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Bergman bundles and applications to the geometry of compact complex manifolds

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Dedicated to Professor Bernard Shiffman on the occasion of his retirement

Abstract: We introduce the concept of Bergman bundle attached to a hermitian manifold X, assuming the manifold X to be compact – although the results are local for a large part. The Bergman bundle is some sort of infinite dimensional very ample Hilbert bundle whose fibers are isomorphic to the standard L^2 Hardy space on the complex unit ball; however the bundle is locally trivial only in the real analytic category, and its complex structure is strongly twisted. We compute the Chern curvature of the Bergman bundle, and show that it is strictly positive. As a potential application, we investigate a long standing and still unsolved conjecture of Siu on the invariance of plurigenera in the general situation of polarized families of compact Kähler manifolds.

Keywords: Bergman metric, Hardy space, Stein manifold, Grauert tubular neighborhood, Hermitian metric, Hilbert bundle, very ample vector bundle, compact Kähler manifold, invariance of plurigenera.

0. Introduction

Projective varieties are characterized, almost by definition, by the existence of an ample line bundle. By the Kodaira embedding theorem [10], they are also characterized among compact complex manifolds by the existence of a positively curved holomorphic line bundle, or equivalently, of a Hodge metric, namely a Kähler metric with rational cohomology class. On the other hand, general compact Kähler manifolds, and especially general complex tori, fail

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to have a positive line bundle. Still, compact Kähler manifolds possess topological complex line bundles of positive curvature, that are in some sense arbitrary close to being holomorphic, see e.g. [11] and [18]. It may nevertheless come as a surprise that every compact complex manifold carries some sort of very ample holomorphic vector bundle, at least if one accepts certain Hilbert bundles of infinite dimension. Motivated by geometric quantization, Lempert and Szőke [12] have introduced and discussed a more general concept of "field of Hilbert spaces" which is similar in spirit.

0.1 Theorem. *Every compact complex manifold X carries a locally trivial real analytic Hilbert bundle $B_\varepsilon \to X$ of infinite dimension, defined for $0 < \varepsilon \le \varepsilon_0$, equipped with an integrable $(0,1)$-connection $\overline{\partial} = \nabla^{0,1}$ (in a generalized sense), that is a closed densely defined operator in the space of L^2 sections, in such a way that the sheaf $\mathcal{B}_\varepsilon = \mathcal{O}_{L^2}(B_\varepsilon)$ of $\overline{\partial}$-closed locally L^2 sections is "very ample" in the following sense.*

(a) *$H^q(X, \mathcal{B}_\varepsilon \otimes_{\mathcal{O}} \mathcal{F}) = 0$ for every (finite rank) coherent sheaf $\mathcal{F}$ on X and every $q \ge 1$.*
(b) *Global sections of the Hilbert space $\mathbb{H} = H^0(X, \mathcal{B}_\varepsilon)$ provide an embedding of X into a certain Grassmannian of closed subspaces of infinite codimension in $\mathbb{H}$.*
(c) *The bundle B_ε carries a natural Hilbert metric h such that the curvature tensor $\mathrm{i}\Theta_{B_\varepsilon,h}$ is Nakano positive (and even Nakano positive unbounded!).*

Parts (a) and (b) are proved by considering the (pre)sheaf structure of $\mathcal{B}_\varepsilon$, and observing that there is a related L^2 Dolbeault complex on which Hörmander's L^2 estimates [8] can be applied. The case of Stein manifolds is sufficient, and the corresponding Hilbert bundle B_ε is not involved in the arguments. Technically, the proof is given in Proposition 2.5 and Remark 2.6. Part (c) deals with the geometry of B_ε, and is treated in section 3.

We start by explaining a little bit more the relationship between the "Hilbert bundles" involved here, and the more familiar concept of locally trivial holomorphic Hilbert bundle: such a bundle $E \to X$ is required to be trivial on sufficiently small open sets $V \subset X$, and such that $E_{|V} \simeq V \times \mathcal{H}$ where $\mathcal{H}$ is a complex Hilbert space. The gluing transition automorphism with another local trivialization $E_{|V'} \simeq V' \times \mathcal{H}$ should then be of the form $(z, \xi) \mapsto (z, g(z) \cdot \xi)$ where g is a holomorphic map from $V \cap V'$ to the open set $\mathrm{GL}(\mathcal{H})$ of invertible continuous operators in $\mathrm{End}(\mathcal{H})$. Smooth and real analytic locally trivial Hilbert bundles can be defined in a similar manner by requiring g to be in C^∞, resp. in C^ω. A smooth hermitian structure on

E is a smooth family of hermitian metrics $h(z)$ on the fibers, given in the trivializations by smooth maps $h_V \in C^\infty(V, \mathrm{Herm}_+(\mathcal{H}))$, where $\mathrm{Herm}_+(\mathcal{H})$ is the set of positive definite (coercive) hermitian forms on $\mathcal{H}$. The usual formalism of Chern connections still applies: one gets a unique connection $\nabla_h = \nabla_h^{1,0} + \nabla_h^{0,1}$ acting on $C^\infty(X, E)$ in such a way that h is ∇_h parallel and $\nabla_h^{0,1} = \overline{\partial}$; moreover, the kernel of $\nabla_h^{0,1}$ coincides with the sheaf of holomorphic sections $\mathcal{O}_X(E)$. This connection is given by exactly the same formulas as in the finite dimensional case, namely $\nabla_h^{1,0} \simeq h_V^{-1} \circ \partial \circ h_V$ over V, with a curvature tensor $\nabla_h^2 = \Theta_{E,h}$ given by $\Theta_{E,h} \simeq \overline{\partial}(h_V^{-1}\partial h_V)$ (if one views $h_V(z)$ as an endomorphism of $\mathcal{H}$); locally, $\Theta_{E,h}$ can thus be seen as a smooth $(1,1)$-form with values in the space of continuous endomorphisms $\mathrm{End}(\mathcal{H})$. In general, if E is a smooth Hilbert bundle (defined as above, but with gluing automorphisms $g \in C^\infty(V, \mathrm{GL}(\mathcal{H}))$), a smooth $(0,1)$-connection $\nabla_A^{0,1}$ is an order 1 linear differential operator that is locally of the form $\overline{\partial} + A_V$ where $A_V \in C^\infty(V, \Lambda^{0,1}T_X^* \otimes \mathrm{End}(\mathcal{H}))$. It is said to be integrable if $(\nabla_A^{0,1})^2 = 0$, i.e. $\overline{\partial}A_V + A_V \wedge A_V = 0$ on each trivializing chart V. The following equivalence of categories is well known, and follows e.g. from Malgrange [13, chap. X, Theorem 1], although the statement is expressed there in more concrete terms.

0.2 Theorem (Malgrange [13]). *The category of holomorphic vector bundles on X is equivalent to the category of smooth bundles equipped with smooth integrable $(0,1)$-connections $\nabla_A^{0,1}$, the holomorphic structure being obtained by taking the kernel sheaf of $\nabla_A^{0,1}$.*

Notice that the usual finite dimensional proofs apply essentially unchanged to the case of locally trivial Hilbert bundles. For instance, one can adapt Malgrange's inductive proof [13] based on the Cauchy formula in one variable (for holomorphic functions with values in a Banach space, depending smoothly on some other parameters), or use a Nash–Moser process along with the Bochner–Martinelli kernel (see e.g. [23]), or an infinite dimensional version of Hörmander's L^2 estimates (the latter do not depend on the rank of bundles and are thus valid for Hilbert bundles equipped with integrable smooth $(0,1)$-connections; the solution of minimal L^2 norm can be used to find local $\nabla_A^{0,1}$-closed sections generating fibers of the bundle). The result is also valid for the category of real analytic Hilbert bundles, assuming E and $\nabla_A^{0,1}$ to be real analytic; the resulting holomorphic structure on E is then compatible with the originally given real analytic structure. Now, when X is compact hermitian, the standard L^2 techniques of PDE theory lead to considering the space $L^2(X, E)$ of L^2 sections. A smooth $(0,1)$-connection then gives rise to a closed densely defined operator

$$\nabla_A^{0,1} : L^2(X, E) \longrightarrow L^2(X, \Lambda^{0,1}T_X^* \otimes E) \tag{0.3}$$

that is never continuous. In our situation, the bundles come in a natural way as a family of smooth (and even real analytic) Hilbert bundles E_ε, $0 < \varepsilon \le \varepsilon_0$, associated with a family $\mathcal{H}_\varepsilon$ of Hilbert spaces which form a "scale", in the sense that there are continuous injections with dense image $\mathcal{H}_{\varepsilon'} \hookrightarrow \mathcal{H}_\varepsilon$, $0 < \varepsilon < \varepsilon' \le \varepsilon_0$. The transition automorphisms defining the E_ε's are supposed to come from invertible automorphisms of $\mathcal{H}_{>0} = \bigcup_{\varepsilon>0} \mathcal{H}_\varepsilon$ preserving each $\mathcal{H}_\varepsilon$ (here $\mathcal{H}_{>0}$ is just an inductive limit of Hilbert spaces). Then it makes sense to consider generalized $(0,1)$-connections that are locally of the form

$$\nabla_A^{0,1} \simeq \overline{\partial} + A_V, \qquad A_V \in C^\infty(\Lambda^{0,1}T_X^* \otimes \mathrm{End}(\mathcal{H}_{>0})), \tag{0.4}$$

where we actually have $A_{V|\mathcal{H}_{\varepsilon'}} \in C^\infty(\Lambda^{0,1}T_X^* \otimes \mathrm{Hom}(\mathcal{H}_{\varepsilon'}, \mathcal{H}_\varepsilon))$ for all $0 < \varepsilon < \varepsilon' \le \varepsilon_0$. By our assumptions, such connections still induce densely defined operators on each of the spaces $L^2(X, E_\varepsilon)$, and we declare them to be integrable when $(\nabla_A^{0,1})^2 = 0$. The usual algebraic formalism for extending the connection to higher degree forms and calculating the curvature tensor still applies in this setting.

However, it may happen, and this will be the case for the Chern connection matrices of our bundles B_ε of Theorem 0.1, that the A_V do not induce continuous endomorphisms of $\mathcal{H}_\varepsilon$ (for any value of $\varepsilon > 0$), although the kernel of $\nabla^{0,1}$ in $L^2(X, B_\varepsilon)$ looks very much like a space of holomorphic sections. In this context, the associated curvature tensor $\Theta_{E_\varepsilon,h}$ need not either take values in the continous endomorphisms. Then Malgrange's theorem implies that such bundles do not correspond to locally trivial holomorphic bundles as defined above, even under the integrability assumption. At the end of Section 3 we will briefly discuss in which sense B_ε can still be considered to be some sort of infinite dimensional complex space, in a way that the projection map $B_\varepsilon \to X$ becomes holomorphic.

The construction of B_ε is made by embedding X diagonally in $X \times \overline{X}$ and taking a Stein tubular neighborhood U_ε of the diagonal, according to a well known technique of Grauert [6]. When U_ε is chosen to be a geodesic neighborhood with respect to some real analytic hermitian metric, one can arrange that the first projection $p : U_\varepsilon \to X$ is a real analytic bundle whose fibers are biholomorphic to hermitian balls. One then takes B_ε to be a "Bergman bundle", consisting of holomorphic n-forms $f(z,w)\,dw_1 \wedge \ldots \wedge dw_n$ that are L^2 on the fibers $p^{-1}(z) \simeq B(0,\varepsilon)$. The fact that U_ε is Stein and real analytically locally trivial over X then implies Theorem 0.1, using the corresponding Bergman type Dolbeault complex.

In [1], given a holomorphic fibration $\pi : X \to Y$ and a positive hermitian holomorphic line bundle $L \to X$, Berndtsson has introduced a formally similar L^2 bundle $Y \ni t \mapsto A_t^2$, whose fibers consist of sections of the adjoint bundle $K_{X/Y} \otimes L$ on the fibers $X_t = \pi^{-1}(t)$ of π, equipped with the corresponding Bergman metric. In the situation considered by Berndtsson, the major application is the case when π is proper, so that A_t^2 is finite dimensional, and is the holomorphic bundle associated with the direct image sheaf $\pi_*(\mathcal{O}_X(K_{X/Y} \otimes L))$. The main result of [1] is a calculation of the curvature, and a proof that the direct image is a Nakano positive vector bundle. On the other hand, when $\pi : X \to Y$ is non proper, and especially when (X_t) is a smooth family of smoothly bounded Stein domains, the corresponding spaces A_t^2 are infinite dimensional Hilbert spaces. The curvature of the corresponding Hilbert bundle has been obtained by Wang Xu [22] in this general setting. Our curvature calculations can be seen as the very special case where the fibers are smoothly varying hermitian balls and the centers vary antiholomorphically. The calculation can then be made in a very explicit way, by first considering the model case of balls of constant radius in $\mathbb{C}^n$, and then by using an osculation and suitable Taylor expansions, in the case of varying hermitian metrics (a similar osculating technique has been used in [24] for the study of Bargmann–Fock spaces). As a consequence, we get

0.5 Proposition. *The curvature tensor of (B_ε, h) admits an asymptotic expansion*

$$\langle(\Theta_{B_\varepsilon,h}\,\xi)(v, Jv), \xi\rangle_h = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, \xi \otimes v),$$

where, in suitable normal coordinates, the leading term $Q_0(z, \xi \otimes v)$ is exactly equal to the curvature tensor of the Bergman bundle associated with the translation invariant tubular neighborhood

$$U_\varepsilon = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |z - \overline{w}| < \varepsilon\},$$

in the "model case" $X = \mathbb{C}^n$. That term Q_0 is an unbounded quadratic hermitian form.

The potential geometric applications we have in mind are for instance the study of Siu's conjecture on the Kähler invariance of plurigenera (see 4.1 below), where the algebraic proof ([20], [17]) uses an auxiliary ample line bundle A. In the Kähler case at least, one possible idea would be to replace A by the infinite dimensional Bergman bundle B_ε. The proof works to some extent, but some crucial additional estimates seem to be missing to get the conclusion, see §4. Another question where Bergman bundles could potentially be useful is the conjecture on transcendental Morse inequalities for real $(1,1)$-cohomology classes α in the Bott–Chern cohomology group $H^{1,1}_{\mathrm{BC}}(X, \mathbb{C})$. In that situation, multiples $k\alpha$ can be approximated by a sequence of integral classes α_k corresponding to topological line bundles $L_k \to X$ that are closer and closer to being holomorphic, see e.g. [11]. However, on the Stein tubular neighborhood U_ε, the pull-back p^*L_k can be given a structure of a genuine holomorphic line bundle with curvature form very close to $k\,p^*\alpha$. Our hope is that an appropriate Bergman theory of "Hilbert dimension" (say, in the spirit of Atiyah's L^2 index theory) can be used to recover the expected Morse inequalities. There seem to be still considerable difficulties in this direction, and we wish to leave this question for future research.

