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On Calabi–Yau fractional complete intersections

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Abstract: In this article, we study mirror symmetry for pairs of *singular* Calabi–Yau varieties which are double covers of toric manifolds. Their period integrals can be seen as certain 'fractional' analogues of those of ordinary complete intersections. This new structure can then be used to solve their Riemann–Hilbert problems. The latter can then be used to answer definitively questions about mirror symmetry for this class of Calabi–Yau varieties.

Keywords: Calabi–Yau, mirror symmetry, fractional complete intersections.

1. Introduction

1.1. Motivation

Mirror symmetry from physics has successfully made numerous predictions in algebraic geometry and attracted lots of attentions in the past thirty years. Roughly, mirror symmetry asserts that for a Calabi–Yau space X there exists a Calabi–Yau space $X^{\vee}$ such that $A(X) \cong B(X^{\vee})$ and $B(X) \cong A(X^{\vee})$. Here $A(X)$, the A model of X, is taken to be the genus zero Gromov–Witten theory whereas $B(X)$, the B model of X, is the variation of Hodge structures.

Various examples of mirror pairs have been constructed. The first mirror pair was given by Greene and Plesser [10], leading to the spectacular prediction of genus zero Gromov–Witten invariants for quintic threefolds [6]. Batyrev generalized the construction to the case of Calabi–Yau hypersurfaces in Gorenstein Fano toric varieties by making use of reflexive polytopes [2], leading to similar predictions of Gromov–Witten invariants for general Calabi–Yau toric hypersurfaces [15]. Later, Batyrev and Borisov gave a general recipe to construct mirror pairs in the case of Calabi–Yau complete intersections in Gorenstein Fano toric varieties by nef-partitions [3]. At the same time, Bershadsky et al. [4] developed their fair reaching theory of topological strings which led to predictions of Gromov–Witten invariants in higher genera.

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Thanks to Torelli theorem, the B model is locally completely determined by period integrals of the Calabi–Yau families. It is known that the period integrals satisfy a set of partial differential equations, known as the Picard–Fuchs equations. Batyrev observed in [1] that the period integrals of a family of Calabi–Yau hypersurfaces or complete intersections in a fixed Gorenstein Fano toric variety satisfy a generalized hypergeometric system introduced by Gel'fand, Kapranov, and Zelevinskiĭ [8], which is called the GKZ A-hypergeometric system nowadays. For a family of Calabi–Yau hypersurfaces or complete intersections in toric varieties, we attempt to understand its period integrals through the GKZ A-hypergeometric systems associated with it.

Hosono et al. observed that the Gröbner basis with respect to the typical weight for the toric ideal determines a finite set of differential operators for the local solutions to the GKZ A-hypergeometric system [15]. For such a GKZ A-hypergemetric system, they also proved that there exists a special boundary point called a maximal degeneracy point on a resolution of the secondary fan compactication of the moduli [16]. It is a point over which for all but one period integrals can not be extended holomorphically; namely, up to a constant, there exists a unique holomorphic period at that point. To study the moduli locally near such a special boundary point, inspired by mirror symmetry, the *generalized Frobenius method* was developed in [11, 15]. Starting with the holomorphic period, the method allows us to produce other period integrals. The generalized Frobenius method gives a uniform treatment to describe the local solutions near a maximal degeneracy point in the moduli.

The works in [14, 17] shed light on a new construction of a mirror pair of *singular* Calabi–Yau varieties. Hosono, Takagi and the last two authors investigated the family of $K3$ surfaces arising from double covers branched over six lines in $\mathbb{P}^2$ and proposed a singular version of mirror symmetry. Recently, together with Hosono, the authors gave a general recipe to construct pairs of *singular* Calabi–Yau varieties $(Y, Y^\vee)$ and showed that they are topological mirror pairs in dimension three [13]; in other words, we have $h^{p,q}(Y) = h^{3-p,q}(Y^\vee)$ for all $0 \le p, q \le 3$.

1.2. Statements of main results

The aim of this note is to straighten the results in [15, 16] to our singular topological mirror pairs.

Consider a nef-partition $(\Delta, \{\Delta_i\}_{i=1}^r)$ and its dual nef-partition $(\nabla, \{\nabla_i\}_{i=1}^r)$ in the sense of Batyrev and Borisov. Let $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$ be the toric varieties defined by Δ and ∇. Let $X \to \mathbf{P}_\Delta$ and $X^\vee \to \mathbf{P}_\nabla$ be maximal projective crepant partial resolutions (MPCP resolutions for short hereafter) of $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$. The nef-partitions on $\mathbf{P}_\Delta$ and $\mathbf{P}_\nabla$ determine nef-partitions on X and $X^\vee$. Let $E_1, \ldots, E_r$ and $F_1, \ldots, F_r$ be the sum of toric divisors representing nef-partitions on X and $X^\vee$. We assume that both X and $X^\vee$ are *smooth* throughout this note.

For a nef-partition $F_1 + \cdots + F_r$ on $X^\vee$, we can define a family $\mathcal{Y}^\vee \to V$ of singular Calabi–Yau varieties as follows. For each i, let $s_{i,1}, s_{i,2} \in \mathrm{H}^0(X^\vee, F_i)$ be sections such that $\mathrm{div}(s_{i,1}) \equiv F_i$ and $\mathrm{div}(s_{i,2})$ is smooth. Let $Y^\vee \to X^\vee$ be a double cover branched over $\cup_{i=1}^r \cup_{j=1}^2 \mathrm{div}(s_{i,j})$. Let

$$V \subset W^\vee := \mathrm{H}^0(X^\vee, F_1) \times \cdots \times \mathrm{H}^0(X^\vee, F_r)$$

be an open subset such that $\sum_{i=1}^r \sum_{j=1}^2 \mathrm{div}(s_{i,j})$ is a simple normal crossing divisor. Deforming $s_{i,2}$ in V, we obtain the said family of double covers of X, which is called *the gauge fixed double covers family* in this paper. Similarly, the dual nef partition $E_1 + \cdots + E_r$ gives another family $\mathcal{Y} \to U$.

To state our main results, let us introduce some notation. Let $N \simeq \mathbb{Z}^n$ be a lattice in which the fan of X sits and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let Σ be the fan defining X. The nef-partition $E_1 + \cdots + E_r$ on X determines a decomposition $\sqcup_{k=1}^r I_k$ of $\Sigma(1)$, the set of 1-cones in Σ. We can write

$$\Sigma(1) = \{\rho_{i,j} \colon \rho_{i,j} \in I_i \text{ for } \ 1 \le i \le r,\ 1 \le j \le n_i = \# I_i\}.$$

The primitive generator for the 1-cone $\rho_{i,j}$ is again denoted by $\rho_{i,j}$. For $1 \le i \le r$ and $1 \le j \le n_i$, we put $\nu_{i,j} := (\rho_{i,j}, \delta_{1,i}, \ldots, \delta_{r,i})$ and additionally $\nu_{i,0} := (\mathbf{0}, \delta_{1,i}, \ldots, \delta_{r,i})$, where $\delta_{i,j}$ is the Kronecker delta. Let

$$A_{\mathrm{ext}} := \begin{bmatrix} \nu_{1,0}^{\intercal} & \cdots & \nu_{r,n_r}^{\intercal} \end{bmatrix} \in \mathrm{Mat}_{(n+r)\times(p+r)}(\mathbb{Z}),\ p = n_1 + \cdots + n_r.$$

It turns out that the *affine period integrals* (For a precise definition, see §2.5.)

$$\Pi_\gamma(x) := \int_\gamma \frac{1}{s_{1,2}^{1/2} \cdots s_{r,2}^{1/2}} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n} \tag{1}$$

for the gauge fixed double cover family $\mathcal{Y}^\vee \to V$ satisfy a GKZ A-hypergeometric system associated with the matrix A_{ext} and a *fractional* exponent

$$\beta = \begin{bmatrix} \mathbf{0} & -1/2 & \cdots & -1/2 \end{bmatrix}^{\intercal} \in \mathbb{Q}^{n+r}.$$

Note that for ordinary complete intersections the exponents appearing in the denominator in the affine period integrals would be integers. But for gauge fixed double cover families, the exponents become half integers (Hence 'fractional' complete intersections).

