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Abstract noncommutative Fourier series on $\Gamma\backslash SE(2)$

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Dedicated to Professor Bernard Shiffman on the Occasion of his 75th Birthday

Abstract: This paper begins with a systematic study of abstract noncommutative Fourier series on $\Gamma\backslash SE(2)$, where Γ is a discrete co-compact subgroup of $SE(2)$, the group of all handedness-preserving isometries of the Euclidean plane. Let μ be the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$, normalized with respect to Weil's formula. The analytic aspects of the proposed method works for any given (discrete) basis of the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$. The paper concludes with some convolution results.

Keywords: Special Euclidean group, non-commutative Fourier series, coset space, discrete subgroup, crystallographic subgroup.

1. Introduction

The special Euclidean group $SE(2)$ is a noncompact non-Abelian finite dimensional real Lie group which describes rigid body motions on $\mathbb{R}^2$. The group $SE(2)$ is a building block in coherent states, quantum mechanics, and geometric harmonic analysis [2, 28, 35, 37]. Over the last few decades, various computational aspects of constructive approximation techniques using harmonic analysis of functions on the unimodular non-Abelian group $SE(2)$ have attracted considerable attention in postmodern applications, see [1, 5, 7, 8, 9, 13, 21, 22, 32, 33, 46, 47].

The right coset space of discrete and co-compact subgroups in $SE(2)$, such as $\mathbb{Z}^2\backslash SE(2)$, appears as the configuration space in many recent applications in computational science and engineering including robotics, computer vision, computational biology, mathematical crystallography, and material

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science [4, 6, 9, 10, 11, 12, 30, 40, 43, 44]. Invoking the algebraic structure of $SE(2)$, the subgroup $\mathbb{Z}^2$ is not normal in $SE(2)$ and hence the right coset space $\mathbb{Z}^2\backslash SE(2)$ is not a group. Therefore, the classical notion of noncommutative Fourier transform and thus Fourier type expansion on this compact 3-dimensional manifold are meaningless. However, there are still both algebraic and geometric structures on it, namely the transitive right action of the Lie group $SE(2)$ on the manifold $\mathbb{Z}^2\backslash SE(2)$, which makes the right coset space into a homogeneous space.

Homogeneous spaces can be viewed as group-like structures with applications in differential geometry, geometric analysis, mathematical physics, and coherent state (covariant) transforms, see [14, 28, 38, 45]. The mathematical theory of abstract Fourier analysis on homogeneous spaces of compact groups, including left coset spaces of compact groups, have been studied at depth in [16, 17] and references therein. In addition, in the case of canonical homogeneous spaces of semi-direct product groups with Abelian normal factor, an approach to the relative Fourier analysis is discussed in [18]. The later theories strongly benefit from some assumptions about the group which do not hold for $SE(2)$ and hence a different approach is required for the right coset space $\mathbb{Z}^2\backslash SE(2)$.

The present article focuses on constructive aspects of non-commutative Fourier expansions on the right coset space $\Gamma\backslash SE(2)$, when Γ is a discrete co-compact subgroup of $SE(2)$. Our aim is to further develop a non-commutative Fourier-type reconstruction on the homogeneous space $\Gamma\backslash SE(2)$, based on the algebraic structure of the coset space, which has not been studied as extensively as the Fourier expansions on the group $SE(2)$. We shall also address analytic aspects of the proposed expansion as a constructive approximation, using tools from abstract harmonic analysis and representation theory.

This article which contains 4 sections, is organized as follows. Section 2 is devoted to fixing notation and gives a brief summary of noncommutative Fourier analysis on the unimodular group $SE(2)$ and classical analysis on the right coset space $\Gamma\backslash SE(2)$. In Section 3, we present the general theory of noncommutative Fourier series for a class of L^2-functions defined on the right coset space $\Gamma\backslash SE(2)$, for any given (discrete) basis of the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$, where μ is the finite $SE(2)$-invariant measure on $\Gamma\backslash SE(2)$, normalized with respect to Weil's formula. As the main result we present a constructive series which can be viewed as reconstruction formula for a class of functions on the right coset space $\Gamma\backslash SE(2)$, using the non-Abelian Fourier integral on $SE(2)$. We then present the notion of convolution of functions on $SE(2)$ by functions on $\Gamma\backslash SE(2)$, also called as the canonical module action of the Banach algebra $L^1(SE(2))$ on the Banach function space

$L^1(\Gamma\backslash SE(2),\mu)$. We shall also study different analytic aspects of noncommutative Fourier series for approximating the convolution functions on the right coset space $\Gamma\backslash SE(2)$. As applications for noncommutative Fourier series of convolution functions, the paper is concluded by presenting Plancherel type formulas for a class of L^2-functions on the right coset space $\Gamma\backslash SE(2)$.

2. Preliminaries and notation

Throughout this section, we review some preliminaries and fix the notations. For more details, we refer the reader to see [21, 22] and references therein. The 2D special orthogonal group $SO(2)$ is defined as

$$SO(2) := \{\mathbf{R} \in O(2) : \det(\mathbf{R}) = 1\},$$

where $O(2)$ is the orthogonal group in dimension 2. It is worthwhile to mention that every $\mathbf{R} \in SO(2)$ can be parameterized via

$$\mathbf{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in (0, 2\pi]$.

The special Euclidean group in dimension 2, denoted by $SE(2)$, is defined as the semi-direct product of the Abelian group $\mathbb{R}^2$ (the Plane) with the Abelian group $SO(2)$, which is

$$SE(2) = \mathbb{R}^2 \rtimes SO(2).$$

The group element $g \in SE(2)$ denoted by the ordered pair $g = (\mathbf{x}, \mathbf{R})$ with $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$.

For every $g = (\mathbf{x}, \mathbf{R})$ and $g' = (\mathbf{x}', \mathbf{R}')$ the group law of $SE(2)$ is given by

$$g \circ g' = (\mathbf{x} + \mathbf{R}\mathbf{x}', \mathbf{R}\mathbf{R}'),$$

and

$$g^{-1} = (-\mathbf{R}^T\mathbf{x}, \mathbf{R}^T),$$

where $\mathbf{R}^T = \mathbf{R}^{-1}$.

In addition, the special Euclidean group $SE(2)$ can be represented as the set of homogeneous transformation matrices

$$g(x_1, x_2, \theta) := \begin{pmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

with $\theta \in (0, 2\pi]$ and $x_1, x_2 \in \mathbb{R}$, or

$$g(a,\phi,\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & a\cos\phi \\ \sin\theta & \cos\theta & a\sin\phi \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

with $\theta, \phi \in (0, 2\pi]$ and $a \geq 0$.

The non-Abelian group $SE(2)$ is unimodular and the normalized Haar measure on $SE(2)$ is given by

$$dg = \frac{1}{4\pi^2}dx_1dx_2d\theta = \frac{1}{4\pi^2}adad\phi d\theta.$$

The convolution integral of functions $f_1, f_2 \in L^1(SE(2))$ is defined by

$$(f_1 \star f_2)(g) = \int_{SE(2)} f_1(h)f_2(h^{-1} \circ g)dh,$$

for $g \in SE(2)$.

The Lie group $SE(2)$ is solvable. Therefore, one can apply classical techniques for characterizing unitary irreducible representations (dual space) of solvable Lie groups, see [3, 23, 39, 41, 42]. The set of all unitary equivalence classes of (continuous) irreducible unitary representations (dual space) of the non-Abelian group $SE(2)$, denoted by $\widehat{SE(2)}$, which is also known as the spectrum of $SE(2)$, can be constructed as

$$\widehat{SE(2)} = \{\chi_n : n \in \mathbb{Z}\} \cup \{U_p : p > 0\}. \tag{3}$$

For every integer n, the character (one-dimensional continuous unitary representation) $\chi_n : SE(2) \to \mathbb{T}$ is defined by

$$\chi_n(g) := e^{\mathrm{i}n\theta}, \tag{4}$$

where $g = (\mathbf{x}, \mathbf{R}_\theta) \in SE(2)$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

For every $p > 0$, the irreducible unitary representation $U_p : SE(2) \to \mathcal{U}(L^2(\mathbb{S}^1))$ is defined by $g \mapsto U_p(g)$, where the unitary linear operator $U_p(g) : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is

$$[U_p(g)v](\mathbf{u}) := e^{-\mathrm{i}p\langle \mathbf{u},\mathbf{x}\rangle}v(\mathbf{R}_\theta^T\mathbf{u}), \tag{5}$$

for $g = (\mathbf{x}, \mathbf{R}_\theta) \in SE(2)$, $v \in L^2(\mathbb{S}^1)$, and $\mathbf{u} \in \mathbb{S}^1 := \{\mathbf{t} \in \mathbb{R}^2 : \|\mathbf{t}\|_2 = 1\}$.