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1. Exponential map and tubular neighborhoods

Let X be a compact n-dimensional complex manifold and $Y \subset X$ a smooth totally real submanifold, i.e. such that $T_Y \cap JT_Y = \{0\}$ for the complex structure J on X. By a well known result of Grauert [6], such a Y always admits a fundamental system of Stein tubular neighborhoods $U \subset X$ (this would be even true when X is noncompact, but we only need the compact case here). In fact, if (Ω_α) is a finite covering of X such that $Y \cap \Omega_\alpha$ is a smooth complete intersection $\{z \in \Omega_\alpha\,;\; x_{\alpha,j}(z) = 0\}$, $1 \le j \le q$ (where $q = \mathrm{codim}_{\mathbb{R}} Y \ge n$), then one can take $U = U_\varepsilon = \{\varphi(z) < \varepsilon\}$ where

$$\varphi(z) = \sum_\alpha \theta_\alpha(z) \sum_{1\le j\le q} (x_{\alpha,j}(z))^2 \ge 0 \tag{1.1}$$

where (θ_α) is a partition of unity subordinate to (Ω_α). The reason is that φ is strictly plurisubharmonic near Y, as

$$\mathrm{i}\partial\overline{\partial}\varphi_{|Y} = 2i \sum_\alpha \theta_\alpha(z) \sum_{1\le j\le q} \partial x_{\alpha,j} \wedge \overline{\partial} x_{\alpha,j}$$

and $(\partial x_{\alpha,j})_j$ has rank n at every point of Y, by the assumption that Y is totally real.

Now, let $\overline{X}$ be the complex conjugate manifold associated with the integrable almost complex structure $(X, -J)$ (in other words, $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$); we denote by $x \mapsto \overline{x}$ the identity map $\mathrm{Id} : X \to \overline{X}$ to stress that it is conjugate holomorphic. The underlying real analytic manifold $X^{\mathbb{R}}$ can be embedded diagonally in $X \times \overline{X}$ by the diagonal map $\delta : x \mapsto (x, \overline{x})$, and the image $\delta(X^{\mathbb{R}})$

is a totally real submanifold of $X \times \overline{X}$. In fact, if $(z_{\alpha,j})_{1\le j\le n}$ is a holomorphic coordinate system relative to a finite open covering (Ω_α) of X, then the $\overline{z}_{\alpha,j}$ define holomorphic coordinates on $\overline{X}$ relative to $\overline{\Omega}_\alpha$, and the "diagonal" $\delta(X^{\mathbb{R}})$ is the totally real submanifold of pairs (z,w) such that $w_{\alpha,j} = \overline{z}_{\alpha,j}$ for all α, j. In that case, we can take Stein tubular neighborhoods of the form $U_\varepsilon = \{\varphi < \varepsilon\}$ where

$$\varphi(z,w) = \sum_\alpha \theta_\alpha(z)\theta_\alpha(w) \sum_{1\le j\le q} |\overline{w}_{\alpha,j} - z_{\alpha,j}|^2. \tag{1.2}$$

Here, the strict plurisubharmonicity of φ near $\delta(X^{\mathbb{R}})$ is obvious from the fact that

$$|w_{\alpha,j} - \overline{z}_{\alpha,j}|^2 = |z_{\alpha,j}|^2 + |w_{\alpha,j}|^2 - 2\,\mathrm{Re}(z_{\alpha,j}w_{\alpha,j}).$$

For $\varepsilon > 0$ small, the first projection $\mathrm{pr}_1 : U_\varepsilon \to X$ gives a complex fibration whose fibers are C^∞-diffeomorphic to balls, but they need not be biholomorphic to complex balls in general. In order to achieve this property, we proceed in the following way. Pick a real analytic hermitian metric γ on X ; take e.g. the $(1,1)$-part $\gamma = g^{(1,1)} = \frac{1}{2}(g + J^*g)$ of the Riemannian metric obtained as the pull-back $g = \delta^*(\sum_j i df_j \wedge d\overline{f}_j)$, where the $(f_j)_{1\le j\le N}$ provide a holomorphic immersion of the Stein neighborhood U_ε into $\mathbb{C}^N$. Let $\exp : T_X \to X$, $(z,\xi) \mapsto \exp_z(\xi)$ be the exponential map associated with the metric γ, in such a way that $\mathbb{R} \ni t \mapsto \exp_z(t\xi)$ are geodesics $\frac{D}{dt}(\frac{du}{dt}) = 0$ for the the Chern connection D on T_X (see e.g. [4, (2.6)]). Then exp is real analytic, and we have Taylor expansions

$$\exp_z(\xi) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta}(z)\xi^\alpha\overline{\xi}^\beta, \qquad \xi \in T_{X,z}$$

with real analytic coefficients $a_{\alpha\beta}$, where $\exp_z(\xi) = z + \xi + O(|\xi|^2)$ in local coordinates. The real analyticity means that these expansions are convergent on a neighborhood $|\xi|_\gamma < \varepsilon_0$ of the zero section of T_X. We define the fiber-holomorphic part of the exponential map to be

$$\mathrm{exph} : T_X \to X, \qquad (z,\xi) \mapsto \mathrm{exph}_z(\xi) = \sum_{\alpha\in\mathbb{N}^n} a_{\alpha 0}(z)\xi^\alpha. \tag{1.3}$$

It is uniquely defined, is convergent on the same tubular neighborhood $\{|\xi|_\gamma < \varepsilon_0\}$, has the property that $\xi \mapsto \mathrm{exph}_z(\xi)$ is holomorphic for $z \in X$ fixed, and satisfies again $\mathrm{exph}_z(\xi) = z + \xi + O(\xi^2)$ in coordinates. By the implicit function, theorem, the map $(z,\xi) \mapsto (z, \mathrm{exph}_z(\xi))$ is a real analytic diffeomorphism from a neighborhood of the zero section of T_X onto a neighborhood V of the diagonal in $X \times X$. Therefore, we get an inverse real analytic mapping $X \times X \supset V \to T_X$, which we denote by $(z, w) \mapsto (z, \xi)$, $\xi = \mathrm{logh}_z(w)$, such that $w \mapsto \mathrm{logh}_z(w)$ is holomorphic on $V \cap (\{z\} \times X)$, and $\mathrm{logh}_z(w) = w - z + O((w - z)^2)$ in coordinates. The tubular neighborhood

$$U_{\gamma,\varepsilon} = \{(z, w) \in X \times \overline{X}\,;\ |\,\mathrm{logh}_z(\overline{w})|_\gamma < \varepsilon\}$$

is Stein for $\varepsilon > 0$ small; in fact, if $p \in X$ and $(z_1, \ldots, z_n)$ is a holomorphic coordinate system centered at p such that $\gamma_p = i \sum dz_j \wedge d\overline{z}_j$, then $|\,\mathrm{logh}_z(\overline{w})|^2_\gamma = |\overline{w} - z|^2 + O(|\overline{w} - z|^3)$, hence $\mathrm{i}\partial\overline{\partial}|\,\mathrm{logh}_z(\overline{w})|^2_\gamma > 0$ at $(p, \overline{p}) \in X \times \overline{X}$. By construction, the fiber $\mathrm{pr}_1^{-1}(z)$ of $\mathrm{pr}_1 : U_{\gamma,\varepsilon} \to X$ is biholomorphic to the ε-ball of the complex vector space $T_{X,z}$ equipped with the hermitian metric γ_z. In this way, we get a locally trivial real analytic bundle $\mathrm{pr}_1 : U_{\gamma,\varepsilon}$ whose fibers are complex balls; it is important to notice, however, that this ball bundle need not – and in fact, will never – be holomorphically locally trivial.

2. Bergman bundles and Bergman Dolbeault complex

Let X be a n-dimensional compact complex manifold equipped with a real analytic hermitian metric γ, $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ the ball bundle considered in §1 and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections. We introduce what we call the "Bergman direct image sheaf"

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})). \tag{2.1}$$

By definition, its space of sections $\mathcal{B}_\varepsilon(V)$ over an open subset $V \subset X$ consists of holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$ that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$, i.e.

$$\int_{p^{-1}(K)} i^{n^2} f \wedge \overline{f} \wedge \gamma^n < +\infty, \quad \forall K \Subset V. \tag{2.2}$$

Then $\mathcal{B}_\varepsilon$ is clearly a sheaf of infinite dimensional Fréchet $\mathcal{O}_X$-modules. In the case of finitely generated sheaves over $\mathcal{O}_X$, there is a well known equivalence of categories between holomorphic vector bundles G over X and locally free $\mathcal{O}_X$-modules $\mathcal{G}$. As is well known, the correspondence is given by

$G \mapsto \mathcal{G} := \mathcal{O}_X(G) =$ sheaf of germs of holomorphic sections of G, and the converse functor is $\mathcal{G} \mapsto G$, where G is the holomorphic vector bundle whose fibers are $G_z = \mathcal{G}_z/\mathfrak{m}_z\mathcal{G}_z = \mathcal{G}_z \otimes_{\mathcal{O}_{X,z}} \mathcal{O}_{X,z}/\mathfrak{m}_z$ where $\mathfrak{m}_z \subset \mathcal{O}_{X,z}$ is the maximal ideal. In the case of $\mathcal{B}_\varepsilon$, we cannot take exactly the same route, mostly because the desired "holomorphic Hilbert bundle" B_ε will not even be locally trivial in the complex analytic sense. Instead, we define directly the fibers $B_{\varepsilon,z}$ as the set of holomorphic sections f of $K_{\overline{X}}$ on the fibers $U_{\varepsilon,z} = p^{-1}(z)$, such that

$$\int_{U_{\varepsilon,z}} i^{n^2} f \wedge \overline{f} < +\infty. \tag{2.2$_z$}$$

Since $U_{\varepsilon,z}$ is biholomorphic to the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$, the fiber $B_{\varepsilon,z}$ is isomorphic to the Hilbert space $\mathcal{H}^2(\mathbb{B}_n)$ of L^2 holomorphic n-forms on $\mathbb{B}_n$. In fact, if we use orthonormal coordinates $(w_1, \ldots, w_n)$ provided by exph acting on the hermitian space $(T_{X,z}, \gamma_z)$ and centered at $\overline{z}$, we get a biholomorphism $\mathbb{B}_n \to p^{-1}(z)$ given by the homothety $\eta_\varepsilon : w \mapsto \varepsilon w$, and a corresponding isomorphism

$$B_{\varepsilon,z} \longrightarrow \mathcal{H}^2(\mathbb{B}_n), \qquad f \longmapsto g = \eta_\varepsilon^* f, \quad \text{i.e. with } I = \{1, \ldots, n\}, \tag{2.3}$$

$$f_I(w)\, dw_1 \wedge \ldots \wedge dw_n \longmapsto \varepsilon^n f_I(\varepsilon w)\, dw_1 \wedge \ldots \wedge dw_n, \quad w \in \mathbb{B}_n, \tag{2.3$'$}$$

$$\|g\|^2 = \int_{\mathbb{B}_n} 2^{-n} i^{n^2} g \wedge \overline{g}, \quad g = g(w)\, dw_1 \wedge \ldots \wedge dw_n \in \mathcal{H}^2(\mathbb{B}_n). \tag{2.3$''$}$$

As $U_\varepsilon \to X$ is real analytically locally trivial over X, it follows immediately that $B_\varepsilon \to X$ is also a locally trivial real analytic Hilbert bundle of typical fiber $\mathcal{H}^2(\mathbb{B}_n)$, with the natural Hilbert metric obtained by declaring (2.3) to be an isometry. Since $\mathrm{Aut}(\mathbb{B}_n)$ is a real Lie group, the gauge group of $B_\varepsilon \to X$ can be reduced to real analytic sections of $\mathrm{Aut}(\mathbb{B}_n)$ and we have a well defined class of real analytic connections on B_ε. In this context, one should pay attention to the fact that a section f in $\mathcal{B}_\varepsilon(V)$ does not necessarily restrict to L^2 holomorphic sections $f_{U_{\varepsilon,z}} \in B_{\varepsilon,z}$ for all $z \in V$, although this is certainly true for almost all $z \in V$ by the Fubini theorem; this phenomenon can already be seen through the fact that one does not have a continuous restriction morphism $\rho_n : \mathcal{H}^2(\mathbb{B}_n) \to \mathcal{H}^2(\mathbb{B}_{n-1})$ to the hyperplane $z_n = 0$. In fact, the function $(1 - z_1)^{-\alpha}$ is in $\mathcal{H}^2(\mathbb{B}_n)$ if and only if $\alpha < (n+1)/2$, so that $(1 - z_1)^{-n/2}$ is outside of the domain of ρ_n. As a consequence, the morphism $\mathcal{B}_{\varepsilon,z} \to B_{\varepsilon,z}$ (stalk of sheaf to vector bundle fiber) only has a dense domain of definition, containing e.g. $\mathcal{B}_{\varepsilon',z}$ for any $\varepsilon' > \varepsilon$. This is a familiar situation in Von Neumann's theory of operators.

We now introduce a natural "Bergman version" of the Dolbeault complex, by introducing a sheaf $\mathcal{F}^q_\varepsilon$ over X of (n,q)-forms which can be written locally over small open sets $V\subset X$ as

$$(2.4)\quad f(z,w)=\sum_{|J|=q} f_J(z,w)\,dw_1\wedge\ldots\wedge dw_n\wedge d\overline{z}_J,\qquad (z,w)\in U_\varepsilon\cap(V\times\overline{X}),$$

where the $f_J(z,w)$ are $L^2_{\rm loc}$ smooth functions on $U_\varepsilon\cap(V\times\overline{X})$ such that $f_J(z,w)$ is holomorphic in w (i.e. $\overline{\partial}_w f=0$) and both f and $\overline{\partial} f=\overline{\partial}_z f$ are in $L^2(p^{-1}(K))$ for all compact subsets $K\Subset V$ (here $\overline{\partial}$ operators are of course taken in the sense of distributions). By construction, we get a complex of sheaves $(\mathcal{F}^\bullet_\varepsilon,\overline{\partial})$ and the kernel $\operatorname{Ker}\overline{\partial}:\mathcal{F}^0_\varepsilon\to\mathcal{F}^1_\varepsilon$ coincides with $\mathcal{B}_\varepsilon$. In that sense, if we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of $L^2_{\rm loc}$ sections f of B_ε such that $\overline{\partial} f=0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_\varepsilon)=\mathcal{B}_\varepsilon$ as a sheaf. For $z\in V$, the restriction map $\mathcal{B}_\varepsilon(V)=\mathcal{O}_{L^2}(B_\varepsilon)(V)\to B_{\varepsilon,z}$ is an unbounded closed operator with dense domain, and the kernel is the closure of $\mathfrak{m}_z\mathcal{B}_\varepsilon(V)$, which need not be closed. If one insists on getting continuous fiber restrictions, one could consider the subsheaf

$$\mathcal{O}_{C^k}(B_\varepsilon)(V):=\mathcal{O}_{L^2}(B_\varepsilon)(V)\cap\mathcal{C}^k(B_\varepsilon)(V)$$

where $\mathcal{C}^k(B_\varepsilon)$ is the sheaf of sections f such that $\nabla^\ell f$ is continuous in the Hilbert bundle topology for all real analytic connections ∇ on B_ε and all $\ell=0,1,\ldots,k$. For these subsheaves (and any $k\ge 0$), we do get continuous fiber restrictions $\mathcal{O}_{C^k}(B_\varepsilon)(V)\to B_{\varepsilon,z}$ for $z\in V$. In the same way, we could introduce the Dolbeault complex $\mathcal{F}^\bullet_\varepsilon\cap\mathcal{C}^\infty$ and check that it is a resolution of $\mathcal{O}\cap\mathcal{C}^\infty(B_\varepsilon)$, but we will not need this refinement. However, a useful observation is that the closed and densely defined operator $\mathcal{O}_{L^2}(B_\varepsilon)(V)\to B_{\varepsilon,z}$ is surjective, in fact it is even true that $H^0(X,\mathcal{O}_{L^2}(B_\varepsilon))\to B_{\varepsilon,z}$ is surjective by the Ohsawa–Takegoshi extension theorem [16] applied on the Stein manifold U_ε. We are going to see that $\mathcal{B}_\varepsilon$ can somehow be seen as an infinite dimensional very ample sheaf. This is already illustrated by the following result.

