Pure and Applied Mathematics Quarterly
Volume 18, Number 1, 305–316, 2022

Compactness of Kähler–Ricci solitons on Fano manifolds*

Bin Guo, Duong H. Phong, Jian Song, and Jacob Sturm

Abstract: In this short paper, we improve the result of Phong–Song–Sturm on degeneration of Fano Kähler–Ricci solitons by removing the assumption on the uniform bound of the Futaki invariant. Let $\mathcal{KR}(n)$ be the space of Kähler–Ricci solitons on n-dimensional Fano manifolds. We show that after passing to a subsequence, any sequence in $\mathcal{KR}(n)$ converge in the Gromov–Hausdorff topology to a Kähler–Ricci soliton on an n-dimensional $\mathbb{Q}$-Fano variety with log terminal singularities.

Keywords: Kähler–Ricci solitons, Fano manifolds.

1. Introduction

The Ricci solitons on compact and complete Riemannian manifolds naturally arise as models of singularities for the Ricci flow [7]. The existence and uniqueness of Ricci solitons has been extensively studied. A gradient Ricci soliton is a Riemannian metric satisfying the following soliton equation

$$Ric(g) = \lambda g + \nabla^2 u \tag{1.1}$$

for some smooth function f with $\lambda = -1, 0, 1$. Such a soliton is called a gradient shrinking Ricci soliton if $\lambda > 0$. If we let the vector field $\mathcal{V}$ be defined by $\mathcal{V} = \nabla u$, the soliton equation becomes

$$Ric(g) = \lambda g + L_{\mathcal{V}} g, \tag{1.2}$$

where $L_{\mathcal{V}}$ is the Lie derivative along $\mathcal{V}$.

A Kähler metric g on a Kähler manifold X is called a Kähler–Ricci soliton if it satisfies the soliton equation (1.1) or equation (1.2) for $\mathcal{V} = \nabla u$. Any shrinking Kähler–Ricci soliton on a compact Kähler manifold X must be a

Received February 27, 2020.
*Work supported in part by National Science Foundation grants DMS-1711439, DMS-12-66033 and DMS-1710500.

gradient Ricci soliton and such a Kähler manifold must be a Fano manifold, i.e. $c_1(X) > 0$. The vector field $\mathcal{V}$ must be holomorphic and it can be expressed in terms of the Ricci potential u, with

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}}, \quad u_{ij} = u_{\bar{i}\bar{j}} = 0, \; \mathcal{V}^i = -g^{i\bar{j}} u_{\bar{j}}. \tag{1.3}$$

The well-known Futaki invariant associated to the Kähler–Ricci soliton $(X, g, \mathcal{V})$ on a Fano manifold X is given by

$$\mathcal{F}_X(\mathcal{V}) = \int_X |\nabla u|^2 dV_g = \int_X |\mathcal{V}|^2 dV_g \geq 0.$$

Let $\mathcal{KR}(n, F)$ be the set of compact Kähler–Ricci solitons (X, g) of complex dimension n with

$$Ric(g) = g + L_{\mathcal{V}} g, \; \mathcal{F}_X(\mathcal{V}) \leq F.$$

It is proved by Tian–Zhang [17] that $\mathcal{KR}(n, F)$ is compact in the Gromov–Hausdorff topology with an additional uniform upper volume bound. In [11], Phong–Song–Sturm established a partial C^0-estimate on $\mathcal{KR}(n, F)$, generalizing the celebrated result of Donaldson–Sun [6] for the space of uniformly non-collapsed Kähler manifolds with uniform Ricci curvature bounds. An immediate consequence of the partial C^0-estimate in [11] is that the limiting metric space must be a $\mathbb{Q}$-Fano variety equipped with a Kähler–Ricci soliton metric.

The purpose of this paper is to remove the assumption in [11] on the bound of the Futaki invariant.

Definition 1.1. Let $\mathcal{KR}(n)$ be the set of compact Kähler–Ricci solitons $(X, g, \mathcal{V})$ of complex dimension n with

$$Ric(g) = g + L_{\mathcal{V}} g.$$

The following is the main result of the paper.