The operator-valued Fourier integral of $f \in L^1(SE(2))$ is defined by the following operator-valued integral

$$\widehat{f}(p) := \int_{SE(2)} f(g)U_p(g^{-1})dg \qquad \text{for } p > 0, \tag{6}$$

where the operator-valued integral (6) is considered in the weak sense.

The corresponding Fourier convolution property for $f_1, f_2 \in L^1(SE(2))$ is given by

$$(\widehat{f_1 \star f_2})(p) = \widehat{f_2}(p)\widehat{f_1}(p), \tag{7}$$

for every $p > 0$.

For every $q \geq 1$, suppose that $\mathcal{H}^q(0,\infty)$ is the Banach space consisting of all measurable fields of bounded linear operators F on $(0,\infty)$ with

$$\|F\|_{\mathcal{H}^q(0,\infty)} := \left(\int_0^\infty \|F(p)\|_q^q p dp\right)^{1/q} < \infty,$$

where for a bounded linear operator T, the Schatten q-norm of T is $\|T\|_q := \mathrm{tr}[|T|^q]^{1/q}$ and $|T| := (T^*T)^{1/2}$, see [34].

The non-commutative Fourier Parseval/Plancherel formula on the unimodular group $SE(2)$ is given by

$$\int_{SE(2)} |f(g)|^2 dg = \int_0^\infty \|\widehat{f}(p)\|_2^2 p dp,$$

and the non-Abelian Fourier reconstruction integral on $SE(2)$ is

$$f(g) = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p)U_p(g)\right] p dp, \tag{8}$$

for every $f \in L^1 \cap L^2(SE(2))$ and $g \in SE(2)$, see [41]. It should be noted that the set $\{\chi_n : n \in \mathbb{Z}\}$ consists of all characters of $SE(2)$, has no role in the non-commutative Fourier integral reconstruction formula (8).

Let Γ be a discrete co-compact subgroup of $SE(2)$ with the counting measure as the Haar measure. Consequentially, the right coset space $\Gamma\backslash SE(2) := \{\Gamma g : g \in SE(2)\}$ is compact as a homogeneous space which the Lie group $SE(2)$ acts on from the right. Each discrete co-compact subgroup Γ of $SE(2)$ is isomorphic to a group of the form $\mathbb{Z}^2 \rtimes \mathbb{P}$, where $\mathbb{P}$ is a finite subgroup of $SO(2)$. Such a Γ must necessarily belongs to one of the five handedness preserving wallpaper groups.

The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces studied in [14, 19, 24, 25, 38] and references therein. Suppose $\mathcal{C}_c(SE(2))$ is the space of all continuous functions on $SE(2)$ with compact supports. The function space $\mathcal{C}(\Gamma\backslash SE(2))$, that is the set of all continuous functions on $\Gamma\backslash SE(2)$, consists of all functions $\widetilde{f}$, where $f \in \mathcal{C}_c(SE(2))$ and

$$\widetilde{f}(\Gamma g) := \sum_{\gamma\in\Gamma} f(\gamma \circ g), \tag{9}$$

for all $g \in SE(2)$.

Let μ be a Radon measure on the right coset space $\Gamma\backslash SE(2)$ and $h \in SE(2)$. The right translation μ_h of μ is defined by $\mu_h(E) := \mu(E \circ h)$, for all Borel subsets E of $\Gamma\backslash SE(2)$, where $E \circ h := \{\Gamma g \circ h : \Gamma g \in E\}$. The measure μ is called $SE(2)$-invariant if $\mu_h = \mu$, for all $h \in SE(2)$. Since $SE(2)$ is unimodular, Γ is discrete and $\Gamma\backslash SE(2)$ is compact, the right coset space $\Gamma\backslash SE(2)$ has a finite $SE(2)$-invariant measure μ, which satisfies the following Weil's formula

$$\int_{\Gamma\backslash SE(2)} \widetilde{f}(\Gamma g)d\mu(\Gamma g) = \int_{SE(2)} f(g)dg, \tag{10}$$

and hence the linear map $f \mapsto \widetilde{f}$ is norm-decreasing from $L^1(SE(2))$ into $L^1(\Gamma\backslash SE(2), \mu)$, that is

$$\|\widetilde{f}\|_{L^1(\Gamma\backslash SE(2),\mu)} \leq \|f\|_{L^1(SE(2))},$$

for all $f \in L^1(SE(2))$.

3. Abstract noncommutative Fourier series on $\Gamma\backslash SE(2)$

This section investigates the notion of noncommutative Fourier series for square integrable functions on the right coset space of discrete and co-compact subgroups in $SE(2)$. Throughout this section we assume that Γ is a discrete co-compact subgroup of $SE(2)$ and μ is the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$ which is normalized with respect to Weil's formula (10).

First, we need some preliminary results concerning L^2-function space on the right coset space $\Gamma\backslash SE(2)$.

Proposition 3.1. *Let $f \in L^1(SE(2))$ with $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2), \mu)$. Then $f \in L^2(SE(2))$ and*

$$\|f\|_{L^2(SE(2))} \leq \|\widetilde{|f|}\|_{L^2(\Gamma\backslash SE(2),\mu)}.$$

Proof. Let $f \in L^1(SE(2))$ such that $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2),\mu)$. We then claim that $f \in L^2(SE(2))$. To see this, we first note that

$$\sum_{\gamma\in\Gamma} |f(\gamma\circ g)|^2 \leq \left(\sum_{\gamma\in\Gamma} |f(\gamma\circ g)|\right)^2,$$

for $g \in SE(2)$. Then, using Weil's formula, we get

$$\begin{aligned}
\|f\|^2_{L^2(SE(2))} &= \int_{SE(2)} |f(g)|^2 dg \\
&= \int_{\Gamma\backslash SE(2)} \sum_{\gamma\in\Gamma} |f(\gamma\circ g)|^2 d\mu(\Gamma g) \\
&\leq \int_{\Gamma\backslash SE(2)} \left(\sum_{\gamma\in\Gamma} |f(\gamma\circ g)|\right)^2 d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \left(\sum_{\gamma\in\Gamma} |f|(\gamma\circ g)\right)^2 d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \widetilde{|f|}(\Gamma g)^2 d\mu(\Gamma g) = \|\widetilde{|f|}\|^2_{L^2(\Gamma\backslash SE(2),\mu)},
\end{aligned}$$

which implies that $f \in L^2(SE(2))$. □

Corollary 3.1. *Let $f \in L^1(SE(2))$ with $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2),\mu)$. Then $\widetilde{f} \in L^2(\Gamma\backslash SE(2),\mu)$ and*

$$\|\widetilde{f}\|_{L^2(\Gamma\backslash SE(2),\mu)} \leq \|\widetilde{|f|}\|_{L^2(\Gamma\backslash SE(2),\mu)}.$$

Proof. Let $f \in L^1(SE(2))$ with $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2),\mu)$. Then, for $g \in SE(2)$, we have

$$|\widetilde{f}(\Gamma g)| = \left|\sum_{\gamma\in\Gamma} f(\gamma\circ g)\right| \leq \sum_{\gamma\in\Gamma} |f(\gamma\circ g)| = \widetilde{|f|}(\Gamma g).$$