2.5 Proposition. *Assume here that $\varepsilon>0$ is taken so small that $\psi(z,w):=|\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon\subset X\times\overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet_\varepsilon,\overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E\to X$ and every $q\ge 1$ we have*

$$H^q(X,\mathcal{B}_\varepsilon\otimes\mathcal{O}(E))=H^q(\Gamma(X,\mathcal{F}^\bullet_\varepsilon\otimes\mathcal{O}(E)),\overline{\partial})=0.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$, in the sense that $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) \to B_{\varepsilon,z} \otimes E_z$ is a closed and densely defined operator with surjective image.

Proof. By construction, we can equip U_ε with the the associated Kähler metric $\omega = \mathrm{i}\partial\overline{\partial}\psi$ which is smooth and strictly positive on $\overline{U}_\varepsilon$. We can then take an arbitrary smooth hermitian metric h_E on E and multiply it by $e^{-C\psi}$, $C \gg 1$, to obtain a bundle with arbitrarily large positive curvature tensor. The exactness of $\mathcal{F}^\bullet_\varepsilon$ and cohomology vanishing then follow from the standard Hörmander L^2 estimates applied either locally on $p^{-1}(V)$ for small Stein open sets $V \subset X$, or globally on U_ε. The global generation of fibers is again a consequence of the Ohsawa–Takegoshi L^2 extension theorem. □

2.6 Remark. The same result holds for an arbitrary coherent sheaf $\mathcal{E}$ instead of a locally free sheaf $\mathcal{O}(E)$, the reason being that $p^*\mathcal{E}$ admits a resolution by (finite dimensional) locally free sheaves $\mathcal{O}_{U_{\varepsilon'}}^{\oplus N}$ on a Stein neighborhood $U_{\varepsilon'}$ of $\overline{U}_\varepsilon$.

2.7 Remark. A strange consequence of these results is that we get some sort of "holomorphic embedding" of an arbitrary complex manifold X into a "Hilbert Grassmannian", mapping every point $z \in X$ to the closed subspace S_z in the Hilbert space $\mathbb{H} = \mathcal{B}_\varepsilon(X)$, consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_{\varepsilon,z}$, i.e. $f_{|p^{-1}(z)} = 0$. However, the fact that the restriction morphisms $f \mapsto f_{|p^{-1}(z)}$ are not continuous in L^2 norm implies that the map $z \mapsto S_z$ is not even continuous in the strong topology, i.e. the metric topology for which the distance of two fibers S_{z_1}, S_{z_2} is the Hausdorff distance of their unit balls in the L^2 norm of $\mathcal{B}_\varepsilon(X)$.

3. Curvature tensor of Bergman bundles

3.1. Calculation in the model case ($\mathbb{C}^n$, std)

In the model situation $X = \mathbb{C}^n$ with its standard hermitian metric, we consider the tubular neighborhood

$$U_\varepsilon := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |\overline{w} - z| < \varepsilon\} \tag{3.1}$$

and the projections

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \quad (z,w) \mapsto z,$$
$$\overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \quad (z,w) \mapsto w.$$

If one insists on working on a compact complex manifold, the geometry is locally identical to that of a complex torus $X = \mathbb{C}^n/\Lambda$ equipped with a constant hermitian metric γ.

3.2 Remark. We check here that the Bergman bundle B_ε is not holomorphically locally trivial, even in the above situation where we have invariance by translation. However, in the category of real analytic bundles, there is a global trivialization of $B_\varepsilon \to \mathbb{C}^n$ given by the map

$$\tau : B_\varepsilon \xrightarrow{\simeq} \mathbb{C}^n \times \mathcal{H}^2(\mathbb{B}_n), \quad B_{\varepsilon,z} \ni f_z \longmapsto \tau(f) = (z, g_z),$$

where $g_z(w) := f_z(\varepsilon w + \overline{z})$, $w \in \mathbb{B}_n$, in other words, for any open set $V \subset \mathbb{C}^n$ and any $k \in \mathbb{N} \cup \{\infty, \omega\}$, we have isomorphisms

$$C^k(V, B_\varepsilon) \to C^k(V, \mathcal{H}^2(\mathbb{B}^n)), \quad f \mapsto g, \quad g(z,w) = f(z, \varepsilon w + \overline{z}),$$

where f, g are C^k in (z,w), holomorphic in w, and the derivatives $z \mapsto D_z^\alpha g(z, \bullet)$, $|\alpha| \le k$, define continuous maps $V \to \mathcal{H}^2(\mathbb{B}_n)$. The complex structures of these bundles are defined by the $(0,1)$-connections $\overline{\partial}_z$ of the associated Dolbeault complexes, but obviously $\overline{\partial}_z f$ and $\overline{\partial}_z g$ do not match. In fact, if we write

$$g(z,w) = u(z,w)\, dw_1 \wedge \ldots \wedge dw_n \in C^\infty(V, \mathcal{H}^2(\mathbb{B}_n)) = C^\infty(V) \mathbin{\widehat{\otimes}} \mathcal{H}^2(\mathbb{B}_n)$$

where $\widehat{\otimes}$ is the ε or π-topological tensor product in the sense of [7], we get

$$\begin{aligned} &f(z,w) = g(z, (w - \overline{z})/\varepsilon) = \varepsilon^{-n} u(z, (w-\overline{z})/\varepsilon)\, dw_1 \wedge \ldots \wedge dw_n, \\ &\overline{\partial}_z f(z,w) = \\ &\quad \varepsilon^{-n}\Big(\overline{\partial}_z u(z, (w-\overline{z})/\varepsilon) - \varepsilon^{-1} \sum_{1 \le j \le n} \frac{\partial u}{\partial w_j}(z, (w-\overline{z})/\varepsilon)\, d\overline{z}_j\Big) \wedge dw_1 \wedge \ldots \wedge dw_n. \end{aligned}$$

Therefore the trivialization $\tau_* : f \mapsto u$ yields at the level of $\overline{\partial}$-connections an identification

$$\tau_* : \overline{\partial}_z f \overset{\simeq}{\longmapsto} \overline{\partial}_z u + Au$$

where the "connection matrix" $A \in \Gamma(V, \Lambda^{0,1} T_X^* \otimes_{\mathbb{C}} \mathrm{End}(\mathcal{H}^2(\mathbb{B}_n)))$ is the constant unbounded Hilbert space operator $A(z) = A$ given by

$$A : \mathcal{H}^2(\mathbb{B}_n) \to \Lambda^{0,1} T_X^* \otimes_{\mathbb{C}} \mathcal{H}^2(\mathbb{B}_n), \quad u \mapsto Au = -\varepsilon^{-1} \sum_{1 \le j \le n} \frac{\partial u}{\partial w_j}\, d\overline{z}_j.$$

We see that the holomorphic structure of B_ε is given by a $(0,1)$-connection that differs by the matrix A from the trivial $(0,1)$-connection, and as A is unbounded, there is no way we can make it trivial by a real analytic gauge change with values in Lie algebra of continuous endomorphisms of $\mathcal{H}^2(\mathbb{B}_n)$. □

We are now going to compute the curvature tensor of the Bergman bundle B_ε. For the sake of simplicity, we identify here $\mathcal{H}^2(\mathbb{B}_n)$ to the Hardy space of L^2 holomorphic functions via $u \mapsto g = u(w)\,dw_1\wedge\ldots\wedge dw_n$. After rescaling, we can also assume $\varepsilon = 1$, and at least in a first step, we perform our calculations on B_1 rather than B_ε. Let us write $w^\alpha = \prod_{1\le j\le n} w_j^{\alpha_j}$ for a multiindex $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, and denote by λ the Lebesgue measure on $\mathbb{C}^n$. A well known calculation gives

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = \pi^n \frac{\alpha_1!\ldots\alpha_n!}{(|\alpha|+n)!}, \quad |\alpha| = \alpha_1+\cdots+\alpha_n.$$

In fact, by using polar coordinates $w_j = r_j e^{i\theta_j}$ and writing $t_j = r_j^2$, we get

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = (2\pi)^n \int_{r_1^2+\cdots+r_n^2<1} r^{2\alpha}\, r_1dr_1\ldots r_ndr_n = \pi^n I(\alpha)$$

with

$$I(\alpha) = \pi^n \int_{t_1+\cdots+t_n<1} t^\alpha\, dt_1\ldots dt_n.$$

Now, an induction on n together with the Fubini formula gives

$$\begin{aligned} I(\alpha) &= \int_0^1 t_n^{\alpha_n} dt_n \int_{t_1+\cdots+t_{n-1}<1-t_n} (t')^{\alpha'} dt_1\ldots dt_{n-1} \\ &= I(\alpha') \int_0^1 (1-t_n)^{\alpha_1+\cdots+\alpha_{n-1}+n-1} t_n^{\alpha_n} dt_n \end{aligned}$$

where $t' = (t_1,\ldots,t_{n-1})$ and $\alpha' = (\alpha_1,\ldots,\alpha_{n-1})$. As

$$\int_0^1 x^a(1-x)^b dt = \frac{a!\,b!}{(a+b+1)!},$$

we get inductively

$$I(\alpha) = \frac{(|\alpha'|+n-1)!\alpha_n!}{(|\alpha|+n)!} I(\alpha') \quad \Rightarrow \quad I(\alpha) = \frac{\alpha_1!\ldots\alpha_n!}{(|\alpha|+n)!}.$$

Such formulas were already used by Shiffman and Zelditch [21] in their study of zeros of random sections of positive line bundles. They imply that a Hilbert (orthonormal) basis of $\mathcal{O} \cap L^2(\mathbb{B}_n) \simeq \mathcal{H}^2(\mathbb{B}_n)$ is

$$e_\alpha(w) = \pi^{-n/2}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\, w^\alpha. \tag{3.3}$$

As a consequence, and quite classically, the Bergman kernel of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is

$$K_n(w) = \sum_{\alpha\in\mathbb{N}^n} |e_\alpha(w)|^2 = \pi^{-n} \sum_{\alpha\in\mathbb{N}^n} \frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}\,|w^\alpha|^2 = n!\,\pi^{-n}(1-|w|^2)^{-n-1}. \tag{3.4}$$

If we come back to U_ε for $\varepsilon > 0$ not necessarily equal to 1 (and do not omit any more the trivial n-form $dw_1 \wedge \ldots \wedge dw_n$), we have to use a rescaling $(z, w) \mapsto (\varepsilon^{-1}z, \varepsilon^{-1}w)$. This gives for the Hilbert bundle B_ε a real analytic orthonormal frame

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha\, dw_1\wedge\ldots\wedge dw_n \tag{3.5}$$

A germ of holomorphic section $\sigma \in \mathcal{O}_{L^2}(B_\varepsilon)$ near $z = 0$ (say) is thus given by a convergent power series

$$\sigma(z,w) = \sum_{\alpha\in\mathbb{N}^n} \xi_\alpha(z)\, e_\alpha(z,w)$$

such that the functions ξ_α are real analytic on a neighborhood of 0 and satisfy the following two conditions:

$$|\sigma(z)|_h^2 := \sum_{\alpha\in\mathbb{N}^n} |\xi_\alpha(z)|^2 \ \text{ converges in } L^2 \text{ near } 0, \tag{3.6}$$

$$\overline{\partial}_{z_k}\sigma(z,w) = \sum_{\alpha\in\mathbb{N}^n} \overline{\partial}_{z_k}\xi_\alpha(z)\, e_\alpha(z,w) + \xi_\alpha(z)\,\overline{\partial}_{z_k}e_\alpha(z,w) \equiv 0. \tag{3.7}$$

Let $c_k = (0,\ldots,1,\ldots,0)$ be the canonical basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$. A straightforward calculation from (3.5) yields

$$\overline{\partial}_{z_k}e_\alpha(z,w) = -\varepsilon^{-1}\sqrt{\alpha_k(|\alpha|+n)}\, e_{\alpha-c_k}(z,w).$$

We have the slight problem that the coefficients are unbounded as $|\alpha| \to +\infty$, and therefore the two terms occurring in (3.7) need not form convergent series when taken separately. However if we take $\sigma \in \mathcal{O}_{L^2}(B_{\varepsilon'})$ in a slightly bigger tubular neighborhood ($\varepsilon' > \varepsilon$), the L^2 condition implies that $\sum_\alpha (\varepsilon''/\varepsilon)^{2|\alpha|}|\xi_\alpha|^2$ is uniformly convergent for every $\varepsilon'' \in \,]\varepsilon, \varepsilon'[\,$, and this is more than enough to ensure convergence, since the growth of $\alpha \mapsto \sqrt{\alpha_k(|\alpha|+n)}$ is at most linear; we can even iterate as many derivatives as we want. For a smooth section $\sigma \in C^\infty(B_{\varepsilon'})$, the coefficients ξ_α are smooth, with $\sum(\varepsilon'/\varepsilon)^{2|\alpha|}|\partial_z^\beta \overline{\partial}_z^\gamma \xi_\alpha|^2$ convergent for all β, γ, and we get

$$\begin{aligned}
\overline{\partial}_{z_k}\sigma(z,w) &= \sum_{\alpha\in\mathbb{N}^n} \overline{\partial}_{z_k}\xi_\alpha(z)\, e_\alpha(z,w) + \xi_\alpha(z)\, \overline{\partial}_{z_k} e_\alpha(z,w) \\
&= \sum_{\alpha\in\mathbb{N}^n} \overline{\partial}_{z_k}\xi_\alpha(z)\, e_\alpha(z,w) - \varepsilon^{-1}\sqrt{\alpha_k(|\alpha|+n)}\;\xi_\alpha(z)\; e_{\alpha-c_k}(z,w) \\
&= \sum_{\alpha\in\mathbb{N}^n} \big(\overline{\partial}_{z_k}\xi_\alpha(z) - \varepsilon^{-1}\sqrt{(\alpha_k+1)(|\alpha|+n+1)}\;\xi_{\alpha+c_k}(z)\big)\; e_\alpha(z,w),
\end{aligned}$$

after replacing α by $\alpha + c_k$ in the terms containing ε^{-1}. The $(0,1)$-part $\nabla_h^{0,1}$ of the Chern connection ∇_h of (B_ε, h) with respect to the orthonormal frame (e_α) is thus given by

$$\nabla_h^{0,1}\sigma = \sum_{\alpha\in\mathbb{N}^n} \Big(\overline{\partial}\xi_\alpha - \sum_k \varepsilon^{-1}\sqrt{(\alpha_k+1)(|\alpha|+n+1)}\;\xi_{\alpha+c_k}\; d\overline{z}_k\Big) \otimes e_\alpha. \tag{3.8}$$

