.3.** *Let $U \subset \mathbb{R}^7$ be a coassociative subspace, a subspace of $\mathbb{R}^7$ which is a coassociative submanifold. Denote by $U^\perp$ the orthogonal complement of $U$. We identify $\Lambda^k U^*$ with the subspace of $\Lambda^k(\mathbb{R}^7)^*$ by*

$$\Lambda^k U^* = \{\, \alpha \in \Lambda^k(\mathbb{R}^7)^* \mid i(v)\alpha = 0 \text{ for any } v \in U^\perp \,\}.$$

*Then $\alpha \in \Lambda^2 U^*$ is anti-self-dual with respect to the induced metric if and only if $\alpha \wedge *\varphi = 0$.*

*Proof.* Since $U$ is coassociative, there is an orthonormal basis $\{ e_i \}_{i=1}^7$ with its dual $\{ e^i \}_{i=1}^7$ such that $U$ is spanned by $e_4, \cdots, e_7$, which is positively oriented, $U^\perp$ is spanned by $e_1, \cdots, e_3$ and (3.1) holds. Setting $\omega_1 = e^{45} + e^{67}, \omega_2 = e^{46} - e^{57}$ and $\omega_3 = -(e^{47} + e^{56})$, we have $*\varphi = e^{4567} + \sum_{k \in \mathbb{Z}/3} e^{k,k+1} \wedge \omega_{k+2}$. Then, it follows that

$$\alpha \wedge *\varphi = \sum_{k \in \mathbb{Z}/3} e^{k,k+1} \wedge \alpha \wedge \omega_{k+2} = \sum_{k \in \mathbb{Z}/3} e^{k,k+1} \wedge \langle \alpha, \omega_{k+2} \rangle e^{4567}.$$

Since $\{e_4, \cdots, e_7\}$ is positively oriented, $\{ \omega_1, \omega_2, \omega_3 \}$ is a basis of the space of self-dual 2-forms on $U$. Hence, the proof is completed. $\square$

By this lemma and results in [6], we can also prove Proposition 6.2 without using Theorem 5.7.

## 7. Compatibilities with other connections

In this section, we post some evidences showing that Definition 1.3 we suggest is compatible with deformed Donaldson–Thomas (dDT) connections for a $G_2$-manifold and deformed Hermitian Yang–Mills (dHYM) connections of a Calabi–Yau 4-manifold.

Use the notation (and identities) of Subsection 3.4. Let $X^8$ be a compact connected 8-manifold with a Spin(7)-structure $\Phi$ and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Set

$$\mathcal{A}_0 = \{ \nabla \mid \text{a Hermitian connection of } (L,h) \} = \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L$$

for a fixed connection $\nabla \in \mathcal{A}_0$. We regard the curvature 2-form $F_\nabla$ of $\nabla$ as a $\sqrt{-1}\mathbb{R}$-valued closed 2-form on $X$.

Define maps $\mathcal{F}^1_{\mathrm{Spin}(7)} : \mathcal{A}_0 \to \sqrt{-1}\Omega^2_7$ and $\mathcal{F}^2_{\mathrm{Spin}(7)} : \mathcal{A}_0 \to \Omega^4_7$ by

$$\begin{aligned}
\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla) &= \pi^2_7 \left( F_\nabla + \frac{1}{6} * F^3_\nabla \right) \\
&= \frac{1}{4} \left( F_\nabla + \frac{1}{6} * F^3_\nabla + * \left( \left( F_\nabla + \frac{1}{6} * F^3_\nabla \right) \wedge \Phi \right) \right), \\
\mathcal{F}^2_{\mathrm{Spin}(7)}(\nabla) &= \pi^4_7 (F^2_\nabla).
\end{aligned}$$

Then, a Hermitian connection $\nabla$ of $(L,h)$ satisfying

$$\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla) = 0 \quad \text{and} \quad \mathcal{F}^2_{\mathrm{Spin}(7)}(\nabla) = 0$$

is a Spin(7)-dDT connection defined in Definition 1.3.

**Lemma 7.1.** *Let* $(Y^7, \varphi, g)$ *be a* $G_2$*-manifold with the Hodge dual* $*_7\varphi \in \Omega^4$*. Then,* $X^8 = S^1 \times Y^7$ *is a* Spin(7)*-manifold. Let* $L \to Y$ *be a smooth complex line bundle with a Hermitian metric* $h$*. Identify a connection* $\nabla$ *on* $Y^7$ *with that on* $X^8$ *by the pullback. Then, the following are equivalent.*

1. $\nabla$ *is a dDT connection in the sense of* $G_2$*, that is,* $\mathcal{F}_{G_2}(\nabla) = *_7\varphi \wedge F_\nabla + F_\nabla^3/6 = 0$.
2. $\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla) = 0$.
3. $\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla) = \mathcal{F}^2_{\mathrm{Spin}(7)}(\nabla) = 0$.

*Proof.* Recall that the induced Spin(7)-structure on $X^8$ is given by

$$\Phi = dx \wedge \varphi + *_7\varphi,$$

where $x$ is a coordinate of $S^1$ and $*_7$ is the Hodge star on $Y^7$. By (3.8), we have

$$*F_\nabla = dx \wedge *_7 F_\nabla, \quad *F_\nabla^3 = dx \wedge (*_7 F_\nabla^3),$$

where $* = *_8$ is the Hodge star on $X^8$. Then, we have

$$\begin{aligned}
&4 * \mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla)\\
=&dx \wedge *_7 F_\nabla + \frac{1}{6}F_\nabla^3 + F_\nabla \wedge (dx \wedge \varphi + *_7\varphi) + \frac{1}{6} dx \wedge (*_7 F_\nabla^3) \wedge *_7\varphi\\
=&dx \wedge \left( *_7 F_\nabla + \varphi \wedge F_\nabla + \frac{1}{6}(*_7 F_\nabla^3) \wedge *_7\varphi \right) + *_7\varphi \wedge F_\nabla + \frac{1}{6}F_\nabla^3.
\end{aligned}$$

Thus, we see that (1) and (2) are equivalent by [6, Lemma 3.2].