Hence, we get

$$\begin{aligned}
\|\widetilde{f}\|^2_{L^2(\Gamma\backslash SE(2),\mu)} &= \int_{\Gamma\backslash SE(2)} |\widetilde{f}(\Gamma g)|^2 d\mu(\Gamma g) \\
&\leq \int_{\Gamma\backslash SE(2)} \widetilde{|f|}(\Gamma g)^2 d\mu(\Gamma g) = \|\widetilde{|f|}\|^2_{L^2(\Gamma\backslash SE(2),\mu)}.
\end{aligned}$$

□

Suppose that K is a subset of $SE(2)$ and $\varphi : \Gamma\backslash SE(2) \to \mathbb{C}$ is a function with

$$\int_K |\varphi(\Gamma g)|dg < \infty.$$

For $p > 0$, define the bounded linear operator $Q_K^\varphi(p) : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ by

$$Q_K^\varphi(p) := \int_K \varphi(\Gamma g)U_p(g)dg. \tag{11}$$

The operator-valued integral (11) is considered in the weak sense. In details, for every $u \in L^2(\mathbb{S}^1)$, the function $Q_K^\varphi(p)u$ is defined by explicit $L^2(\mathbb{S}^1)$-inner product with an arbitrary $v \in L^2(\mathbb{S}^1)$ which is given by

$$\langle Q_K^\varphi(p)u, v\rangle := \int_K \varphi(\Gamma g)\langle U_p(g)u, v\rangle dg. \tag{12}$$

Since $g \mapsto \langle U_p(g)u, v\rangle$ is a bounded and continuous function on $SE(2)$ and $g \mapsto \varphi(\Gamma g)$ is integrable on K, the integral on the right side of (12) is the ordinary integral of a function on K. Also, $(u, v) \mapsto \langle Q_K^\varphi(p)u, v\rangle$ is a sesquilinear form on $L^2(\mathbb{S}^1)$. In addition, we have

$$\begin{aligned}|\langle Q_K^\varphi u, v\rangle| &= \left|\int_K \varphi(\Gamma g)\langle U_p(g)u, v\rangle dg\right| \leq \int_K |\varphi(\Gamma g)||\langle U_p(g)u, v\rangle|dg \\ &\leq \int_K |\varphi(\Gamma g)|\|U_p(g)u\|\|v\|dg = \|u\|_{L^2(\mathbb{S}^1)}\|v\|_{L^2(\mathbb{S}^1)}\left(\int_K |\varphi(\Gamma g)|dg\right),\end{aligned}$$

implying that $(u, v) \mapsto \langle Q_K^\varphi(p)u, v\rangle$ is a bounded sesquilinear form on the Hilbert space $L^2(\mathbb{S}^1)$. Therefore, using Theorem 2.3.6. of [36], we conclude that $Q_K^\varphi(p)$ defines a bounded linear operator on the Hilbert space $L^2(\mathbb{S}^1)$ with the operator norm

$$\|Q_K^\varphi(p)\| \leq \int_K |\varphi(\Gamma g)|dg.$$

Remark 3.1. Let $p > 0$. There are canonical feasible scenarios for well-defined $Q_K^\varphi(p)$.

(i) If K is compact then each $\varphi \in \mathcal{C}(\Gamma\backslash SE(2))$, defines the bounded linear operator $Q_K^\varphi(p)$ on $L^2(\mathbb{S}^1)$ with the operator norm

$$\|Q_K^\varphi(p)\| \leq \int_K |\varphi(\Gamma g)|dg.$$

(ii) If Ω is a fundamental domain of Γ in $SE(2)$ then every $\varphi \in L^1(\Gamma\backslash SE(2), \mu)$, defines the bounded linear operator $Q_\Omega^\varphi(p)$ on $L^2(\mathbb{S}^1)$ with the operator norm

$$\|Q_\Omega^\varphi(p)\| \leq \|\varphi\|_{L^1(\Gamma\backslash SE(2),\mu)}.$$

Theorem 3.1. *Let Γ be a discrete co-compact subgroup of $SE(2)$. Suppose $K \subset SE(2)$ is compact and $\varphi \in \mathcal{C}(\Gamma\backslash SE(2))$. Assume $f \in \mathcal{C}_c(SE(2))$ is supported in K with $\widehat{f} \in \mathcal{H}^1(0,\infty)$. Then*

$$\langle \widetilde{f}, \varphi \rangle = \int_0^\infty \operatorname{tr}\left[\widehat{f}(p) Q_K^{\overline{\varphi}}(p)\right] p dp, \tag{13}$$

where

$$Q_K^{\overline{\varphi}}(p) := \int_K \overline{\varphi(\Gamma g)} U_p(g) dg,$$

for $p > 0$.

Proof. We have $f \in L^1 \cap L^2(SE(2))$ and $\widetilde{f} \in \mathcal{C}(\Gamma\backslash SE(2)) \subset L^2(\Gamma\backslash SE(2), \mu)$. Invoking Equation (8), we have

$$f(g) = \int_0^\infty \operatorname{tr}[\widehat{f}(p) U_p(g)] p dp, \tag{14}$$

for $g \in SE(2)$. Therefore, using Weil's formula and (14), we get

$$\begin{aligned}
\langle \widetilde{f}, \varphi \rangle &= \int_{\Gamma\backslash SE(2)} \widetilde{f}(\Gamma g) \overline{\varphi(\Gamma g)} d\mu(\Gamma g) = \int_{SE(2)} f(g) \overline{\varphi(\Gamma g)} dg \\
&= \int_K f(g) \overline{\varphi(\Gamma g)} dg = \int_K \left(\int_0^\infty \operatorname{tr}[\widehat{f}(p) U_p(g)] p dp \right) \overline{\varphi(\Gamma g)} dg.
\end{aligned}$$

Since $\widehat{f} \in \mathcal{H}^1(0,\infty)$, we conclude that

$$\begin{aligned}
\int_K \int_0^\infty |\operatorname{tr}[\widehat{f}(p) U_p(g)]| |\varphi(\Gamma g)| p dp dg &\leq \int_K \int_0^\infty \|\widehat{f}(p) U_p(g)\|_1 |\varphi(\Gamma g)| p dp dg \\
&\leq \int_K \int_0^\infty \|\widehat{f}(p)\|_1 \|U_p(g)\| |\varphi(\Gamma g)| p dp dg \\
&= \int_K \int_0^\infty \|\widehat{f}(p)\|_1 |\varphi(\Gamma g)| p dp dg \\
&= \|\widehat{f}\|_{\mathcal{H}^1(0,\infty)} \left(\int_K |\varphi(\Gamma g)| dg \right) < \infty,
\end{aligned}$$

which guarantees that

$$\int_K \left(\int_0^\infty \mathrm{tr}[\widehat{f}(p)U_p(g)]pdp \right) \overline{\varphi(\Gamma g)}dg = \int_0^\infty \left(\int_K \mathrm{tr}[\widehat{f}(p)U_p(g)]\overline{\varphi(\Gamma g)}dg \right) pdp.$$

Thus, we achieve

$$\langle \widetilde{f}, \varphi \rangle = \int_0^\infty \left(\int_K \mathrm{tr}[\widehat{f}(p)U_p(g)]\overline{\varphi(\Gamma g)}dg \right) pdp. \tag{15}$$