The $(1,0)$-part can be derived from the identity

$$\partial|\sigma|_h^2 = \langle \nabla_h^{1,0}\sigma, \sigma\rangle_h + \langle \sigma, \nabla_h^{0,1}\sigma\rangle_h.$$

However

$$\begin{aligned}
\partial_{z_j}|\sigma|_h^2 = \partial_{z_j} \sum_{\alpha\in\mathbb{N}^n} \xi_\alpha\overline{\xi}_\alpha &= \sum_{\alpha\in\mathbb{N}^n} (\partial_{z_j}\xi_\alpha)\,\overline{\xi}_\alpha + \xi_\alpha\,(\overline{\overline{\partial}_{z_j}\xi_\alpha}\,) \\
&= \sum_{\alpha\in\mathbb{N}^n} \Big(\,\partial_{z_j}\xi_\alpha + \varepsilon^{-1}\sqrt{\alpha_j(|\alpha|+n)}\;\xi_{\alpha-c_j}\Big)\,\overline{\xi}_\alpha \\
&\quad + \sum_{\alpha\in\mathbb{N}^n} \xi_\alpha\Big(\,\overline{\overline{\partial}_{z_j}\xi_\alpha - \varepsilon^{-1}\sqrt{(\alpha_j+1)(|\alpha|+n+1)}\;\xi_{\alpha+c_j}}\,\Big).
\end{aligned}$$

For $\sigma \in C^\infty(B_{\varepsilon'})$, it follows from there that

$$\nabla_h^{1,0}\sigma = \sum_{\alpha\in\mathbb{N}^n} \Big(\partial\xi_\alpha + \varepsilon^{-1}\sum_j \sqrt{\alpha_j(|\alpha|+n)}\;\xi_{\alpha-c_j}dz_j\Big) \otimes e_\alpha. \tag{3.9}$$

Finally, to find the curvature tensor of (B_ε, h), we only have to compute the $(1,1)$-form $(\nabla_h^{1,0}\nabla_h^{0,1} + \nabla_h^{0,1}\nabla_h^{1,0})\sigma$ and take the terms that contain no differentiation at all, especially in view of the usual identity $\partial\overline\partial+\overline\partial\partial = 0$ and the fact that we also have here $(\nabla_h^{1,0})^2 = 0$, $(\nabla_h^{0,1})^2 = 0$. As $(\alpha-c_j)_k = \alpha_k-\delta_{jk}$ and $(\alpha + c_k)_j = \alpha_j + \delta_{jk}$, we are left with

$$
\begin{aligned}
&(\nabla_h^{1,0}\nabla_h^{0,1} + \nabla_h^{0,1}\nabla_h^{1,0})\sigma\\
&= -\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j(|\alpha|+n)}\,\sqrt{(\alpha_k-\delta_{jk}+1)(|\alpha|+n)}\,\xi_{\alpha-c_j+c_k}\,dz_j\wedge d\overline{z}_k\otimes e_\alpha\\
&\quad+\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{(\alpha_j+\delta_{jk})(|\alpha|+n+1)}\times\\
&\qquad\qquad\sqrt{(\alpha_k+1)(|\alpha|+n+1)}\,\xi_{\alpha-c_j+c_k}\,dz_j\wedge d\overline{z}_k\otimes e_\alpha\\
&= -\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{(\alpha_j-\delta_{jk})(\alpha_k-\delta_{jk})}\,(|\alpha|+n-1)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k}\\
&\quad+\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j\alpha_k}\,(|\alpha|+n)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k}.\\
&= \quad\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j\alpha_k}\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k}\\
&\quad+\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_{j}(|\alpha|+n-1)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_j\otimes e_{\alpha-c_j},
\end{aligned}
$$

where the last summation comes from the subtraction of the diagonal terms $j = k$. By changing α into $\alpha + c_j$ in that summation, we obtain the following expression of the curvature tensor of (B_ε, h).

3.10 Theorem. *The curvature tensor of the Bergman bundle* (B_ε, h) *is given by*

$$
\langle\Theta_{B_\varepsilon,h}\sigma(v,Jv),\sigma\rangle_h = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\left(\Big|\sum_j\sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2+\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\right)
$$

for every $\sigma = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, *and every tangent vector* $v = \sum v_j\,\partial/\partial z_j$.

The above curvature hermitian tensor is positive definite, and even positive definite unbounded if we view it as a hermitian form on $T_X \otimes B_\varepsilon$ rather than on $T_X \otimes B_{\varepsilon'}$. This is not so surprising since the connection matrix was already an unbounded operator. Philosophically, the very ampleness of the sheaf $\mathcal{B}_\varepsilon$ was also a strong indication that the curvature of the corresponding vector bundle B_ε should have been positive. Observe that we have in fact

$$\begin{aligned}
&\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\\
(3.11)\qquad &\leq \langle\Theta_{B_\varepsilon,h}\sigma(v,Jv),\sigma\rangle_h\leq 2\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2,
\end{aligned}$$

thanks to the Cauchy–Schwarz inequality

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{N}^n}\Big|\sum_j\sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2\\
&\quad\leq\sum_\ell|v_\ell|^2\sum_{\alpha\in\mathbb{N}^n}\sum_j\alpha_j|\xi_{\alpha-c_j}|^2=\sum_\ell|v_\ell|^2\sum_j\sum_{\alpha\in\mathbb{N}^n}\alpha_j|\xi_{\alpha-c_j}|^2\\
&\quad=\sum_\ell|v_\ell|^2\sum_j\sum_{\alpha\in\mathbb{N}^n}(\alpha_j+1)|\xi_\alpha|^2=\sum_\ell|v_\ell|^2\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2.
\end{aligned}$$

3.2. Curvature of Bergman bundles on compact hermitian manifolds

We consider here the general situation of a compact hermitian manifold (X,γ) described in §1, where γ is real analytic and exph is the associated partially holomorphic exponential map. Fix a point $x_0\in X$, and use a holomorphic system of coordinates $(z_1,\ldots,z_n)$ centered at x_0, provided by

$$\mathrm{exph}_{x_0}:T_{X,x_0}\supset V\to X.$$

If we take γ_{x_0} orthonormal coordinates on T_{X,x_0}, then by construction the fiber of $p:U_\varepsilon\to X$ over x_0 is the standard ε-ball in the coordinates $(w_j)=(\overline{z}_j)$. Let $T_X\to V\times\mathbb{C}^n$ be the trivialization of T_X in the coordinates (z_j), and

$$X\times X\to T_X,\quad (z,w)\mapsto\xi=\mathrm{logh}_z(w)$$

the expression of logh near $(x_0,\overline{x}_0)$, that is, near $(z,w)=(0,0)$. By our choice of coordinates, we have $\mathrm{logh}_0(w)=w$ and of course $\mathrm{logh}_z(z)=0$, hence we get a real analytic expansion of the form

$$\begin{aligned}
\mathrm{logh}_z(w)&=w-z+\sum z_ja_j(w-z)+\sum\overline{z}_ja'_j(w-z)\\
&\quad+\sum z_jz_kb_{jk}(w-z)+\sum\overline{z}_j\overline{z}_kb'_{jk}(w-z)+\sum z_j\overline{z}_kc_{jk}(w-z)+O(|z|^3)
\end{aligned}$$

with holomorphic coefficients a_j, a'_j, b_{jk}, b'_{jk}, c_{jk} vanishing at 0. In fact by [4], we always have $da'_j(0)=0$, and if γ is Kähler, the equality $da_j(0)=0$

also holds; we will not use these properties here. In coordinates, we then have locally near $(0,0) \in \mathbb{C}^n \times \mathbb{C}^n$

$$U_{\varepsilon,z} = \big\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |\Psi_z(w)| < \varepsilon\big\}$$

where $\Psi_z(w) = \overline{\mathrm{logh}_z(\overline{w})}$ has a similar expansion

$$\tag{3.12} \begin{aligned}\Psi_z(w) &= w - \overline{z} + \sum z_j a_j(w-\overline{z}) + \sum \overline{z}_j a'_j(w-\overline{z})\\ &+ \sum z_j z_k b_{jk}(w-\overline{z}) + \sum \overline{z}_j\overline{z}_k b'_{jk}(w-\overline{z}) + \sum z_j\overline{z}_k c_{jk}(w-\overline{z}) + O(|z|^3)\end{aligned}$$

(when going from logh to Ψ, the coefficients a_j, a'_j and b_j, b'_j get twisted, but we do not care and keep the same notation for Ψ, as we will not refer to logh any more). In this situation, the Hilbert bundle B_ε has a real analytic normal frame given by $\widetilde{e}_\alpha = \Psi^* e_\alpha$ where

$$\tag{3.13} e_\alpha(w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\ w^\alpha\ dw_1\wedge\ldots\wedge dw_n$$

and the pull-back $\Psi^* e_\alpha$ is taken with respect to $w \mapsto \Psi_z(w)$ (z being considered as a parameter). For a local section $\sigma = \sum_\alpha \xi_\alpha \widetilde{e}_\alpha \in C^\infty(B_{\varepsilon'})$, $\varepsilon > \varepsilon$, we can write

$$\overline{\partial}_{z_k}\sigma(z,w) = \sum_{\alpha\in\mathbb{N}^n} \overline{\partial}_{z_k}\xi_\alpha(z)\,\widetilde{e}_\alpha(z,w) + \xi_\alpha(z)\,\overline{\partial}_{z_k}\widetilde{e}_\alpha(z,w).$$

Near $z=0$, by taking the derivative of $\Psi^* e_\alpha(z,w)$, we find

$$\begin{aligned}\overline{\partial}_{z_k}\widetilde{e}_\alpha(z,w) &= -\varepsilon^{-1}\sqrt{\alpha_k(|\alpha|+n)}\,\widetilde{e}_{\alpha-c_k}(z,w)\\ &+\varepsilon^{-1}\sum_m \sqrt{\alpha_m(|\alpha|+n)}\Big(a'_{k,m}(w-\overline{z}) + \sum_j z_j c_{jk,m}(w-\overline{z})\Big)\,\widetilde{e}_{\alpha-c_m}(z,w)\\ &+\sum_m\Big(\frac{\partial a'_{k,m}}{\partial w_m}(w-\overline{z}) + \sum_j z_j\frac{\partial c_{jk,m}}{\partial w_m}(w-\overline{z})\Big)\,\widetilde{e}_\alpha(z,w) + O(\overline{z},|z|^2),\end{aligned}$$

where the last sum comes from the expansion of $dw_1\wedge\ldots\wedge dw_n$, and $a'_{k,m}$, $c_{jk,m}$ are the m-th components of a'_k and c_{jk}. This gives two additional terms in comparison to the translation invariant case, but these terms are "small" in the sense that the first one vanishes at $(z,w) = (0,0)$ and the second one does not involve ε^{-1}. If $\nabla^{0,1}_{h,0}$ is the $\overline{\partial}$-connection associated with the standard tubular neighborhood $|\overline{w}-z|<\varepsilon$, we thus find in terms of the local trivialization $\sigma \simeq \xi = \sum \xi_\alpha \widetilde{\varepsilon}_\alpha$ an expression of the form

$$\nabla_h^{0,1}\sigma \simeq \nabla_{h,0}^{0,1}\xi + A^{0,1}\xi,$$

where

$$\begin{aligned}
A^{0,1}\Big(\sum_\alpha \xi_\alpha \widetilde{e}_\alpha\Big) &= \sum_{\alpha\in\mathbb{N}^n}\sum_k \\
&\xi_\alpha\bigg(\varepsilon^{-1}\sum_m \sqrt{\alpha_m(|\alpha|+n)}\,\Big(a'_{k,m}(w)+\sum_j z_j c_{jk,m}(w)\Big)\,d\overline{z}_k\otimes \widetilde{e}_{\alpha-c_m}\\
&\quad+\sum_m\Big(\frac{\partial a'_{k,m}}{\partial w_m}(w)+\sum_j z_j\frac{\partial c_{jk,m}}{\partial w_m}(w)\Big)\,d\overline{z}_k\otimes\widetilde{e}_\alpha\bigg)+O(\overline{z},|z|^2).
\end{aligned}$$

The corresponding $(1,0)$-parts satisfy

$$\nabla_h^{1,0}\sigma \simeq \nabla_{h,0}^{1,0}\xi + A^{1,0}\xi, \qquad A^{1,0} = -(A^{0,1})^*,$$

and the corresponding curvature tensors are related by

$$\Theta_{\beta_\varepsilon,h} = \Theta_{\beta_\varepsilon,h,0} + \partial A^{0,1} + \overline{\partial}A^{1,0} + A^{1,0}\wedge A^{0,1} + A^{0,1}\wedge A^{1,0}. \tag{3.14}$$

At $z=0$ we have

$$\begin{aligned}
A^{0,1}\xi &= \sum_{\alpha\in\mathbb{N}^n}\sum_k \xi_\alpha\bigg(\varepsilon^{-1}\sum_m\sqrt{\alpha_m(|\alpha|+n)}\;a'_{k,m}(w)\;d\overline{z}_k\otimes\widetilde{e}_{\alpha-c_m}\\
&\qquad\qquad+\sum_m\frac{\partial a'_{k,m}}{\partial w_m}(w)\;d\overline{z}_k\otimes\widetilde{e}_\alpha\bigg),\\
\partial A^{0,1}\xi &= \sum_{\alpha\in\mathbb{N}^n}\sum_k \xi_\alpha\bigg(\varepsilon^{-1}\sum_{j,m}\sqrt{\alpha_m(|\alpha|+n)}\;c_{jk,m}(w)\;dz_j\wedge d\overline{z}_k\otimes\widetilde{e}_{\alpha-c_m}\\
&\qquad\qquad+\sum_{j,m}\frac{\partial c_{jk,m}}{\partial w_m}(w)\;dz_j\wedge d\overline{z}_k\otimes\widetilde{e}_\alpha\bigg),
\end{aligned}$$

and $A^{1,0}$, $\overline{\partial}A^{1,0}$ are, up to the sign, the adjoint endomorphisms of $A^{0,1}$ and $\partial A^{0,1}$. The unboundedness comes from the fact that we have unbounded factors $\sqrt{(\alpha_m+1)(|\alpha|+n+1)}$; it is worth noticing that multiplication by a holomorphic factor $u(w)$ is a continuous operator on the fibers $B_{\varepsilon,z}$, whose norm remains bounded as $\varepsilon \to 0$. In this setting, it can be seen that the only term in (3.14) that is (a priori) not small with respect to the main term $\Theta_{B_\varepsilon,h,0}$ is the term involving ε^{-2} in $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$, and that the other terms appearing in the quadratic form $\langle\Theta_{B_\varepsilon,h}\xi,\xi\rangle$ are $O(\varepsilon^{-1}\sum(|\alpha|+n)|\xi_\alpha|^2)$ or smaller. In order to check this, we expand $c_{jk,m}(w)$ into a power series $\sum_\mu c_{jk,m,\mu}\, g_\mu(w)$ where