Theorem 1.1. *Let $\{(X_i, g_i, \mathcal{V}_i)\}_{i=1}^{\infty}$ be a sequence in $\mathcal{KR}(n)$ with $n \geq 2$. Then after possibly passing to subsequence, (X_i, g_i) converges in the Gromov–Hausdorff topology to a compact metric length space (X_∞, d_∞) satisfying the following.*

1. *The singular set Σ_∞ of the metric space (X_∞, d_∞) is a closed set of Hausdorff dimension no greater than $2n - 4$.*

2. $(X_i, g_i, \mathcal{V}_i)$ *converges smoothly to a Kähler–Ricci soliton* $(X_\infty \setminus \Sigma_\infty, g_\infty, \mathcal{V}_\infty)$ *satisfying*

$$Ric(g_\infty) = g_\infty + L_{\mathcal{V}_\infty} g_\infty, \tag{1.4}$$

where $\mathcal{V}_\infty$ *is a holomorphic vector field on* $X_\infty \setminus \Sigma_\infty$*.*

3. (X_∞, d_∞) *coincides with the metric completion of* $(X_\infty \setminus \Sigma_\infty, g_\infty)$ *and it is a projective* $\mathbb{Q}$*-Fano variety with log terminal singularities. The soliton Kähler metric* g_∞ *extends to a Kähler current on* X_∞ *with bounded local potential and* $\mathcal{V}_\infty$ *extends to a global holomorphic vector field on* X_∞*.*

The assumption on the bound of the Futaki invariant in [11] is used to obtain a uniform lower bound of Perelman's μ-functional. We use the recent deep result of Birkar [1] in birational geometry and show that there exists $\epsilon(n) > 0$ such that for any n-dimensional Fano manifold X, there exists a Kähler metric g with $Ric(g) \geq \epsilon g$. In particular, the μ-functional for (X, g) is bounded below by a uniform constant that only depends on n. Then for any Kähler–Ricci soliton $(X, g) \in \mathcal{KR}(n)$, the μ-functional for (X, g) is uniformly bounded below because the soliton metric is the limit of the Kähler–Ricci flow. The proof of Theorem 1.1 also implies a uniform bound for the scalar curvature and the Futaki invariant for all $(X, g) \in \mathcal{KR}(n)$.

Corollary 1.1. *There exist* $F = F(n)$*,* $D = D(n)$ *and* $K = K(n) > 0$ *such that for any* $(X, g, u) \in \mathcal{KR}(n)$*, the Futaki invariant, the diameter and scalar curvature* R *of* (X, g) *satisfy*

$$\mathcal{F}_X \leq F, \ diam(X, g) \leq D, \ 0 < R \leq K.$$

We also derive some general compactness for compact or complete gradient shrinking solitons assuming a uniform lower bound of Perelman's μ-functional (see Section 3). For any closed or complete gradient shrinking soliton (M, g, u), one can always normalize u such that $\int_M e^{-u} dV_g = 1$. We define $\mathcal{RS}(n, A)$ to be the space of closed or complete shrinking gradient soliton (M, g, u) of real dimension $n \geq 4$ satisfying

$$\mu(g) \geq -A. \tag{1.5}$$

Then for any $A \geq 0$ and any sequence $(M_j, g_j, u_j, p_j) \in \mathcal{RS}(n, A)$ with p_j being the minimal point of u_j, after passing to a subsequence, it converges in the pointed Gromov–Hausdorff topology to a compact or complete metric space (M_∞, d_∞) of dimension n with smooth convergence to a shrinking gradient Ricci soliton outside the closed singular set of dimension no greater than $n-4$.

2. Proof of Theorem 1.1

Let us first recall the α-invariant introduced by Tian on a Fano manifold [14].