Since $\widehat{f} \in \mathcal{H}^1(0,\infty)$, we can also conclude that the bounded linear operator $\widehat{f}(p)$ has a finite trace-class norm for a.e. $p > 0$. Suppose that $\{\mathbf{e}_m : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{S}^1)$. Then,

$$\begin{aligned}
&\int_K \sum_{m\in\mathbb{Z}} |\varphi(\Gamma g)||\langle \widehat{f}(p)U_p(g)\mathbf{e}_m, \mathbf{e}_m\rangle|dg \\
&\quad = \int_K |\varphi(\Gamma g)| \left(\sum_{m\in\mathbb{Z}} |\langle \widehat{f}(p)U_p(g)\mathbf{e}_m, \mathbf{e}_m\rangle| \right) dg \\
&\quad \leq \int_K |\varphi(\Gamma g)| \|\widehat{f}(p)U_p(g)\|_1 dg \\
&\quad \leq \|\widehat{f}(p)\|_1 \left(\int_K |\varphi(\Gamma g)|dg \right) < \infty,
\end{aligned}$$

for a.e. $p > 0$. Therefore, we obtain

$$\begin{aligned}
\mathrm{tr}\left[\widehat{f}(p)Q_K^{\overline{\varphi}}(p)\right] &= \sum_{m\in\mathbb{Z}} \langle \widehat{f}(p)Q_K^{\overline{\varphi}}(p)\mathbf{e}_m, \mathbf{e}_m\rangle \\
&= \sum_{m\in\mathbb{Z}} \langle Q_K^{\overline{\varphi}}(p)\mathbf{e}_m, \widehat{f}(p)^*\mathbf{e}_m\rangle \\
&= \sum_{m\in\mathbb{Z}} \int_K \overline{\varphi(\Gamma g)}\langle U_p(g)\mathbf{e}_m, \widehat{f}(p)^*\mathbf{e}_m\rangle dg \\
&= \sum_{m\in\mathbb{Z}} \int_K \overline{\varphi(\Gamma g)}\langle \widehat{f}(p)U_p(g)\mathbf{e}_m, \mathbf{e}_m\rangle dg \\
&= \int_K \sum_{m\in\mathbb{Z}} \overline{\varphi(\Gamma g)}\langle \widehat{f}(p)U_p(g)\mathbf{e}_m, \mathbf{e}_m\rangle dg \\
&= \int_K \overline{\varphi(\Gamma g)} \left(\sum_{m\in\mathbb{Z}} \langle \widehat{f}(p)U_p(g)\mathbf{e}_m, \mathbf{e}_m\rangle \right) dg \\
&= \int_K \overline{\varphi(\Gamma g)}\mathrm{tr}[\widehat{f}(p)U_p(g)]dg,
\end{aligned}$$

for a.e. $p \in (0,\infty)$. Then, using (15), we get

$$\langle \widetilde{f},\varphi\rangle = \int_0^\infty \left(\int_K \mathrm{tr}[\widehat{f}(p)U_p(g)]\overline{\varphi(\Gamma g)}dg\right) pdp = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^{\overline{\varphi}}(p)\right] pdp.$$

□

We now present the following constructive noncommutative Fourier type approximation for a class of L^2-functions on the right coset space $\Gamma\backslash SE(2)$ with respect to orthonormal bases of continuous functions.

Proposition 3.2. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $K \subset SE(2)$ is compact and $f \in \mathcal{C}_c(SE(2))$ is supported in K with $\widehat{f} \in \mathcal{H}^1(0,\infty)$. Then*

$$\widetilde{f} = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^\ell(p)\right] pdp\right)\psi_\ell, \tag{16}$$

where

$$Q_K^\ell(p) := \int_K \overline{\psi_\ell(\Gamma g)}U_p(g)dg,$$

for $p > 0$ and $\ell \in \mathbb{I}$.

Proof. Since $\{\psi_\ell : \ell \in \mathbb{I}\}$ is an orthonormal basis for $L^2(\Gamma\backslash SE(2),\mu)$, we get

$$\widetilde{f} = \sum_{\ell\in\mathbb{I}} \langle \widetilde{f},\psi_\ell\rangle\psi_\ell. \tag{17}$$

Since each ψ_ℓ is continuous, by applying Equation (13), we have

$$\langle \widetilde{f},\psi_\ell\rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^\ell(p)\right] pdp,$$

which implies that

$$\widetilde{f} = \sum_{\ell\in\mathbb{I}} \langle \widetilde{f},\psi_\ell\rangle\psi_\ell = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^\ell(p)\right] pdp\right)\psi_\ell.$$

□

Remark 3.2. For instance if $\Gamma := \mathbb{Z}^2$, one of the classical ways to construct an orthonormal basis for the Hilbert function space $L^2(\mathbb{Z}^2\backslash SE(2),\mu)$ is to consider an orthonormal basis for $L^2(\Omega)$ with $\Omega = [0,1)^2 \times [0,2\pi)$. To this end, for each integral vector $\ell = (k_1, k_2, k_3)^T \in \mathbb{Z}^3$, define the function $\psi_\ell : \Omega \to \mathbb{C}$ by

$$\psi_\ell(x, y, \theta) := \exp\left(2\pi i(k_1 x + k_2 y)\right) \exp(ik_3\theta), \tag{18}$$

for all $(x, y, \theta) \in \Omega$.

We can also conclude the following results for functions supported in a fundamental domain of Γ in $SE(2)$.

Proposition 3.3. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\varphi \in L^2(\Gamma\backslash SE(2), \mu)$. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Assume $f \in \mathcal{C}_c(SE(2))$ is supported in Ω with $\widehat{f} \in \mathcal{H}^1(0, \infty)$. Then*

$$\langle \widetilde{f}, \varphi \rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_\Omega^{\overline{\varphi}}(p)\right] pdp, \tag{19}$$

where

$$Q_\Omega^{\overline{\varphi}}(p) := \int_\Omega \overline{\varphi(\Gamma g)} U_p(g) dg,$$

for $p > 0$.

Proof. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Since $\varphi \in L^2(\Gamma\backslash SE(2), \mu)$ and hence $\varphi \in L^1(\Gamma\backslash SE(2), \mu)$, $Q_\Omega^\ell(p)$ defines a bounded linear operator on $L^2(\mathbb{S}^1)$, for every $p > 0$. Then, using a similar method as we used in Theorem 3.1, we get

$$\langle \widetilde{f}, \varphi \rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_\Omega^{\overline{\varphi}}(p)\right] pdp,$$

where

$$Q_\Omega^{\overline{\varphi}}(p) := \int_\Omega \overline{\varphi(\Gamma g)} U_p(g) dg,$$

for $p > 0$. □

We then present the following constructive noncommutative Fourier type approximation for a class of L^2-functions on the right coset space $\Gamma\backslash SE(2)$.

Theorem 3.2. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Assume $f \in \mathcal{C}_c(SE(2))$ is supported in Ω with $\widehat{f} \in \mathcal{H}^1(0, \infty)$. Then*

$$\widetilde{f} = \sum_{\ell \in \mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_\Omega^\ell(p)\right] pdp \right) \psi_\ell, \tag{20}$$

where

$$Q_\Omega^\ell(p) := \int_\Omega \overline{\psi_\ell(\Gamma g)} U_g(p) dg,$$

for $p > 0$ and $\ell \in \mathbb{I}$.

Proof. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Since each $\psi_\ell \in L^2(\Gamma\backslash SE(2), \mu)$ and hence $\psi_\ell \in L^1(\Gamma\backslash SE(2), \mu)$, $Q_\Omega^\ell(p)$ defines a bounded linear operator on $L^2(\mathbb{S}^1)$, for every $p > 0$. Let $f \in \mathcal{C}_c(SE(2))$ be supported in Ω with $\widehat{f} \in \mathcal{H}^1(0,\infty)$. Since $\{\psi_\ell : \ell \in \mathbb{I}\}$ is an orthonormal basis for $L^2(\Gamma\backslash SE(2), \mu)$, we get

$$\widetilde{f} = \sum_{\ell\in\mathbb{I}} \langle \widetilde{f}, \psi_\ell\rangle \psi_\ell. \tag{21}$$

Applying Equation (19) we get

$$\widetilde{f} = \sum_{\ell\in\mathbb{I}} \langle \widetilde{f}, \psi_\ell\rangle \psi_\ell = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_\Omega^\ell(p)\right] pdp \right) \psi_\ell. \qquad \square$$

Absolutely convergent Fourier series on $\boldsymbol{\Gamma\backslash SE(2)}$ Let Γ be a discrete co-compact subgroup of $SE(2)$ and μ be the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$ which is normalized with respect to Weil's formula (10). Suppose $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ is a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subseteq \mathcal{C}(\Gamma\backslash SE(2))$. Let $\mathcal{A}(\mathcal{E})$ be the linear subspace of $L^2(\Gamma\backslash SE(2), \mu)$ given by

$$\mathcal{A}(\mathcal{E}) := \left\{ \psi \in L^2(\Gamma\backslash SE(2), \mu) : \sum_{\ell\in\mathbb{I}} |\langle \psi, \psi_\ell\rangle| \|\psi_\ell\|_{\sup} < \infty \right\}.$$

We then deduce the following observations concerning the function space $\mathcal{A}(\mathcal{E})$.