The affine period integrals of $\mathcal{Y}^\vee \to V$ form a local system on $W^\vee \setminus \mathcal{D}$ for some closed subset $\mathcal{D}$. Let $T_M := \mathrm{Hom}(N, \mathbb{C}^*)$. The space $W^\vee$ is equipped with a $T_M \times (\mathbb{C}^*)^r$ action via the inclusion $T_M \times (\mathbb{C}^*)^r \hookrightarrow (\mathbb{C}^*)^{\dim W^\vee}$ and the affine periods are invariant under this action. In other words, the periods descend to local sections of a locally constant sheaf on $S_{W^\vee}$, where $S_{W^\vee}$ is the image of $(\mathbb{C}^*)^{\dim W^\vee} \setminus \mathcal{D}$ under

$$(\mathbb{C}^*)^{\dim W^\vee} \to (\mathbb{C}^*)^{\dim W^\vee}/T_M \times (\mathbb{C}^*)^r.$$

Following the idea in [16], we compactify $(\mathbb{C}^*)^{\dim W^\vee}/T_M \times (\mathbb{C}^*)^r$ into a toric variety via the secondary fan $S\Sigma$ and the Gröbner fan $G\Sigma$. Our first theorem in this note is

Theorem 1.1 (=Theorem 3.12)**.** *For every toric resolution $X_{G\Sigma'} \to X_{G\Sigma}$, there exists at least one maximal degeneracy point in $X_{G\Sigma'}$.*

The precise definition of maximal degeneracy points is given in Definition 3.1. The secondary fan $S\Sigma$ is natural from combinatorics whereas the Gröbner fan $G\Sigma$ contains more information about our GKZ system. The proof of Theorem 1.1 is parallel to the proof given in [16].

Let $L_{\mathrm{ext}} := \ker(A_{\mathrm{ext}} \colon \mathbb{Z}^{p+r} \to \mathbb{Z}^{n+r})$. Note that the Mori cone $\overline{\mathrm{NE}}(X)$ is a cone in $L_{\mathrm{ext}} \otimes \mathbb{R}$. Pick an $\alpha \in \mathbb{C}^{p+r}$ such that $A_{\mathrm{ext}}(\alpha) = \beta$. As observed in [15], after a renormalization, a solution to the GKZ system is given by

$$(2) \qquad \sum_{\ell \in L_{\mathrm{ext}}} \frac{\prod_{i=1}^r \Gamma(-\ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^r \Gamma(-\alpha_{i,0}) \prod_{i=1}^r \prod_{j=1}^{n_i} \Gamma(\ell_{i,j} + \alpha_{i,j} + 1)} (-1)^{\sum_i \ell_{i,0}} x^{\ell+\alpha}.$$

Here the components of $L_{\mathrm{ext}} \subset \mathbb{Z}^{p+r}$ are labeled by (i,j) with $1 \le i \le r$ and $0 \le j \le n_i$. The variables $x_{i,j}$ (again $1 \le i \le r$ and $0 \le j \le n_i$) are the coordinates for the GKZ A-hypergeometric system associated with $\mathcal{Y}^\vee \to V$.

Let $D_{i,j}$ be the Weil divisor associated with $\rho_{i,j}$. Combining 2 with these cohomology classes, we introduce a cohomology-valued power series

$$(3) \qquad B_X^\alpha(x) := \left(\sum_{\ell \in \overline{\mathrm{NE}}(X) \cap L_{\mathrm{ext}}} \mathcal{O}_\ell^\alpha x^{\ell+\alpha} \right) \exp\left(\sum_{i=1}^r \sum_{j=0}^{n_i} (\log x_{i,j}) D_{i,j} \right),$$

where

$$\mathcal{O}_\ell^\alpha := \frac{\prod_{i=1}^r (-1)^{\ell_{i,0}} \Gamma(-D_{i,0} - \ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^r \Gamma(-\alpha_{i,0}) \prod_{i=1}^r \prod_{j=1}^{n_i} \Gamma(D_{i,j} + \ell_{i,j} + \alpha_{i,j} + 1)}$$

and $D_{i,0} := -\sum_{j=1}^{n_i} D_{i,j}$.

The cohomology-valued series (3) was introduced by Hosono et al. in [15] (a.k.a. Givental's I-function up to an overall Γ-factor [9]) which encodes the information from the A model and the B model for a Calabi–Yau mirror pair.

We regard $B_X^{\alpha}(x)$ as an element in $\mathbb{C}[\![x_{i,j}]\!]\otimes_{\mathbb{C}}\mathrm{H}^{\bullet}(X,\mathbb{C})$. Our second result in this note is the following theorem.

Theorem 1.2 (=Corollary 4.4)**.** *When $h\in\mathrm{H}^{\bullet}(X,\mathbb{C})^{\vee}$ runs through a basis of $\mathrm{H}^{\bullet}(X,\mathbb{C})^{\vee}$, the pairings $\langle B_X^{\alpha}(x),h\rangle$ give a complete set of solution to the GKZ A-hypergeometric system associated with $\mathcal{Y}^{\vee}\to V$.*

A direct calculation shows that $\langle B_X^{\alpha}(x),h\rangle$ is a solution to the GKZ A-hypergeometric system associated with $\mathcal{Y}^{\vee}\to V$. See also [5]. The dimension of the solution space to this GKZ system is given by the normalized volume of A_{ext}, which turns out to be equal to the dimension of $\mathrm{H}_n(Y^{\vee},\mathbb{C})$ if n is odd for a generic fiber $Y^{\vee}$.

Theorem 1.2 solves the Riemann–Hilbert problem for the periods of the family of Calabi–Yau varieties $\mathcal{Y}^{\vee}$. It gives a complete description for the Picard–Fuchs system of the periods of this family in terms of a GKZ system.

2. Preliminaries

We begin with some notation and terminologies.

- Let $N=\mathbb{Z}^n$ be a rank n lattice and $M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ be its dual lattice. Let $N_{\mathbb{R}}:=N\otimes_{\mathbb{Z}}\mathbb{R}$ and $M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}$.
- Let Σ be a fan in $N_{\mathbb{R}}$ and X_{Σ} be the toric variety determined by Σ. Let $T\subset X_{\Sigma}$ be its maximal torus with coordinates $t_1,\ldots,t_n$.
- We denote by $\Sigma(k)$ the set of k-dimensional cones in Σ. In particular, $\Sigma(1)$ is the set of 1-cones in Σ. Similarly, for a cone $\sigma\in\Sigma$, we denote by $\sigma(1)$ the set of 1-cones belonging to σ. By abuse of the notation, we also denote by ρ the primitive generator of the corresponding 1-cone.
- Each ρ determines a T-invariant Weil divisor on X_{Σ}, which is denoted by D_{ρ} hereafter. Any T-invariant Weil divisor D is of the form $D=\sum_{\rho\in\Sigma(1)}a_{\rho}D_{\rho}$. The polyhedron of D is defined to be
$$\Delta_D:=\{m\in M_{\mathbb{R}}\colon\langle m,\rho\rangle\geq -a_{\rho}\text{ for all }\rho\}\,.$$
The integral points $M\cap\Delta_D$ gives rise to a canonical basis of $\mathrm{H}^0(X_{\Sigma},D)$.
- A *nef-partition* on X_{Σ} is a decomposition of $\Sigma(1)=\sqcup_{k=1}^{r}I_k$ such that $E_k:=\sum_{\rho\in I_k}D_{\rho}$ is nef for each k. Recall that a divisor D is called nef if $D.C\geq 0$ for any irreducible complete curve $C\subset X_{\Sigma}$. We also have $E_1+\cdots+E_r=-K_{X_{\Sigma}}$.

