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The exponential map of the complexification of the group of analytic Hamiltonian diffeomorphisms

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Dedicated to Bernie Shiffman, with fond admiration

Abstract: Let (M, ω, J) be a Kähler manifold and $\mathcal{K} = Ham(M, \omega)$ its group of Hamiltonian symplectomorphisms. Complexifications of $\mathcal{K}$ have been introduced by Semmes and then Donaldson which are not groups, only "formal Lie group" in a precise sense. However, it still makes sense to talk about the exponential map in the complexification. In this note we give a geometric construction of the exponential map (for small time), in case the initial data are real-analytic. (A more general analytic description has been given by Semmes.) The construction is motivated by, but does not use, semiclassical analysis and quantum coherent states. We use this geometric construction to solve the equations of motion in several basic examples and recapture a case already considered in the physics community where the quantum analogue of our system is explicitly solvable, showing a potential relation to non-Hermitian quantum mechanics. Finally, in the case of geodesics in the space of Kähler metrics on a Kähler manifold originally studied variously by Mabuchi, Semmes and Donaldson, we derive an infinitesimal obstruction to the completeness of Mabuchi geodesic rays in the space of smooth Kähler metrics.

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1. Introduction

A real Lie algebra can be easily complexified by tensoring it with the complex numbers over the field of real numbers, and extending the Lie bracket bilinearly. Complexifying a Lie group is a much more subtle problem, for which a solution does not always exist. All compact Lie groups admit a complexification, but the proof of that is not trivial (see, for example [21], §106).

In this paper we present a geometric construction of the exponential map for the formal complexification of the group of Hamiltonian diffeomorphisms of a Kähler manifold. As noted, an earlier analytic version was presented by Semmes in [18].

1.1. The complexification of the group of Hamiltonian symplectomorphisms

Let (M, ω, J) be a compact and connected Kähler manifold, and let $\mathcal{K}$ denote the group $Ham(M, \omega)$ of Hamiltonian symplectomorphisms of (M, ω), with Lie algebra $C^\infty(M, \mathbb{R})/\mathbb{R}$. $\mathcal{K}$ is known to be "morally" an infinite-dimensional analogue of a compact group, and one can wonder whether it has a complexification. From a physical point of view this would correspond to finding a sensible way to associate a dynamical system to a complex-valued Hamiltonian, $h : M \to \mathbb{C}$, in a manner that extends the notion of Hamilton flow in case h is real-valued.

In [6, 7] Donaldson discusses a model for the complexification of $\mathcal{K}$, first discussed by Semmes in [18] (see also Mabuchi's work, [14]), which we now describe. Let $\mathrm{Diff}^0(M)$ be the space of diffeomorphisms $\phi : M \to M$ that are isotopic to the identity, and for such a ϕ let $J_\phi := d\phi \circ J \circ d\phi^{-1}$ be the push-forward by ϕ of the complex structure J. Then the model of the complexification of $\mathcal{K}$ is

$$\mathcal{G} = \left\{ \phi \in \mathrm{Diff}^0(M) \mid \omega \text{ and } J_\phi \text{ are compatible} \right\}, \tag{1.1}$$

where compatibility means that ω is of J_ϕ-type $(1,1)$ and also $g_\phi(u, v) := \omega(u, J_\phi(v))$ defines a positive definite metric on M. (Note that $\mathcal{G}$ is *not* closed under composition.) Note that the condition for $\phi \in \mathrm{Diff}^0$ to be in $\mathcal{G}$ is equivalent to:

$$\phi^*\omega \text{ is compatible with } J. \tag{1.2}$$

To explain in what sense (1.1) can be called a complexificaton of $\mathcal{K}$, we first explain in what sense $\mathcal{G}$ is a "formal Lie group". We begin by identifying the tangent vectors to $\mathcal{G}$, in the following Lemma, which is implicit in the work of Semmes and Donaldson, but that we include for completeness (all the proofs are deferred to §3):

Lemma 1.1. *Let $(-\epsilon, \epsilon) \ni t \mapsto \phi_t \in \mathcal{G}$ be a smooth curve in $\mathcal{G}$ (meaning that $(-\epsilon, \epsilon) \times M \ni (t,p) \mapsto \phi_t(p) \in M$ is smooth), let $\phi = \phi_0$ and $\forall p \in M$ $\dot{\phi}(p) = \frac{d}{dt}\phi_t(p)|_{t=0} \in T_{\phi(p)}M$. Then there exists $h \in C^\infty(M, \mathbb{C})$, unique up to an additive constant, such that the field*

$$\dot{\phi} \circ \phi^{-1} \in \mathfrak{X}(M)$$

is given by

$$\dot{\phi} \circ \phi^{-1} = \Xi_{\Re h} + J_\phi(\Xi_{\Im h}) =: \Theta_{h,\phi}, \tag{1.3}$$

where for any $f \in C^\infty(M, \mathbb{R})$ Ξ_f denotes the Hamilton field of f with respect to ω ($\iota_{\Xi_f}\omega = -df$), and $\Re h, \Im h$ denote the real and complex parts of the complex valued function h.

Thinking, formally[1], that $\dot{\phi} \circ \phi^{-1} \in T_\phi \mathcal{G}$ is an arbitrary tangent vector to $\mathcal{G}$ at ϕ, the relation (1.3) induces a trivialization

$$T\mathcal{G} \cong \mathcal{G} \times C^\infty(M, \mathbb{C})/\mathbb{C}, \tag{1.4}$$

namely $(\phi, \dot{\phi} \circ \phi^{-1}) \mapsto (\phi, [h])$, which is to be thought of as the trivialization of the tangent bundle of a Lie group by left translations. Continuing with this analogy, any $h \in C^\infty(M, \mathbb{C})$ defines a "left-invariant" vector field $h^\sharp$ on $\mathcal{G}$ by:

$$h^\sharp_\phi = \Theta_{h,\phi}. \tag{1.5}$$

In [7] §4 Donaldson justifies that the map

$$C^\infty(M, \mathbb{C}) \ni h \mapsto h^\sharp \in \mathfrak{X}(\mathcal{G}) \tag{1.6}$$

is a Lie algebra morphism, intertwining the Poisson bracket (extended bilinearly to complex-valued functions) to the Lie bracket of vector fields on $\mathcal{G}$. The structure on $\mathcal{G}$ that we just described makes it a "formal Lie group" with Lie algebra $C^\infty(M, \mathbb{C})$, mod constants.

The sense in which $\mathcal{G}$ is a complexification of $\mathcal{K}$ is as follows. First, note that $\mathcal{K} \subset \mathcal{G}$ (this is clear from (1.2)), and $C^\infty(M, \mathbb{C})/\mathbb{C}$ is the complexification

[1] Although we won't do so here, this may be made rigorous using the language of diffeologies; see [11] for a general introduction to this formalism.

of the Lie algebra $C^\infty(M,\mathbb{R})/\mathbb{R}$. Moreover, for real Hamiltonians h, $h^\sharp_\phi = \Xi_h$ is the standard Hamilton field of h, independently of ϕ, and $h^\sharp$ is tangent to $\mathcal{K}$. These properties would characterize the complexification of a compact group in finite dimensions.

1.2. The exponential map and quantum mechanics

Continuing with the analogy with Lie groups, we define

Definition 1.2. Given $h \in C^\infty(M,\mathbb{C})$, by its exponential we will mean the integral curve of the field $h^\sharp$ starting at $\phi = \mathrm{Id}|_M$ the identity.

In concrete terms, the exponential of h is the smooth curve $t \mapsto \phi_t \in \mathcal{G}$ such that $\phi_0 = \mathrm{Id}|_M$ and

$$\dot\phi_t \circ \phi_t^{-1} = \Xi_{\Re h} + J_t(\Xi_{\Im h}), \tag{1.7}$$

where $J_0 = J$ and

$$\dot J_t = -\mathcal{L}_{\Xi_{\Re h}+J_t(\Xi_{\Im h})} J_t. \tag{1.8}$$

These equations and their initial conditions guarantee that $J_t = J_{\phi_t}$. Given a solution of (1.8), equation (1.7) is a system of time-dependent first order ODEs, solved for the time derivatives.

Let us analyze in more detail equation (1.8). Using the fact that the Lie derivative is a derivation, one obtains the very general identity: $\forall X, Y \in \mathfrak{X}(M)$ and any $J \in \mathrm{End}(TM)$ endomorphism of the tangent bundle, $(\mathcal{L}_X J)Y = [X, JY] - J[X,Y]$. Therefore, equation (1.8) is equivalent to

$$\forall Y \in \mathfrak{X}(M) \qquad \dot J_t(Y) = [JY, \Theta_t] - J[Y,\Theta_t], \tag{1.9}$$

where

$$\Theta_t := \Xi_{\Re h} + J_t(\Xi_{\Im h}) \tag{1.10}$$

is the right-hand side of (1.7). From these expressions it is clear that (1.8) is a self-standing PDE (it is not coupled to (1.7)), quadratic in the components of J in local coordinates, which we write in detail in (3.17). Note that if one is only interested in writing (1.7), one only needs to know the field $J_t(\Xi_{\Im h})$, which satisfies:

$$\frac{d}{dt} J_t(\Xi_{\Im h}) = [J_t\Xi_{\Im h}, \Xi_{\Re h}] + J_t[\Xi_{\Re h}, \Xi_{\Im h}] + J_t[J_t\Xi_{\Im h}, \Xi_{\Im h}]. \tag{1.11}$$

As a separate motivation for equations (1.7, 1.8), we conjecture that they are related to non-Hermitian quantum mechanics in a way that we will now describe.

Suppose that the cohomology class $[\omega]/2\pi$ is integral, so that there exists a holomorphic Hermitian line bundle $\pi : L \to M$ whose curvature is $-2\pi i \omega$. More generally, we consider the tensor powers $L^k \to M$ of L. According to geometric quantization, the space $\mathcal{H}_k$ of holomorphic sections of L^k, with its natural Hermitian inner product, is a quantization of M with $k = 1/\hbar$ playing the role of the inverse of Planck's constant. Denote by $P_k : L^2(L^k) \to \mathcal{H}_k$ the Bergman (orthogonal) projector. The Schwartz kernel of P_k is a section $\mathcal{P}_k \in H^0(M \times M, L^k \boxtimes L^{-k})$. Given $p \in M$, choose a non-zero vector $e_p \in \pi^{-1}(p)$ and let $\psi_p^k = \mathcal{P}_k(\cdot, p)(e_p) \in \mathcal{H}_k$ (ψ_p^k is defined up to a multiplicative constant depending on the choice of e_p.) The sequence (ψ_p^k) is called the *standard coherent state centered at* p. The well-known semiclassical estimates on the Bergman kernel, [1], imply that the functions $|\psi_p^k|^2$ (the so-called Husimi functions) concentrate in a neighborhood of size $O(1/\sqrt{k})$ of p as $k \to \infty$.