Definition 2.1. On a Fano manifold (X,ω) with $\omega \in c_1(X)$, the α-invariant is defined as

$$\alpha(X) = \sup\{\alpha > 0 |\ \exists C_\alpha \text{ s.t. } \int_X e^{-\alpha(\varphi - \sup_X \varphi)}\omega^n \le C_\alpha, \forall \varphi \in PSH(X,\omega)\}.$$

It is obvious that the $\alpha(X)$ does not depend on the choice $\omega \in c_1(X)$.

Definition 2.2. Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-Cartier divisor, the pair (X,Δ) is said to be log canonical if the coefficients of components of Δ are no greater than 1 and there exists a log resolution $\pi : Y \to X$ such that $\pi^{-1}(\text{supp}\Delta) \cup \text{exc}(\pi)$ is a divisor with normal crossings satisfying

$$K_Y = \pi^*(K_X + \Delta) + \sum_j a_j F_j, \quad \mathbb{Q} \ni a_j \ge -1, \forall\ j.$$

Definition 2.3. Let X be a projective manifold and D be a $\mathbb{Q}$-Cartier divisor. The *log canonical threshold* of D is defined by

$$\text{lct}(X,D) = \sup\{t \in \mathbb{R} \mid (X,tD) \text{ is log canonical}\}.$$

It is proved by Demailly that the α-invariant is related to the log canonical thresholds of anti-canonical divisors through the following formula (see Theorem A.3. in the Appendix A of [5]).

Theorem 2.1. *For any Fano manifold X,*

$$\alpha(X) = \inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|-mK_X|} \text{lct}(X, m^{-1}D)$$

The following lemma is known among birational geometers on the uniform positive lower bound for the α-invariant. The result was first derived by Chenyang Xu using upper semi-continuity of log canonical thresholds and we refer the readers to [9] (Proposition 2.4) for the very short proof.

Lemma 2.1. *There exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that for any n-dimensional Fano manifold X*

$$\alpha(X) \geq \varepsilon_0(n).$$

We remark that the above lemma also follows from recent work of Birkar (Theorem 1.4 of [1]).

From the Harnack inequality in [14], for any fixed Kähler metric $\omega \in c_1(X)$, the curvature equation for ω_t along the continuity method

$$Ric(\omega_t) = t(\omega_t) + (1-t)\omega \tag{2.1}$$

can be solved for all $t \in [0, (n+1)\alpha(X)/n)$. As a consequence, we have the following corollary.

Corollary 2.1. *There exists $\varepsilon_1 = \varepsilon_1(n) > 0$ such that for any n-dimensional Fano manifold X, there exists a Kähler metric $\hat{\omega} \in c_1(X)$ satisfying*

$$Ric(\hat{\omega}) \geq \varepsilon_1 \hat{\omega}. \tag{2.2}$$

We can also assume that $\hat{\omega}$ is invariant under the group action of the maximal compact subgroup G of $Aut(X)$ by choosing a G-invariant Kähler metric ω in the equation (2.1).

The greatest Ricci lower bound $R(X)$ for a Fano manifold X is introduced in [16, 12] and is defined by

$$\mathcal{R}(X) = \sup\{t \in \mathbb{R} \mid \text{there exits } \omega \in c_1(X) \text{ such that } Ric(\omega) \geq t\omega\}.$$

Immediately one has the following corollary.

Corollary 2.2. *There exists an $r_0 = r_0(n) > 0$ such that for any n-dimensional Fano manifold X,*

$$\mathcal{R}(X) \geq r_0. \tag{2.3}$$

Now let us recall Perelman's entropy functional for a Fano manifold (X, g) with the associated Kähler form $\omega_g \in c_1(X)$. The $\mathcal{W}$-functional is defined by

$$\mathcal{W}(g, f) = \frac{1}{V}\int_X (R + |\nabla f|^2 + f - 2n)e^{-f} dV_g,$$

where $V = c_1^n(X)$, and the μ-functional is defined by

$$\mu(g) = \inf_f \left\{ \mathcal{W}(g, f) \;\middle|\; \frac{1}{V}\int_X e^{-f} dV_g = 1 \right\}.$$