Proposition 3.4. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Then,*

1. *The sequence $\{\|\psi_\ell\|_{\sup}^{-1} : \ell \in \mathbb{I}\}$ is bounded.*
2. *For each sequence $\{a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^1(\mathbb{I})$, we have $\{\|\psi_\ell\|_{\sup}^{-1} a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^2(\mathbb{I})$.*
3. *For each complex valued sequence $\{a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^1(\mathbb{I})$, we have*

$$\sum_{\ell\in\mathbb{I}} \|\psi_\ell\|_{\sup}^{-1} a_\ell \psi_\ell \in \mathcal{A}(\mathcal{E}).$$

Proof. (1) Let $\ell \in \mathbb{I}$ be given. Since $\Gamma\backslash SE(2)$ is compact and hence has finite volume, we get

$$\|\psi_\ell\|^2_{L^2(\Gamma\backslash SE(2),\mu)} = \int_{L^2(\Gamma\backslash SE(2),\mu)} |\psi_\ell(\Gamma g)|^2 d\mu(\Gamma g) \leq \|\psi_\ell\|^2_{\sup}\mu(\Gamma\backslash SE(2)).$$

Therefore, we have

$$\|\psi_\ell\|^{-1}_{\sup} \leq \|\psi_\ell\|^{-1}_{L^2(\Gamma\backslash SE(2),\mu)}\sqrt{\mu(\Gamma\backslash SE(2))} = \mu(\Gamma\backslash SE(2))^{1/2}.$$

(2) Suppose $\{a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^1(\mathbb{I})$ is given. Using boundedness of $\{\|\psi_\ell\|^{-1}_{\sup} : \ell \in \mathbb{I}\}$, we get $\{\|\psi_\ell\|^{-1}_{\sup}a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^1(\mathbb{I})$. Since $\ell^1(\mathbb{I}) \subseteq \ell^2(\mathbb{I})$, we achieve $\{\|\psi_\ell\|^{-1}_{\sup}a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^2(\mathbb{I})$.
(3) Suppose $\{a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^1(\mathbb{I})$ is given. Since $\{\|\psi_\ell\|^{-1}_{\sup}a_\ell : \ell \in \mathbb{I}\} \in \boldsymbol{\ell}^2(\mathbb{I})$, we have $\sum_{\ell\in\mathbb{I}} \|\psi_\ell\|^{-1}_{\sup}a_\ell\psi_\ell \in L^2(\Gamma\backslash SE(2),\mu)$. Let $\psi := \sum_{\ell\in\mathbb{I}} \|\psi_\ell\|^{-1}_{\sup}a_\ell\psi_\ell$ and $\ell' \in \mathbb{I}$. We then have $\langle\psi,\psi_{\ell'}\rangle = \|\psi_{\ell'}\|^{-1}_{\sup}a_{\ell'}$. Therefore, we get

$$\sum_{\ell'\in\mathbb{I}} |\langle\psi,\psi_{\ell'}\rangle|\|\psi_{\ell'}\|_{\sup} = \sum_{\ell'\in\mathbb{I}} |a_{\ell'}| < \infty,$$

which implies that $\sum_{\ell\in\mathbb{I}} \|\psi_\ell\|^{-1}_{\sup}a_\ell\psi_\ell \in \mathcal{A}(\mathcal{E})$. □

Theorem 3.3. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $\psi \in \mathcal{A}(\mathcal{E})$ is given. Then,*

1. *The series $\sum_{\ell\in\mathbb{I}}\langle\psi,\psi_\ell\rangle\psi_\ell$ converges uniformly on $\Gamma\backslash SE(2)$.*
2. *For μ-a.e. $\Gamma g \in \Gamma\backslash SE(2)$, we have*

$$\psi(\Gamma g) = \sum_{\ell\in\mathbb{I}}\langle\psi,\psi_\ell\rangle\psi_\ell(\Gamma g). \tag{22}$$

3. *If ψ is continuous then the reconstruction formula (22) holds pointwise.*

Proof. (1) Invoking the Weierstrass M-test, the series $\sum_{\ell\in\mathbb{I}}\langle\psi,\psi_\ell\rangle\psi_\ell$ converges uniformly on $\Gamma\backslash SE(2)$.
(2) Using (1), the series $\sum_{\ell\in\mathbb{I}}\langle\psi,\psi_\ell\rangle\psi_\ell(\Gamma g)$ converges for each $g \in SE(2)$, denoted by $\varphi(\Gamma g)$. Then, $\varphi : \Gamma\backslash SE(2) \to \mathbb{C}$ given by $\Gamma g \mapsto \varphi(\Gamma g)$ is a well-defined complex valued bounded function. Since each ψ_ℓ is continuous, we deduce that φ is continuous as well. Thus, we conclude that $\varphi \in L^2(\Gamma\backslash SE(2),\mu)$.

Suppose $\ell' \in \mathbb{I}$ is arbitrary. We then have

$$
\begin{aligned}
\langle \varphi, \psi_{\ell'} \rangle &= \int_{\Gamma\backslash SE(2)} \varphi(\Gamma g)\overline{\psi_{\ell'}(\Gamma g)} d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \left(\sum_{\ell\in\mathbb{I}} \langle \psi, \psi_\ell \rangle \psi_\ell(\Gamma g) \right) \overline{\psi_{\ell'}(\Gamma g)} d\mu(\Gamma g) \\
&= \sum_{\ell\in\mathbb{I}} \langle \psi, \psi_\ell \rangle \left(\int_{\Gamma\backslash SE(2)} \psi_\ell(\Gamma g)\overline{\psi_{\ell'}(\Gamma g)} d\mu(\Gamma g) \right) \\
&= \sum_{\ell\in\mathbb{I}} \langle \psi, \psi_\ell \rangle \langle \psi_\ell, \psi_{\ell'} \rangle = \sum_{\ell\in\mathbb{I}} \langle \psi, \psi_\ell \rangle \delta_{\ell,\ell'} = \langle \psi, \psi_{\ell'} \rangle,
\end{aligned}
$$

which implies that $\varphi = \psi$ in $L^2(\Gamma\backslash SE(2), \mu)$. This implies that Equation (22) holds for μ-a.e. $\Gamma g \in \Gamma\backslash SE(2)$.
(3) is straightforward from (2). □

Theorem 3.4. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $K \subset SE(2)$ is compact and $f \in \mathcal{C}_c(SE(2))$ is supported in K such that $\widehat{f} \in \mathcal{H}^1(0, \infty)$ and $\widetilde{f} \in \mathcal{A}(\mathcal{E})$. Then, for every $\Gamma h \in \Gamma\backslash SE(2)$, we have*

$$
\widetilde{f}(\Gamma h) = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_K^\ell(p) \right] p dp \right) \psi_\ell(\Gamma h), \tag{23}
$$

where

$$
Q_K^\ell(p) := \int_K \overline{\psi_\ell(\Gamma g)} U_p(g) dg,
$$

for $p > 0$ and $\ell \in \mathbb{I}$.