(3.15)

$$g_\mu(w) = s_\mu^{-1}w^\mu, \quad \text{with } s_\mu = \sup_{|w|\le 1}|w^\mu| = \prod_{1\le j\le n}\left(\frac{\mu_j}{|\mu|}\right)^{\mu_j/2} = \frac{\prod \mu_j^{\mu_j/2}}{|\mu|^{|\mu|/2}},$$

so that $\sup_{|w|\le\varepsilon}|g_\mu(w)| = \varepsilon^{|\mu|}$. We get from the term $\langle\partial A^{0,1}\xi,\xi\rangle$ a summation

$$\Sigma(\xi) = \varepsilon^{-1}\sum_{j,k,m}\ \sum_{\alpha\in\mathbb{N}^n}\sqrt{\alpha_m(|\alpha|+n)}\ \sum_{\mu\in\mathbb{N}^n} c_{jk,m,\mu}\, dz_j\wedge d\overline{z}_k\otimes\langle\xi_\alpha\, g_\mu\widetilde{e}_\alpha,\xi\rangle.$$

At $z=0$, $g_\mu\widetilde{e}_\alpha = g_\mu e_\alpha$ is proportional to $e_{\alpha+\mu}$, and by (3.15) and the definition of the L^2 norm, we have $\|g_\mu\widetilde{e}_\alpha\|\le\varepsilon^{|\mu|}$ and $|\langle\xi_\alpha\, g_\mu\widetilde{e}_\alpha,\xi\rangle|\le\varepsilon^{|\mu|}\,|\xi_\alpha||\xi_{\alpha+\mu}|$. We infer

$$\big|\Sigma(\xi)\big| \le \varepsilon^{-1}\sum_{j,k,m}\ \sum_{\alpha\in\mathbb{N}^n}\sqrt{\alpha_m(|\alpha|+n)}\ \sum_{\mu\in\mathbb{N}^n}|c_{jk,m,\mu}|\,\varepsilon^{|\mu|}\,|\xi_\alpha||\xi_{\alpha+\mu}|.$$

Let r be the infimum of the radius of convergence of $w\mapsto\Psi_z(w)$ over all $z\in X$. Then for $\varepsilon<r$ and $r'\in\,]\varepsilon,r[$, we have a uniform bound $|c_{jk,m,\mu}|\le C(1/r')^{|\mu|}$, hence

$$\big|\Sigma(\xi)\big| \le C'\varepsilon^{-1}\sum_{\alpha\in\mathbb{N}^n}\sum_{\mu\in\mathbb{N}^n}\left(\frac{\varepsilon}{r'}\right)^{|\mu|}\sqrt{\alpha_m(|\alpha|+n)}\,|\xi_\alpha||\xi_{\alpha+\mu}|.$$

If we write

$$\begin{aligned}\sqrt{\alpha_m(|\alpha|+n)}\,|\xi_\alpha||\xi_{\alpha+\mu}| &\le \frac{1}{2}(|\alpha|+n)\big(|\xi_\alpha|^2+|\xi_{\alpha+\mu}|^2\big)\\ &\le \frac{1}{2}\big((|\alpha|+n)|\xi_\alpha|^2+(|\alpha+\mu|+n)|\xi_{\alpha+\mu}|^2\big),\end{aligned}$$

the above bound implies

$$\big|\Sigma(\xi)\big| \le C'\varepsilon^{-1}(1-\varepsilon/r')^{-n}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2 = O\Big(\varepsilon^{-1}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2\Big).$$

We now come to the more annoying term $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$, and especially to the part containing ε^{-2} (the other parts can be treated as above or are smaller). We compute explicitly that term by expanding $a'_{k,m}(w)$ into a power series $\sum_\mu a'_{k,m,\mu}\,g_\mu(w)$ as above. Let us write $g_m(w)=s_\mu^{-1}w^\mu$. As $a'_{k,m}(0)=0$, the relevant term in $A^{0,1}$ is

$$\varepsilon^{-1}\sum_{k,m}\ \sum_{\mu\in\mathbb{N}^n\smallsetminus\{0\}} a'_{k,m,\mu}s_\mu^{-1}\,d\overline{z}_k\otimes W^\mu D_m$$

where D_m and $W^\mu=W_1^{\mu_1}\ldots W_n^{\mu_n}$ are operators on the Hilbert space $\mathcal{H}^2(B_{\varepsilon,0})$, defined by

$$D_m\widetilde{e}_\alpha=\sqrt{\alpha_m(|\alpha|+n)}\,\widetilde{e}_{\alpha-c_m},\quad W_m(f)=w_mf.$$

The corresponding term in $A^{1,0}$ is the opposite of the adjoint, namely

$$-\varepsilon^{-1}\sum_{j,\ell}\ \sum_{\lambda\in\mathbb{N}^n\smallsetminus\{0\}} a'_{k,\ell,\lambda}s_\lambda^{-1}dz_j\otimes D_\ell^*W^{*\lambda}$$

and the annoying term in $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$ is

$$Q=-\,\varepsilon^{-2}\sum_{j,k,\ell,m}\ \sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} a'_{k,\ell,\lambda}s_\lambda^{-1}\,a'_{k,m,\mu}s_\mu^{-1}\,dz_j\wedge d\overline{z}_k\otimes \Big(D_\ell^*W^{*\lambda}W^\mu D_m-W^\mu D_mD_\ell^*W^{*\lambda}\Big). \tag{3.16}$$

We have here $\|W^\mu\|\le s_\mu\,\varepsilon^{|\mu|}$ (as W^μ is the multiplication by $w^\mu=s_\mu\,g_\mu(w)$, and $|g_\mu|\le\varepsilon^{|\mu|}$ on $B_{\varepsilon,0}$). The operators D_ℓ^* and D_m are unbounded, but the important point is that their commutators have substantially better continuity than what could be expected a priori. We have for instance

$$\begin{aligned}
&D_m\widetilde{e}_\alpha=\sqrt{\alpha_m(|\alpha|+n)}\,\widetilde{e}_{\alpha-c_m},\quad D_\ell^*(\widetilde{e}_\alpha)=\sqrt{(\alpha_\ell+1)(|\alpha|+n+1)}\,\widetilde{e}_{\alpha+c_\ell},\\
&[D_\ell^*,D_m](\widetilde{e}_\alpha)=\Big(\sqrt{(\alpha_\ell+1-\delta_{\ell m})\alpha_m}\,(|\alpha|+n)\\
&\qquad\qquad-\sqrt{(\alpha_\ell+1)(\alpha_m+\delta_{\ell m})}\,(|\alpha|+n+1)\Big)\,\widetilde{e}_{\alpha+c_\ell-c_m}
\end{aligned}$$

and the coefficient between braces is controlled by $2(|\alpha|+n)$, as one sees by considering separately the two cases $\ell\ne m$, where we get $-\sqrt{(\alpha_\ell+1)\alpha_m}$, and $\ell=m$, where we get $\alpha_\ell(|\alpha|+n)-(\alpha_\ell+1)(|\alpha|+n+1)$. Therefore

$\|[D_\ell^*, D_m](\widetilde{e}_\alpha)\| \le 2(|\alpha|+n)$. We obtain similarly

$$W_m(\widetilde{e}_\alpha) = \varepsilon\sqrt{\frac{\alpha_m+1}{|\alpha|+n+1}}\,\widetilde{e}_{\alpha+c_m}, \quad W_\ell^*(\widetilde{e}_\alpha) = \varepsilon\sqrt{\frac{\alpha_\ell}{|\alpha|+n}}\,\widetilde{e}_{\alpha-c_\ell},$$

$$[W_\ell^*, W_m](\widetilde{e}_\alpha) = \varepsilon^2\bigg(\frac{\sqrt{(\alpha_\ell+\delta_{\ell m})(\alpha_m+1)}}{|\alpha|+n+1} - \frac{\sqrt{\alpha_\ell(\alpha_m+1-\delta_{\ell m})}}{|\alpha|+n}\bigg)\,\widetilde{e}_{\alpha-c_\ell+c_m},$$

and it is easy to see that the coefficient between large braces is bounded for $\ell \ne m$ by $\sqrt{\alpha_\ell(\alpha_m+1)}/((|\alpha|+n)(|\alpha|+n+1)) \le (|\alpha|+n)^{-1}$, and for $\ell = m$ we have as well

$$\left|\frac{(\alpha_\ell+1)(|\alpha|+n) - \alpha_\ell(|\alpha|+n+1)}{(|\alpha|+n)(|\alpha|+n+1)}\right| \le (|\alpha|+n)^{-1}.$$

Therefore $\|[W_\ell^*, W_m](\widetilde{e}_\alpha)\| \le \varepsilon^2(|\alpha|+n)^{-1}$. Finally

$$[W_\ell^*, D_m](\widetilde{e}_\alpha) = \varepsilon\Bigg(\sqrt{\frac{(\alpha_\ell-\delta_{\ell m})\alpha_m(|\alpha|+n)}{|\alpha|+n-1}} - \sqrt{\frac{\alpha_\ell(\alpha_m-\delta_{\ell m})(|\alpha|+n-1)}{|\alpha|+n}}\Bigg)\,\widetilde{e}_{\alpha-c_\ell-c_m}$$

with a coefficient between braces less than 1, thus $\|[W_\ell^*, D_m](\widetilde{e}_\alpha)\| \le \varepsilon$. By adjunction, the same is true for $[D_\ell^*, W_m]$, and we can summarize our estimates as follows:

$$(3.17)\quad \begin{cases} \|W^\mu\| \le s_\mu \varepsilon^{|\mu|}, \quad \|W^{*\lambda}\| \le s_\lambda \varepsilon^{|\lambda|}, \\ \|D_\ell^*(\widetilde{e}_\alpha)\| \le |\alpha|+n+1, \quad \|D_m(\widetilde{e}_\alpha)\| \le |\alpha|+n, \\ \|[D_\ell^*, D_m](\widetilde{e}_\alpha)\| \le 2(|\alpha|+n), \quad \|[W_\ell^*, W_m](\widetilde{e}_\alpha)\| \le (|\alpha|+n)^{-1}, \\ \|[W_\ell^*, D_m](\widetilde{e}_\alpha)\| \le \varepsilon, \quad \|[D_\ell^*, W_m](\widetilde{e}_\alpha)\| \le \varepsilon \end{cases}$$

Now, we observe that both $D_\ell^* W^{*\lambda} W^\mu D_m(\widetilde{e}_\alpha)$ and $W^\mu D_m D_\ell^* W^{*\lambda}(\widetilde{e}_\alpha)$ are multiples of $\widetilde{e}_{\alpha+c_\ell-c_m-\lambda+\mu}$. By considering the second product $W^\mu D_m D_\ell^* W^{*\lambda}$ and permuting successively its factors $D_m D_\ell^*$, $D_m W^{*\lambda}$, $W^\mu D_\ell^*$, $W^\mu W^{*\lambda}$, the difference with $D_\ell^* W^{*\lambda} W^\mu D_m$ is expressed as a sum of $1+|\lambda|+|\mu|+|\lambda||\mu|$ terms involving commutators. We derive from our estimates (3.17) precise bounds for the image of $\widetilde{e}_\alpha$ by the commutators. For instance, when we arrive at $D_\ell^* W^\mu W^{*\lambda} D_m$ and permute $W^\mu W^{*\lambda}$, we go through intermediate steps $D_\ell W^{*\lambda'} W^{\mu'} W_k W_j^* W^{\mu''} W^{*\lambda''} D_m$ with $\lambda = \lambda' + \lambda'' + c_j$, $\mu = \mu' + \mu'' + c_k$, $|\lambda| = |\lambda'| + |\lambda''| + 1$, $|\mu| = |\mu'| + |\mu''| + 1$, and have to evaluate the commutators

$$D_\ell W^{*\lambda'} W^{\mu'} [W_j^*, W_k] W^{\mu''} W^{*\lambda''} D_m(\widetilde{e}_\alpha).$$

By (3.17), the norm of these $|\lambda||\mu|$ terms is bounded by

$$\begin{aligned} &((|\alpha| - |\lambda| + |\mu| - 1)_+ + n + 1) s_{\lambda'} \varepsilon^{|\lambda'|} s_{\mu'} \varepsilon^{|\mu'|} \times \\ &\qquad\qquad ((|\alpha| - |\lambda''| + |\mu''|)_+ + n)^{-1} s_{\mu''} \varepsilon^{|\mu''|} s_{\lambda''} \varepsilon^{|\lambda''|} (|\alpha| + n) \\ &\le s_{\lambda'} s_{\lambda''} s_{\mu'} s_{\mu''}\, \varepsilon^{|\lambda| + |\mu|}\, \frac{((|\alpha| - |\lambda| + |\mu| - 1)_+ + n + 1)(|\alpha| + n)}{(|\alpha| - |\lambda|)_+ + n}. \end{aligned} \tag{3.18}$$

The remaining commutators are easier, they lead to bounds

$$\begin{cases} s_\lambda s_\mu\, \varepsilon^{|\lambda| + |\mu|}\, 2((|\alpha| - |\lambda|)_+ + n) & \text{(once)}, \\ s_{\lambda'} s_{\lambda''} s_\mu\, \varepsilon^{|\lambda| + |\mu|}\, (|\alpha| - |\lambda| - 1)_+ + n + 1 & (|\lambda| \text{ times}), \\ s_\lambda s_{\mu'} s_{\mu''}\, \varepsilon^{|\lambda| + |\mu|}\, (|\alpha| + n) & (|\mu| \text{ times}). \end{cases} \tag{3.19}$$

In the final estimates, we will have to bound some combinatorial factors of the form

$$\frac{s_{\lambda'} s_{\lambda''}}{s_\lambda} \frac{s_{\mu'} s_{\mu''}}{s_\mu} \quad \text{(worst case)}, \tag{3.20}$$

and we want the ratios $s_{\lambda'} s_{\lambda''} / s_\lambda$ to be as small as possible (clearly they are at least equal to 1). For this, we try to keep the proportions $\lambda'_j / |\lambda'|$, $\lambda''_j / |\lambda''|$ as close as possible to $\lambda_j / |\lambda|$ by selecting carefully which factor W_ℓ^* (and W_m) we exchange at each step. After a permutation of the coordinates, we may assume than $\lambda_n \ge \max_{j<n} \lambda_j$, hence $\lambda_n \ge \frac{1}{n} |\lambda|$. If $t' = |\lambda'| / |\lambda|$ and $t'' = |\lambda''| / |\lambda| = 1 - t' - 1/|\lambda|$, we take $\lambda'_j = \lfloor t' \lambda_j \rfloor$, $\lambda''_j = \lfloor t'' \lambda_j \rfloor$ for $j \le n - 1$ and compensate by taking ad hoc values of λ'_n, λ''_n and $c_\ell = \lambda - (\lambda' + \lambda'')$. Then $t' \lambda_j - 1 < \lambda'_j \le t' \lambda_j$ for $j < n$ and

$$\lambda'_n = |\lambda'| - \sum_{j<n} \lambda'_j \quad \begin{cases} < t'|\lambda| - \sum_{j<n} t' \lambda_j + n - 1 = t' \lambda_n + n - 1, \\ \ge t'|\lambda| - \sum_{j<n} t' \lambda_j = t' \lambda_n. \end{cases}$$