- A polytope in $M_{\mathbb{R}}$ is called a *lattice polytope* if its vertices belong to M. For a lattice polytope Δ in $M_{\mathbb{R}}$, we denote by Σ_{Δ} the normal fan of Δ. The toric variety determined by Δ is denoted by $\mathbf{P}_{\Delta}$, i.e., $\mathbf{P}_{\Delta} = X_{\Sigma_{\Delta}}$.
- A *reflexive polytope* $\Delta \subset M_{\mathbb{R}}$ is a lattice polytope containing the origin $\mathbf{0} \in M_{\mathbb{R}}$ in its interior and such that the polar dual $\Delta^{\vee}$ is again a lattice polytope. If Δ is a reflexive polytope, then $\Delta^{\vee}$ is also a lattice polytope and satisfies $(\Delta^{\vee})^{\vee} = \Delta$. The normal fan of Δ is the face fan of $\Delta^{\vee}$ and vice versa.

2.1. The Batyrev–Borisov duality construction

We briefly recall the construction of the dual nef-partition [3]. Let $I_1, \ldots, I_r$ be a nef-partition on $\mathbf{P}_{\Delta}$. This gives rise to a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$, where $\Delta_i = \Delta_{E_i}$ is the section polytope of E_i. Following Batyrev–Borisov, let ∇_k be the convex hull of $\{\mathbf{0}\} \cup I_k$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ be their Minkowski sum. One can prove that ∇ is a reflexive polytope in $N_{\mathbb{R}}$ whose polar dual is $\nabla^{\vee} = \mathrm{Conv}(\Delta_1, \ldots, \Delta_r)$ and $\nabla_1 + \cdots + \nabla_r$ corresponds to a nef-partition on $\mathbf{P}_{\nabla}$, called the *dual nef-partition*. The corresponding nef toric divisors are denoted by $F_1, \ldots, F_r$. Then the section polytope of F_j is ∇_j.

Let $X \to \mathbf{P}_{\Delta}$ and $X^{\vee} \to \mathbf{P}_{\nabla}$ be maximal projective crepant partial (MPCP for short hereafter) resolutions for $\mathbf{P}_{\Delta}$ and $\mathbf{P}_{\nabla}$. Via pullback, the nef-partitions on $\mathbf{P}_{\Delta}$ and $\mathbf{P}_{\nabla}$ determine nef-partitions on X and $X^{\vee}$ and they determine the families of Calabi–Yau complete intersections in X and $X^{\vee}$ respectively.

Recall that the section polytopes Δ_i and ∇_j correspond to E_i on $\mathbf{P}_{\Delta}$ and F_j on $\mathbf{P}_{\nabla}$, respectively. To save the notation, the corresponding nef-partitions and toric divisors on X and $X^{\vee}$ will be still denoted by Δ_i, ∇_j and E_i, F_j respectively.

2.2. Calabi–Yau double covers

We briefly review the construction of Calabi–Yau double covers in [13]. Let $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ be a dual pair of nef-partitions representing $E_1 + \cdots + E_r$ on $-K_{\mathbf{P}_{\Delta}}$ and $F_1 + \cdots + F_r$ on $-K_{\mathbf{P}_{\nabla}}$ respectively. Let X and $X^{\vee}$ be the MPCP resolution of $\mathbf{P}_{\Delta}$ and $\mathbf{P}_{\nabla}$ respectively. Hereafter, we will simply call the decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ a nef-partition on X for short with understanding the nef-partition $E_1 + \cdots + E_r$ and likewise for the decomposition $\nabla = \nabla_1 + \cdots + \nabla_r$. Unless otherwise stated, we assume that

$$X \text{ and } X^\vee \text{ are both smooth.}$$

Equivalently, we assume that both Δ and ∇ admit uni-modular triangulations. From the duality, we have

$$\mathrm{H}^0(X^\vee, F_i) \simeq \bigoplus_{\rho \in \nabla_i \cap N} \mathbb{C} \cdot t^\rho \text{ and } \mathrm{H}^0(X, E_i) \simeq \bigoplus_{m \in \Delta_i \cap M} \mathbb{C} \cdot t^m.$$

Here we use the same notation $t = (t_1, \ldots, t_n)$ to denote the coordinates on the maximal torus of $X^\vee$ and X.

A double cover $Y^\vee \to X^\vee$ has trivial canonical bundle if and only if the branch locus is linearly equivalent to $-2K_{X^\vee}$. Let $Y^\vee \to X^\vee$ be the double cover constructed from the section $s = s_1 \cdots s_r$ with

$$(s_1, \ldots, s_r) \in \mathrm{H}^0(X^\vee, 2F_1) \times \cdots \times \mathrm{H}^0(X^\vee, 2F_r).$$

We assume that $s_i \in \mathrm{H}^0(X^\vee, 2F_i)$ is of the form $s_i = s_{i,1}s_{i,2}$ with $s_{i,1}, s_{i,2} \in \mathrm{H}^0(X^\vee, F_i)$. We further assume that $s_{i,1}$ is the section corresponding to the lattice point $\mathbf{0} \in \nabla_i \cap N$, i.e., the scheme-theoretic zero of $s_{i,1}$ is F_i, and that the scheme-theoretic zero of $s_{i,2}$ is non-singular. Deforming $s_{i,2}$, we obtain a subfamily of double covers of $X^\vee$ branched over the nef-partition parameterized by an open subset

$$V \subset \mathrm{H}^0(X^\vee, F_1) \times \cdots \times \mathrm{H}^0(X^\vee, F_r).$$

Definition 2.1. Given a decomposition $\nabla = \nabla_1 + \cdots + \nabla_r$ representing a nef-partition $F_1 + \cdots + F_r$ on $X^\vee$, the subfamily $\mathcal{Y}^\vee \to V$ constructed above is called the *family of gauge fixed double covers of $X^\vee$ branched over the nef-partition* or simply the *gauge fixed double cover family* if no confuse occurs.

Given a decomposition $\nabla = \nabla_1 + \cdots + \nabla_r$ representing a nef-partition $F_1 + \cdots + F_r$ on $X^\vee$ as above, we denote by $\mathcal{Y}^\vee \to V$ the gauge fixed double cover family. A parallel construction is applied for the dual decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ representing the dual nef-partition $E_1 + \cdots + E_r$ over X and this yields another family $\mathcal{Y} \to U$, where U is an open subset in

$$\mathrm{H}^0(X, E_1) \times \cdots \times \mathrm{H}^0(X, E_r).$$

2.3. Notation and conventions

Let us fix the notation and conventions we are going to use throughout this note. We resume the situation and notation in §2.1.