The equivalence of (2) and (3) follows from [7, Proposition 3.3] since $F_\nabla^4 = 0$. This equivalence can also be proved by [6, Remark 3.3]. □

**Lemma 7.2.** *Let* $(X^8, J, g, \omega, \Omega)$ *be a Calabi–Yau 4-manifold and* $L \to X$ *be a complex line bundle with a Hermitian metric* $h$*. Equip* $X$ *with a* Spin(7)*-structure* $\Phi$ *given by*

$$\Phi = \frac{1}{2}\omega^2 + \mathrm{Re}\,\Omega.$$

*Suppose that* $\nabla$ *is a Hermitian connection such that the* $(0,2)$*-part* $F_\nabla^{0,2}$ *of* $F_\nabla$ *vanishes. Then, we have* $\mathcal{F}^2_{\mathrm{Spin}(7)}(\nabla) = 0$*. Moreover,* $\nabla$ *is a dHYM connection with phase* 1 *on* $X^8$*, that is,* $\mathrm{Im}\,(\omega + F_\nabla)^4 = 0$*, if and only if* $\nabla$ *is a* Spin(7)*-dDT connection, that is,* $\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla) = 0$*.*

*Proof.* By [12, Proposition 2], $\Lambda^4_7 T^*X$ is contained in the space of $(3,1)$, $(1,3)$, $(4,0)$ and $(0,4)$-forms. Since $F_\nabla^{0,2}=0$, $F_\nabla^2$ is a real $(2,2)$-form, which implies that $\mathcal{F}^2_{\mathrm{Spin}(7)}(\nabla)=\pi^4_7(F_\nabla^2)=0$.

Next, we show the second statement. By [12, Proposition 2], we have $\Lambda^2_7 T^*X=\mathbb{R}\omega\oplus A_+$, where $A_+$ is a subspace of $\Lambda^{2,0}T^*X\oplus\Lambda^{0,2}T^*X$. Then, we have

$$\langle A_+, \mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla)\rangle = \langle A_+, F_\nabla + *F_\nabla^3/6\rangle = 0$$

since $F_\nabla$ is a $(1,1)$-form. Thus, $\mathcal{F}^1_{\mathrm{Spin}(7)}(\nabla)=0$ if and only if $\langle \omega, F_\nabla + *F_\nabla^3/6\rangle = 0$. Since $*\omega=\omega^3/6$, we have

$$\begin{aligned}\left\langle \omega, F_\nabla + \frac{1}{6} * F_\nabla^3 \right\rangle \mathrm{vol} &= *\omega\wedge F_\nabla + \frac{1}{6}\omega\wedge F_\nabla^3 \\ &= \frac{1}{6}\left(\omega^3\wedge F_\nabla + \omega\wedge F_\nabla^3\right) = \frac{\sqrt{-1}}{24}\mathrm{Im}\,(\omega+F_\nabla)^4.\end{aligned}$$

Hence, the proof is completed. □

**Remark 7.3.** Note that dHYM connections do not depend on the holomorphic volume form $\Omega$. Then, since $\left(X^8, J, g, \omega, e^{-\sqrt{-1}\theta}\Omega\right)$ is again a Calabi–Yau manifold for $\theta\in\mathbb{R}$, Lemma 7.2 implies that for a Hermitian connection $\nabla$ with $F_\nabla^{0,2}=0$, $\nabla$ is a dHYM connection with phase 1 if and only if $\nabla$ is a Spin(7)-dDT connection with respect to $\Phi_\theta=\omega^2/2+\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$.

## Appendix A. Notation

We summarize the notation used in this paper. We use the following for a manifold $X$ with a $G_2$- or Spin(7)-structure. Denote by $g$ the associated Riemannian metric.

| Notation | Meaning |
|---|---|
| $i(\,\cdot\,)$ | The interior product |
| $\Gamma(X,E)$ | The space of all smooth sections of a vector bundle $E\to X$ |
| $\Omega^k$ | $\Omega^k=\Omega^k(X)=\Gamma(X,\Lambda^kT^*X)$ |
| $v^\flat\in T^*X$ | $v^\flat=g(v,\,\cdot\,)$ for $v\in TX$ |
| $\alpha^\sharp\in TX$ | $\alpha=g(\alpha^\sharp,\,\cdot\,)$ for $\alpha\in T^*X$ |
| vol | The volume form induced from $g$ |
| $\Lambda^k_\ell T^*X$ | The subspace of $\Lambda^kT^*X$ corresponding to an $\ell$-dimensional irreducible subrepresentation as in Subsection 3.4 |
| $\Omega^k_\ell$ | $\Omega^k_\ell=\Gamma(X,\Lambda^k_\ell T^*X)$ |
| $\pi^k_\ell$ | The projection $\Lambda^kT^*X\to\Lambda^k_\ell T^*X$ or $\Omega^k\to\Omega^k_\ell$ |

## Acknowledgements

The authors would like to thank anonymous referees for the careful reading of an earlier version of this paper and many useful comments which helped to improve the quality of the paper.

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