Therefore

$$\frac{\lambda'_j}{|\lambda'|} \le \frac{\lambda_j}{|\lambda|} \quad \text{for } j<n, \qquad \frac{\lambda'_n}{|\lambda'|} \le \frac{t'\lambda_n+n-1}{t'|\lambda|} = \frac{\lambda_n}{|\lambda|}\Big(1+\frac{n-1}{t'\lambda_n}\Big) \quad \text{if } \lambda_n>0.$$

These inequalities imply respectively

$$\Big(\frac{\lambda'_j}{|\lambda'|}\Big)^{\lambda'_j/2} \le \Big(\frac{\lambda_j}{|\lambda|}\Big)^{(t'\lambda_j-1)/2},$$
$$\Big(\frac{\lambda'_n}{|\lambda'|}\Big)^{\lambda'_n/2} \le \Big(\frac{\lambda_n}{|\lambda|}\Big)^{t'\lambda_n/2}\Big(1+\frac{n-1}{t'\lambda_n}\Big)^{(t'\lambda_n+n-1)/2}.$$

In the last inequality we have $t'\lambda_n \ge \frac{1}{|\lambda|}\lambda_n \ge \frac{1}{n}$ unless $\lambda'=0$. Thus, if $\lambda'\ne 0$, we get

$$\begin{aligned}\Big(1+\frac{n-1}{t'\lambda_n}\Big)^{(t'\lambda_n+n-1)/2} &\le \exp\Big(\frac12\frac{n-1}{t'\lambda_n}(t'\lambda_n+n-1)\Big)\\ &\le \exp\Big(\frac12\big(n-1+n(n-1)^2\big)\Big),\end{aligned}$$

and by taking the product over all $j\in\{1,\ldots,n\}$ we find

$$s_{\lambda'} \le e^{n^3/2}\prod_{j\le n}\Big(\frac{\lambda_j}{|\lambda|}\Big)^{t'\lambda_j/2}\prod_{j<n}\Big(\frac{|\lambda|}{\lambda_j}\Big)^{1/2} \le e^{n^3/2}\,(\sigma_\lambda)^{t'}\,|\lambda|^{(n-1)/2}$$

(notice that for $\lambda_j=0$ we also have $\lambda'_j=0$, and the corresponding factors are then equal to 1). Notice also that

$$(s_\lambda)^{-1/|\lambda|} = \prod_{j\le n}\Big(\frac{|\lambda|}{\lambda_j}\Big)^{\lambda_j/2|\lambda|} \le \prod_{j\le n}|\lambda|^{\lambda_j/2|\lambda|} = |\lambda|^{1/2}.$$

For $\lambda',\lambda''\ne 0$, this implies

$$\text{(3.21)}\qquad s_{\lambda'}s_{\lambda''} \le e^{n^3}(s_\lambda)^{t'+t''}\,|\lambda|^{n-1} = e^{n^3}(s_\lambda)^{1-1/|\lambda|}\,|\lambda|^{n-1} \le e^{n^3}s_\lambda\,|\lambda|^n,$$

and our combinatorial factor (3.20) is less than $e^{2n^3}|\lambda|^n|\mu|^n$. When $\lambda'=0$ or $\lambda''=0$ (say $\lambda''=0$), we have $\lambda'=\lambda-c_j$ for some j and $s_{\lambda''}=1$, thus

$$s_{\lambda'}s_{\lambda''} = s_{\lambda'} = s_\lambda\Big(\frac{|\lambda|}{|\lambda|-1}\Big)^{(|\lambda|-1)/2}|\lambda|^{1/2}\,\frac{(\lambda_j-1)^{(\lambda_j-1)/2}}{\lambda_j^{\lambda_j/2}} \le e^{1/2}\,s_\lambda\,|\lambda|^{1/2}$$

and inequality (3.21) still holds. We now put all our bounds together. For all $r' < r =$ radius of convergence of $w \mapsto \Psi_z(w)$, the coefficients $a'_{k,\ell,\lambda}$ satisfy $|a'_{k,\ell,\lambda}| \le C_0(1/r')^{|\lambda|}$ with $C_0 = C_0(r') > 0$, and for every $\xi = \sum_\alpha \xi_\alpha \widetilde{e}_\alpha$, (3.16)–(3.21) imply a bound of the form

$$|\langle Q(\xi),\xi\rangle| \le \varepsilon^{-2} \sum_{\alpha\in\mathbb{N}^n} \sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} C_1 \left(\frac{\varepsilon}{r'}\right)^{|\lambda|+|\mu|} (2+|\lambda|+|\mu|+|\lambda||\mu|) \times \\ |\lambda|^n|\mu|^n \frac{((|\alpha|-|\lambda|+|\mu|-1)_+ + n+1)(|\alpha|+n)}{(|\alpha|-|\lambda|)_+ + n} |\xi_\alpha||\xi_{\alpha-\lambda+\mu}|.$$

Here $|\lambda|+|\mu| \ge 2$, and for $\delta > 0$ arbitrary, there exists $C_2 = C_2(\delta)$ such that

$$(2+|\lambda|+|\mu|+|\lambda||\mu|)\ |\lambda|^n|\mu|^n \le C_2\ (1+\delta)^{|\lambda|+|\mu|-2},$$

thus

$$|\langle Q(\xi),\xi\rangle| \le \frac{C_1C_2}{r'^2} \sum_{\alpha\in\mathbb{N}^n} \sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} \Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2} \times \\ \frac{((|\alpha|-|\lambda|+|\mu|)_+ + n)(|\alpha|+n)}{(|\alpha|-|\lambda|)_+ + n} |\xi_\alpha||\xi_{\alpha-\lambda+\mu}|.$$

Now, we split the summation with respect to (λ,μ) between the two subsets $|\lambda|+|\mu| \le (|\alpha|+n)/2$ and $|\lambda|+|\mu| > (|\alpha|+n)/2$. We find respectively

$$\frac{((|\alpha|-|\lambda|+|\mu|)_+ + n)(|\alpha|+n)}{(|\alpha|-|\lambda|)_+ + n} \le \begin{cases} \sqrt{6}\sqrt{|\alpha|+n)((|\alpha|-|\lambda|+|\mu|)_+ + n)} \\ \frac{6}{n}(|\lambda|+|\mu|)^2. \end{cases}$$

In the first case, we use the inequality

$$2\sqrt{|\alpha|+n)((|\alpha|-|\lambda|+|\mu|)_+ + n)}\ |\xi_\alpha||\xi_{\alpha-\lambda+\mu}| \\ \le (|\alpha|+n)|\xi_\alpha|^2 + (|\alpha|-|\lambda|+|\mu|)_+ + n)|\xi_{\alpha-\lambda+\mu}|^2,$$

and in the second case we content ourselves with the simpler bound

$$2|\xi_\alpha||\xi_{\alpha-\lambda+\mu}| \le |\xi_\alpha|^2 + |\xi_{\alpha-\lambda+\mu}|^2.$$

For $\varepsilon \in \,]0,r[$, the series

$$\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} \Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2} \quad \text{and} \quad \sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} \Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2} (|\lambda|+|\mu|)^2$$

can be made convergent by choosing $r' = (r+\varepsilon)/2 \in \,]\varepsilon, r[$ and $1+\delta = \sqrt{r'/\varepsilon}$, thus there exists a positive continuous and increasing function $\varepsilon \mapsto C(\varepsilon)$ on $]0, r[$ such that

$$|\langle Q(\xi), \xi\rangle| \le C(\varepsilon) \sum_{\alpha\in\mathbb{N}^n} (|\alpha| + n)|\xi_\alpha|^2 \quad \text{for all } \xi \in B_\varepsilon,$$

which is what we wanted. This bound, together with Theorem 3.10 and the estimates from the preliminary discussion yield the following result.

3.22 Theorem. *Let (X, γ) be a compact hermitian manifold equipped with a real analytic metric, and let r we the supremum of the radii r' of the ball bundles $\{\|\zeta\|_\gamma < r'\}$ on which the related exponential map*

$$\mathrm{exph} = \mathrm{exph}_\gamma : \{\|\zeta\|_\gamma < r'\} \subset T_X \to X \times X$$

defines a real analytic diffeomorphism $(z, \zeta) \mapsto (z, \mathrm{exph}_z(\zeta))$. Then, for all $\varepsilon < r$, the curvature tensor of the Bergman bundle (B_ε, h) satisfies an estimate

$$\begin{aligned}&\langle(\Theta_{B_\varepsilon,h}\,\xi)(v, Jv), \xi\rangle_h \\ &\qquad = \varepsilon^{-2} \sum_{\alpha\in\mathbb{N}^n} \left(\Big| \sum_j \sqrt{\alpha_j}\; \xi_{\alpha-c_j} v_j \Big|^2 + (1 + O(\varepsilon)) \sum_j (|\alpha| + n)\; |\xi_\alpha|^2 |v_j|^2 \right)\end{aligned}$$

for every $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, and every tangent vector $v = \sum v_j\, \partial/\partial z_j$, where $O(\varepsilon) = \varepsilon\; C(\varepsilon)$ for a continuous increasing function $\varepsilon \mapsto C(\varepsilon)$ on $]0, r[$. In particular $\Theta_{B_\varepsilon,h}$ is positive definite (and even coercive unbounded) for $\varepsilon < \varepsilon_0$ small enough.

3.23 Remark. Under our real analyticity assumptions, the proof makes clear that there exists an asymptotic expansion

$$\langle(\Theta_{B_\varepsilon,h}\,\xi)(v, Jv), \xi\rangle_h = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, \xi \otimes v),$$

where

$$Q_0(z, \xi \otimes v) = Q_0(\xi \otimes v) = \sum_{\alpha\in\mathbb{N}^n} \left(\Big| \sum_j \sqrt{\alpha_j}\; \xi_{\alpha-c_j} v_j \Big|^2 + \sum_j (|\alpha| + n)\; |\xi_\alpha|^2 |v_j|^2 \right)$$

corresponds to the model case $X = \mathbb{C}^n$. The terms Q_j can be derived from the Taylor expansion of exph associated with the metric γ, and they a priori depend on the coefficients of the torsion and curvature tensor and their derivatives. In the Kähler case, cf. for instance [3, (8.5)], one has $\mathrm{exph}_z(\xi) = z + \xi + O(\overline{z}\xi^2)$ and one can check from the above calculations that $Q_1 = 0$. It would be interesting to identify more precisely Q_1 and Q_2 in general. It is very likely that Q_1 involves the torsion form $d\omega$ and that Q_2 is strongly related to the curvature tensor of (T_X, ω).

3.24 Remark. Although we have already observed that B_ε cannot be a locally trivial holomorphic Hilbert bundle, as follows from Remark 3.2 and the discussion made in the introduction, one can still endow the total space of B_ε and of its Hilbert dual $B_\varepsilon^\vee$ with some sort of weird infinite dimensional complex space structure, for which the projections $\pi : B_\varepsilon \to X$ and $\pi^\vee : B_\varepsilon^\vee \to X$ are holomorphic. Let us start with $B_\varepsilon^\vee$. This space has a lot of global "holomorphic functions", that actually separate all points of $B_\varepsilon^\vee$ except those of the zero section. In fact, every global holomorphic function $F \in \mathcal{B}_{\varepsilon'}(X)$, $\varepsilon' > \varepsilon$, gives rise to a function $\ell_F : B_\varepsilon^\vee \to \mathbb{C}$ where $\ell_F(\xi) = F_{|B_{\varepsilon,z}} \cdot \xi$ for $\xi \in B_{\varepsilon,z}^\vee \subset B_\varepsilon^\vee$. More generally, one can define a presheaf $\mathcal{O}_{B_\varepsilon^\vee}$ of "holomorphic functions" on $B_\varepsilon^\vee$ as follows: if $V \subset B_\varepsilon^\vee$ is an open set, we take $\mathcal{O}_{B_\varepsilon^\vee}(V)$ to be the closure in locally uniform topology in V of the algebra generated by the pull-backs $u \circ \pi^\vee$, $u \in \mathcal{O}_X(\pi^\vee(V))$, and by the functions ℓ_F, $F \in \mathcal{B}_{\varepsilon'}(\pi^\vee(V))$, which are linear on the fibers of $B_\varepsilon^\vee$. One then gets a genuine sheaf $\mathcal{O}_{B_\varepsilon^\vee}$ by sheafifying the above presheaf. The construction of $\mathcal{O}_{B_\varepsilon}$ is made by reversing the roles of B_ε and $B_\varepsilon^\vee$ (the $\overline{\partial}$ operator of $B_\varepsilon^\vee$ being the Von Neumann adjoint of the $(1,0)$ part of the Chern connection of $\nabla^{1,0}$ on B_ε, and the sheaf of "holomorphic sections" of $B_\varepsilon^\vee$ being its kernel).

4. On the invariance of plurigenera for polarized Kähler families

An important unsolved problem of Kähler geometry is the invariance of plurigenera for compact Kähler manifolds, which can be stated as follows.

4.1 Conjecture. *Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that π admits local polarizations, i.e. every point $t_0 \in S$ has a neighborhood V such that $\pi^{-1}(V)$ carries a Kähler metric ω. Then the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$.*

This conjecture has been affirmatively settled by Y.-T. Siu [19] in the case of projective varieties of general type (in which case the proof has been translated into a purely algebraic form by Y. Kawamata [9]), and then by [20] and Păun [17] in the case of arbitrary projective varieties; remarkably, no algebraic proof of the result is known beyond the case proved by Kawamata. Here, we wish to study such results in the Kähler context. This requires a priori substantial modifications of Siu's proof, since the technique involves in a crucial manner the use of an auxiliary ample line bundle. In the light of the previous sections, a potential replacement would be to use the "very ample" Bergman bundles just constructed. Conjecture 4.1 would be a consequence of the following more technical statement.

4.2 Conjecture (Generalized version of the Claudon–Păun theorem)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a polarized family of compact Kähler manifolds over a disc $\Delta \subset \mathbb{C}$, and let $(\mathcal{L}_j, h_j)_{0\le j\le N-1}$ be (singular) hermitian line bundles with semi-positive curvature currents $i\Theta_{\mathcal{L}_j,h_j} \ge 0$ on $\mathcal{X}$. Assume that*

(a) *the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*
(b) *the multiplier ideal sheaf $\mathcal{I}(h_{j|X_0})$ is trivial for $1 \le j \le N-1$.*

Then any section σ of $\mathcal{O}(NK_{\mathcal{X}} + \sum \mathcal{L}_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ over the central fiber X_0 extends into a section $\widetilde{\sigma}$ of $\mathcal{O}(NK_{\mathcal{X}} + \sum \mathcal{L}_j)$ over a certain neighborhood $\mathcal{X}' = \pi^{-1}(\Delta')$ of X_0, where $\Delta' \subset \Delta$ is a sufficienty small disc centered at 0.