- Let $X \to \mathbf{P}_\Delta$ be a MPCP resolution and Σ be the fan defining X. We will assume throughout this note that both X and $X^\vee$ are *smooth*.
- Let $I_1, \ldots, I_r$ be the induced nef-partition on X as before. We label the elements in I_k by $i_{k,1}, \ldots, i_{k,n_k}$ where $n_k = \#I_k$. We define $p = n_1 + \cdots + n_r$. We will write
$$\Sigma(1) = \{\rho_{i,j}\}_{1 \le i \le r,\ 1 \le j \le n_i}\,.$$
For convenience, we will also write $D_{i,j}$ for the Weil divisor associated with $\rho_{i,j}$.
- Let $\nu_{i,j} := (\rho_{i,j}, \delta_{1,i}, \ldots, \delta_{r,i}) \in N \times \mathbb{Z}^r$ be the lifting of $\rho_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. We additionally put $\nu_{i,0} := (\mathbf{0}, \delta_{1,i}, \ldots, \delta_{r,i}) \in N \times \mathbb{Z}^r$ for $1 \le i \le r$.
- We define an order on the set of double indexes by declaring $(i,j) \preceq (i',j')$ if and only if $i \le i'$ or $i = i'$ and $j \le j'$. Recall that $\#\{(i,j) \colon 1 \le i \le r,\ 0 \le j \le n_i\} = p + r$. There are unique bijections
$$\begin{aligned} J &:= \{(i,j) \colon 1 \le i \le r,\ 0 \le j \le n_i\} \to \{1, \ldots, p+r\} \subset (\mathbb{Z}, \le), \\ I &:= \{(i,j) \colon 1 \le i \le r,\ 1 \le j \le n_i\} \to \{1, \ldots, p\} \subset (\mathbb{Z}, \le), \end{aligned}$$
preserving the order.
- For a positive integer s and a matrix $A_{\mathrm{ext}} \in \mathrm{Mat}_{s \times (p+r)}(\mathbb{Z})$ (resp. $A \in \mathrm{Mat}_{s \times p}(\mathbb{Z})$), we will label the columns of A_{ext} by J (resp. the columns of A by I) and speak the $(k,l)^{\mathrm{th}}$ column of A_{ext} instead of the $(\sum_{1 \le i \le k-1}(n_i + 1) + l + 1)^{\mathrm{th}}$ column of A_{ext} (resp. the $(k,l)^{\mathrm{th}}$ column of A instead of the $(\sum_{1 \le i \le k-1} n_i + l)^{\mathrm{th}}$ column of A). For instance, for $A_{\mathrm{ext}} \in \mathrm{Mat}_{s \times (p+r)}(\mathbb{Z})$, the $(1,0)^{\mathrm{th}}$ column of A_{ext} is the 1^{st} column of A_{ext}. The $(r, n_r)^{\mathrm{th}}$ column of A_{ext} is the last column of A_{ext}.
- Define the matrices
$$\begin{aligned} A &:= \begin{bmatrix} \nu_{1,1}^{\mathsf{T}} & \cdots & \nu_{r,n_r}^{\mathsf{T}} \end{bmatrix} \in \mathrm{Mat}_{(n+r) \times p}(\mathbb{Z}), \\ A_{\mathrm{ext}} &:= \begin{bmatrix} \nu_{1,0}^{\mathsf{T}} & \cdots & \nu_{r,n_r}^{\mathsf{T}} \end{bmatrix} \in \mathrm{Mat}_{(n+r) \times (p+r)}(\mathbb{Z}). \end{aligned}$$
According to our convention, the columns of A are labeled by I and the columns of A_{ext} are labeled by J. We have the following commutative

diagram

$$\begin{array}{ccc} \mathbb{Z}^{p+r} & \xrightarrow{A_{\text{ext}}} & \mathbb{Z}^{n+r} \\ \downarrow & & \downarrow \\ \mathbb{Z}^{p} & \xrightarrow{A} & \mathbb{Z}^{n}. \end{array}$$

The left vertical map is given by forgetting the $(i,0)^{\text{th}}$ component for all $1 \leq i \leq r$. The right vertical map is given by projecting to the first n coordinates. By assumption, A_{ext} and A are surjective. Let $L_{\text{ext}} := \ker(A_{\text{ext}})$ and $L = \ker(A)$. We then have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\text{ext}} & \longrightarrow & \mathbb{Z}^{p+r} & \xrightarrow{A_{\text{ext}}} & \mathbb{Z}^{n+r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^{p} & \xrightarrow{A} & \mathbb{Z}^{n} & \longrightarrow & 0 \end{array}$$

where the leftmost vertical arrow is an isomorphism.

- Each element $\ell \in \mathbb{Z}^s$ can be uniquely written as $\ell^+ - \ell^-$ where $\ell^\pm \in \mathbb{Z}^s_{\geq 0}$ whose supports are disjoint.

2.4. GKZ A-hypergeometric systems

We adapt the notation in §2.3. For $1 \leq i \leq r$, let $W_i = \mathbb{C}^{n_i+1}$. Let $x_{i,0}, \ldots, x_{i,n_i}$ be a fixed coordinate system on the dual space $W_i^\vee$. Set $\partial_{i,j} = \partial/\partial x_{i,j}$. Given the matrix A_{ext} as above and a parameter $\beta \in \mathbb{C}^{n+r}$, the A-hypergeometric ideal $I(A_{\text{ext}}, \beta)$ is the left ideal of the Weyl algebra $\mathscr{D} = \mathbb{C}[x, \partial]$ on the *dual* vector space $W^\vee := W_1^\vee \times \cdots \times W_r^\vee$ generated by the following two types of operators

- The "box operators": $\partial^{\ell^+} - \partial^{\ell^-}$, where $\ell^\pm \in \mathbb{Z}^{p+r}_{\geq 0}$ satisfy $A_{\text{ext}}\ell^+ = A_{\text{ext}}\ell^-$. Here the multi-index convention is used.
- The "Euler operators": $\mathscr{E}_k - \beta_k$, where $\mathscr{E}_k = \sum_{(i,j)\in J} \langle \nu_{i,j}, \mathrm{e}_k\rangle x_{i,j}\partial_{i,j}$. Here $\mathrm{e}_k = (\delta_{k,1}, \ldots, \delta_{k,n+r}) \in \mathbb{Z}^{n+r}$.

The A-hypergeometric system $\mathcal{M}(A_{\text{ext}}, \beta)$ is the cyclic $\mathscr{D}$-module $\mathscr{D}/I(A_{\text{ext}}, \beta)$. As shown by Gel'fand et al. [8], $\mathcal{M}(A_{\text{ext}}, \beta)$ is a holonomic $\mathscr{D}$-module.

2.5. Affine period integrals

Let $\mathcal{Y}^\vee \to V$ be the gauge fixed double cover family constructed in §2.2. Fix a reference fiber $Y^\vee = \mathcal{Y}^\vee_\bullet$ and let R be the branch locus of the cover $\pi\colon Y^\vee \to X^\vee$. Instead of looking at the integral of the holomorphic top form on $Y^\vee$ over cycles in $\mathrm{H}_n(Y^\vee, \mathbb{C})$, we will work over the maximal torus and consider affine period integrals.

Definition 2.2. For a gauge fixed double cover family $\mathcal{Y}^\vee \to V$ as above, we define *affine period integrals* to be

$$\Pi_\gamma(x) := \int_\gamma \frac{1}{s_{1,2}^{1/2} \cdots s_{r,2}^{1/2}} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}, \tag{4}$$

where $\gamma \in \mathrm{H}_n(X^\vee \setminus R, \mathscr{E})$ and $s_{i,2} = x_{i,0} + \sum_{j=1}^{n_i} x_{i,j} t^{\rho_{i,j}} \in \mathrm{H}^0(X^\vee, F_i) = W_i^\vee$ is the universal section. Here $\mathscr{E}$ is the local system over $X^\vee \setminus R = (\mathbb{C}^*)^n \setminus \cup_{i=1}^r \{s_{i,2} = 0\}$ whose monodromy exponent around $\{s_{i,2} = 0\}$ is $1/2$. We also define the *normalized affine period integrals* to be $\overline{\Pi}_\gamma(x) := (\prod_{i=1}^r x_{i,0})^{1/2} \Pi_\gamma(x)$.

Note that the integrand is also multi-valued. The precise meaning of the integral (4) is explained in [7, §2.2]. Set $\mathcal{A}_{\mathrm{ext}} = \{\nu_{i,j} \colon (i,j) \in J\}$. We identify $\mathbb{C}^{\mathcal{A}_{\mathrm{ext}}}$ with

$$W^\vee := W_1^\vee \times \cdots \times W_r^\vee = \mathrm{H}^0(X^\vee, F_1) \times \cdots \times \mathrm{H}^0(X^\vee, F_r).$$

Then the affine period integrals (4) form a local system on $\mathbb{C}^{\mathcal{A}_{\mathrm{ext}}} \setminus \mathcal{D}$ for some closed subset $\mathcal{D}$ and in general have monodromies.