Proof. Invoking Theorem 3.3(2), we get

$$
\widetilde{f}(\Gamma h) = \sum_{\ell\in\mathbb{I}} \langle \widetilde{f}, \psi_\ell \rangle \psi_\ell(\Gamma h), \tag{24}
$$

for μ-a.e. $\Gamma h \in \Gamma\backslash SE(2)$. Also, using Theorem 3.1, we have

$$
\langle \widetilde{f}, \psi_\ell \rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f}(p) Q_K^\ell(p) \right] p dp, \tag{25}
$$

for all $\ell \in \mathbb{I}$. Applying (25) in (24) we get

$$\begin{aligned}\widetilde{f}(\Gamma h) &= \sum_{\ell\in\mathbb{I}}\langle\widetilde{f},\psi_\ell\rangle\psi_\ell(\Gamma h)\\ &= \sum_{\ell\in\mathbb{I}}\left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^\ell(p)\right]pdp\right)\psi_\ell(\Gamma h),\end{aligned}$$

for μ-a.e. $\Gamma h \in \Gamma\backslash SE(2)$. Since f is continuous, $\widetilde{f}$ is continuous and using Theorem 3.3(3), we conclude that the function $\widetilde{f}$, satisfies Equation (26), for all $\Gamma h \in \Gamma\backslash SE(2)$. □

Corollary 3.2. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Assume $f \in \mathcal{C}_c(SE(2))$ is supported in Ω with $\widehat{f} \in \mathcal{H}^1(0,\infty)$ and $\widetilde{f} \in \mathcal{A}(\mathcal{E})$. Then, for every $h \in \Omega$, we have*

$$f(h) = \sum_{\ell\in\mathbb{I}}\left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_\Omega^\ell(p)\right]pdp\right)\psi_\ell(\Gamma h), \tag{26}$$

where

$$Q_\Omega^\ell(p) := \int_\Omega \overline{\psi_\ell(\Gamma g)}U_p(g)dg,$$

for $p > 0$ and $\ell \in \mathbb{I}$.

4. Abstract noncommutative Fourier series of convolutions on $\Gamma\backslash SE(2)$

Throughout this section we still assume that Γ is a discrete co-compact subgroup of $SE(2)$ and μ is the finite $SE(2)$-invariant measure on the right coset space $\Gamma\backslash SE(2)$ which is normalized with respect to Weil's formula (10).

We then introduce the notion of convolution integral of functions on $SE(2)$ by functions on $\Gamma\backslash SE(2)$. Also, we shall study different aspects of noncommutative Fourier series for approximating the convolution functions on the right coset space $\Gamma\backslash SE(2)$, using the non-Abelian Fourier integral operator on $SE(2)$. As applications for noncommutative Fourier series of convolution functions, we discuss noncommutative Plancherel type formulas for functions on the right coset space $\Gamma\backslash SE(2)$.

Convolution of functions The structure of convolution function algebras on left coset spaces of compact subgroups in locally compact groups introduced in [20] and studied in details using an operator theoretic approach in [15, 19]. This theory can be canonically reformulated for right coset spaces of compact subgroups in locally compact groups and hence be employed for convolution integrals on right coset space of compact subgroups in $SE(2)$ which is not the case for $\Gamma\backslash SE(2)$ if Γ is not a finite subgroup. We here extend convolution structure of functions for the case of the right coset space of $\Gamma\backslash SE(2)$.

Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2), \mu)$. Then, define the convolution of f with ψ as the function $\psi \oslash f : \Gamma\backslash SE(2) \to \mathbb{C}$ via

$$(\psi \oslash f)(\Gamma g) := \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ g) dh, \tag{27}$$

for $g \in SE(2)$.

For each $\gamma \in \Gamma$ and $g \in SE(2)$, we can write

$$\begin{aligned}\int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ \gamma \circ g) dh &= \int_{SE(2)} \psi(\Gamma\gamma \circ h) f((\gamma \circ h)^{-1} \circ \gamma \circ g) d(\gamma \circ h) \\ &= \int_{SE(2)} \psi(\Gamma h) f((\gamma \circ h)^{-1} \circ \gamma \circ g) dh \\ &= \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ \gamma^{-1} \circ \gamma \circ g) dh \\ &= \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ g) dh,\end{aligned}$$

hence we deduce that

$$\Gamma g \mapsto \int_{SE(2)} \psi(\Gamma h) f(h^{-1} \circ g) dh,$$

is well-defined as a function on the right coset space $\Gamma\backslash SE(2)$.

Next result shows that $L^1(\Gamma\backslash SE(2), \mu)$ equipped with the module action of $L^1(SE(2))$ given by (27) is a Banach module.

Theorem 4.1. *Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2), \mu)$. Then, $\psi \oslash f \in L^1(\Gamma\backslash SE(2), \mu)$ with*

$$\|\psi \oslash f\|_{L^1(\Gamma\backslash SE(2),\mu)} \leq \|f\|_{L^1(SE(2))} \|\psi\|_{L^1(\Gamma\backslash SE(2),\mu)}.$$

Proof. Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2),\mu)$. We then have

$$\begin{aligned}
&\|\psi \oslash f\|_{L^1(\Gamma\backslash SE(2),\mu)} \\
&= \int_{\Gamma\backslash SE(2)} \left| \int_{SE(2)} \psi(\Gamma h) f(h^{-1}\circ g)dh \right| d\mu(\Gamma g) \\
&\leq \int_{\Gamma\backslash SE(2)} \int_{SE(2)} \left|\psi(\Gamma h) f(h^{-1}\circ g)\right| dh d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \int_{SE(2)} |\psi(\Gamma h)||f(h^{-1}\circ g)| dh d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \int_{SE(2)} |\psi(\Gamma g\circ h)||f((g\circ h)^{-1}\circ g)| d(g\circ h) d\mu(\Gamma g) \\
&= \int_{\Gamma\backslash SE(2)} \int_{SE(2)} |\psi(\Gamma g\circ h)||f(h^{-1})| dh d\mu(\Gamma g) \\
&= \int_{SE(2)} \left(\int_{\Gamma\backslash SE(2)} |\psi(\Gamma g\circ h)| d\mu(\Gamma g)\right) |f(h^{-1})| dh \\
&= \int_{SE(2)} \left(\int_{\Gamma\backslash SE(2)} |\psi(\Gamma g)| d\mu(\Gamma g\circ h^{-1})\right) |f(h^{-1})| dh \\
&= \int_{SE(2)} \left(\int_{\Gamma\backslash SE(2)} |\psi(\Gamma g)| d\mu(\Gamma g)\right) |f(h^{-1})| dh \\
&= \|\psi\|_{L^1(\Gamma\backslash SE(2),\mu)} \left(\int_{SE(2)} |f(h^{-1})| dh \right) \\
&= \|f\|_{L^1(SE(2))} \|\psi\|_{L^1(\Gamma\backslash SE(2),\mu)}. \qquad \square
\end{aligned}$$

Proposition 4.1. *Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2),\mu)$. Then*

$$(\psi \oslash f)(\Gamma g) = \int_{\Gamma\backslash SE(2)} \psi(\Gamma h) \left(\sum_{\gamma\in\Gamma} f(h^{-1}\circ\gamma^{-1}\circ g)\right) d\mu(\Gamma h). \tag{28}$$

Proof. Let $f \in L^1(SE(2))$ and $\psi \in L^1(\Gamma\backslash SE(2),\mu)$. Let $g \in SE(2)$. Using Weil's formula, we obtain

$$\begin{aligned}
(\psi \oslash f)(\Gamma g) &= \int_{SE(2)} \psi(\Gamma h) f(h^{-1}g) dh \\
&= \int_{\Gamma\backslash SE(2)} \left(\sum_{\gamma\in\Gamma} \psi(\Gamma\gamma\circ h) f((\gamma\circ h)^{-1}\circ g)\right) d\mu(\Gamma h)
\end{aligned}$$

$$= \int_{\Gamma\backslash SE(2)} \left(\sum_{\gamma\in\Gamma} \psi(\Gamma h) f((\gamma \circ h)^{-1} \circ g) \right) d\mu(\Gamma h)$$

$$= \int_{\Gamma\backslash SE(2)} \psi(\Gamma h) \left(\sum_{\gamma\in\Gamma} f(h^{-1} \circ \gamma^{-1} \circ g) \right) d\mu(\Gamma h). \qquad \square$$

Theorem 4.2. *Let $f_1, f_2 \in L^1(SE(2))$ and $g \in SE(2)$. Then*

$$(29) \qquad (\widetilde{f_1 \star f_2})(\Gamma g) = (\widetilde{f_1} \oslash f_2)(\Gamma g).$$

Proof. Let $f_k \in L^1(SE(2))$ with $k \in \{1, 2\}$ and $g \in SE(2)$. We then have

$$\begin{aligned}
(\widetilde{f_1 \star f_2})(\Gamma g) &= \sum_{\gamma\in\Gamma} (f_1 \star f_2)(\gamma \circ g) \\
&= \sum_{\gamma\in\Gamma} \left(\int_{SE(2)} f_1(h) f_2(h^{-1} \circ \gamma \circ g) dh \right) \\
&= \sum_{\gamma\in\Gamma} \left(\int_{SE(2)} f_1(\gamma \circ h) f_2(h^{-1} \circ g) dh \right) \\
&= \int_{SE(2)} \left(\sum_{\gamma\in\Gamma} f_1(\gamma \circ h) \right) f_2(h^{-1} \circ g) dh \\
&= \int_{SE(2)} \widetilde{f_1}(\Gamma h) f_2(h^{-1} \circ g) dh = (\widetilde{f_1} \oslash f_2)(\Gamma g). \qquad \square
\end{aligned}$$

We then present some constructive expansions for the coefficients of convolution of $L^1(SE(2))$ on $L^1(\Gamma\backslash SE(2), \mu)$ in the L^2-sense.