The invariance of plurigenera is the special case of Conjecture 4.2 when all line bundles $\mathcal{L}_j$ and their metrics h_j are trivial. Since the dimension $t \mapsto h^0(X_t, mK_{X_t})$ is always upper semicontinuous and since Conjecture 4.2 implies the lower semicontinuity, we conclude that the dimension must be constant along analytic discs, hence along the irreducible base S, by joining any two points through a chain of analytic discs. □

4.3 Remark. A standard cohomological argument shows that we can in fact take $\mathcal{X}' = \mathcal{X}$ in the conclusion of Conjecture 4.2, because the direct image sheaf $\mathcal{E} = \pi_*\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ is coherent, and the restriction $\mathcal{E} \to \mathcal{E} \otimes (\mathcal{O}_\Delta/\mathfrak{m}_0\mathcal{O}_\Delta)$ induces a surjective map at the H^0 level on the Stein space Δ, so we can extend $\widetilde{\sigma}$ mod $\pi^*\mathfrak{m}_0$ to $\mathcal{X}$.

We now indicate how the technology of Bergman bundles could possibly be used to approach the conjectures.

4.4 Lemma. *Let $\mathcal{X}' = \pi^{-1}(\Delta') \to \Delta'$ be the restriction of $\pi : \mathcal{X} \to \Delta$ to a disc $\Delta' \Subset \Delta$ centered at 0, of radius $R' < R$. For $\varepsilon \le \varepsilon_0 = \varepsilon_0(R')$ small enough, one can find a Stein open subset $\mathcal{U}'_\varepsilon \subset \mathcal{X}' \times \overline{\mathcal{X}}$, such that the projection $\mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ is a complex ball bundle over $\mathcal{X}'$ that is locally trivial real analytically.*

Proof. The arguments are very similar to those of §1, except for the fact that $\mathcal{X}$ is no longer compact, but this is not a problem since $\mathcal{X} \to \Delta$ is proper, and since we can always shrink Δ a little bit to achieve uniform bounds (would they be needed). Let γ be a real analytic hermitian metric on $\mathcal{X}$, and $\mathrm{exph} : T_{\mathcal{X}} \to \mathcal{X}$ be the corresponding real analytic and fiber-holomorphic exponential map associated with γ, as in §1. The map exph is no longer everywhere defined, but if we restrict it to the ε-tubular neighborhood of the zero section in $T_{\mathcal{X}'}$, we get for $\varepsilon > 0$ small enough a real analytic diffeomorphism $(z, \xi) \mapsto (z, \mathrm{exph}_z(\xi))$ onto a tubular neighborhood of the diagonal of $\mathcal{X}' \times \mathcal{X}'$. The rest of the proof is identical to what we did in §1, taking

$$\mathcal{U}'_\varepsilon = \{(z, w) \in \mathcal{X}' \times \overline{\mathcal{X}};\ |\mathrm{logh}_z(\overline{w})|_\gamma < \varepsilon\}. \tag{4.5}$$ □

In order to study Conjecture 4.2, we first state a technical extension theorem needed for the proof, which is a special case of the well-known and extremely powerful Ohsawa–Takegoshi theorem [16], see also [14], [15], [5], the general Kähler case stated below being due to [2].

4.6 Proposition. *Let $\pi : \mathcal{Z} \to \Delta$ be a smooth and proper morphism from a* (*non compact*) *Kähler manifold $\mathcal{Z}$ to a disc $\Delta \subset \mathbb{C}$ and let $(\mathcal{L}, h)$ be a* (*singular*) *hermitian line bundle with semi-positive curvature current $\mathrm{i}\Theta_{\mathcal{L},h} \geq 0$ on $\mathcal{Z}$. Let ω be a global Kähler metric on $\mathcal{Z}$, and let $dV_{\mathcal{Z}}$, dV_{Z_0} the respective induced volume elements on $\mathcal{Z}$ and $Z_0 = \pi^{-1}(0)$. Assume that h_{Z_0} is well defined* (*i.e. almost everywhere finite*)*. Then any holomorphic section s of $\mathcal{O}(K_{\mathcal{Z}} + \mathcal{L}) \otimes \mathcal{I}(h_{|Z_0})$ extends into a section $\widetilde{s}$ over $\mathcal{Z}$ satisfying an L^2 estimate*

$$\int_{\mathcal{Z}} \|\widetilde{s}\|^2_{\omega\otimes h} dV_{\mathcal{Z}} \leq C_0 \int_{Z_0} \|s\|^2_{\omega\otimes h} dV_{Z_0},$$

where $C_0 \geq 0$ is some universal constant (*depending on* $\dim \mathcal{Z}$ *and* $\operatorname{diam} \Delta$, *but otherwise independent of $\mathcal{Z}$, $\mathcal{L}$, ...*).

4.7 Remark. The assumptions of Proposition 4.6 imply that $\mathcal{Z}$ is holomorphically convex and complete Kähler, thus, as an alternative to the technique used in [2], the regularization arguments explained in [3] would also apply to yield the result. We leave motivated readers eventually complete such a proof.

Attempt of proof of Conjecture 4.2. Let $p = \mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ be as in Lemma 4.4, and $q = \mathrm{pr}_2 : \mathcal{U}'_\varepsilon \to \overline{\mathcal{X}}$. We take $\varepsilon < \varepsilon_0$ and use on $\mathcal{Z} := \mathcal{U}'_\varepsilon$ a Kähler metric ω_0 defined on the Stein manifold $\mathcal{U}'_{\varepsilon_0}$. On can define e.g. ω_0 as the $\mathrm{i}\partial\overline{\partial}$

of a strictly plurisubharmonic exhaustion function on $\mathcal{U}'_{\varepsilon_0}$, but we can also take the restriction of $\mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\overline{\omega}_{|\overline{\mathcal{X}}}$ where ω is the Kähler metric on the total space $\mathcal{X}$, and $\overline{\omega} = -\,\omega$ the corresponding Kähler metric on the conjugate space $\overline{\mathcal{X}}$.

First step: construction of a sequence of extensions on $\mathcal{Z} = \mathcal{U}'_\varepsilon$ *via the Ohsawa–Takegoshi extension theorem.*

The strategy is to apply iteratively the special case 4.6 of the Ohsawa–Takegoshi extension theorem on the total space of the fibration

$$\pi' = \pi \circ p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}' \to \Delta',$$

and to extend sections of ad hoc pull-backs $p^*\mathcal{G}$ from the zero fiber $Z_0 = \pi'^{-1}(0) = p^{-1}(X_0)$ to the whole of $\mathcal{Z} = \mathcal{U}'_\varepsilon$. We write $h_j = e^{-\varphi_j}$ in terms of local plurisubharmonic weights, and define inductively a sequence of line bundles $\mathcal{G}_m$ by putting $\mathcal{G}_0 = \mathcal{O}_{\mathcal{X}'}$ and

$$\mathcal{G}_m = \mathcal{G}_{m-1} + K_{\mathcal{X}'} + \mathcal{L}_r \qquad \text{if } m = Nq + r,\ \ 0 \le r \le N-1.$$

By construction we have

$$\begin{aligned} \mathcal{G}_m &= mK_{\mathcal{X}'} + \mathcal{L}_1 + \cdots + \mathcal{L}_m, && \text{for } 1 \le m \le N-1\,,\\ \mathcal{G}_{m+N} - \mathcal{G}_m = \mathcal{G}_N &= NK_{\mathcal{X}'} + \mathcal{L}_0 + \cdots + \mathcal{L}_{N-1}\,, && \text{for all } m \ge 0. \end{aligned}$$

The game is to construct inductively families of sections, say $\{\widetilde{f}_j^{(m)}\}_{j=1,\ldots,J(m)}$, of $p^*\mathcal{G}_m$ over $\mathcal{Z}$, together with ad hoc L^2 estimates, in such a way that

(4.8) for $m = 0, \ldots, N-1$, $p^*\mathcal{G}_m$ is generated by L^2 sections $\{\widetilde{f}_j^{(m)}\}_{j=1,\ldots,J(m)}$ on $\mathcal{U}'_{\varepsilon_0}$;

(4.9) we have the m-periodicity relations $J(m+N) = J(m)$ and $\widetilde{f}_j^{(m)}$ is an extension of $f_j^{(m)} := (p^*\sigma)^q f_j^{(r)}$ over $\mathcal{Z}$ for $m = Nq + r$, where $f_j^{(r)} := \widetilde{f}^{(r)}_{j|Z_0}$, $0 \le r \le N-1$.

Property (4.8) can certainly be achieved since $\mathcal{U}'_{\varepsilon_0}$ is Stein, and for $m = 0$ we can take $J(0) = 1$ and $\widetilde{f}_1^{(0)} = 1$. Now, by induction, we equip $p^*\mathcal{G}_{m-1}$ with the tautological metric $|\xi|^2/\sum_\ell |\widetilde{f}_\ell^{(m-1)}(x)|^2$, and

$$\widetilde{\mathcal{G}}_m := p^*\mathcal{G}_m - K_{\mathcal{Z}} = p^*\mathcal{G}_m - (p^*K_{\mathcal{X}'} + q^*K_{\overline{\mathcal{X}}}) = p^*(\mathcal{G}_{m-1} + \mathcal{L}_r) - q^*K_{\overline{\mathcal{X}}}$$

with that metric multiplied by $p^*h_r = e^{-p^*\varphi_r}$ and a fixed smooth metric $e^{-\psi}$ of positive curvature on $(-q^*K_{\overline{\mathcal{X}}})_{|\mathcal{U}'_{\varepsilon_0}}$ (remember that $\mathcal{U}'_{\varepsilon_0}$ is Stein!). It is clear that these metrics have semi-positive curvature currents on $\mathcal{Z}$ (by adjusting ψ, we could even take them to be strictly positive if we wanted). In this setting, we apply the Ohsawa–Takegoshi theorem to the line bundle $K_{\mathcal{Z}} + \widetilde{\mathcal{G}}_m = p^*\mathcal{G}_m$, and extend in this way $f_j^{(m)}$ into a section $\widetilde{f}_j^{(m)}$ over $\mathcal{Z}$. By construction the pointwise norm of that section in $p^*\mathcal{G}_{m|Z_0}$ in a local trivialization of the bundles involved is the ratio

$$\frac{|f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r-\psi},$$

up to some fixed smooth positive factor depending only on the metric induced by ω_0 on $K_{\mathcal{Z}}$. However, by the induction relations, we have

$$\frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r} = \begin{cases} \dfrac{\sum_j |f_j^{(r)}|^2}{\sum_\ell |f_\ell^{(r-1)}|^2} e^{-p^*\varphi_r} & \text{for } m = Nq + r,\ 0 < r \le N-1, \\ \dfrac{\sum_j |f_j^{(0)}|^2}{\sum_\ell |f_\ell^{(N-1)}|^2} |p^*\sigma|^2 e^{-p^*\varphi_0} & \text{for } m \equiv 0 \bmod N,\ m > 0. \end{cases}$$

Since the sections $\{f_j^{(r)}\}_{0\le r<N}$ generate their line bundle on $\mathcal{U}_{\varepsilon_0} \supset \overline{\mathcal{U}'_\varepsilon}$, the ratios involved are positive functions without zeroes and poles, hence smooth and bounded [possibly after shrinking a little bit the base disc Δ', as is permitted]. On the other hand, assumption 4.2 (b) and the fact that σ has coefficients in the multiplier ideal sheaf $\mathcal{I}(h_{0|X_0})$ tell us that $e^{-p^*\varphi_r}$, $1 \le r < m$ and $|p^*\sigma|^2 e^{-p^*\varphi_0}$ are locally integrable on Z_0. It follows that there is a constant $C_1 = C_1(\varepsilon) \ge 0$ such that

$$\int_{Z_0} \frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r-\psi} dV_{\omega_0} \le C_1$$

for all $m \ge 1$ (of course, the integral certainly involves finitely many trivializations of the bundles involved, whereas the integrand expression is just local in each chart). Inductively, the L^2 extension theorem produces sections $\widetilde{f}_j^{(m)}$ of $p^*\mathcal{G}_m$ over $\mathcal{Z}$ such that

$$\int_{\mathcal{Z}} \frac{\sum_j |\widetilde{f}_j^{(m)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(m-1)}|^2} e^{-p^*\varphi_r-\psi} dV_{\omega_0} \le C_2 = C_0 C_1.$$

Second step: applying the Hölder inequality. Put $k = Nq(k) + r(k)$ with $0 \le r(k) < N$, and take $m = Nq(m)$ to be a multiple of N. The Hölder inequality $|\int \prod_{1\le k\le m} u_k d\mu| \le \prod_{1\le k\le m}(\int |u_k|^m d\mu)^{1/m}$ applied to the measure $\mu = dV_{\omega_0}$ and to the product of functions

$$\left(\frac{\sum_j |\widetilde f_j^{(m)}|^2}{\sum_\ell |\widetilde f_\ell^{(0)}|^2}\right)^{1/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} = \prod_{1\le k\le m}\left(\frac{\sum_j |\widetilde f_j^{(k)}|^2}{\sum_\ell |\widetilde f_\ell^{(k-1)}|^2} e^{-p^*\varphi_{r(k)}-\psi}\right)^{1/m}$$

in which $\sum_\ell |\widetilde f_\ell^{(0)}|^2 = |\widetilde f_1^{(0)}|^2 = 1$ and $\sum_j |\widetilde f_j^{(m)}|^2 = |\widetilde f_1^{(m)}|^2$, implies that

$$\int_{\mathcal Z} \big|\widetilde f_1^{(m)}\big|^{2/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} dV_{\omega_0} \le C_2. \tag{4.10}$$

As the functions $\varphi_{r(k)}$ and ψ are locally bounded from above, we infer from this the weaker inequality

$$\int_{\mathcal Z} \big|\widetilde f_1^{(m)}\big|^{2/m} dV_{\omega_0} \le C_3. \tag{4.10$'$}$$

The last inequality is to be understood as an inequality that holds in fact only locally over $\mathcal X'$, on sets of the form $p^{-1}(V)$, where $V \Subset \mathcal X'$ are small coordinate open sets where our line bundles are trivial, so that the section $\widetilde f_1^{(m)}$ of $q(m)\, p^*(NK_{\mathcal X'} + \sum \mathcal L_j)$ can be viewed as a holomorphic function on $p^{-1}(V)$.

Third step: construction of singular hermitian metrics on $NK_{\mathcal X'} + \sum \mathcal L_j$. The rough idea is to extract a weak limit of the m-th root occurring in (4.10), (4.10′), combined with an integration on the fibers of $p : \mathcal Z = \mathcal U'_\varepsilon \to \mathcal X'$, to get a singular hermitian metric on $NK_{\mathcal X'} + \sum \mathcal L_j$. This is the crucial step in the proof, and the place where the Kähler setup will require new arguments; especially, the integration on fibers makes the weak limit argument much less obvious than in the projective setup. Our (yet incomplete) attempt involves the results of §2, §3 on Bergman bundles.