From the explicit form in (4), it is straightforward to see that

Proposition 2.1. *The affine period integrals satisfy the GKZ system $\mathcal{M}(A_{\mathrm{ext}}, \beta)$ with*

$$\beta = \begin{bmatrix} \mathbf{0} & -1/2 & \cdots & -1/2 \end{bmatrix}^\intercal \in \mathbb{C}^{n+r}.$$

In the region $\mathcal{R} := \{x \in \mathbb{C}^{\mathcal{A}_{\mathrm{ext}}} \colon |x_{i,0}| \gg \max_j\{|x_{i,j}|\} \text{ for all } i = 1, \ldots, r\}$, by making use of the power series expansion

$$\frac{1}{\sqrt{1-w}} = \sum_{k \geq 0} r_k w^k, \text{ for } |w| \ll 1$$

we can write

$$\left(\frac{x_{i,0}}{s_{i,2}}\right)^{1/2} = \sum_{k\geq 0} \frac{r_k}{x_{i,0}^k} (-x_{i,1} t^{\rho_{i,1}} - \cdots - x_{i,n_i} t^{\rho_{i,n_i}})^k.$$

The *normalized affine period integrals* $\bar{\Pi}_\gamma(x)$ become

$$\bar{\Pi}_\gamma(x) = \int_\gamma \left(\prod_{i=1}^{r} \sum_{k\geq 0} \frac{r_k}{x_{i,0}^k} (-x_{i,1} t^{\rho_{i,1}} - \cdots - x_{i,n_i} t^{\rho_{i,n_i}})^k \right) \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}. \tag{5}$$

Consider the cycle $\gamma_0 := \{|t_1| = \cdots = |t_n| = \epsilon\}$. We can compute $\bar{\Pi}_{\gamma_0}(x)$. Using the residue formula, over the region $\mathcal{R}$, we have

$$\bar{\Pi}_{\gamma_0}(x) = (2\pi\sqrt{-1})^n \sum_{\ell \in \mathfrak{L}} C_\ell \cdot (-1)^{\sum_{i=1}^{r} \ell_{i,0}} x^\ell \tag{6}$$

where $\mathfrak{L} := \{\ell \in L_{\mathrm{ext}} \colon \ell_{i,j} \geq 0 \text{ for all } j \neq 0\}$ and

$$C_\ell = \prod_{i=1}^{r} \frac{r_{-\ell_{i,0}} \Gamma(-\ell_{i,0} + 1)}{\Gamma(\ell_{i,1} + 1) \cdots \Gamma(\ell_{i,n_i} + 1)}. \tag{7}$$

Remark 2.3. The sheaf $\pi_* \mathbb{C}_{Y^\vee}$ (resp. $\pi_* \mathbb{C}_{Y^\vee \setminus R}$) is decomposed into eigensheaves

$$\pi_* \mathbb{C}_{Y^\vee} = \mathscr{G}_{\chi_0} \oplus \mathscr{G}_{\chi_1} \text{ (resp. } \pi_* \mathbb{C}_{Y^\vee \setminus R} = \mathscr{L}_{\chi_0} \oplus \mathscr{L}_{\chi_1}).$$

Here $\chi_k(a) = a^k$ where a is the generator of the multiplicative group $\mathbb{Z}/2\mathbb{Z}$. Let $i \colon R \to Y^\vee$ and $j \colon Y^\vee \setminus R \to Y^\vee$ be the closed and open embedding. Consider the standard triangle in the derived category

$$j_! \mathbb{C}_{Y^\vee \setminus R} \to \mathbb{C}_{Y^\vee} \to i_+ \mathbb{C}_R.$$

Applying the functor $R\pi_*$ to the above sequence, one can show that $\mathscr{G}_{\chi_1}|_{X^\vee \setminus R} \simeq \mathscr{L}_{\chi_1}$ and that $\mathrm{H}_\mathrm{c}^n(X^\vee, \mathscr{G}_{\chi_1}) \simeq \mathrm{H}_\mathrm{c}^n(X^\vee \setminus R, \mathscr{L}_{\chi_1})$.

Moreover, we have

$$\begin{aligned} \mathrm{H}_\mathrm{c}^n(Y^\vee, \mathbb{C}) &= \mathrm{H}_\mathrm{c}^n(X^\vee, \mathscr{G}_{\chi_0}) \oplus \mathrm{H}_\mathrm{c}^n(X^\vee, \mathscr{G}_{\chi_1}) \\ &= \mathrm{H}_\mathrm{c}^n(X^\vee, \mathbb{C}) \oplus \mathrm{H}_\mathrm{c}^n(X^\vee, \mathscr{G}_{\chi_1}) \\ &\simeq \mathrm{H}_\mathrm{c}^n(X^\vee, \mathbb{C}) \oplus \mathrm{H}_\mathrm{c}^n(X^\vee \setminus R, \mathscr{L}_{\chi_1}). \end{aligned}$$

If n is odd, $\mathrm{H}_\mathrm{c}^n(X^\vee, \mathbb{C}) = 0$ since $X^\vee$ is a smooth toric variety.

3. Existence of maximal degeneracy points

In this section, we study the maximal degeneracy problem and show that the GKZ system associated with the gauge fixed double cover family $\mathcal{Y}^\vee \to V$

admits a maximal degeneracy point on a resolution of the secondary fan compactification of the moduli. This extends the results in [16] to our case. The proof presented here is parallel to the one given in [16].

3.1. The maximal degeneracy points

From the discussion in §2.5, the affine period integrals (4) are sections of a local system defined on $\mathbb{C}^{\mathcal{A}_{\text{ext}}} \setminus \mathcal{D}$. Recall that $\mathcal{A}_{\text{ext}} = \{\nu_{i,j} \colon (i,j) \in J\}$, $W_i = \mathrm{H}^0(X^\vee, F_i)^\vee$, and $W = \prod_{i=1}^r W_i$. We also identify $\mathbb{C}^{\mathcal{A}_{\text{ext}}}$ with $W^\vee$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the sequence

$$0 \to L_{\text{ext}} \to \mathbb{Z}^{p+r} \equiv \mathbb{Z}^{\mathcal{A}_{\text{ext}}} \to \mathbb{Z}^{n+r} \equiv N \times \mathbb{Z}^r \to 0,$$

we obtain a short exact sequence of algebraic tori ($T_M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*)$)

$$1 \to T_M \times (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{\mathcal{A}_{\text{ext}}} \to \mathrm{Hom}(L_{\text{ext}}, \mathbb{C}^*) \to 1.$$

Let $S_{\mathcal{A}_{\text{ext}}}$ be the image of $(\mathbb{C}^*)^{\mathcal{A}_{\text{ext}}} \setminus \mathcal{D}$ under the map

$$(\mathbb{C}^*)^{\mathcal{A}_{\text{ext}}} \to (\mathbb{C}^*)^{\mathcal{A}_{\text{ext}}} / T_M \times (\mathbb{C}^*)^r \xrightarrow{\phi} \mathrm{Hom}_{\mathbb{Z}}(L_{\text{ext}}, \mathbb{C}^*).$$

Here the isomorphism ϕ is given by

$$\phi(x)(\ell) = (-1)^{\sum_{i=1}^r \ell_{i,0}} x^\ell \text{ where } \ell \in L_{\text{ext}}.$$

Any complete fan F in $L_{\text{ext}}^\vee \otimes \mathbb{R}$ gives rise to a complete toric variety X_F which compactifies the torus

$$\mathrm{Hom}_{\mathbb{Z}}(L_{\text{ext}}, \mathbb{C}^*) \simeq \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}^*)$$

and $S_{\mathcal{A}_{\text{ext}}}$ as well. Since the normalized affine period integrals $\bar{\Pi}_\gamma(x)$ are $T_M \times (\mathbb{C}^*)^r$ invariant, they descend to local sections of a locally constant sheaf on $S_{\mathcal{A}_{\text{ext}}}$.

Definition 3.1. We call a smooth boundary point $p \in X_F \setminus \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}^*)$ a *maximal degeneracy point* if near p there is exactly one normalized affine period integral $\bar{\Pi}_\gamma$ (up to a constant) extends over p holomorphically.

3.2. Triangulations, secondary fans and Gröbner fans

To proceed, let us retain the notation in §2.3 and recall the following terminologies.