Lemma 2.2. *There exists $A = A(n) > 0$ such that for the Riemannian metric $\hat{g}$ associated to the form $\hat{\omega}$ in (2.2)*

$$\mu(\hat{g}) \geq -A.$$

Proof. Since $Ric(\hat{g})$ is bounded from below by a uniform positive constant $\varepsilon_1(n)$, by Myers' theorem and volume comparison,

$$\mathrm{Vol}(X, \hat{g}) \leq C(n), \quad \mathrm{diam}(X, \hat{g}) \leq C(n).$$

On the other hand, since $\hat{\omega} \in c_1(X)$ is in an integral cohomology class, in particular $\mathrm{Vol}(X, \hat{g}) \geq c(n) > 0$. By Croke's theorem, the Sobolev constant C_S of $(X, \hat{g})$ is uniformly bounded. It is well-known that a Sobolev inequality implies the lower bound of μ-functional. For completeness, we provide a proof below.

For any $f \in C^\infty$ with $\int_X e^{-f} dV_{\hat{g}} = V$, we write $e^{-f/2} = \phi$. By Jensen's inequality

$$\begin{aligned} \frac{1}{V} \int_X \phi^2 \log \phi^{\frac{2}{n-1}} &\leq \log \Big(\frac{1}{V} \int_X \phi^{\frac{2n}{n-1}} \Big) \\ &\leq \log \Big(C_S \int_X (|\nabla \phi|^2 + \phi^2) \Big) \\ &\leq \frac{4}{n-1} \int_X |\nabla \phi|^2 + C(n). \end{aligned}$$

So

$$\mathcal{W}(\hat{g}, f) = \frac{1}{V} \int_X (R\phi^2 + 4|\nabla\phi|^2 - \phi^2 \log \phi^2) dV_{\hat{g}} - 2n \geq -C(n). \qquad \square$$

Let $(X, g) \in \mathcal{KR}(n)$ be a gradient shrinking Kähler–Ricci soliton which satisfies the equation

$$Ric(\omega_g) + \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} u = \omega_g, \quad \nabla\nabla u = 0. \tag{2.4}$$

Let $G \subset Aut(X)$ be the compact one-parameter subgroup generated by the holomorphic vector field $\mathrm{Im}(\nabla u)$. As we mentioned before, the metric $\hat{\omega}$ in (2.2) can be taken to be G-invariant.

Corollary 2.3. *For any $(X, g) \in \mathcal{KR}(n)$, we have*

$$\mu(g) \geq -A,$$

where $A = A(n)$ is the constant in Lemma 2.2.

Proof. We consider the normalized Kähler–Ricci flow with initial metric $\hat{\omega}$ in (2.2) G-invariant.

$$\frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + \omega(t), \quad \omega(0) = \hat{\omega}.$$

By the convergence theorem for Kähler–Ricci flow ([17, 19]), $\omega(t)$ converges smoothly to ω_g, modulo some diffeomorphisms. So $\lim_{t\to\infty} \mu(g(t)) = \mu(g)$.

On the other hand, $\mu(g(t))$ is monotonically non-decreasing along the Kähler–Ricci flow ([10]). The lower bound of $\mu(g)$ follows from this monotonicity and the lower bound of $\mu(\hat{g})$ established in Lemma 2.2. □

Now we can apply the same argument as in [11] because the assumption of the uniform bound for the Futaki invariant in [11] is to obtain a uniform lower bound for the μ-functional. This will complete the proof of Theorem 1.1. The argument in [11] also implies the uniform bound for the scalar curvature and diameter of $(X, g, u) \in \mathcal{KR}(n)$ as well as $|\nabla u|^2$ and hence the Futaki invariant of (X, g). This implies Corollary 1.1.

3. Generalizations

We generalize our previous discussion to Riemannian complete gradient shrinking Ricci solitons (M^n, g, u) satisfying the equation

$$Ric(g) + \nabla^2 u = \frac{1}{2} g.$$

By [3] we can always normalize u such that $\int_M e^{-u} dV_g = 1$.