4.11 Proposition. *Assume that the sections $\widetilde f_1^{(m)}$ have been constructed on $\mathcal Z = \mathcal U'_\varepsilon \to \mathcal X'$, $\varepsilon \le \varepsilon_0(R')$, and let us shrink these sections to a smaller neighborhood $\mathcal U'_{\rho\varepsilon}$, $\rho < 1$. Then there exists a subsequence $m \in M_0 \subset \mathbb N$ such that, with respect to local trivializations of the $\mathcal L_j$ and local holomorphic sections $dw = dw_1 \wedge \ldots \wedge dw_{n+1}$ of $K_{\overline{\mathcal X}}$), we have a well defined limit*

$$\theta(z) = \lim_{\substack{m\in M_0\\ m\to+\infty}} \frac{1}{m}\log \int_{w\in\mathcal U'_{\rho\varepsilon,z}} \big|\widetilde f_1^{(m)}(z,w)\big|^2\, i^{(n+1)^2} dw\wedge d\overline w, \quad z \in \mathcal X'$$

that exists almost everywhere on $\mathcal{X}'$, and $H = e^{-N\theta}$ defines a singular hermitian metric on $p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$ satisfying the following estimates:

(a) $|\sigma|^2_H = |\sigma|^2 e^{-N\theta} = 1$ *on* $X_0 \subset \mathcal{X}'$;

(b) $\displaystyle\int_{\mathcal{X}'} e^{-\theta} e^{-\frac{1}{N}(\varphi_0+\ldots+\varphi_{N-1})} dV_\omega < \infty$;

(c) *there are constants* $C_4, C_5 > 0$ *such that* $\theta \le C_4$ *and* $\displaystyle \mathrm{i}\partial\overline{\partial}\theta \ge -\frac{C_5}{\varepsilon^2\rho^2}(C_4-\theta)\,\omega.$

Proof. First notice that the choice of the w local coordinates on $\overline{\mathcal{X}}$ is irrelevant in the definition of θ (the L^2 integrals may eventually change by bounded multiplicative factors, which get killed as $m \to +\infty$). We apply the mean value inequality for plurisubharmonic functions, applied on ω_0-geodesic balls of $\mathcal{Z}$ centered at points $(z,w) \in \mathcal{U}'_{\rho\varepsilon}$ and of radius $\frac{1}{2}(1-\rho)\varepsilon$ (say). As $\dim \mathcal{Z} = 2(n+1)$, we obtain by (4.10′) a uniform upper bound

$$
\begin{aligned}
\sup_{\mathcal{U}'_{\rho\varepsilon,z}} |\widetilde{f}_1^{(m)}|^{2/m} &\le \frac{C_6}{((1-\rho)\varepsilon)^{4(n+1)}} \int_{\mathcal{U}'_\varepsilon} |\widetilde{f}_1^{(m)}|^{2/m}\, i^{(n+1)^2}\, dw \wedge d\overline{w} \\
&\le \frac{C_7}{((1-\rho)\varepsilon)^{4(n+1)}}, \quad \forall z \in \mathcal{X}'.
\end{aligned}
\tag{4.12}
$$

Here our sections can be seen as functions only locally over trivializing open sets of the line bundles in $\mathcal{X}'$, but we can arrange that there are only finitely many of these; hence the transition automorphisms only involve bounded constants, after raising to power $1/m$. At this point, we consider the Bergman bundle $B_\varepsilon \to \mathcal{X}'$, and write locally over $\mathcal{X}'$

$$
\widetilde{f}_1^{(m)}(z,w)\, dw = \sum_{\alpha\in\mathbb{N}^{n+1}} \xi_{m,\alpha}(z)\, \widetilde{e}_\alpha(z,w) \otimes g(z)^{q(m)}, \quad z \in \mathcal{X}',\ w \in \mathcal{U}'_{\varepsilon,z}
$$

in terms of an orthonormal frame $(\widetilde{e}_\alpha)_{\alpha\in\mathbb{N}^{n+1}}$ of B_ε, of the corresponding Hilbert space coefficients $\xi_m = (\xi_{m,\alpha})_{\alpha\in\mathbb{N}^{n+1}}$ as defined in §2, and of a local holomorphic generator g of $\mathcal{O}_{\mathcal{X}}(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$. If we put $dw = dw_1 \wedge \ldots \wedge dw_{n+1}$ in local coordinates, we get an equality

$$
\begin{aligned}
\theta_{m,\rho}(z) &:= \frac{1}{m} \log \int_{w\in\mathcal{U}'_{\rho\varepsilon,z}} \big|\widetilde{f}_1^{(m)}(z,w)\big|^2\, i^{(n+1)^2} dw \wedge d\overline{w} \\
&= \frac{1}{m} \log\Bigg(\sum_{\alpha\in\mathbb{N}^{n+1}} \rho^{2(|\alpha|+n+1)}\, |\xi_{m,\alpha}(z)|^2\Bigg),
\end{aligned}
\tag{4.13}
$$

and by (4.12), we obtain an upper bound

(4.14)
$$\theta_{m,\rho}(z) \le \frac{1}{m}\log\frac{C_8\;(\rho\varepsilon)^{2(n+1)}\;C_7^m}{((1-\rho)\varepsilon)^{4m(n+1)}} \le C_9 + 4(n+1)\;\log\frac{1}{(1-\rho)\varepsilon} =: C_{10,\rho,\varepsilon}.$$

The sum $\sum_{\alpha\in\mathbb{N}^{n+1}}\rho^{2(|\alpha|+n+1)}\;|\xi_{m,\alpha}(z)|^2 = e^{m\theta_{m,\rho}(z)}$ is nothing else than the square of the norm of the section $\widetilde{f}_1^{(m)}$, expressed with respect to the natural hermitian metric $\langle\bullet,\bullet\rangle_\rho$ of the Bergman bundle $B_{\rho\varepsilon}$. The inequalities (4.12) show that the series converges uniformly over the whole of $\mathcal{X}'$. As $\nabla^{0,1}\xi = 0$, a standard calculation with respect to the Bergman connection $\nabla = \nabla^{1,0}+\nabla^{0,1}$ of $B_{\rho\varepsilon}$ implies

$$\begin{aligned} \mathrm{i}\partial\overline{\partial}\theta_{m,\rho} &= \frac{\mathrm{i}}{m\,\|\xi_m\|_\rho^2}\bigg(\langle\nabla^{1,0}\xi_m,\nabla^{1,0}\xi_m\rangle_\rho - \langle\Theta_{B_{\rho\varepsilon}}\xi_m,\xi_m\rangle_\rho - \\ &\qquad\qquad \frac{\langle\nabla^{1,0}\xi_m,\xi_m\rangle_\rho\wedge\overline{\langle\nabla^{1,0}\xi_m,\xi_m\rangle_\rho}}{\|\xi_m\|_\rho^2}\bigg) \\ &\ge -\frac{1}{m}\,\frac{\langle\mathrm{i}\Theta_{B_{\rho\varepsilon}}\xi_m,\xi_m\rangle_\rho}{\|\xi_m\|_\rho^2} \qquad (4.15) \end{aligned}$$

by the Cauchy–Schwarz inequality. On the other hand, as the orthonormal coordinates expressed in $B_{\rho\varepsilon}$ are the $(\rho^{|\alpha|+n+1}\xi_{m,\alpha})$, the curvature bound obtained in §2 yields

$$\langle\mathrm{i}\Theta_{B_{\rho\varepsilon}}\xi_m,\xi_m\rangle_\rho \le (2+O(\rho\varepsilon))(\rho\varepsilon)^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n+1)\,\rho^{2(|\alpha|+n+1)}\;|\xi_{m,\alpha}|^2\,\omega.$$

The last two inequalities imply the fundamental estimate

$$\begin{aligned} &\mathrm{i}\partial\overline{\partial}\theta_{m,\rho} \\ (4.16)\qquad &\ge -\frac{(2+O(\rho\varepsilon))(\rho\varepsilon)^{-2}}{m}\;\frac{\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n+1)\,\rho^{2(|\alpha|+n+1)}\;|\xi_{m,\alpha}|^2}{\sum_{\alpha\in\mathbb{N}^{n+1}}\rho^{2(|\alpha|+n+1)}\;|\xi_{m,\alpha}|^2}\,\omega \\ (4.16')\qquad &\ge -\frac{1+O(\rho\varepsilon)}{\varepsilon^2\rho}\Big(\frac{\partial}{\partial\rho}\theta_{m,\rho}\Big)\,\omega. \end{aligned}$$

From its definition, we see that $\theta_{m,\rho}$ is a convex function of $\log\rho$. Therefore, for $\rho \le \rho_1 < 1$, we have

$$\rho\,\frac{\partial}{\partial\rho}\theta_{m,\rho} \le \frac{\theta_{m,\rho_1}-\theta_{m,\rho}}{\log\rho_1-\log\rho} \le \frac{C_{9,\rho_1,\varepsilon}-\theta_{m,\rho}}{\log\rho_1},$$

by (4.14), and (4.16′) implies

$$\mathrm{i}\partial\overline{\partial}\theta_{m,\rho} \ge -\frac{C_{11}}{\varepsilon^2\rho^2}(C_{10,\rho_1,\varepsilon}-\theta_{m,\rho})\,\omega.$$

A straightforward calculation yields

$$-\mathrm{i}\partial\overline{\partial}\log(C_{10,\rho_1,\varepsilon}+1-\theta_{m,\rho}) \ge -\frac{C_{11}}{\varepsilon^2\rho^2}\,\omega,$$

hence the functions $u_m = -\log(C_{10,\rho_1,\varepsilon}+1-\theta_{m,\rho}) \le 0$ have Hessian forms that are uniformly bounded from below. Also, by construction (cf. 4.9), $\theta_{m,\rho}$ converges to $\frac{1}{N}\log|\sigma|$ on X. Standard results of pluripotential theory imply that we can find a subsequence of (u_m) that converges in L^p topology (for every $p\in[1,+\infty[$) and pointwise almost everywhere. Therefore we can find a limit $\theta_{m,\rho}\to\theta$ satisfying the Hessian estimates

$$\mathrm{i}\partial\overline{\partial}\theta \ge -\frac{C_{11}}{\varepsilon^2\rho^2}(C_{10,\rho_1,\varepsilon}-\theta)\,\omega, \qquad -\mathrm{i}\partial\overline{\partial}\log(C_{10,\rho_1,\varepsilon}+1-\theta) \ge -\frac{C_{11}}{\varepsilon^2\rho^2}\,\omega$$

Proposition 4.11 is proved, as estimate (b) follows from (4.10).

Fourth step: applying Ohsawa–Takegoshi once again with the singular hermitian metric produced in the third step. Assume that we can replace estimate 4.11 (c) by the stronger fact that the curvature form of $H = e^{-N\theta}$ is positive in the sense of currents, i.e.

$$-\mathrm{i}\partial\overline{\partial}\log H = N\mathrm{i}\partial\overline{\partial}\theta \ge 0. \tag{4.17}$$

This means that $NK_{\mathcal{X}'}+\sum\mathcal{L}_j$ possesses a hermitian metric H such that $\|\sigma\|_H\le 1$ on X_0 and $\Theta_H\ge 0$ on $\mathcal{X}'$. In order to conclude, we proceed as Siu and Păun, and equip the bundle

$$\mathcal{E} = (N-1)K_{\mathcal{X}'}+\sum\mathcal{L}_j$$

with the metric $\eta = H^{1-1/N}\prod h_j^{1/N}$, and $NK_{\mathcal{X}'}+\sum\mathcal{L}_j = K_{\mathcal{X}'}+\mathcal{E}$ with the metric $\omega\otimes\eta$. It is important here that $\mathcal{X}$ possesses a global Kähler polarization ω, otherwise the required estimates would not be valid. Clearly η has a semi-positive curvature current on $\mathcal{X}'$ and in a local trivialization we have

$$\|\sigma\|^2_{\omega\otimes\eta} \le C|\sigma|^2 \exp\Big(-(N-1)\theta - \frac{1}{N}\sum \varphi_j\Big) \le C\Big(|\sigma|^2 \prod e^{-\varphi_j}\Big)^{1/N}$$

on X_0. Since $|\sigma|^2 e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r > 0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega\otimes\eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E}, \eta)$ implies that σ can be extended to $\mathcal{X}'$. Conjecture 4.2 would then be proved.

Fifth step: final discussion. Unfortunately, estimate (4.17) will a priori hold only in the case where ε can be taken arbitrarily large (in the sense that the exponential map is at least everywhere an immersion – one can then argue on the "unfolded neighborhood" $\widetilde{U}_\varepsilon$ diffeomorphic to the ε-tubular neighborhood of the 0 section in T_X, equipped with the complex structure obtained by pulling back the complex structure of $X \times \overline{X}$ via exph. This condition is met e.g. when X is a complex torus or a ball quotient. However, it is doubtful that all compact Kähler manifolds with K_X pseudo-effective satisfy this property. The main issue is that the unboundedness of $\Theta_{B_\varepsilon,h}$ does not a priori imply that the right hand side of (4.15) converges weakly to 0, while this is obviously true in the algebraic situation where we use instead a given ample line bundle A on $\mathcal{X}$. One possible way to circumvent this difficulty is to observe that the term $\langle \mathrm{i}\Theta_{B_{\rho\varepsilon}}\xi_m, \xi_m\rangle_\rho$ is controlled by $\|\xi_m\|_\rho \|\xi_m\|'_\rho$ where

$$\begin{aligned}\|\xi_m\|'^2_\rho :&= \sum_{\alpha\in\mathbb{N}^{n+1}} (|\alpha|+n+1)^2\, \rho^{2(|\alpha|+n+1)}\, |\xi_{m,\alpha}|^2 \\ &\sim \int_{w\in\mathcal{U}'_{\rho\varepsilon,z}} \big|\widetilde{f}^{(m)}_1(z,w)\big|^2 + \big|D_w \widetilde{f}^{(m)}_1(z,w)\big|^2,\end{aligned}$$

and it would be sufficient to find extensions $\widetilde{f}^{(m)}_1$ satisfying the additional estimate

$$\int_{w\in\mathcal{U}'_{\rho\varepsilon,z}} \big|D_w \widetilde{f}^{(m)}_1(z,w)\big|^2 \le K_m \int_{w\in\mathcal{U}'_{\rho\varepsilon,z}} \big|\widetilde{f}^{(m)}_1(z,w)\big|^2 \tag{4.18}$$

where K_m grows subquadratically, i.e. $\frac{1}{m^2}K_m \to 0$. Getting such an estimate, e.g. a bound $K_m = O(m)$ in the general situation, does not appear to be completely implausible, since the main inductive step consists of extending a section multiplied by $p^*\sigma(z,w) = \sigma(z)$, which is therefore independent of w on $\mathcal{X}'_0$. In this process, one might hope to obtain an appropriate L^2 extension theorem taking care of "vertical derivatives" with respect to a given morphism $\mathcal{Y} \to \mathcal{X} \to \Delta$ (namely, $\mathcal{U}'_\varepsilon \to \mathcal{X}' \to \Delta'$ in this circumstance). We will try to investigate these questions in the near future.

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