- Let $\mathcal{A}_{\text{ext}} = \{\nu_{i,j} \colon (i,j) \in J\}$ be the set points in $\mathbb{Z}^{n+r}$. We denote by $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$ the convex hull generated by $\mathcal{A}_{\text{ext}}$.
- A triangulation $\mathscr{T}$ of $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$ is a collection of $(r+n-1)$-dimensional simplices whose vertices are in $\mathcal{A}_{\text{ext}}$ such that the intersection of two such simplices is a face of both and that their union is $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$.
- A continuous function h on the cone over $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$ is called $\mathscr{T}$-*piecewise linear* if it is linear on the cone over each simplex in $\mathscr{T}$. A $\mathscr{T}$-piecewise linear function h is called *convex* if $h(a+b) \leq h(a)+h(b)$ for arbitrary a, b and is called *strictly convex* if it is convex and $h|_\sigma \neq h|_\tau$ for any large cones $\sigma \neq \tau$.
- Each point $x \in \mathbb{R}^{\mathcal{A}_{\text{ext}}}$ (components are labeled by $(i,j) \in J$) determines a $\mathscr{T}$-piecewise linear function, which is denoted by h_x. Let $\mathcal{C}(\mathscr{T})$ be the set of all $x \in \mathbb{R}^{\mathcal{A}_{\text{ext}}}$ such that h_x is convex and that $h_x(\nu_{i,j}) \leq x_{i,j}$ for a non vertex $\nu_{i,j} \in \mathcal{A}_{\text{ext}}$. Note that $\mathcal{C}(\mathscr{T})$ is a rational polyhedral cone in $\mathbb{R}^{\mathcal{A}_{\text{ext}}}$ but not strongly convex.
- A triangulation $\mathscr{T}$ is called *regular* if $\mathcal{C}(\mathscr{T})$ contains an interior point, i.e., there exists an $x \in \mathbb{R}^{\mathcal{A}_{\text{ext}}}$ such that h_x is a strictly convex function.

Definition 3.2. The collection of the cones $\mathcal{C}(\mathscr{T})$ with $\mathscr{T}$ regular, together with all of their faces form a generalized fan in $\mathbb{R}^{\mathcal{A}_{\text{ext}}}$. Note that each cone in $\mathcal{C}(\mathscr{T})$ contains $M_{\mathbb{R}} \times \mathbb{R}^r$ as a linear subspace via $A_{\text{ext}}^{\intercal} \colon M_{\mathbb{R}} \times \mathbb{R}^r \hookrightarrow \mathbb{R}^{\mathcal{A}_{\text{ext}}}$. We can project the generalized fan $\mathcal{C}(\mathscr{T})$ along the subspace and get a complete fan in $L_{\text{ext}}^{\vee} \otimes \mathbb{R}$. The resulting fan $S\Sigma$ is called the *secondary fan* of $\mathcal{A}_{\text{ext}}$.

Each $\omega \in \mathbb{R}^{\mathcal{A}_{\text{ext}}}$ determines a polyhedral subdivision on $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$. Let $C = \operatorname{Cone}\{(\nu_{i,j}, \omega_{i,j}) \in \mathcal{A}_{\text{ext}} \times \mathbb{R} \colon \nu_{i,j} \in \mathcal{A}_{\text{ext}}\}$. Recall that the lower hull of C is a collection of facets of C whose last coordinate in the inward normal vector is positive. Projecting down the facets in the lower hull gives rises to a polyhedral subdivision of $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$ if $\dim C = n + r$. For generic ω, the subdivision $\mathscr{T}_\omega$ is a triangulation. One can show that a triangulation $\mathscr{T}$ of $\operatorname{Conv}(\mathcal{A}_{\text{ext}})$ is regular if and only if $\mathscr{T} = \mathscr{T}_\omega$ for some $\omega \in \mathbb{R}^{\mathcal{A}_{\text{ext}}}$.

Consider a polynomial ring $\mathbb{C}[y] := \mathbb{C}[y_{i,j} \colon (i,j) \in J]$ and the toric ideal

$$I_{\mathcal{A}_{\text{ext}}} = \left\langle y^{l^+} - y^{l^-} : l = l^+ - l^- \in L_{\text{ext}} \right\rangle.$$

Each $\omega \in \mathbb{R}_{\geq 0}^{\mathcal{A}_{\text{ext}}}$ determines a weight on $\mathbb{C}[y]$ by defining

$$\operatorname{in}_\omega(y^n) := \sum_{i,j} \omega_{i,j} n_{i,j}, \text{ where } y^n = \prod_{i,j} y_{i,j}^{n_{i,j}}.$$

Let $\mathrm{LT}_\omega(I_{\mathcal{A}_{\mathrm{ext}}})$ be the leading term ideal with respect to in_ω. We say that $\omega,\ \omega' \in \mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}_{\geq 0}$ are equivalent if $\mathrm{LT}_\omega(I_{\mathcal{A}_{\mathrm{ext}}}) = \mathrm{LT}_{\omega'}(I_{\mathcal{A}_{\mathrm{ext}}})$. We can extend the equivalence relation to $\mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}$ by the homogeneity of $I_{\mathcal{A}_{\mathrm{ext}}}$.

Definition 3.3 (Cf. [18, 19])**.** The equivalence classes of vectors in $\mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}$ form a fan. Projecting along the linear subspace $A^{\intercal}_{\mathrm{ext}} \colon M_{\mathbb{R}} \times \mathbb{R}^r \hookrightarrow \mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}$, we obtain a fan in $L^\vee_{\mathrm{ext}} \otimes \mathbb{R}$. The resulting fan $G\Sigma$ is called the *Gröbner fan* of $\mathcal{A}_{\mathrm{ext}}$. An interior point in a large cone in $G\Sigma$ is called a *term order* of $I_{\mathcal{A}_{\mathrm{ext}}}$.

Remark 3.4. Although the secondary fan and the Gröbner fan (cf. Definition 3.2 and Definition 3.3) depend not only on Σ but also on the nef-partition, we still denote them by $S\Sigma$ and $G\Sigma$ respectively for simplicity. We also remark that $\mathrm{Conv}(\mathcal{A}_{\mathrm{ext}})$ projects to $\mathrm{Conv}(\nabla_1, \ldots, \nabla_r)$ under the canonical projection $N_{\mathbb{R}} \times \mathbb{R}^r \to N_{\mathbb{R}}$.

Remark 3.5. Sturmfels [18] showed that the Gröbner fan $G\Sigma$ refines the secondary fan $S\Sigma$. The two fans coincide if $\mathcal{A}_{\mathrm{ext}}$ is unimodular. In particular, if $\omega \in \mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}$ is a term order, then $\mathscr{T}_\omega$ is a triangulation of $\mathrm{Conv}(\mathcal{A}_{\mathrm{ext}})$.

3.3. The cohomology ring of toric manifolds

We resume the notation in §2.3 and the situation there. Recall that a primitive collection of Σ is a subset $\mathcal{P} \subset \Sigma(1)$ such that the full set $\mathcal{P}$ does not form a cone in Σ but any proper subset does.

For a projective *smooth* toric variety X_Σ, the cohomology ring $\mathrm{H}^\bullet(X_\Sigma, \mathbb{Z})$ is given by $\mathbb{Z}[a_{i,j} \colon (i,j) \in I]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by

(a) $a_{\mathcal{P}} := \prod_{(i,j)\in\mathcal{P}} a_{i,j}$, where $\mathcal{P}$ is a primitive collection in Σ;
(b) $\sum_{(i,j)\in I} \langle m, \rho_{i,j}\rangle a_{i,j}$ for all $m \in M$.

The ideal generated by (a) is called the *Stanley–Reisner ideal* of Σ.

For a primitive collection $\mathcal{P}$, we can define the *primitive relation* of $\mathcal{P}$ as follows. By completeness of Σ, the vector $\sum_{(i,j)\in\mathcal{P}} \rho_{i,j}$ must lie in the relative interior of some cone σ uniquely in Σ. We may write

$$\sum_{(i,j)\in\mathcal{P}} \rho_{i,j} = \sum_{(i,j)\in\sigma(1)} c_{i,j}\rho_{i,j},\ \ c_{i,j} \in \mathbb{Z}_{>0}.$$

Equivalently, we have

$$\sum_{(i,j)\in\mathcal{P}} \rho_{i,j} - \sum_{(i,j)\in\sigma(1)} c_{i,j}\rho_{i,j} = \sum_{(i,j)\in I} b_{i,j}\rho_{i,j} = 0,\ \ \text{with } b_{i,j} \in \mathbb{Z}.$$

Under the inclusion $L \hookrightarrow \mathbb{R}^p$, the vector $(b_{i,j}) \in \mathbb{R}^p$ is an element in L, called the *primitive relation of* $\mathcal{P}$, and is denoted by $\ell(\mathcal{P})$. We can identify $L\otimes\mathbb{R}$ with $\mathrm{N}_1(X_\Sigma)$, the real vector space of 1-cycles on X_Σ modulo numerical equivalence.