Definition 3.1. We denote $\mathcal{RS}(n, A)$ to the set of n-dimensional closed or complete shrinking gradient Ricci solitons (M, g, u) satisfying

$$\mu(g) \geq -A$$

with the normalization condition $\int_M e^{-u} dV_g = 1$.

The following proposition is the main result of this section and most results in the proposition are straightforward applications of the compactness results [20, 21] with Bakry–Emery Ricci curvature bounded below.

Proposition 3.1. *Let* $\{(M_i, g_i, u_i, p_i)\}_{i=1}^{\infty}$ *be a sequence in* $\mathcal{RS}(n, A)$ *with* $n \geq 4$*, where* p_i *be a minimal point of* u_i*. Then after possibly passing to subsequence,* (M_i, g_i, u_i, p_i) *converges in the Gromov–Hausdorff topology to a metric length space* $(M_\infty, d_\infty, u_\infty)$ *satisfying the following.*

1. *The singular set* Σ_∞ *of the metric space* (M_∞, d_∞) *is a closed set of Hausdorff dimension no greater than* $n-4$.
2. (M_i, g_i, u_i) *converges smoothly to a gradient shrinking Ricci soliton* $(M_\infty \backslash \Sigma_\infty, g_\infty, u_\infty)$ *satisfying*

$$Ric(g_\infty) = \frac{1}{2} g_\infty + \nabla^2 u_\infty.$$

3. (M_∞, d_∞) *coincides with the metric completion of* $(M_\infty \setminus \Sigma_\infty, g_\infty)$.

Furthermore, if there exists $V > 0$ *such that* $Vol_{g_i}(M_i) \leq V$ *for all* $i = 1, 2, \ldots$, *the limiting metric space* (M_∞, d_∞) *is compact.*

Proof. For any $(M, g, u) \in \mathcal{RS}(n, A)$, $R = n/2 - \Delta u \geq 0$ ([23]), the potential function u satisfies

$$\Delta u - |\nabla u|^2 + u = a, \quad a = \int_M u e^{-u} dV_g.$$

We denote $\tilde{u} = u - a$. From $\Delta u \leq n/2$ and immediately we have $|\nabla \tilde{u}|^2 \leq n/2 + \tilde{u}$. By [3], the minimum of $\tilde{u}$ is achieved at some finite point $p \in M$, so $\min \tilde{u} = \tilde{u}(p) \geq -n/2$. Applying maximum principle to $\tilde{u}$ which satisfies $\Delta \tilde{u} - |\nabla \tilde{u}|^2 + \tilde{u} = 0$ at a minimum point $p \in M$, we obtain that $\min_M \tilde{u} = \tilde{u}(p) \leq 0$.

From $|\nabla \tilde{u}|^2 \leq \tilde{u} + n/2$, we have $|\nabla \sqrt{\tilde{u} + n/2}| \leq \frac{1}{2}$. Thus for any $x \in M$

$$\tilde{u}(x) \leq \frac{1}{2} d(p,x)^2 + \tilde{u}(p) + C(n) \leq \frac{1}{2} d(p,x)^2 + C(n). \tag{3.1}$$

Immediately we have

$$|\nabla \tilde{u}|^2(x) \leq \frac{1}{2} d(p,x)^2 + C(n), \tag{3.2}$$

and

$$-n/2 \leq -\Delta \tilde{u}(x) \leq \frac{1}{2} d(p,x)^2 + C(n). \tag{3.3}$$

When (M, g) is closed and $\mathrm{Vol}(M, g) \leq V$. We note by Jensen's inequality $a \leq \log V$. The Ricci soliton (M, g, u) gives rise to a Ricci flow $g(t) = \varphi_t^* g$ with initial metric $g(0) = g$, where φ_t is the diffeomorphism group generated by ∇u, $\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + g(t)$. Combining with the fact that $R(g) \geq 0$ and Perelman's non-collapsing theorem, we see that (M, g) is non-collapsed in the sense that if $R \leq r^{-2}$ on $B_r(x)$, then $\mathrm{Vol}(B_r(x)) \geq \kappa(n, A)r^n$, for all $r \in (0, \bar{r}(n, A)]$. With this non-collapsing and equations (3.1), (3.2) and (3.3), we can apply the same argument of Perelman as in Section 3 of [13] to show that there exists a uniform constant $C(n, A, V) > 0$ such that for any closed $(M, g, u) \in \mathcal{RS}(n, A)$ with the additional assumption $\mathrm{Vol}(M, g) \leq V$,