Now let $h \in C^\infty(M, \mathbb{C})$, and form the (sequence of) Berezin–Toeplitz operators

$$\mathrm{Op}_k(h) : \mathcal{H}_k \to \mathcal{H}_k, \quad \mathrm{Op}_k(h)(\psi) := P_k(h\psi). \tag{1.12}$$

One can then pose the initial value problem for Schrödinger's equation

$$i\frac{d}{dt}\psi = k\mathrm{Op}_k(h)(\psi), \quad \psi|_{t=0} = \psi_p^{(k)}. \tag{1.13}$$

The solution clearly exists for each k, as $\mathcal{H}_k$ is finite-dimensional.

Conjecture: The solution to (1.13) at time t, after L^2 normalization, concentrates as $k \to \infty$ on the trajectory $\phi_t(p)$ where ϕ_t is a solution of (1.7, 1.8).

When h is real-valued, this has been known for a long time in the setting when M is a cotangent bundle, see for example the book [5]. In the current setting, still with h real, the conjecture follows fairly easily from the techniques in [3] or [4]. The conjecture has been verified algebraically by Graefe and Schubert [9] in case $M = \mathbb{C}^n$ and $h : \mathbb{R}^{2n} \to \mathbb{C}$ is a quadratic form, in which case the exact solution to (1.13) is known. Numerical calculations of Wasim Rehman [15] in the example (4.34) support the conjecture, see figure 2. We hope to return to this conjecture in the future.

1.3. Description of our results

In this paper we present a geometric construction of solutions to the equations (1.7) and (1.8) when all the data are real-analytic, see Theorem 2.2. Our approach is to complexify the manifold M (regarded as a real-analytic manifold), which Semmes had already considered in [18] and [19], and consider holomorphic Hamiltonian flows on its complexification. As we will see, the complexification X of M comes with a natural projection $\Pi : X \to M$ whose fibers are the leaves of a holomorphic foliation $\mathcal{F}$ with Lagrangian fibers. To construct the exponential of $h \in C^\omega(M, \mathbb{C})$, we first holomorphically extend h to $H : \mathcal{U} \to \mathbb{C}$, where $\mathcal{U}$ is a neighborhood of M in X. We then take the Hamilton flow Φ of H on X, and use the leaves of $\mathcal{F}_t = \Phi_t(\mathcal{F})$ to project trajectories of Φ back to M. We will refer to this as the "foliation method" for tracing out the motion ϕ_t on M, i.e. solving equations (1.7) and (1.8). This method was also found and used independently by Graefe and Schubert in [8], for quadratic Hamiltonians in Euclidean space.

We mention the work of other authors on the problem of exponentiating real-valued Hamiltonians but with complex time. Hall and Kirwin [10], developing an earlier observation of Thiemann [20], use imaginary time dynamics to alter the complex structure on the classical phase space T^*M, generalizing Grauert tube constructions. Kirwin, Mourão, and Nunes [12] used complexified dynamics, especially on toric varieties, to study the relation between real and complex polarizations in geometric quantization.

The paper is organized as follows. Details of the geometric construction are presented in the next section. In §3 we write (1.8) in local coordinates, for explicit comparison with the physics literature, and provide the proofs of the basic results. The foliation method allows one to find the solutions to (1.7, 1.8) in several examples, some of which we discuss in §4. In §4.1 we check the motivating classical case of a compact group action on a compact Kähler manifold, as in [9], e.g., while in §4.2 we treat the case of $M = \mathbb{C}^n$, connecting with the work [8] of Graefe and Schubert. §4.3 treats a geometrically global, yet tractable, case, the coadjoint orbits of compact groups and their complexifications. In §4.4 we specialize this discussion to the simplest case, $M = \mathbb{CP}^1$, which provides examples solvable on a compact manifold. Wasim Rehman has recently made numerical calculations of the quantum side of examples of non-Hermitian evolution on $\mathbb{CP}^1$, and his pictures, an example of which is included in §4.4, appear consistent with the conjecture discussed above. Finally, in §5 we consider the case when h is purely imaginary. We confirm that the exponential map produces Mabuchi geodesics in the space of Kähler potentials, a much studied topic, and then motivated by the Hamiltonian dynamics in the complexification, we derive an infinitesimal obstruction to the extension of a Mabuchi geodesic from studying the flow at a critical point of the function h. Since this obstruction is infinitesimal, it holds true for $\mathcal{C}^\infty$ solutions of the geodesic equations.

2. The main construction and theorem

2.1. Complexifying a complex manifold

We begin by recalling a result attributed to Bruhat and Whitney, [2]: If M is a real-analytic manifold, then there exists a complex manifold X, of twice the dimension of M, together with an embedding $\iota : M \hookrightarrow X$ and an anti-holomorphic involution $\tau : X \to X$ whose fixed point set is precisely $\iota(M)$. In the present context we will need the following version of this result:

Proposition 2.1. *Let (M, J, ω) be a real analytic Kähler manifold of real dimension $2n$. There exists a holomorphic complex symplectic manifold (X, I) of complex dimension $2n$ and an inclusion $\iota : M \hookrightarrow X$ such that $\iota^*\Omega = \omega$, and with the following additional structure:*

1. *An anti-holomorphic involution $\tau : X \to X$ whose fixed point set is the image of ι and such that $\tau^*\Omega = \overline{\Omega}$.*
2. *A holomorphic projection $\Pi : X \to M$, $\Pi \circ \iota = \mathrm{Id}_M$, whose fibers are holomorphic Lagrangian submanifolds.*

The local existence is simple: We take X to be a neighborhood of the diagonal in $M \times M$, with the complex structure $I = (J, -J)$. Ω is the analytic extension of ω and $\tau(z, w) = (w, z)$. Finally, the projection is simply $\Pi(z, w) = z$.

We note that there often exist natural complexifications that make our results below much more global in some cases. For example, if M is a generic coadjoint orbit of a compact simply connected Lie group G_0 (one through the interior of a Weyl chamber) then, one can take for X the orbit of the complexification, G, of G_0 through the same element (see §4 for details).

The fibers of Π are the leaves of a holomorphic foliation $\mathcal{F}$ of X that plays a central role in what follows.

2.2. Main results

Given a function $h : M \to \mathbb{C}$ whose real and imaginary parts are real analytic, there is a holomorphic extension $H : X \to \mathbb{C}$ perhaps only defined near $\iota(M)$, but we will not make a notational distinction between X and such a neighborhood, as our results are mainly local in time. Denote by $\Phi_t : X \to X$ the Hamilton flow of $\Re H$ with respect to the real part of Ω. We denote by $\mathcal{F}_t$ the image of the foliation $\mathcal{F}$ under Φ_t (so that $\mathcal{F}_0 = \mathcal{F}$). We assume that there exists an open interval $E \subset \mathbb{R}$ containing the origin such that for all

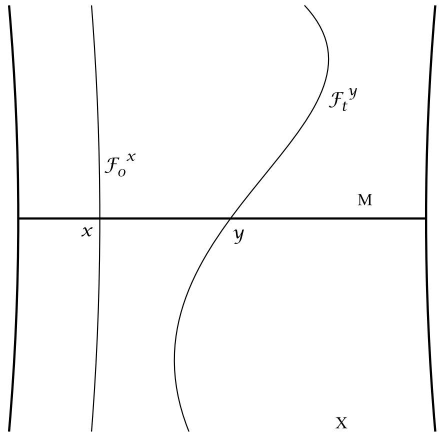

Figure 1: In this figure M is represented by the horizontal segment. Under Φ_t the fibers of the foliation $\mathcal{F}_0$ are transformed into the fibers of $\mathcal{F}_t$, and $y = \phi_t(x)$.

$t \in E$, the leaves of $\mathcal{F}_t$ are the fibers of a projection $\Pi_t : X \to M$. We will denote

$$\mathcal{F}_t^x := \Pi_t^{-1}(x)$$

the fiber of $\mathcal{F}_t$ over x.

Theorem 2.2. *Let $E \subset \mathbb{R}$ be an open set as above. Let $\phi_t : M \to M$ be defined by*

$$\phi_t := \Pi_t \circ \Phi_t \circ \iota. \tag{2.1}$$

Then, $\forall t \in E$ there is a complex structure $J_t : TM \to TM$ such that $J_t \circ d\Pi_t = d\Pi_t \circ I$, and J_t and ϕ_t satisfy (1.7) and (1.8).

We now explain the geometry behind the construction of ϕ_t summarized by (2.1). To find the image of $x \in M$ under ϕ_t, one flows the leaf $\mathcal{F}_0^x = \Pi^{-1}(x)$ of the foliation $\mathcal{F} = \mathcal{F}_0$ by Φ_t, and intersects the image leaf with M. In other words, (2.1) can be stated equivalently as:

$$\{\phi_t(x)\} = \Phi_t\left(\Pi^{-1}(x)\right) \cap M. \tag{2.2}$$

The definition of ϕ is summarized in figure 1, where $\mathcal{F}_t^y := \Pi_t^{-1}(y)$.

Equivalently still, $\forall x, y \in M$

$$y = \phi_t(x) \ \Leftrightarrow \ \exists w \in \Pi^{-1}(x) \ y = \Phi_t(x). \tag{2.3}$$

The following can be thought of as a means to computing ϕ_t by using a fixed projection:

Corollary 2.3. *Let* Π *be as in proposition 2.1, and let*

$$f_t = \Pi \circ \Phi_t \circ \iota : M \to M. \tag{2.4}$$

Then if f_{-t} *is invertible,*

$$\phi_t = (f_{-t})^{-1}, \tag{2.5}$$

and if ϕ_t *is a one-parameter subgroup of diffeomorphisms one has* $f_t = \phi_t$*. Moreover, if* ω_t *is the symplectic form defined by*

$$\omega = f_t^* \omega_t, \tag{2.6}$$

then

$$\dot{f}_t \circ f_t^{-1} = \Xi^{\omega_t}_{\Re h \circ f_t^{-1}} + J\left(\Xi^{\omega_t}_{\Im h \circ f_t^{-1}}\right), \tag{2.7}$$

where $\Xi^{\omega_t}_{\Re h \circ f_t^{-1}}$ *denotes the Hamilton vector field of* $\Re h$ *with respect to* ω_t*, etc.*

Going back to the discussion of the quantum propagation of coherent states, each leaf of the foliation $\mathcal{F}_0$ corresponds to a coherent state centered at a point on the leaf, that is, an element in the Hilbert space that semiclassically concentrates at the intersection of the leaf with M (after normalization). The fact that Π is holomorphic says the coherent states are associated to the metric of (M, J, ω). On the quantum side, the evolution of a coherent state remains a coherent state, whose Lagrangian is simply the image of the one at time $t = 0$ by the complexified classical flow. As explained above, our maps $\{\phi_t\}$ simply follow the evolution of the real center.

3. Proofs and calculations

In this section we present most of the proofs of the main results, as well as making the equations derived above as explicit as possible in local coordinates for comparison with the physics literature.

3.1. Proofs

We gather here the short proofs of most the previous statements.