Proposition 3.1 (Toric cone theorem)**.** *Let* $\mathrm{NE}(X_\Sigma) \subset L\otimes\mathbb{R}$ *be the cone generated by classes of irreducible complete curves in* X_Σ*. We have*

$$\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{NE}(X_\Sigma) = \sum_{\mathcal{P}} \mathbb{R}_{\geq 0}\ell(\mathcal{P}), \tag{8}$$

where the summation runs over all primitive collections $\mathcal{P}$*.*

Lemma 3.2. *Under our smoothness assumption, we have* $\mathcal{P}\cap\sigma(1) = \emptyset$ *where* $\sigma(1)$ *is the set of* 1*-cones contained in* σ*.*

Proof. See [16, Proposition 4.7]. □

We can lift the primitive relations to obtain relations among $\nu_{i,j}$. For a primitive collection $\mathcal{P}$, we have correspondingly a cone σ in Σ as above. We can thus write

$$\sum_{(i,j)\in\mathcal{P}} \nu_{i,j} = \sum_{(i,j)\in\sigma(1)} c_{i,j}\nu_{i,j} + \sum_{i=1}^{r} c_{i,0}\nu_{i,0}. \tag{9}$$

Corollary 3.3. $c_{i,0} \geq 0$ *for all* $i = 1,\ldots,r$*.*

Proof. $\ell(\mathcal{P})$ represents a curve class. The assertion follows from the fact that $I_1 \sqcup\cdots\sqcup I_r$ is a nef-partition and $c_{i,0}$ is the intersection number of $\ell(\mathcal{P})$ with E_i. □

Let $\ell(\mathcal{P})$ be a primitive relation and $\ell_{\mathrm{ext}}(\mathcal{P})$ be the corresponding element in L_{ext} under the identification $L \simeq L_{\mathrm{ext}}$. We can rewrite (9) into

$$0 = \sum_{(i,j)\in\mathcal{P}} \nu_{i,j} - \sum_{(i,j)\in\sigma(1)} c_{i,j}\nu_{i,j} - \sum_{i=1}^{r} c_{i,0}\nu_{i,0} = \sum_{(i,j)\in J} d_{i,j}\nu_{i,j}. \tag{10}$$

Corollary 3.4. *The vector* $(d_{i,j})_{(i,j)\in J}$ *is equal to* $\ell_{\mathrm{ext}}(\mathcal{P})$ *as elements in* $\mathbb{R}^{\mathcal{A}_{\mathrm{ext}}}$ *and* $\ell^{\pm}_{\mathrm{ext}}(\mathcal{P})$ *is given by the left-hand and the right-hand side of* (9)*.*

3.4. Indicial ideals of Picard–Fuchs equations

Our aim in this paragraph is to describe the indicial rings attached to the GKZ system. The arguments here are almost along the same line in [16]. In this subsection, unless otherwise stated, $X = X_\Sigma$ is a smooth projective toric variety defined by a fan Σ as in §2.3.

Definition 3.6. For $\ell \in L_{\mathrm{ext}} = \ker(A_{\mathrm{ext}})$, we define

$$I_\ell(\alpha) := x^{-\alpha} x^{\ell^+} (\partial_x)^{\ell^+} x^\alpha \in \mathbb{C}[\alpha] := \mathbb{C}[\alpha_{i,j} \colon (i,j) \in J].$$

Let us recall the definition of indicial ideals.

Definition 3.7. For a cone $\tau \subset L_{\mathrm{ext}}^\vee \otimes \mathbb{R}$ and an exponent $\beta \in \mathbb{C}^{n+r}$, the *indicial ideal* $\mathrm{Ind}(\tau, \beta)$ is the ideal in $\mathbb{C}[\alpha_{i,j} \colon (i,j) \in J]$ generated by

- $I_\ell(\alpha)$ where $0 \neq \ell \in \tau^\vee \cap L_{\mathrm{ext}}$;
- $\sum_{(i,j)\in J} \langle \bar{m}, \nu_{i,j} \rangle \alpha_{i,j} - \langle \bar{m}, \beta \rangle$ for all $\bar{m} \in M \times \mathbb{Z}^r$.

There is a canonical triangulation on $\mathrm{Conv}(\mathcal{A}_{\mathrm{ext}})$. It is given by the maximal cones in the fan defining the toric variety W, the total space of the rank r vector bundle over X whose sheaf of sections is $\oplus_{i=1}^r \mathcal{O}_X(-E_i)$. (Recall that $E_1 + \ldots + E_r$ is the nef-partition on X.) We call this triangulation the *maximal triangulation* of $\mathrm{Conv}(\mathcal{A}_{\mathrm{ext}})$ and is denoted by $\mathscr{T}_{\max}$.

For a smooth variety X, the Kähler cone of X is denoted by $\mathrm{Käh}(X)$. If X is a *smooth projective* toric variety, then $\mathrm{Käh}(X)$ is a cone sitting inside $\mathrm{H}^2(X, \mathbb{R})$ whose closure coincides with the the closure of the ample cone $\mathrm{Amp}(X)$. This is a large cone since X is projective. Let $\overline{\mathrm{Käh}(X)}$ be the closure of the Kähler cone of X.

Since $E_1 + \cdots + E_r$ is a nef-partition, we have $\ell_{\mathrm{ext}}(\mathcal{P})_{i,0} \leq 0$ for all i and [16, Proposition 6.1] still holds. Combining with [loc. cit., Corollary 6.2 and Corollary 6.3], we obtain the following corollaries.

Corollary 3.5. *The leading term ideal* $\mathrm{LT}_\omega(I_{\mathcal{A}_{\mathrm{ext}}})$ *with respect to the term order* ω *such that* $\mathscr{T}_\omega = \mathscr{T}_{\max}$ *is the Stanley–Reisner ideal of* Σ.

Corollary 3.6. $\overline{\mathit{Käh}(X)} \in G\Sigma$.

Taking such a term order ω (one can take any element in the ample cone to achieve this), we see that the leading term ideal of

$$\left\{ y^{\ell_{\mathrm{ext}}^+(\mathcal{P})} - y^{\ell_{\mathrm{ext}}^-(\mathcal{P})} \colon \mathcal{P} \text{ is a primitive collection} \right\}$$

is nothing but the Stanley–Reisner ideal. Indeed, since $\mathcal{P}$ is primitive and ω is ample, $\omega.\ell(\mathcal{P}) > 0$. Consequently, $y^{\ell^+_{\mathrm{ext}}(\mathcal{P})}$ is the leading term with respect to ω. Now use Corollary 3.4.

Corollary 3.7. *The set*

$$\left\{ y^{\ell^+_{\mathrm{ext}}(\mathcal{P})} - y^{\ell^-_{\mathrm{ext}}(\mathcal{P})} \colon \mathcal{P} \text{ is a primitive collection} \right\}$$

is a minimal Gröbner basis of the toric ideal $I_{\mathcal{A}_{\mathrm{ext}}}$ for any term order ω with $\mathscr{T}_\omega = \mathscr{T}_{\max}$. Consequently, the polynomial operators in the GKZ system $\mathcal{M}(A_{\mathrm{ext}}, \beta)$ are generated by the box operators associated with $\ell_{\mathrm{ext}}(\mathcal{P})$, where $\mathcal{P}$ is a primitive collection of Σ.

Remark 3.8. From the corollaries above, we see that one can also use the Gröbner basis with respect to a term order ω with $\mathscr{T}_\omega = \mathscr{T}_{\max}$ to approximate the indicial ideal $\mathrm{Ind}(\tau, \beta)$ as well as the GKZ system in our fractional case.

Let $\ell_{\mathrm{ext}}(\mathcal{P})$ be the lifting of $\ell(\mathcal{P})$ under the isomorphism $L_{\mathrm{ext}} \simeq L$ as before.