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty(M,g)} + \|R\|_{L^\infty} + \mathrm{diam}(M, g) \leq C(n, A, V). \tag{3.4}$$

The non-collapsing of (M, g) also implies a uniform lower bound on $\mathrm{Vol}(M, g)$. Now we can apply the main theorem of [22].

In general, when (M, g) is complete, applying [10] to the Ricci flow associated to (M, g), there exists a $\kappa = \kappa(A, n)$ such that (M, g) is κ-noncollapsed. In particular, $\mathrm{Vol}(B(p, 1)) \geq c(A, n) > 0$. On any geodesic ball $B(p, r)$ with p being the minimal point of u, $|\nabla u| \leq \frac{1}{2}r^2 + C(n, A)$. By the Cheeger–Colding theory for Bakry–Emery Ricci tensor $Ric(g) + \nabla^2 u$ ([20, 21]), for any sequence of $(M_i, g_i, u_i, p_i) \in \mathcal{RS}(n, A)$ converges (up to a subsequence) in pointed Gromov–Hausdorff topology to a metric space $(M_\infty, d_\infty, p_\infty)$. Here we choose p_i to be a minimum point of u_i. M_∞ has the regular-singular decomposition $M_\infty = \mathcal{R} \cup \Sigma$. Recall a point $y \in \mathcal{R}$ if all tangent cone of (M_∞, d_∞) at y is isometric to $\mathbb{R}^n$. From [21] we know the singular set Σ is closed and of Hausdorff dimension at most $n - 4$ and d_∞ on $\mathcal{R}$ is induced by a C^α metric g_∞. For any $y \in \mathcal{R}$ and $M_i \ni y_i \xrightarrow{GH} y$, when i is large enough there exists a uniform $r_0 = r_0(y)$ such that $(B_{g_i}(y_i, r_0), g_i)$ has uniform C^α bound (Theorem 1.2 of [21]). By choosing r_0 even smaller if possible, we may assume the isoperimetric constant of $(B_{g_i}(y_i, r_0), g_i)$ is very small so that we can apply Perelman's pseudo-locality theorem ([10]) to the associated Ricci flow to derive uniform higher order estimates of g_i nearby y_i, which in turn gives local estimates of u_i. So locally near y_i, the convergence is smooth and we conclude that the metric g_∞ in a small ball around y is a Ricci soliton. □

We remark that in the compact case, a compactness result is obtained earlier by Zhang [22] assuming a uniform upper bound for the diameter and a uniform lower bound for the volume.

Acknowledgements

The authors would like to thank Xiaowei Wang and Chenyang Xu for valuable discussions. After our paper was posted on the arXiv, we were informed by Eiji Inoue that he had also derived the compactness result for Kähler–Ricci solitons in [8] by an algebraic approach using the boundedness result of Kollar–Mori–Mukai. Our proof is essentially analytic. Both proofs by us and Inoue are built on our earlier compactness result in [11].