For completeness, we begin with a proof of Lemma 1.1:

Proof. Let ϕ_t be any smooth curve in $\mathcal{G}$, and consider $\omega_t := \phi_t^*\omega$. Let $\Theta = \dot{\phi}_t \circ \phi_t^{-1}|_{t=0}$. Since ω_t is compatible with J, $\dot{\omega} = \mathcal{L}_\Theta\omega = d(\iota_\Theta\omega)$ is of type $(1,1)$. Therefore, if we write $\iota_\Theta\omega = \eta + \overline{\eta}$, where η is of type $(1,0)$, we must have $\partial\eta = 0$. By the topological assumptions we are making on M, there exists $h \in C^\infty(M, \mathbb{C})$, unique up to a constant, such that $\eta = \partial h$. Writing that $\iota_\Theta\omega = \partial h + \overline{\partial h}$ in terms of real and imaginary parts of h yields the result. $\square$

Next we establish some lemmas needed in the proof of Theorem 2.2. Let ξ denote the infinitesimal generator of Φ_t, that is

$$\forall x \in X \qquad \xi_x = \frac{d}{dt}\Phi_t(x)|_{t=0} \in T_xX.$$

The following is easy to check in a local trivialization of the foliation of X by fibers of Π_t:

Lemma 3.1. *Let E be the interval in Theorem 2.2, fix $t \in E$ and $x \in M$. Let $y = \phi_t(x)$ and let ξ denote the infinitesimal generator of the flow Φ. Then $\dot{\phi}_t(x) \in T_yM$ is*

$$\dot{\phi}_t(x) = d\left(\Pi_t\right)_y(\xi_y). \tag{3.1}$$

Proof. Introduce coordinates (u, v) in a neighborhood $U \subset X$ centered at y so that $U \cap M$ is defined by $v = 0$ and the projection Π_t is just $\Pi_t(u, v) = u$. Note that, since Φ is a one-parameter local subgroup of diffeomorphisms, for s small enough

$$\phi_{t+s}(x) = \Pi_{t+s} \circ \Phi_s(y).$$

For s near zero denote, the map Φ_s in coordinates by

$$\Phi_s(u, v) = (U(s, u, v), V(s, u, v))$$

(in a smaller neighborhood of y). For each s the image of the v-axis under Φ_s locally parametrize the fiber $\mathcal{F}^y_{t+s}$, namely

$$v \mapsto (U(s, 0, v), V(s, 0, v)).$$

Therefore we can write $\phi_{t+s}(x) = U(s, 0, v(s))$, where $v(s)$ is defined implicitly by $V(s, 0, v(s)) = 0$ and $v(0) = 0$. It follows that

$$\dot{\phi}_t(x) = \dot{U}(0,0,0) + \frac{\partial U}{\partial v}(0,0,0) \cdot \dot{v}(0).$$

However Φ_0 is the identity, so that $U(0, u, v) = u$ and, therefore, $\frac{\partial U}{\partial v}(0,0,0) = 0$. □

To proceed further we will need some notation. We regard X as a real manifold of real dimension $4n$ with an integrable complex structure $I : TX \to TX$. Let us write

$$\Omega = \omega_1 + i\omega_2$$

for the real and imaginary parts of Ω. Thus the ω_j are real symplectic forms on X and M is ω_1-symplectic and ω_2-Lagrangian. Let us write

$$H = F + iG \tag{3.2}$$

for the real and imaginary parts of H. Recall that, by definition, ξ is the Hamilton field of F with respect to ω_1.

Lemma 3.2. $\Xi^{\Omega}_{2H} = \xi - iI(\xi)$ *is the holomorphic vector field on* X *associated to the Hamiltonian* $2H$ *with respect to the form* Ω*. Therefore* Φ_t *is a holomorphic automorphism of* (X, Ω)*.*

Proof. We first note that, since Ω is of type $(2, 0)$,

$$\omega_1 \rfloor I\xi = -\omega_2 \rfloor \xi \quad \text{and} \quad \omega_2 \rfloor I\xi = \omega_1 \rfloor \xi. \tag{3.3}$$

From this it follows easily that $\Omega \rfloor (\xi - iI(\xi)) = 2dH$. For the final statement just use that Ω and H are holomorphic. □

For future reference we note the relations

$$\omega_1 \rfloor \xi = d\Re H \quad \text{and} \quad \omega_1 \rfloor I(\xi) = -d\Im H \tag{3.4}$$

that follow from (3.3).

Lemma 3.3. *Suppose that* $h := \iota^* H$ *is real. Then* ξ *is tangent to* M*, and its restriction to* M *is the Hamilton field of* h *with respect to* ω*.*

Proof. If h is real then $\tau^* H = \overline{H}$ (by uniqueness of analytic continuation of h), so $\Re H$ is τ-invariant. Since $\tau^*\Omega = \overline{\Omega}$, ω_1 is also τ-invariant, and therefore τ maps ξ to itself and so ξ has to be tangent to the fixed-point set of τ. For the second part just note that $\omega = \iota^*\omega_1$. □

Proof of Theorem 2.2. We take one point at a time:

(1) Since the fibers of Π_t are the leaves of a holomorphic foliation, there is a well-defined complex structure in the abstract normal bundle to the fibers. The inclusion $\iota : M \hookrightarrow X$ realizes M as a cross-section to the foliation and identifies TM with the normal bundle to the foliation along M. Therefore, it inherits a complex structure that makes Π_t holomorphic.

(2) This follows from the interpretation of the structures J_t as arising from the normal bundle structures together with the fact that Φ_t is holomorphic, or can be checked directly as follows. Let $v \in T_xM$, then $I(v) = J_0(v) + w$, where $w \in T_x\mathcal{F}_0^x$. Since $d\Phi_t$ is holomorphic, one has

$$I d\Phi_t(v) = d\Phi_t(J_0(v)) + d\Phi_t(w).$$

But $d\Phi_t(w) \in T\mathcal{F}_t$ since Φ_t maps fibers of $\mathcal{F}_0$ to fibers of $\mathcal{F}_t$. Therefore, the previous relation implies that

$$d(\Pi_t)(d\Phi_t(J_0(v))) = d(\Pi_t)(Id\Phi_t(v)) = J_t d(\Pi_t)(d\Phi_t(v)),$$

which precisely says that ϕ_t is holomorphic.

(3) Omitting the subscript t for simplicity, by Lemma 3.1 we need to show that

$$d\Pi_x(\xi) = \Xi_{\Re h} + \nabla \Im h$$

where:

1. $\Xi_{\Re h}$ is the Hamilton field of the real part of h with respect to ω, and
2. $\nabla \Im h$ is the gradient of the imaginary part of h with respect to the metric (ω, J).

By the previous lemma, if h is real, $d\Pi(\xi_x) = \xi_x$ and there is nothing more to prove. Suppose now that h is purely imaginary. By the second relation in (3.4), $I(\xi)$ is the Hamilton field of $-G$ (see 3.2), and by the lemma 3.3 $I(\xi)$ is tangent to M and its restriction to M is the Hamilton field of $ih = \iota^*(-G+iF)$ with respect to ω. Therefore, in this case, we can write

$$-\Xi_{\Im H} = d\Pi(I(\xi)) = J\, d\Pi(\xi), \tag{3.5}$$

and it suffices to apply J to both sides to get the result. The general case follows by $\mathbb{R}$-linearity of the composition $h \mapsto H \mapsto \xi$, where the first arrow is analytic continuation. This concludes the proof of Theorem 2.2.

Proof of Corollary 2.3. It is not hard to check that $\forall x, y \in M$ and $\forall t \in \mathbb{R}$ such that $\Phi_{\pm t}$ exist,

$$y = \phi_t(x) \;\Leftrightarrow\; f_{-t}(y) = x,$$

from which it follows that $f_t = (\phi_{-t})^{-1}$.

We wish to compute $\dot{f}_t \circ \phi_{-t}$. Differentiating with respect to time the identity $f_t \circ \phi_{-t}(x) = x$, we get

$$\dot{f}_t \circ \phi_{-t}(x) = d(f_t)(\dot{\phi}_{-t}(x)) = d(f_t)\,[\Xi^\omega_{\Re h} + J_{-t}(\Xi^\omega_{\Im h})]_{\phi_{-t}(x)}$$

Now it is not hard to check that $d(f_t)\,[\Xi^\omega_{\Re h}] = \Xi^{\omega_t}_{\Re h \circ f_t^{-1}}$ and that $df_t \circ J_{-t} = J_0 \circ df_t$ (using that $\phi_{-t} : (M, J_0) \to (M, J_{-t})$ is holomorphic). Therefore,

$$\dot{f}_t \circ \phi_{-t}(x) = \left[\Xi^{\omega_t}_{\Re h \circ f_t^{-1}} + J_0(\Xi^{\omega_t}_{\Im h \circ f_t^{-1}})\right]_{f_t^{-1}(x)}.$$

3.2. Computations in coordinates

Our goal in this section is to write down the equations (1.7, 1.8) in Darboux coordinates, as explicitly as possible.

Let $\phi_t : M \to M$ be the exponential of $h = F + iG : M \to \mathbb{C}$, i.e. $\dot{\phi}_t \circ \phi_t^{-1}$ is given by (1.7) where J_t satisfies (1.8). Introduce Darboux coordinates

$$(p, q) = (p_1, \ldots, p_n, q_1, \ldots q_n) = (x_1, \ldots, x_{2n})$$

on an open set of M, so that $\omega = \sum_j dp_j \wedge dq_j$. If we use ∇° to denote the gradient with respect to the flat metric in these coordinates, the field Θ_t of (1.10) is

$$\Theta_t = \Omega\, \nabla^\circ F + \mathbb{J}_t\, \Omega\, \nabla^\circ G \tag{3.6}$$

where $\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $\mathbb{J}_t = \mathbb{J}_t(p, q)$ is the matrix of $J_t : T_{(p,q)}M \to T_{(p,q)}M$ in the basis $\{\partial/\partial x_j\}$. The equations of motion (1.7) are then

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \Omega\, \nabla^\circ F + \mathbb{J}_t\, \Omega\, \nabla^\circ G, \tag{3.7}$$

where the right-hand side is evaluated at $(p(t), q(t))$.