Theorem 4.3. *Let Γ be a discrete co-compact subgroup of $SE(2)$. Suppose $K \subset SE(2)$ is compact and $\varphi \in \mathcal{C}(\Gamma\backslash SE(2))$. Let $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1, 2\}$ be supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset K$. Then*

$$(30) \qquad \langle \widetilde{f_1} \oslash f_2, \varphi \rangle = \int_0^\infty \mathrm{tr} \left[\widehat{f_2}(p) \widehat{f_1}(p) Q_K^{\overline{\varphi}}(p) \right] p dp,$$

where for $p > 0$ and $\ell \in \mathbb{I}$ we have

$$Q_K^{\overline{\varphi}}(p) := \int_K \overline{\varphi(\Gamma g)} U_p(g) dg.$$

Proof. Assume that $f := f_1 \star f_2$. Then, f is supported in K. Since each $f_k \in L^1 \cap L^2(SE(2))$, we conclude that every $\widehat{f_k}(p)$ has a finite Hilbert–Schmidt norm, for a.e. $p > 0$. Hence,

$$\|\widehat{f}(p)\|_1 = \|\widehat{f_1 \star f_2}(p)\|_1 = \|\widehat{f}_2(p)\widehat{f}_1(p)\|_1 \le \|\widehat{f}_2(p)\|_2\|\widehat{f}_1(p)\|_2 < \infty,$$

which guarantees that for a.e. $p > 0$ the bounded linear operator $\widehat{f}(p)$ is of trace-class. In addition, we obtain

$$\begin{aligned}\int_0^\infty \|\widehat{f}(p)\|_1 pdp &\le \int_0^\infty \|\widehat{f}_2(p)\|_2\|\widehat{f}_1(p)\|_2 pdp \\ &\le \left(\int_0^\infty \|\widehat{f}_2(p)\|_2^2 pdp\right)^{1/2}\left(\int_0^\infty \|\widehat{f}_1(p)\|_2^2 pdp\right)^{1/2} \\ &= \|\widehat{f}_2\|_{\mathcal{H}^2(0,\infty)}\|\widehat{f}_1\|_{\mathcal{H}^2(0,\infty)},\end{aligned}$$

implying that $\widehat{f} \in \mathcal{H}^1(0,\infty)$. Therefore, applying Theorem 3.1 for f, we get

$$\begin{aligned}\langle \widetilde{f_1} \oslash f_2, \varphi\rangle &= \langle f, \varphi\rangle \\ &= \int_0^\infty \mathrm{tr}\left[\widehat{f}(p)Q_K^{\overline{\varphi}}(p)\right] pdp \\ &= \int_0^\infty \mathrm{tr}\left[\widehat{f_1 \star f_2}(p)Q_K^{\overline{\varphi}}(p)\right] pdp = \int_0^\infty \mathrm{tr}\left[\widehat{f}_2(p)\widehat{f}_1(p)Q_K^{\overline{\varphi}}(p)\right] pdp.\end{aligned}$$

□

Next we conclude some reconstruction expansions including Fourier coefficients for the convolution of $L^1(SE(2))$ on $L^1(\Gamma\backslash SE(2),\mu)$ in the L^2-sense.

Proposition 4.2. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $K \subset SE(2)$ is compact and $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1,2\}$ are supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset K$. Then*

$$\widetilde{f_1} \oslash f_2 = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}_2(p)\widehat{f}_1(p)Q_K^{\ell}(p)\right] pdp\right)\psi_\ell, \tag{31}$$

where for $p > 0$, and $\ell \in \mathbb{I}$ we have

$$Q_K^\ell(p) := \int_K \overline{\psi_\ell(\Gamma g)}U_p(g)dg.$$

Proof. Suppose that $f := f_1 \star f_2$. Then, f is supported in K. So, $\psi := \widetilde{f}$ is in $L^2(\Gamma\backslash SE(2), \mu)$ as well. Since $\{\psi_\ell : \ell \in \mathbb{I}\}$ is an orthonormal basis, we get

$$\psi = \sum_{\ell\in\mathbb{I}} \langle \psi, \psi_\ell\rangle \psi_\ell. \tag{32}$$

Using Theorem 4.2, we have $\psi = \widetilde{f_1} \oslash f_2$. Therefore, by applying Equation (30) in (32), we get

$$\begin{aligned}\widetilde{f_1} \oslash f_2 &= \sum_{\ell\in\mathbb{I}} \langle \widetilde{f_1} \oslash f_2, \psi_\ell\rangle \psi_\ell \\ &= \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p)\widehat{f_1}(p) Q_K^\ell(p)\right] pdp \right) \psi_\ell.\end{aligned}$$ □

We then conclude the following results for functions supported in a fundamental domain of Γ in $SE(2)$.

Proposition 4.3. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\varphi \in L^2(\Gamma\backslash SE(2), \mu)$. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Let $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1, 2\}$ be supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset \Omega$. Then*

$$\langle \widetilde{f_1} \oslash f_2, \varphi\rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p)\widehat{f_1}(p) Q_\Omega^{\overline{\varphi}}(p)\right] pdp, \tag{33}$$

where for $p > 0$ we have

$$Q_\Omega^{\overline{\varphi}}(p) := \int_\Omega \overline{\varphi(\Gamma g)} U_p(g) dg.$$

Proof. Suppose that Ω is a fundamental domain of Γ in $SE(2)$. Since $\varphi \in L^2(\Gamma\backslash SE(2), \mu)$ and hence $\varphi \in L^1(\Gamma\backslash SE(2), \mu)$, $Q_\Omega^{\overline{\varphi}}(p)$ defines a bounded linear operator on $L^2(\mathbb{S}^1)$, for every $p > 0$. Assume that $f := f_1 \star f_2$. Then, f is supported in Ω. Hence, using a similar method as we used in Theorem 4.3, we get

$$\langle \widetilde{f_1} \oslash f_2, \varphi\rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p)\widehat{f_1}(p) Q_\Omega^{\overline{\varphi}}(p)\right] pdp.$$ □

Corollary 4.1. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$. Suppose Ω is a fundamental domain of Γ in*

$SE(2)$. Let $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1,2\}$ be supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset \Omega$. Then

$$\widetilde{f_1} \oslash f_2 = \sum_{\ell \in \mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p) \widehat{f_1}(p) Q^\ell_\Omega(p) \right] pdp \right) \psi_\ell, \tag{34}$$

where for $p > 0$ and $\ell \in \mathbb{I}$, we have

$$Q^\ell_\Omega(p) := \int_\Omega \overline{\psi_\ell(\Gamma g)} U_p(g) dg.$$

Absolutely convergent Fourier series of convolutions on $\Gamma\backslash SE(2)$ We then deduce the following constructive expansions for the convolution of $L^1(SE(2))$ on $L^1(\Gamma\backslash SE(2), \mu)$ in the almost everywhere and pointwise senses.