References

[1] Birkar, C., Singularities of linear systems and boundedness of Fano varieties, arXiv:1609.05543v1. MR4224714

[2] Cao, H. D., Geometry of Ricci solitons, *Chinese Ann. Math. Ser. B* **27** (2006), no. 2, 121–142. MR2243675

[3] Cao, H. D. and Zhou, D., On complete gradient shrinking Ricci solitons, *J. Diff. Geom.* **85** (2010), no. 2, 175–186. MR2732975

[4] Cheeger, J., Colding, T. H. and Tian, G., On the singularities of spaces with bounded Ricci curvature, *Geom. Funct. Anal.* **12** (2002), 873–914. MR1937830

[5] Cheltsov, I. A. and Shramov, K. A., Log-canonical thresholds for nonsingular Fano threefolds, with an appendix by J.-P. Demailly, *Uspekhi Mat. Nauk* **63** (2008), no. 5(383), 73–180. MR2484031

[6] Donaldson, S. and Sun, S., Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry, *Acta Math.* **213** (2014), no. 1, 63–106. MR3261011

[7] Hamilton, R., The Ricci flow on surfaces. Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, *Contemp. Math.*, **71**, Amer. Math. Soc., Providence, RI, 1988. MR0954419

[8] Inoue, E., The moduli space of Fano manifolds with Kähler–Ricci solitons, *Adv. Math.* **357** (2019), 106841, 65 pp., arXiv:1802.08128. MR4017922

[9] Odaka, Y., On the moduli of Kahler–Einstein Fano manifolds, arXiv:1211.4833.

[10] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159 [math.DG].

[11] Phong, D. H., Song, J. and Sturm, J., Degeneration of Kähler–Ricci solitons on Fano manifolds, *Univ. Iagel. Acta Math.* **52** (2015), 29–43. MR3438282

[12] Szekelyhidi, G., Greatest lower bounds on the Ricci curvature of Fano manifolds, *Compos. Math.* **147** (2011), no. 1, 319–331. MR2771134

[13] Sesum, N. and Tian, G., Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman) and some applications, *Jour. Inst. Math. Juss.* **7** (2008), 575–587. MR2427424

[14] Tian, G., On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* **89** (1987), no. 2, 225–246. MR0894378

[15] Tian, G., On Calabi's conjecture for complex surfaces with positive first Chern class, *Invent. Math.* **101** (1990), no. 1, 101–172. MR1055713

[16] Tian, G., On stability of the tangent bundles of Fano varieties, *Internat. J. Math.* **3** (1992), 401–413. MR1163733

[17] Tian, G. and Zhang, Z., Degeneration of Kähler–Ricci solitons, *Int. Math. Res. Not. IMRN* **5** (2012), 957–985. MR2899957

[18] Tian, G. and Zhu, X., Convergence of Kähler–Ricci flow, *J. Amer. Math. Soc.* **20** (2007), no. 3, 675–699. MR2291916

[19] Tian, G., Zhang, S., Zhang, Z. and Zhu, X., Perelman's entropy and Kähler–Ricci flow on a Fano manifold, *Trans. Amer. Math. Soc.* **365** (2013), no. 12, 6669–6695. MR3105766

[20] Wang, F. and Zhu, X., Structure of spaces with Bakry–Emery Ricci curvature bounded below, arXiv:1304.4490. MR2970058

[21] Zhang, Q. S. and Zhu, M., Bounds on harmonic radius and limits of manifolds with bounded Bakry–Emery Ricci curvature, arXiv:1705.10071v2. MR3969422

[22] Zhang, Z., Degeneration of shrinking Ricci solitons, *Int. Math. Res. Not. IMRN* **21** (2010), 4137–4158. MR2738353

[23] Zhang, Z., On the completeness of gradient Ricci solitons, *Proc. Amer. Math. Soc.* **137** (2009), no. 8, 2755–2759. MR2497489

Bin Guo
Department of Mathematics
Columbia University
New York, NY 10027
USA
Current address:
Department of Mathematics & Computer Science
Rutgers University
Newark, NJ 07102
USA
E-mail: bguo@rutgers.edu

Duong H. Phong
Department of Mathematics
Columbia University
New York, NY 10027
USA
E-mail: phong@math.columbia.edu

Jian Song
Department of Mathematics
Rutgers University
Piscataway, NJ 08854
USA
E-mail: jiansong@math.rutgers.edu

Jacob Sturm
Department of Mathematics & Computer Science
Rutgers University
Newark, NJ 07102
USA
E-mail: sturm@newark.rutgers.edu