The equation for $\dot{\mathbb{J}}_t$, namely $\dot{J}_t = -\mathcal{L}_{\Theta_t} J_t$, more complicated. Let the components of Θ_t be $\Theta_t = \langle \Theta_t^1, \dots, \Theta_t^{2n} \rangle = \Xi_F + J_t \Xi_G$, write $\phi_t = (\phi_t^1, \cdots, \phi_t^{2n})$ the flow in coordinates, and let

$$K_t = \left(\tfrac{\partial \phi_t^i}{\partial x_j}\right)_{|x=x_0} \tag{3.8}$$

denote the Jacobian of ϕ_t. Then that $\phi_t : (M, J) \to (M, J_t)$ is holomorphic becomes:

$$\forall x_0 = (q_0, p_0) \qquad K_t\, \mathbb{J}_0(x_0) = \mathbb{J}_t(\phi_t(x_0))\, K_t. \tag{3.9}$$

From this, it follows that

$$\frac{d}{dt} \mathbb{J}_t(\phi_t(x_0)) = [\dot{K}_t K_t^{-1} \,,\, \mathbb{J}_t(\phi_t(x_0))] \tag{3.10}$$

(matrix commutator). To compute $\dot{K}_t$ we use the equations of motion:

$$\frac{\partial^2 \phi_t^i}{\partial t \partial x_j} = \frac{\partial}{\partial x_j} \Theta_t^i(\phi_t(x)) = \sum_k \frac{\partial \Theta_t^i}{\partial x_k}(\phi_t(x))\, \frac{\partial \phi_t^k}{\partial x_j}(x),$$

which says that

$$\dot{K}_t = \Theta_t'(\phi_t(x_0))\, K_t, \quad \text{where} \quad \Theta_t' = \left(\tfrac{\partial \Theta_t^i}{\partial x_j}\right) \tag{3.11}$$

is the Jacobian matrix of the field Θ_t regarded as a map $\Theta_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ at $\phi_t(x_0)$. Thus we obtain that $\dot{K}_t K_t^{-1} = \Theta_t'$, which then leads to:

$$\frac{d}{dt} \mathbb{J}_t(\phi_t(x_0)) = [\Theta_t'(\phi_t(x_0)) \,,\, \mathbb{J}_t(\phi_t(x_0))]. \tag{3.12}$$

However

$$\frac{d}{dt} \mathbb{J}_t(\phi_t(x_0)) = \dot{\mathbb{J}}_t(\phi_t(x_0)) + \sum_{k=1}^{2n} \Theta_t^k(\phi_t(x_0))\, \frac{\partial \mathbb{J}_t}{x_k}(\phi_t(x_0)).$$

We obtain:

Lemma 3.4. *At each $x \in M$ in the coordinate patch*

$$\dot{\mathbb{J}}_t(x) = [\Theta_t'(x) \,,\, \mathbb{J}_t(x)] - \sum_{k=1}^{2n} \Theta_t^k(x)\, \frac{\partial \mathbb{J}_t}{x_k}(x). \tag{3.13}$$

There's a faster derivation of this: For any torsion free connection on a manifold M and any endomorphism J of TM and any $X, Y \in \mathfrak{X}(M)$,

$$(\mathcal{L}_X J)(Y) = (\nabla_X J)(Y) + J(\nabla_Y X) - \nabla_{JY} X. \tag{3.14}$$

We can apply this to $\nabla = \nabla^\circ$ the flat connection on $\mathbb{R}^{2n}$ and Y a coordinate field to get that the right-hand side of the above formula is $-\mathcal{L}_\Theta J$.

So far we have not used the expression (3.6) for Θ. From it we obtain

$$\Theta_t' = \Omega\, F'' + \mathbb{J}_t\, \Omega\, G'' + \left(\textstyle\sum_{k=1}^{2n} \frac{\partial \Upsilon_{ik}}{\partial x_j} \gamma_k\right) \tag{3.15}$$

where we have introduced the notations

$$F'' = \begin{pmatrix} F_{pp} & F_{pq} \\ F_{qp} & F_{qq} \end{pmatrix} \quad \text{with} \quad F_{pq} = \left(\tfrac{\partial^2 F}{\partial p_j \partial q_i}\right),$$

where i is the row index, etc., and

$$\mathbb{J}_t(x) = \left(\Upsilon_{ij}(t,x)\right), \quad (\gamma_1, \ldots, \gamma_{2n}) = (-G_{q_1}, \cdots, -G_{q_n}, G_{p_1}, \cdots, G_{p_n}).$$

We have proved:

Proposition 3.5. *Let* Γ *be the matrix*

$$\Gamma = \left(\textstyle\sum_{k=1}^{2n} \frac{\partial \Upsilon_{ik}}{\partial x_j} \gamma_k\right). \tag{3.16}$$

Under the evolution of the complex-valued Hamiltonian $h = F + iG : M \to \mathbb{C}$, *the complex structure evolves according to the equation*

$$\dot{\mathbb{J}}_t = [\Omega F'' + \mathbb{J}_t \Omega G'' + \Gamma\,,\, \mathbb{J}_t] - \sum_{k=1}^{2n} \Theta_t^k(x)\, \frac{\partial \mathbb{J}_t}{x_k}(x). \tag{3.17}$$

In case $M = \mathbb{R}^{2n}$ and F, G are quadratic forms, $\mathbb{J}_t$ is independent of the x variables, and (3.17) simplifies to

$$\dot{\mathbb{J}}_t = [\Omega F'' + \mathbb{J}_t \Omega G''\,,\, \mathbb{J}_t], \tag{3.18}$$

which is in exact agreement with equation (48) in [8] (though the latter is written in terms of the metric $\mathbb{G} = \Omega \mathbb{J}_t$).

4. Examples

Here we present examples of complexifications and of exponentials.

4.1. Compact group actions

Suppose G is a compact Lie group acting on M in a Hamiltonian fashion and preserving J. Then, the action extends to a holomorphic action to the complexification $G_{\mathbb{C}}$ (cf. [9], §4). The extended action is as follows: If $a, b : C^\infty(M) \to \mathbb{R}$ are two components of the moment map of the G action, then the infinitesimal action corresponding to $a + ib$ is the vector field $\Xi^\omega_a + J(\Xi^\omega_b)$. The corresponding one-parameter group of diffeomorphisms, $\varphi_t : M \to M$, satisfies (2) and (3) of Theorem 2.2, with $J_t = J_0$ for all t. By the uniqueness part of the previous remark, we must have $\varphi_t = \phi_t$. In other words, our construction is an extension of the process of complexifying the action of a compact group of symmetries of (M, ω, J).

4.2. The case $M = \mathbb{C}^n$

We consider $\mathbb{R}^{2n}$ with the standard symplectic structure and complex coordinates $\zeta_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$, so the symplectic structure is

$$\omega = \sum_{j=1}^n dp_j \wedge dq_j = -i \sum_{j=1}^n d\zeta_j \wedge d\bar{\zeta}_j. \tag{4.1}$$

Given $h : \mathbb{R}^{2n} \to \mathbb{R}$, its Hamilton field Ξ_h is defined by the condition $\omega(\cdot, \Xi_h) = dh$. This gives the usual equations of motion $\dot{q}_j = h_{p_j},\ \dot{p}_j = -h_{q_j}$. One can check that in complex coordinates Hamilton's equations are

$$\dot{\zeta}_j = -i \frac{\partial h}{\partial \bar{\zeta}_j} \tag{4.2}$$

and its complex conjugate (which is redundant).

We complexify $\mathbb{C}^n$ by the anti-diagonal embedding $\iota : \mathbb{C}^n \hookrightarrow \mathbb{C}^n \times \mathbb{C}^n$, $\zeta \mapsto (\zeta, \bar{\zeta})$. The initial projection $\Pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is just projection onto first factor. We denote by (z, w) complex variables on $\mathbb{C}^{2n}$. The symplectic form ω extends analytically to the complex symplectic form

$$\Omega = -i \sum_{j=1}^n dz_j \wedge dw_j.$$

If $H : \mathbb{C}^{2n} \to \mathbb{C}$ is holomorphic, the associated Hamiltonian equations are:

$$\dot{z}_j = -iH_{w_j}, \quad \dot{w}_j = iH_{z_j}. \tag{4.3}$$

We now explain how to implement our scheme for constructing the exponential of $C^\omega(\mathbb{R}^{2n}, \mathbb{C})$, where $\mathbb{R}^{2n} \cong \mathbb{C}^n$ as above. Let $h : \mathbb{R}^{2n} \to \mathbb{C}$ be such that there is a holomorphic $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that

$$h = H \circ \iota.$$

Assume that Hamilton's equations for H can be integrated (take t to be real, for simplicity), to yield a flow

$$\Phi_t : \mathbb{C}^{2n} \to \mathbb{C}^{2n}.$$

For each $\zeta \in \mathbb{C}^n$, let

$$\mathcal{F}^\zeta = \{(\zeta, w) \mid w \in \mathbb{C}^n\}$$

be the fiber over ζ of the projection Π. Then $\phi_t(\zeta) \in \mathbb{C}^n$ is defined by the condition

$$\{(\phi_t(\zeta), \overline{\phi_t(\zeta)})\} \in \Phi_t(\mathcal{F}^\zeta). \tag{4.4}$$

Now we proceed to examples, where we take $n = 1$.

4.2.1. The imaginary harmonic oscillator This is $h = \frac{i}{2}(q^2+p^2) = i\zeta\bar{\zeta}$, so that $H = izw$. Then the equations are $\dot{z} = z$, $\dot{w} = -w$, so

$$\Phi_t(z, w) = (e^t z, e^{-t} w).$$

We must implement (4.4) to find ϕ_t. In this case, this is trivial because Hamilton's equations separate:

$$\{(e^t\zeta, e^{-t}w) \mid w \in \mathbb{C}\} \cap \text{real locus} = \{(e^t\zeta, e^t\overline{\zeta})\}.$$

Therefore, in this case $\phi_t(\zeta) = e^t\zeta$, which is the gradient flow of $\Im(h)$, in agreement with §3.1.

4.2.2. A quadratic, non-Hermitian example Let us take next an example that Graefe and Schubert also discuss in [8], namely

$$h = \frac{i}{2}q^2 = \frac{i}{4}(\zeta + \bar{\zeta})^2.$$

The analytic continuation of this Hamiltonian is just

$$H = \frac{i}{4}(z + w)^2,$$

and the equations of motion in the complexification are

$$\dot{z} = -iH_w = \frac{1}{2}(z+w), \qquad \dot{w} = iH_z = -\frac{1}{2}(z+w).$$

It is clear that $z+w$ is constant in time, and therefore the flow in the complexification is

$$\Psi_t(z,w) = \left(\frac{t}{2}(z+w)+z, -\frac{t}{2}(z+w)+w\right). \tag{4.5}$$

To find the induced map $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, we are to proceed as follows. Fix $\zeta \in \mathbb{C}$, and flow-out under Φ_t the complex line

$$\mathcal{F}^\zeta = \{(\zeta, w)\ ;\ w \in \mathbb{C}\}. \tag{4.6}$$

The result is

$$\mathcal{F}^\zeta_t = \left\{\left(\frac{t}{2}(\zeta+w)+\zeta, -\frac{t}{2}(\zeta+w)+w\right)\ ;\ w \in \mathbb{C}\right\}. \tag{4.7}$$

The points of intersection of this complex line with the real locus are given by the solutions to the equation in w

$$-\frac{t}{2}(\bar{\zeta}+\bar{w})+\bar{w} = \frac{t}{2}(\zeta+w)+\zeta, \tag{4.8}$$

or equivalently

$$\bar{w}-\zeta = t\left(\Re(\zeta)+\Re(w)\right). \tag{4.9}$$

Let us write $\zeta = a+ib$, $w = \alpha+i\beta$. The equation becomes the system

$$\alpha - a = t(\alpha+a), \quad \beta = b. \tag{4.10}$$

Assuming $t \neq 1$ the solution is, $w = \frac{1+t}{1-t}a + ib$, which after some calculations yields

$$\phi_t(a+ib) = \frac{a}{1-t} + ib. \tag{4.11}$$

As $t \to 1$ from the right, the image point tends to infinity.