Theorem 4.4. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $K \subset SE(2)$ is compact. Let $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1,2\}$ be supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset K$ and $\widetilde{f_1} \oslash f_2 \in \mathcal{A}(\mathcal{E})$. Then, for every $\Gamma g \in \Gamma\backslash SE(2)$, we have*

$$(\widetilde{f_1} \oslash f_2)(\Gamma g) = \sum_{\ell \in \mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p) \widehat{f_1}(p) Q^\ell_K(p) \right] pdp \right) \psi_\ell(\Gamma g), \tag{35}$$

where for $p > 0$ and $\ell \in \mathbb{I}$, we have

$$Q^\ell_K(p) := \int_K \overline{\psi_\ell(\Gamma g)} U_p(g) dg.$$

Proof. Since $\widetilde{f_1} \oslash f_2 \in \mathcal{A}(\mathcal{E})$, using Theorem 3.3(3), we have

$$(\widetilde{f_1} \oslash f_2)(\Gamma g) = \sum_{\ell \in \mathbb{I}} \langle \widetilde{f_1} \oslash f_2, \psi_\ell \rangle \psi_\ell(\Gamma g),$$

for every $\Gamma g \in \Gamma\backslash SE(2)$. Invoking Equation (30), we have

$$\langle \widetilde{f_1} \oslash f_2, \psi_\ell \rangle = \int_0^\infty \mathrm{tr}\left[\widehat{f_2}(p) \widehat{f_1}(p) Q^\ell_\Omega(p) \right] pdp,$$

which completes the proof. □

Corollary 4.2. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthonormal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose Ω is a fundamental domain of Γ in $SE(2)$. Let $f_k \in \mathcal{C}_c(SE(2))$ with $k \in \{1, 2\}$ be supported in $B_k \subset SE(2)$ such that $B_1 \circ B_2 \subset K$ and $\widetilde{f_1 \oslash f_2} \in \mathcal{A}(\mathcal{E})$. Then, for every $h \in \Omega$, we have*

$$f_1 \star f_2(h) = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}_2(p)\widehat{f}_1(p)Q_\Omega^\ell(p)\right] pdp \right) \psi_\ell(\Gamma h), \tag{36}$$

where for $p > 0$ and $\ell \in \mathbb{I}$, we have

$$Q_\Omega^\ell(p) := \int_\Omega \overline{\psi_\ell(\Gamma g)} U_p(g) dg.$$

Plancherel formula We then finish the paper by presenting Plancherel type formulas according to the noncommutative Fourier series on $\Gamma\backslash SE(2)$.

The following result presents the canonical connection of L^2-norms on the right coset space $\Gamma\backslash SE(2)$ with convolution of functions.

Proposition 4.4. *Let $f \in L^1(SE(2))$ with $\widetilde{|f|} \in L^2(\Gamma\backslash SE(2), \mu)$ be given. Then,*

$$\|\widetilde{f}\|^2_{L^2(\Gamma\backslash SE(2),\mu)} = (\widetilde{f} \oslash f^*)(\Gamma).$$

Proof. Since $\widetilde{f} \in L^2(\Gamma\backslash SE(2), \mu)$, using Weils formula, we obtain

$$\begin{aligned}
\|\widetilde{f}\|^2_{L^2(\Gamma\backslash SE(2),\mu)} &= \int_{\Gamma\backslash SE(2)} |\widetilde{f}(\Gamma h)|^2 d\mu(\Gamma h) = \int_{\Gamma\backslash SE(2)} \widetilde{f}(\Gamma h)\overline{\widetilde{f}(\Gamma h)} d\mu(\Gamma h) \\
&= \int_{\Gamma\backslash SE(2)} \sum_{\gamma\in\Gamma} \widetilde{f}(\Gamma\gamma \circ h)\overline{f(\gamma \circ h)} d\mu(\Gamma h) \\
&= \int_{SE(2)} \widetilde{f}(\Gamma h)\overline{f(h)} dh \\
&= \int_{SE(2)} \widetilde{f}(\Gamma h) f^*(h^{-1}) dh = (\widetilde{f} \oslash f^*)(\Gamma).
\end{aligned}$$

□

We then have the following general form of Plancherel formula associated to the noncommutative Fourier series on $\Gamma\backslash SE(2)$.

Theorem 4.5. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthogonal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2), \mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose $K \subset SE(2)$*

is compact and $f \in \mathcal{C}_c(SE(2))$ is supported in $B \subset SE(2)$ with $B \circ B^{-1} \subset K$ and $\widetilde{f} \oslash f^* \in \mathcal{A}(\mathcal{E})$. Then

$$\|\widetilde{f}\|^2_{L^2(\Gamma\backslash SE(2),\mu)} = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[|\widehat{f}(p)|^2 Q^\ell_K(p)\right] pdp\right) \psi_\ell(\Gamma), \tag{37}$$

where for $p > 0$ and $\ell \in \mathbb{I}$, we have

$$Q^\ell_K(p) := \int_K \overline{\psi_\ell(\Gamma g)} U_p(g) dg.$$

Proof. Using Equation (35) and Proposition 4.4, we get

$$\begin{aligned}
\|\widetilde{f}\|^2_{L^2(\Gamma\backslash SE(2),\mu)} &= (\widetilde{f} \oslash f^*)(\Gamma) \\
&= \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f^*}(p)\widehat{f}(p) Q^\ell_K(p)\right] pdp\right) \psi_\ell(\Gamma) \\
&= \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[\widehat{f}(p)^*\widehat{f}(p) Q^\ell_K(p)\right] pdp\right) \psi_\ell(\Gamma) \\
&= \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[|\widehat{f}(p)|^2 Q^\ell_K(p)\right] pdp\right) \psi_\ell(\Gamma). \qquad \square
\end{aligned}$$

Corollary 4.3. *Let Γ be a discrete co-compact subgroup of $SE(2)$ and $\mathcal{E}(\Gamma) := \{\psi_\ell : \Gamma\backslash SE(2) \to \mathbb{C} \mid \ell \in \mathbb{I}\}$ be a (discrete) orthogonal basis for the Hilbert function space $L^2(\Gamma\backslash SE(2),\mu)$ with $\mathcal{E}(\Gamma) \subset \mathcal{C}(\Gamma\backslash SE(2))$. Suppose Ω is a fundamental domain for Γ in $SE(2)$. Assume $f \in \mathcal{C}_c(SE(2))$ is supported in $B \subset SE(2)$ with $B \circ B^{-1} \subset \Omega$ and $\widetilde{f} \oslash f^* \in \mathcal{A}(\mathcal{E})$. Then*

$$\|f\|^2_{L^2(SE(2))} = \sum_{\ell\in\mathbb{I}} \left(\int_0^\infty \mathrm{tr}\left[|\widehat{f}(p)|^2 Q^\ell_\Omega(p)\right] pdp\right) \psi_\ell(\Gamma), \tag{38}$$

where for $p > 0$ and $\ell \in \mathbb{I}$, we have

$$Q^\ell_\Omega(p) := \int_\Omega \overline{\psi_\ell(\Gamma g)} U_p(g) dg.$$

Concluding remarks. This paper has developed an abstract theory of Fourier series on the compact space $\Gamma\backslash SE(2)$, where Γ is discrete and co-compact. In recent years, Prof. Bernard Shiffman has worked on geometric aspects of fundamental domains of related spaces. Moreover, substantial work exists on harmonic analysis on spaces $\Gamma\backslash G$ when G is compact or semi-simple, which is not the case for $SE(2)$. This paper therefore provides an example for how to define Fourier series with useful convolution-theorem properties in cases not handled by previous theory.

In some applications in mathematical crystallography and robotics, convolutions of continuous functions on the right coset space $\Gamma\backslash SE(2)$ appear. In contrast to classical unimodular Fourier/Plancherel theory of the non-Abelian group $SE(2)$, where the spectrum of functions with compact support is neither compactly supported nor discrete, in this paper an alternative constructive method is introduced in which noncommutative Fourier reconstruction formulas with discrete spectrum are constructed.

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