4.2.3. The generic element of the "maximal torus" Let us now take $h = \varphi(|\zeta|^2)$ with $\varphi : \mathbb{R} \to \mathbb{C}$ having an analytic extension $F : \mathbb{C} \to \mathbb{C}$. Then

$$H(z, w) = F(zw),$$

and Hamilton's equations are

$$\dot{z} = -izF'(zw), \quad \dot{w} = iwF'(zw).$$

Clearly, the function zw is a constant of motion, and we can integrate:

$$z(t) = e^{-itF'(\zeta w_0)}\zeta, \quad w(t) = e^{itF'(\zeta w_0)}w_0.$$

Therefore $\phi_t(\zeta) = e^{-itF'(\zeta w)}\zeta$ where w solves

$$e^{itF'(\zeta w)}w = \overline{\left(e^{-itF'(\zeta w)}\zeta\right)}. \tag{4.12}$$

This is a transcendental equation. However, we can easily prove:

Lemma 4.1. *Fix H as above. Then ϕ_t commutes with the (usual) harmonic oscillator.*

Proof. Fix $\zeta \in \mathbb{C}$, w solving (4.12), and $s \in \mathbb{R}$. Let $\zeta_s = e^{-is}\zeta$ and $w_s = e^{is}w$. Then

$$\begin{aligned} e^{itF'(\zeta_s w_s)}w_s &= e^{is}\, e^{itF'(\zeta w)}w = e^{is}\, \overline{\left(e^{-itF'(\zeta w)}\zeta\right)} = \\ &= \overline{\left(e^{-itF'(\zeta w)}e^{-is}\zeta\right)} = \overline{\left(e^{-itF'(\zeta_s w_s)}\zeta_s\right)}, \end{aligned}$$

which shows that $\phi_t(\zeta_s) = e^{-itF'(\zeta w)}\zeta_s = e^{-is}\phi_t(\zeta)$. □

4.3. Coadjoint orbits

We start by detailing the remark after Proposition 2.1 above for the case of M a coadjoint orbit of a compact Lie group G_0. For clarity, we restrict to the case of G_0 semi-simple. Let $\mathfrak{g}_0^*$ denote the dual of the Lie algebra $\mathfrak{g}_0$ of G_0, and let $\lambda \in \mathfrak{g}_0^*$. We identify $\mathfrak{g}_0^*$ with $\mathfrak{g}_0$ via the Killing form B. Thus there is a unique vector $\xi_\lambda \in \mathfrak{g}_0$ such that $< \lambda, \eta >= B(\xi_\lambda, \eta)$, for all $\eta \in \mathfrak{g}_0$. The Kostant–Kirilov G_0-invariant symplectic form on the orbit $G_0 \cdot \lambda := \mathcal{O}_\lambda$ is defined at $\lambda \in \mathcal{O}_\lambda$ via

$$\omega_{KK}(\tilde{\xi}, \tilde{\eta}) := \lambda([\xi, \eta]) = B(\xi_\lambda, [\xi, \eta]),$$

for all $\xi, \eta \in \mathfrak{g}_0$, where $\tilde{\xi}$ is the vector field on $\mathcal{O}_\lambda$ induced by $\xi \in \mathfrak{g}_0$, etc. We will drop the subscript on the symplectic form. Let $H_0 \subset G_0$ be the isotropy group of λ, i.e., the centralizer of ξ_λ and let G, resp. H be the complexifications of G_0, resp., H_0. Note that any maximal torus T_0 in H_0 has ξ_λ in its Lie algebra $\mathfrak{t}_0$, and that T_0 is a maximal torus in G_0. Let $\Delta_{\lambda,+}$ be a set of positive roots for $\mathfrak{g}$ for which $-i\alpha(\xi_\lambda) \geq 0$, for all $\alpha \in \Delta_{\lambda,+}$ (i.e., for which λ is in the closure of the corresponding Weyl chamber). Note that

$$\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} = \mathfrak{t} \ \oplus \ \oplus_{\{\alpha \,|\, <\alpha, \xi_\lambda>=0\}} \mathfrak{g}_\alpha.$$

Define the nilpotent algebra

$$\mathfrak{n}_{\lambda+} = \oplus_{\{\alpha \,|\, -i<\alpha,\xi_\lambda>>0\}} \mathfrak{g}_\alpha,$$

let N be the corresponding unipotent group in G, and $P = HN$ denote the associated parabolic subgroup of G. The coadjoint orbit $\mathcal{O}_\lambda = G_0/H_0 = G/P$ has an induced complex structure from the second presentation. Two such structures coming from different sets of positive roots for $\mathfrak{g}$ are equivalent, by an isomorphism induced by the element of the Weyl group which relates these two sets of positive roots and fixes λ. Thus there is an invariant Kähler metric on $\mathcal{O}_\lambda$ such that the corresponding Kähler form is the Kostant–Kirilov form.

Now the complex orbit $\mathcal{O}_{\lambda,\mathbb{C}} = G \cdot \lambda \subset \mathfrak{g}^*$ is a holomorphic symplectic manifold with symplectic form $\omega_\mathbb{C}$ given by the Kostant–Kirilov prescription above, which is the complexification of the real symplectic form $(\mathcal{O}_\lambda, \omega)$. The conjugation τ of $\mathfrak{g}^*$ fixing $\mathfrak{g}_0^*$ fixes $\mathcal{O}_\lambda$ and $\tau^*\omega_\mathbb{C} = \bar{\omega}_\mathbb{C}$. To complete the verification of the criteria (1) and (2) of Proposition 2.1 above for $\mathcal{O}_{\lambda,\mathbb{C}}$, note that since $H \subset P$, the foliation of G by left P-cosets is preserved under right multiplication by elements of H, and hence descends to give a foliation $\mathcal{F}$ of $\mathcal{O}_{\lambda,\mathbb{C}}$ which is invariant under the action of G. The leaves of this foliation are biholomorphic to $P/H \cong N_+ \cong \mathbb{C}^d$, where "$\cong$" denotes isomorphism as algebraic varieties. That the leaves of this foliation are Lagrangian for $\omega_\mathbb{C}$ amounts to the fact that $B(\xi_\lambda, [\xi, \eta]) = 0$, which is because $ad(\xi_\lambda) \circ ad(\zeta)$ is nilpotent on $\mathfrak{g}$ for any $\zeta \in \mathfrak{n}_+$; for example, $\zeta = [\xi, \eta]$, for $\xi, \eta \in \mathfrak{n}_+$. Finally, we have a canonical holomorphic G-equivariant mapping

$$\Pi : \mathcal{O}_{\lambda,\mathbb{C}} \to \mathcal{O}_\lambda,$$

given by

$$\mathcal{O}_{\lambda,\mathbb{C}} \ni gH \to gP \in G/P = \mathcal{O}_\lambda,$$

whose fibers are obviously the leaves of the foliation $\mathcal{F}$.

Proposition 4.2. *Keeping the previous notation, let $\xi \in \mathfrak{g}$ and let $h_\xi : \mathcal{O} \to \mathbb{C}$ be the induced Hamiltonian. Let f_t, $\phi_t : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ be constructed as in §1, using the complexification $\mathcal{O}_{\lambda,\mathbb{C}}$ above. Then $f_t = \phi_t$, and they coincide with the action of* $\exp(t\xi) \in G$ *on* $\mathcal{O}_\lambda$ *(in particular they exist for all $t \in \mathbb{R}$).*

Proof. By definition $f_t = \Pi \circ \Phi_t \circ \iota$, where Φ_t is the action of $\exp(t\xi)$ on $\mathcal{O}_\mathbb{C}$. Since the projection Π is covariant with respect to the action of G, we get that f_t agrees with the action of $\exp(t\xi)$ on $\mathcal{O}$. It therefore exists for all time, and it is a one-parameter group. As already remarked, this implies that $\phi_t = f_t$. □

4.4. The case $M = \mathbb{P}^1$

We now specialize the discussion of coadjoint orbits to the case of SU(2).

4.4.1. Generalities We take the (skew-Hermitian) Pauli matrices as a basis of the Lie algebra,

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.13}$$

(so that $[\sigma_1, \sigma_2] = \sigma_3$, etc.). Give $\mathfrak{su}(2)$ the invariant inner product such that the σ_j are orthogonal and have length 1, namely

$$\forall \alpha,\ \beta \in \mathfrak{su}(2) \qquad (\alpha, \beta) = -2\,\mathrm{Tr}(\alpha\beta) = 2\,\mathrm{Tr}(\alpha\beta^*), \tag{4.14}$$

where $\beta^* = \overline{\beta}^T$, and use it to identify adjoint and coadjoint orbits. We will consider

$$\mathcal{O} = \text{adjoint orbit of } \sigma_3 \cong \mathbb{CP}^1.$$

We let x_k denote the k-th coordinate:

$$x_k(\alpha) = (\alpha, \sigma_k).$$

Explicitly, if

$$\alpha = \frac{i}{2}\begin{pmatrix} a & z \\ \overline{z} & -a \end{pmatrix} \in \mathfrak{su}(2) \tag{4.15}$$

with $a \in \mathbb{R},\ z \in \mathbb{C}$, then

$$x_1(\alpha) = \Re z, \quad x_2(\alpha) = \Im z, \quad x_3(\alpha) = a$$

and $\mathcal{O}$ is the unit sphere, $\sum x_j^2 = 1$.

To complexify $\mathcal{O}$ we introduce $SL(2,\mathbb{C})$ and its Lie algebra. We take the basis (4.13) as a basis over $\mathbb{C}$ of $\mathfrak{sl}(2,\mathbb{C})$. The quadratic form $-2\operatorname{Tr}(\alpha\beta)$ is $G = SL(2,\mathbb{C})$ invariant and non-degenerate, so that we can continue to use it to identify adjoint and coadjoint orbits.

The $SL(2,\mathbb{C})$ orbit through σ_3, $\mathcal{O}_{\mathbb{C}}$, is the complexification of the previous orbit. Let us denote a general element of $\mathfrak{sl}(2,\mathbb{C})$ by

$$m = \frac{i}{2}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a,\, b,\, c \in \mathbb{C}. \tag{4.16}$$

Comparing with (4.15), we see that the real locus is $a \in \mathbb{R}$ and $c = \bar{b}$. The coordinate functions x_j extend holomorphically to the functions

$$Z_1(m) = \frac{b+c}{2}, \quad Z_2(m) = \frac{b-c}{2i}, \quad Z_3(m) = a. \tag{4.17}$$

The equation of the complex orbit, $\mathcal{O}_{\mathbb{C}}$, is

$$1 = \sum_j Z_j^2 = bc + a^2 = 4\det(m),$$

or $\det(m) = \frac{1}{4}$, which corresponds to m having the eigenvalues $\pm\frac{i}{2}$.

The isotropy group of σ_3 in $SL(2,\mathbb{C})$ is

$$H = T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{C}^* \right\},$$

while $H_0 = T_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$. The unipotent group in $SL(2,\mathbb{C})$ corresponding to σ_3 is

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;;\; n \in \mathbb{C} \right\}, \tag{4.18}$$

and the parabolic subgroup $P = TN$ is

$$P = \left\{ \begin{pmatrix} t & n \\ 0 & t^{-1} \end{pmatrix} \;;\; n \in \mathbb{C},\; t \in \mathbb{C}^* \right\}. \tag{4.19}$$

If we now let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;;\; a > 0 \right\}, \tag{4.20}$$

then we have $T = T_0\mathcal{A}$ (polar decomposition of complex numbers) as well as the Iwasawa (or QR) decomposition ($G = SL(2,\mathbb{C})$, $G_0 = SU(2)$)

$$G = G_0AN \tag{4.21}$$

and, therefore, $G/P \cong G_0/T_0$. More specifically, we have the commuting diagram of diffeomorphisms

$$\begin{array}{ccc} G/P & \to & G_0/T_0 \\ & \searrow & \downarrow \\ & & \mathcal{O} \end{array} \tag{4.22}$$

where the top arrow is $gP \mapsto kT_0$ ($g = kan$ the Iwasawa decomposition of g), the vertical arrow is $kT_0 \mapsto k \cdot \sigma_3$.

According to the general discussion of orbits, the projection $\Pi : \mathcal{O}_\mathbb{C} \to \mathcal{O}$ corresponds to the projection $G/T \to G/P$ given by $gT \mapsto gP$. This shows that

$$\Pi^{-1}(\sigma_3) = \{gT\ ;\ g \in AN\}.$$

It is easy to see that this is the orbit of N through σ_3, or, equivalently:

$$\Pi^{-1}(\sigma_3) = \left\{\frac{1}{2}\begin{pmatrix} i & n \\ 0 & -i\end{pmatrix}\ ;\ n \in \mathbb{C}\right\}. \tag{4.23}$$

This line is one of the two lines which are the intersection of the plane $X_3 = 1$ with the quadric. By equivariance, we can conclude:

Lemma 4.3. *For $\lambda \in \mathcal{O}$, let $\ell_\lambda : \mathcal{O}_\mathbb{C} \to \mathbb{C}$ be the function $\ell_\lambda(m) = \langle m, \lambda\rangle$. Then the fiber $\Pi^{-1}(\lambda)$ is one of the two lines whose union is $\ell_\lambda^{-1}(1)$.*

4.4.2. An alternate model There is an alternate model for the pair $(\mathcal{O}, \mathcal{O}_\mathbb{C})$ in which the fibers of the projection $\Pi : \mathcal{O}_\mathbb{C} \to \mathcal{O}$ are easier to describe. Let us define

$$M := \{(\ell\,, \ell^\perp)\ ;\ \ell \subset \mathbb{C}^2\ \ \text{1-dimensional subspace}\} \cong \mathbb{P}^1 \tag{4.24}$$

where $\mathbb{P}^1$ is the complex projective line, and

$$X := \{(\ell_+, \ell_-) \in \mathbb{P}^1 \times \mathbb{P}^1\ ;\ \ell_+ \cap \ell_- = 0\} \cong \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \setminus \mathbb{P}_\Delta, \tag{4.25}$$

where $\mathbb{P}_\Delta$ is the diagonal. We can identify $\mathcal{O}_\mathbb{C} \cong X$ by

$$\mathcal{O}_\mathbb{C} \ni A \mapsto (\ell_+\,,\, \ell_-) \in X, \quad \ell_\pm = \pm\frac{i}{2}\ \text{eigenspace of}\ A. \tag{4.26}$$

Under this identification $\mathcal{O} \subset \mathcal{O}_\mathbb{C}$ gets identified with $M \subset X$.

Lemma 4.4. *Under the identification (4.26), the projection* $\Pi : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}$ *is simply*

$$\Pi : X \to M \quad \Pi(\ell_+,\, \ell_-) = (\ell_+\,, \ell_+^{\perp}).$$

Therefore, a leaf of the foliation of $\mathcal{O}_{\mathbb{C}}$ *consists of all elements in* $\mathcal{O}_{\mathbb{C}}$ *with a common* $i/2$ *eigenspace.*

Proof. By (4.23), the leaf through σ_3 consists of the elements in $\mathcal{O}_{\mathbb{C}}$ having $e_1 := \langle 1, 0\rangle$ as an $i/2$ eigenvector. Therefore, in our new model $\Pi : X \to M$

$$\Pi^{-1}(\mathbb{C}e_1,\, \mathbb{C}e_2) = \{(\mathbb{C}e_1,\, m)\ ;\ e_1 \notin m\}.$$

The statement follows by $SL(2, \mathbb{C})$ equivariance of the projection. ☐

For future reference, let us now use this lemma to find the leaf of the foliation of $\mathcal{O}_{\mathbb{C}}$ consisting of all matrices having the vector $\langle 1, \kappa\rangle$, $\kappa \in \mathbb{C}$, as an $i/2$ eigenvector. If $\kappa = 0$ the leaf is just the leaf over σ_3, that is (4.23). If $\kappa \neq 0$, one can easily check that

$$\frac{i}{2}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 1 \\ \kappa \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 1 \\ \kappa \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} b = \frac{1-a}{\kappa} \\ c = \kappa(1+a). \end{cases}$$

Therefore, the leaf in question is

$$\mathcal{L}_\kappa := \left\{ \frac{i}{2}\begin{pmatrix} a & \frac{1-a}{\kappa} \\ \kappa(1+a) & -a \end{pmatrix}\ ;\ a \in \mathbb{C} \right\}. \tag{4.27}$$

For each κ, $a \in \mathbb{C}$ parametrizes the leaf $\mathcal{L}_\kappa$.

A calculation shows that the intersection of $\mathcal{L}_\kappa$ with the real locus $\mathcal{O}$ is the matrix

$$A_\kappa := \frac{i}{2(1+|\kappa|^2)}\begin{pmatrix} 1-|\kappa|^2 & 2\overline{\kappa} \\ 2\kappa & |\kappa|^2-1 \end{pmatrix}. \tag{4.28}$$

We see that $A_1 = \sigma_1$, $A_{-i} = \sigma_2$ and $A_0 = \sigma_3$. In fact the only point in $\mathcal{O}$ which is not of the form A_κ for some $\kappa \in \mathbb{C}$ is $(-\sigma_3)$ (the only element in $\mathcal{O}$ having $\langle 0, 1\rangle$ as $i/2$ eigenvector). In fact, we can take κ as a (stereographic) coordinate on $\mathcal{O} \setminus \{-\sigma_3\}$, centered at σ_3.

4.4.3. An example of dynamics We will now use the calculations above to find the trajectory of A_κ under the Hamiltonian $\frac{i}{2}x_3^2$. Recall (4.17), the holomorphic extension of $x_3 : M \to \mathbb{R}$ is

$$X_3\left[\frac{i}{2}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] = a.$$

The Hamilton flow Ψ of X_3 is conjugation by $\exp(t\sigma_3)$, that is

$$\begin{aligned}\Psi_t\left[\frac{i}{2}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] &= \frac{i}{2}\begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \\ &= \frac{i}{2}\begin{pmatrix} a & e^{it}b \\ e^{-it}c & -a \end{pmatrix}.\end{aligned}$$

To find the Hamilton flow Φ of $\frac{i}{2}X_3^2$ on $\mathcal{O}_\mathbb{C}$ we simply replace t by $itX_3 = ita$, using the fact that the Hamilton field of $\frac{i}{2}X_3^2$ is itX_3 times the Hamilton field of X_3. We obtain

$$\Phi_t\left[\frac{i}{2}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right] = \frac{i}{2}\begin{pmatrix} a & e^{-ta}b \\ e^{ta}c & -a \end{pmatrix}. \tag{4.29}$$

Let $\phi_t : M \to M$ be the exponential of $\frac{i}{2}x_3^2$. To find $\phi_t(A_\kappa)$ with $\kappa \neq 0$ we look for the element $\varpi \in \mathcal{L}_\kappa$ such that $\Phi_t(\varpi) \in M$, and then $\phi_t(A_\kappa) = \Phi(\varpi)$. Let ϖ be as in the right-hand side of (4.27), so that

$$\Phi_t(\varpi) = \frac{i}{2}\begin{pmatrix} a & e^{-ta}\,\frac{1-a}{\kappa} \\ e^{ta}\,\kappa(1+a) & -a \end{pmatrix}. \tag{4.30}$$

For this matrix to be in M, i.e., for it to be skew-Hermitian, we need:

$$a \in \mathbb{R} \quad \text{and} \quad e^{ta}\,\kappa(i+2a) = \overline{e^{-ta}\,\frac{1-a}{\kappa}} = e^{-ta}\,\frac{1-a}{\overline{\kappa}}. \tag{4.31}$$

This is equivalent to

$$e^{-2ta} = |\kappa|^2\,\frac{1+a}{1-a} \tag{4.32}$$

with $a \in \mathbb{R}$. If $t = 0$ the solution is $a(0) = \frac{1-|\kappa|^2}{1+|\kappa|^2}$.

Lemma 4.5. *If $\kappa = 0$ or $|\kappa| = 1$, A_κ is a fixed point of the exponential of $\frac{i}{2}x_3^2$. If $\kappa \neq 0$, the exponential of $\frac{i}{2}x_3^2$ with initial condition A_κ exists for all time. More specifically, $\forall t$ (4.32) has a unique (real) solution $a(t)$ which depends smoothly on t, and*

$$\phi_t(A_\kappa) = \frac{i}{2}\begin{pmatrix} a(t) & e^{-ta(t)}\,\frac{1-a(t)}{\kappa} \\ e^{ta(t)}\,\kappa(1+a(t)) & -a(t) \end{pmatrix}. \tag{4.33}$$

In terms of the coordinate κ, ϕ_t is given by

$$\kappa \mapsto \kappa(t) = e^{ta(t)}\kappa \tag{4.34}$$

Proof. The function

$$a \mapsto |\kappa|^2\,\frac{1+a}{1-a}$$

is strictly increasing on $[-1\,,1)$ and maps this interval onto $[0,+\infty)$. On the other hand, for any $t \in \mathbb{R}$ the function $a \mapsto e^{-2ta}$ is positive, monotone and bounded on $[-1\,,1]$. Therefore (4.32) has a unique solution in $(-1,\,1)$. Smoothness follows from the implicit function theorem: $a(t)$ is implicitly defined by the equation $F(t,a) = 0$, where

$$F(t,a) = r^2\,\frac{1+a}{1-a} - e^{-2ta} \tag{4.35}$$

with $r = |\kappa|$. A calculation shows that, on $F = 0$

$$\frac{\partial F}{\partial a} = 2\frac{r^2}{1-a}\left[\frac{1}{1-a} + t(1+a)\right]$$

where we have used the equation $F = 0$ in the form $e^{-2ta} = r^2\,\frac{1+a}{1-a}$. Therefore

$$\frac{\partial F}{\partial a} = 0 \quad \Leftrightarrow \quad t = \frac{1}{a^2-1},$$

and therefore $\frac{\partial F}{\partial a} \neq 0$ for $a \in (-1,1)$.

The final expression (4.34) follows by comparing (4.30) with (4.28). □

The previous lemma says that the trajectories of the exponential of $\frac{i}{2}x_3^2$ exist for all time and are smooth. However, it is not true that, for all t, $\phi_t : \mathcal{O} \to \mathcal{O}$ is a diffeomorphism. We will show this in the next section where we prove that in certain circumstances (that include the present example), the complex structures J_t must degenerate. We can also argue as follows. Regard a, solving (4.32), as a function of t and of $r = |\kappa|$. By implicit differentiation with respect to r this time, one finds that

$$r\left(\frac{1}{1-a^2}+t\right)\frac{\partial a}{\partial r} = -1. \tag{4.36}$$

This equation cannot be satisfied if $t = \frac{1}{a^2-1}$. For such values of t, $t \in (-\infty, -1]$, and, in fact, ϕ_t is a diffeomorphism for $t \in (-1, \infty)$.

We finally remark that, contrary to the example discussed in §4.2.2 where $M = \mathbb{C}$, for $M = \mathcal{O}$ it is not possible to have that Φ_t acts as holomorphic diffeomorphisms of the complexification for all time, and have a leaf of $\mathcal{F}_t$ not intersect the real locus. This is for topological reasons, as we now explain. First, note that the complexification $X = \mathcal{O}_{\mathbb{C}}$ is an affine quadric in $\mathbb{C}^3$. Each leaf of the foliation $\mathcal{F}_0$ is a ruling complex line ℓ of $\mathcal{O}_{\mathbb{C}}$, which closes up to a projective line $\bar{\ell} \subset \overline{\mathcal{O}_{\mathbb{C}}} \subset \mathbb{P}^3$. Suppose that, for some $t \in \mathbb{C}$ $\Phi_t(\ell)$, which is a leaf of $\mathcal{F}_t$, does not intersect $M = S^2$. Then

$$\Phi_t^{-1}(S^2) \cap \ell = \emptyset.$$

Having compactified in $\mathbb{P}^3$, we have homology classes $[S^2] \in H_2(\mathcal{O}_{\mathbb{C}}, \overline{\mathcal{O}_{\mathbb{C}}} \setminus \mathcal{O}_{\mathbb{C}}; \mathbb{Z})$ and $[\bar{\ell}] \in H_2(\overline{\mathcal{O}_{\mathbb{C}}}, \overline{\mathcal{O}_{\mathbb{C}}} \setminus \mathcal{O}_{\mathbb{C}}; \mathbb{Z})$ and, calculating intersection products, we get

$$[S^2] \cdot [\Phi_t \bar{\ell}] = [\Phi_t^{-1}(S^2)] \cdot [\bar{\ell}] = \pm 1,$$

the sign depending on the choice of orientation on S^2. This contradicts $\Phi_t(\ell) \cap S^2 = \emptyset$.

This argument is valid also for the complexifications of any coadjoint orbit of a compact group as described in §4.3.

4.4.4. The quantum version of the previous example We now turn to the quantum version of the dynamics of the Hamiltonian $h = \frac{i}{2}x_3^2$ on $\mathcal{O} \cong \mathbb{CP}^1$, and the conjecture of §1.2.

To $\mathbb{CP}^1$ we associate the sequence of finite-dimensional Hilbert spaces $\mathcal{H}_k = H^0(\mathcal{L}^k)$ where $\mathcal{L} \to \mathbb{CP}^1$ is the hyperplane bundle. Here $1/k = \hbar$ is Planck's costant. The norm of $\mathcal{H}_k$ is

$$\forall \psi \in \mathcal{H}_k \quad \|\psi\|^2 = \int_{\mathbb{CP}^1} |\psi|_k^2 \, dV,$$

where dV is Liouville measure and $|\cdot|_k$ is the Hermitian norm in $\mathcal{L}^k$. The natural action of $\mathrm{SU}(2)$ on $\mathcal{L} \to \mathbb{CP}^1$ induces the irreducible unitary representation of $\mathrm{SU}(2)$ of dimension $k+1$, which complexifies to a representation of

$\mathrm{SL}(2,\mathbb{C})$. Specifically, $\mathcal{H}_k$ is naturally the space of homogeneous polynomials of degree k int two complex variables, and an orthonormal basis of $\mathcal{H}_k$ is

$$|n\rangle = \frac{1}{\pi}\sqrt{\frac{k+1}{2}}\sqrt{\binom{k}{n}}\, z_1^n\, z_2^{k-n}, \quad 0 \le n \le k. \tag{4.37}$$

The (Hermitian) angular momentum operators corresponding to the Pauli matrices (4.13) are

$$\begin{aligned} \widehat{L}_1 &= \tfrac{1}{2}\left(z_1 \tfrac{\partial}{\partial z_2} + z_2 \tfrac{\partial}{\partial z_1}\right), \\ \widehat{L}_2 &= -\tfrac{i}{2}\left(z_1 \tfrac{\partial}{\partial z_2} - z_2 \tfrac{\partial}{\partial z_1}\right), \\ \widehat{L}_3 &= \tfrac{1}{2}\left(z_1 \tfrac{\partial}{\partial z_1} - z_2 \tfrac{\partial}{\partial z_2}\right), \end{aligned} \tag{4.38}$$

and (4.37) are eigenvectors of the operator $\widehat{L}_3$, with corresponding eigenvalues $n - \frac{k}{2}$. The raising operator is

$$J_+ = J_1 - iJ_2 = z_1 \frac{\partial}{\partial z_2},$$

which is the infinitesimal representation of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$ (the generator of the Lie algebra of N), and the associated highest weight vectors are multiples of $|k\rangle$.

Identifying $\mathbb{CP}^1 \cong S^2$, we define the coherent state map (which is the Veronese embedding)

$$V_k : S^2 \to \mathbb{P}\mathcal{H}_k \tag{4.39}$$

by mapping the north pole to the line $\mathbb{C}|k\rangle$ and imposing that V be equivariant with respect to $\mathrm{SU}(2)$. In the physics literature, a non-zero element $\psi_{p,k} \in V_k(p)$, $p \in S^2$ is called a (standard) $\mathrm{SU}(2)$ coherent state centered at p.

Let us now return to the example of the Hamiltonian $h = \frac{i}{2}x_3^2$ above, which we quantize to be the (sequence of) operators

$$\hat{h} : \mathcal{H}_k \to \mathcal{H}_k, \qquad \hat{h} = \frac{i}{2k}\,\widehat{L}_3^2.$$

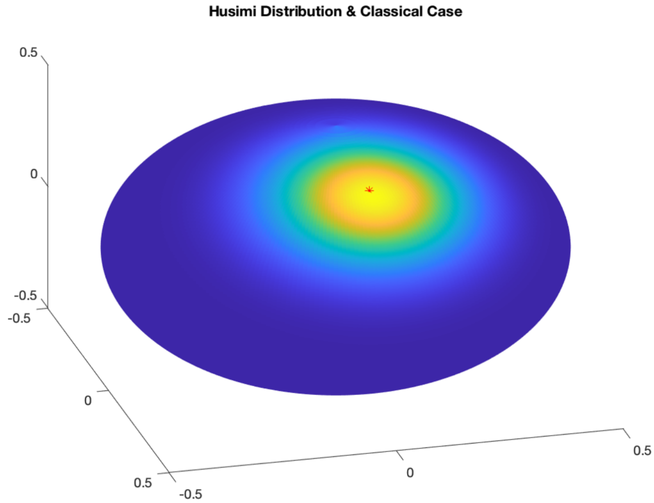

Figure 2: Plot of the Husimi function $|\widetilde{\psi}_k(t)|^2$, where $\widetilde{\psi}_k(t)$ is the normalized solution to (4.40) with initial condition a coherent state at $p = (\sqrt{2}/4, 0, \sqrt{2}/4)$. The asterix is the location of $\phi_t(p)$. $t \cong 0.6597$ and $k = 20$.

The Schrödinger equation is equivalent to

$$\frac{\partial}{\partial t}\psi_k = \frac{1}{2}\,\widehat{L}_3^2\psi_k. \tag{4.40}$$

The conjecture of §1.2 in this case is the statement that the solution of (4.40) with initial condition a coherent state $\psi_{p,k}$ concentrates (after normalization) as $k \to \infty$ at the point $\phi_t(p)$ described by Lemma 4.5. Numerical computations by Wasim Rehman [15] support this conjecture, see figure 2.

5. Geodesics and infinitesimal obstructions

In this section we first check that our trajectories from the exponential map described here agree, for purely imaginary Hamiltonians, with the the geodesics of Mabuchi in the space of Kähler metrics with a given Kähler class. We close by finding an infinitesimal obstruction to the completeness of such geodesics.

5.1. Relation with geodesics in the space of Kähler potentials

To compare with previous work on Mabuchi geodesics, we emphasize here the model of the infinitesimal complexification of $\mathcal{K}$, the Hamiltonian group of M, based on the map f_t (2.4) and its basic equations (2.6) and (2.7) and relation to ϕ_t (2.5). Thus, as in equation (1.2), we may identify

$$\mathcal{G} = \left\{ f \in \mathrm{Diff}^0(M) \mid f^*\omega \text{ and } J \text{ are compatible} \right\}. \tag{5.1}$$

More explicitly, we can view

$$\mathcal{G} = \left\{ (f, \omega_a) \mid f \in \mathrm{Diff}^0(M), a \in \mathcal{H}, f^*\omega_a = \omega \right\}, \tag{5.2}$$

where the space

$$\mathcal{H} := \{a : M \to \mathbb{R} \mid \omega_a := \omega + i\bar{\partial}\partial a > 0\}/\mathbb{R} \tag{5.3}$$

is the space of Kähler potentials for Kähler metrics on M with fundamental class $[\omega]$. A slightly more explicit version of the exponential equation (2.7) is now given by

$$\forall t, \quad \dot{f}_t \circ f_t^{-1} = \Xi^{\omega_{a_t}}_{\Re h \circ f_t^{-1}} + J\left(\Xi^{\omega_{a_t}}_{\Im h \circ f_t^{-1}}\right), \tag{5.4}$$

i.e., $\omega_t = \omega_{a_t}$. $\mathcal{H}$ has a natural Riemannian metric, due to Mabuchi [14], given by

$$\|\delta a\|_z^2 = \int_M |\delta a|^2 d\mu_a$$

where $d\mu_a = \omega_a^n/n!$ The geodesic equation turns out to be

$$\ddot{a} = -\frac{1}{2}|\nabla^a \dot{a}|_a^2, \tag{5.5}$$

as in, e.g., [7], equation (12).

Let (f_t, a_t) be the exponential of a purely-imaginary Hamiltonian, $h = iH$, $H : M \to \mathbb{R}$. Differentiating the pull-back relation in (5.2), equivalently (2.6), we get

$$0 = f_t^*[\dot{\omega}_{a_t} + \mathcal{L}_{V_t}\omega_{a_t}] = f_t^*[i\bar{\partial}\partial\dot{a}_t + \mathcal{L}_{V_t}\omega_{a_t}] = f_t^*[i\bar{\partial}\partial\dot{a}_t + d(i_{V_t}\omega_{a_t})],$$

where $V_t = \dot{f}_t \circ f_t^{-1}$, and the $\mathcal{L}$ derivative is in the M variables only. Meanwhile

(5.4) yields, for purely imaginary h,

$$\begin{aligned} d(i_{V_t}\omega_{a_t}) &= d(d(H\circ f_t^{-1})\circ J) \\ &= d(\partial(H\circ f_t^{-1})\circ J+\bar\partial(H\circ f_t^{-1})\circ J) \\ &= d(i\partial(H\circ f_t^{-1})-i\bar\partial(H\circ f_t^{-1})) \\ &= 2i\bar\partial\partial H\circ f_t^{-1}. \end{aligned}$$

Thus, we get

$$\bar\partial\partial(\dot a_t-2H\circ f_t^{-1})=0.$$

Since M is compact,

$$\dot a_t-2H\circ f_t^{-1}=c(t), \tag{5.6}$$

a constant on M, depending on t. Since a_t is only well-defined up to the addition of a constant, without loss of generality we note the following lemma.

Lemma 5.1. *There is a choice of constant $C(t)$ such that, replacing a_t by $a_t+C(t)$ gives a_t normalized to have*

$$f_t^*\dot a_t\equiv 2H. \tag{5.7}$$

Proof. To get an equation for $C(t)$ we set

$$(a_t+C)'=\dot a_t+C'(t)=2H\circ f_t^{-1}+c(t)+C'(t),$$

so we simply want $C(t)=-\int_0^t c(s)\,ds$. $\square$

Differentiating (5.7) with respect to time gives

$$0=\frac{d}{dt}f_t^*\dot a_t=f_t^*[\ddot a_t+d(\dot a_t)(V_t)],$$

that is,

$$\ddot a_t=-d(\dot a_t)(V_t)=-d(\dot a_t)(\nabla^t(H\circ f_t^{-1})=-d(\dot a_t)(\nabla^t\frac{1}{2}\dot a_t)=-\frac{1}{2}|\nabla^t\dot a_t|_t^2. \tag{5.8}$$

We conclude:

Corollary 5.2. *Let f_t be the exponential of a purely-imaginary Hamiltonian, $h = iH$, $H : M \to \mathbb{R}$. Then the curve of potentials a_t of the metrics (ω_t, J), normalized as in lemma 5.1, is the Mabuchi geodesic on $\mathcal{H}$ with initial conditions*

$$a_0 = 0,\ \dot{a}_0 = 2H.$$

Note: The factor of two in the equation $\dot{a}_0 = 2H$ can be gotten rid of by letting

$$\omega_a = \omega_0 + 2i\bar{\partial}\partial a.$$

5.2. Infinitesimal obstruction to continuing a Mabuchi geodesic

In this section, we examine further the obstructions to continuation of the flows associated to complex Hamiltonians, especially purely imaginary Hamiltonians because of their relationship to Mabuchi geodesics in $\mathcal{H}$ given by ϕ_t generated by a purely imaginary Hamiltonian iH, H real valued and $\mathcal{C}^\omega$ on M. In Donaldson's analysis [6], complete Mabuchi geodesic rays were proposed as a way to prove the uniqueness of constant scalar curvature Kähler metrics. Counterexamples to the extendability of these geodesics were found [13] using geometric constraints associated with such continuations. We point out here that there are infinitesimal obstructions to this continuation centered at one point coming from the infinitesimal dynamics at a critical point of H. This obstruction appears to be new. We do not know its full interpretation, say, with respect to the Semmes–Donaldson method of construction via the homogeneous complex Monge–Ampère equation (see [18], [19] or [6]). For real analytic H, this is readily understood in terms of the Hamiltonian dynamics of its holomorphic extension at the critical point of H. Since the obstruction is infinitesimal, real analyticity of H proves to be immaterial, and the obstruction applies to smooth solutions of the geodesic equation.

We work with f_t as in corollary (5.4). We elaborate here on its relation to the corresponding solution of the geodesic equation. A Mabuchi geodesic is defined on the interval $(t_0, t_1), t_0 < 0 < t_1$, if and only if the corresponding solution $a = a(t)$ to (5.8) is defined, smooth and positive in the sense of (5.3) on that t-interval.

Lemma 5.3. *The Mabuchi geodesic $a(t)$ is defined on the interval (t_0, t_1) if and only if f_t satisfying (2.7) with $H = \dot{a}(0)$ exists and is invertible for $t \in (t_0, t_1)$.*

Proof. One simply refers to equation (2.7), which in the present context (Hamiltonian iH, H real) becomes:

$$\dot{f}_t \circ f_t^{-1} = J_0 \left(\Xi^{\omega_t}_{H \circ f_t^{-1}} \right). \tag{5.9}$$

If the geodesic exists on (t_0, t_1), then (5.9) gives an ODE for f_t where the right hand side of the equation is already defined for $t \in (t_0, t_1)$, since $\omega_t = \omega_{a(t)}$ depends only on fixed data and $a(t)$. Since M is assumed compact, this equation is solvable by simple integration. Similarly, one can, by integration, reverse the argument that leads to (2.7) back to its source, namely $f_t^* \omega_t = \omega_0$. This implies f_t is a local diffeomorphism. Since (t_0, t_1) is connected, and M is compact and connected, then f_t is a global diffeomorphism for $t \in (t_0, t_1)$ since it is so for $t = 0 \in (t_0, t_1)$. □

We will show that there are bounds to the domain of existence of f_t, or its invertibility. We first need a lemma. Suppose f_t is defined and smooth on the interval (t_0, t_1).

Lemma 5.4. *Let H be a smooth real valued function on M with critical point $x_0 \in M$, and let f_t be the solution of (2.7) for Hamiltonian $h = iH$. Then $f_t(x_0) = x_0$ for all $t \in (t_0, t_1)$.*

Proof. Consider again equation (5.9). Let V be any smooth vectorfield in a neighborhood of $f_t(x_0)$. Then

$$\omega_t(V, J_0 \cdot \Xi^{\omega_t}_{H \circ f_t^{-1}}) = -d(H \circ f_t^{-1})(J_0 \cdot V) = dH(x_0)(Df_t^{-1}(J_0 \cdot V)) = 0,$$

since $dH(x_0) = 0$. Since we assume ω_t is non-degenerate and V is arbitrary, we conclude $\dot{f}_t \equiv 0$ on the interval (t_0, t_1). □

Now set $A = A(t) = Df_t(x_0) : T_{x_0}M \to T_{x_0}M$, and let V be a vectorfield in a neighborhood of x_0. Take the Lie derivative with respect to V on both sides of (5.9) and evaluate at x_0. On the left hand side one obtains $\dot{A} \cdot A^{-1}(V)$. On the right, we first rewrite $\Xi^{\omega_t}_{H \circ f_t^{-1}}$ as $\psi_t^{-1} \cdot d(H \circ f_t^{-1})$, where $\psi_t : T_{x_0}M \to T^*_{x_0}M$ is the contraction isomorphism corresponding to ω_t. Taking $\mathcal{L}_V$ gives

$$\begin{aligned} \mathcal{L}_V(J_0 \cdot \Xi^{\omega_t}_{H \circ f_t^{-1}}) &= \mathcal{L}_V(J_0 \cdot \psi_t^{-1} \cdot d(H \circ f_t^{-1})) \\ &= J_0 \cdot \psi_t^{-1} \cdot \mathcal{L}_V(d(H \circ f_t^{-1})), \end{aligned}$$

when evaluated at x_0. Recalling that $f_t^*\omega_t = \omega_0$ from (2.6), we can write at x_0

$$\psi_t = (A^t)^{-1} \cdot \psi_0 \cdot A. \tag{5.10}$$

Finally, we have

$$\begin{aligned}\mathcal{L}_V(d(H \circ f_f^{-1})) = d(\mathcal{L}_V(H \circ f_t^{-1})) =\\ = (A^{-1})^t \cdot \mathit{Hess}(H) \cdot A^{-1}(V)\end{aligned}$$

at x_0, where $\mathit{Hess}(H)$ is the Hessian of H at x_0 viewed as a linear transformation from $T_{x_0}M$ to $T^*_{x_0}M$. Altogether now we see

$$\begin{aligned}\mathcal{L}_V(J_0\left(\Xi^{\omega_t}_{H\circ f_t^{-1}}\right)) = J_0 \cdot A \cdot \psi_0^{-1} \cdot A^t \cdot (A^{-1})^t \cdot \mathit{Hess}(H) \cdot A^{-1}(V) =\\ = J_0 \cdot A \cdot \psi_0^{-1} \cdot \mathit{Hess}(H) \cdot A^{-1}(V)\end{aligned}$$

at x_0. Thus, we have a differential equation for A:

$$\dot{A} = J_0 \cdot A \cdot \psi_0^{-1} \cdot \mathit{Hess}(H). \tag{5.11}$$

This has the solution, for initial value $A(0) = A_0$,

$$A(t) = \sum_{n=0} \frac{t^n}{n!} J_0^n \cdot A_0 \cdot (\psi_0^{-1} \cdot \mathit{Hess}(H))^n,$$

valid for all $t \in \mathbb{R}$. In our case, $A_0 = I$, and the solution is

$$A(t) = \cos(t\,\psi_0^{-1} \cdot \mathit{Hess}(H)) + J_0 \cdot \sin(t\,\psi_0^{-1} \cdot \mathit{Hess}(H)). \tag{5.12}$$

We want to analyze the solutions $A(t)$ of (5.12). Since the relation of $\mathit{Hess}(H)$ to J_0 on $T_{x_0}M$ can be somewhat arbitrary, we make some simplifying assumptions to extract some consequences of (5.12). We use the metric on $T_{x_0}M$ given by ω_0 and J_0 to view $\mathit{Hess}(H)$ as a self-adjoint endomorphism of $T_{x_0}M$, and we assume the existence of a two-dimensional subspace $V \subset T_{x_0}M$ which is invariant under both J_0 and $\mathit{Hess}(H)$. It is easy to check then that V and $V^\perp$ are invariant by $J_0, \mathit{Hess}(H)$ and $\psi_{x_0}^{-1}$, and therefore by $A(t)$, for all t.

Proposition 5.5. *Let H and J_0 satisfy the assumptions above. If $\mathit{Hess}(H)$ restricted to V has rank 1, then $A(t)$ restricted to V becomes singular at time $t = -1/\lambda$, where λ is the non-zero eigenvalue of $\mathit{Hess}(H)$ on V.*

Proof. In the two dimensional space V we can find an orthonormal basis $\{v_1, v_2\}$ in which the endomorphisms J_0 and $Hess(H)$ have the standard forms

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad Hess(H) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $\lambda \neq 0$. Note that with the normalizations made $\psi_{x_0}^{-1}$ is represented by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that

$$\psi_{x_0}^{-1} \cdot Hess(H) = \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix}.$$

Hence

$$\cos(t\psi_{x_0}^{-1} \cdot Hess(H)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sin(t\psi_{x_0}^{-1} Hess(H)) = \begin{bmatrix} 0 & 0 \\ -t\lambda & 0 \end{bmatrix},$$

and so

$$\det A(t) = \det \begin{bmatrix} 1 + t\lambda & 0 \\ 0 & 1 \end{bmatrix} = 1 + t\lambda.$$

□

Theorem 5.6. *Let H be as above. Then the Mabuchi geodesic with initial conditions $a(0) = 0, \dot{a}(0) = 2H$ cannot extend past $t = -\frac{1}{\lambda}$.*

Proof. Indeed, this follows directly from lemma 5.3 and the proposition. □

Note that this is in agreement with the example $h = \frac{i}{2}x_3^2$, or $H = \frac{1}{2}x_3^2$ in section 4.4.3, where the relevant critical points of H are along the equator $\{x_3 = 0\}$.

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