Lemma 3.8. *Let $\tau = \overline{K\ddot{a}h(X)}$. The ideal generated by*

- *(a') $I_{\ell_{\mathrm{ext}}(\mathcal{P})}(\alpha)$ for $\mathcal{P}$ primitive;*
- *(b') $\sum_{(i,j)\in J} \langle \bar{m}, \nu_{i,j} \rangle \alpha_{i,j} - \langle \bar{m}, \beta \rangle$ for all $\bar{m} \in M \times \mathbb{Z}^r$;*

is an ideal contained in $\mathrm{Ind}(\tau, \beta)$*. Moreover, they have the same zero locus.*

Proof. Let $\mathcal{I}'$ be the ideal generated by the elements in (a′) and (b′). Clearly, we have $\mathcal{I}' \subset \mathrm{Ind}(\tau, \beta)$. For any term order $\omega \in \tau$, we have $\mathscr{T}_\omega = \mathscr{T}_{\max}$. Together with Corollary 3.7, it follows from [16, Proposition 5.6] that the zero locus of $\mathrm{Ind}(\tau, \beta)$ is the same as the one defined by $\mathcal{I}'$. □

From this, we can deduce that

Proposition 3.9. *Let $\tau \subset \overline{K\ddot{a}h(X)}$. There is a surjection*

$$\mathrm{H}^\bullet(X, \mathbb{C}) \to \mathbb{C}[\alpha]/\mathrm{Ind}(\tau, \beta),\ D_{i,j} \mapsto \alpha_{i,j} \tag{11}$$

from the cohomology ring of X to the indicial ring of the GKZ A-hypergoemetric system associated with the family $\mathcal{Y}^\vee \to V$.

Proof. Let $\mathcal{I}'$ again be the ideal generated by the elements in (a′) and (b′) in Lemma 3.8. By Corollary 3.4, for a primitive collection $\mathcal{P}$, we have $\ell^+_{\mathrm{ext}}(\mathcal{P})_{i,0} = 0$ for all i, $\ell^+_{\mathrm{ext}}(\mathcal{P})_{i,j} = 1$ for $\rho_{i,j} \in \mathcal{P}$, and $\ell^+_{\mathrm{ext}}(\mathcal{P})_{i,j} = 0$ for $\rho_{i,j} \notin \mathcal{P}$. Consequently,

$$I_{\ell_{\mathrm{ext}}(\mathcal{P})}(\alpha) = \alpha^{\ell^+_{\mathrm{ext}}(\mathcal{P})}.$$

When $\mathcal{P}$ runs through all primitive collections of Σ, the elements $I_{\ell_{\text{ext}}(\mathcal{P})}(\alpha)$ generate exactly the Stanley–Reisner ideal of Σ. From this, we see that

$$\mathrm{H}^{\bullet}(X, \mathbb{C}) \simeq \mathbb{C}[\alpha_{i,j} \colon (i,j) \in J]/\mathcal{I}'.$$

The statement follows from the fact that $\mathcal{I}' \subset \mathrm{Ind}(\overline{\mathrm{K\ddot{a}h}(X)}, \beta) \subset \mathrm{Ind}(\tau, \beta)$. □

In particular, this implies

Corollary 3.10. *Let τ be as in Proposition 3.9. The zero locus of $\mathrm{Ind}(\tau, \beta)$ consists of at most one point $\alpha = (\alpha_{i,j}) \in \mathbb{C}^{p+r}$ where $\alpha_{i,0} = -1/2$ for $1 \leq i \leq r$ and $\alpha_{i,j} = 0$ for other i, j.*

3.5. The existence of maximal degeneracy points

We summarize the results we have obtained in the previous paragraphs. Recall that the secondary fan $S\Sigma$ is a complete fan in $L_{\text{ext}}^{\vee} \otimes \mathbb{R}$ and the toric variety $X_{S\Sigma}$ gives rise to a compactification of the algebraic torus

$$(\mathbb{C}^*)^{\mathcal{A}_{\text{ext}}}/T_M \times (\mathbb{C}^*)^r.$$

The Gröbner fan $G\Sigma$ gives a partial resolution $X_{G\Sigma} \to X_{S\Sigma}$.

Let τ be a *regular* maximal cone in the space $L_{\text{ext}}^{\vee} \otimes \mathbb{R}$. It determines a unique integral basis $\{\ell^{(1)}, \ldots, \ell^{(p-n)}\}$ of L_{ext} in $\tau^{\vee} \cap L_{\text{ext}}$, and hence a set of canonical coordinates $z_{\tau}^{(1)}, \ldots, z_{\tau}^{(p-n)}$ on the smooth affine toric variety $X_{\tau} = \mathrm{Hom}(\tau^{\vee} \cap L_{\text{ext}}, \mathbb{C})$. Explicitly, we have

$$z_{\tau}^{(k)} = (-1)^{\sum_{i=1}^{r} \ell_{i,0}^{(k)}} x^{\ell^{(k)}}, \ 1 \leq k \leq p - n.$$

We can employ the argument in [16, Corollary 5.12] to obtain the following result.

Corollary 3.11. *Let $\tau \subset \overline{\mathrm{K\ddot{a}h}(X)}$ be a regular cone of maximal dimension. The GKZ system $\mathcal{M}(A_{\text{ext}}, \beta)$ has at most one power series solution of the form $x^{\alpha}(1 + g(z))$ with $g(0) = 0$ on X_{τ}. Moreover, if this is a solution, then $\alpha = (\alpha_{i,j})$ with $\alpha_{i,0} = -1/2$ for $1 \leq i \leq r$ and $\alpha_{i,j} = 0$ for other i, j.*

Now we can prove our main result in this section.

Theorem 3.12. *For every toric resolution $X_{G\Sigma'} \to X_{G\Sigma}$, there exists at least one maximal degeneracy point in $X_{G\Sigma'}$.*

Proof. Put $\tau' = \overline{\text{Käh}}(X)$. Then $X_{\tau'}$ is a (possibly singular) affine toric variety. A smooth subdivision F of τ' gives a toric resolution $X_F \to X_{\tau'}$. Let τ be a regular maximal cone in F.

Recall that τ' determines the maximal triangulation $\mathscr{T}_{\max}$. By definition,

$$\tau^\vee \supset \{\ell \in L_{\text{ext}} \colon \ell_{i,j} \geq 0,\ \nu_{i,j} \notin \mathfrak{B}\}$$

for all bases $\mathfrak{B} \in \mathscr{T}_{\max}$. Also, for all $1 \leq i \leq r$ and all $\mathfrak{B} \in \mathscr{T}_{\max}$, we have $\nu_{i,0} \in \mathfrak{B}$. It follows that the range $\mathfrak{L}$ in the summation (6) is contained in $\tau^\vee$. Consequently, for any $\ell \in \mathfrak{L}$, there exist uniquely non-negative integers $m_1, \ldots, m_{p-n}$ such that

$$\ell = \sum_{k=1}^{p-n} m_k \ell^{(k)}.$$

As a function on X_τ, the normalized affine period integral $\bar{\Pi}_{\gamma_0}$ becomes

$$\bar{\Pi}_{\gamma_0}(z) = (2\pi\sqrt{-1})^n \sum_{m \in \mathcal{S}} C_{\sum_{k=1}^{p-n} m_k \ell^{(k)}} z_\tau^m, \tag{12}$$

where $\mathcal{S} = \left\{(m_1, \ldots, m_{p-n}) \in \mathbb{Z}_{\geq 0}^{p-n} \colon \ell_{i,0} \leq 0 \text{ for all } i, \text{ where } \ell = \sum_{k=1}^{p-n} m_k \ell^{(k)}\right\}$ and C_ℓ as well as γ_0 are defined in §2.5.

On one hand, from (12), we see that $\bar{\Pi}_{\gamma_0}$ extends holomorphically to the unique torus fixed point in X_τ. On the other hand, by Corollary 3.11, there are no other normalized affine period integrals with this property. This completes the proof. □

4. Generalized Frobenius methods

The aim of this section is to give a complete set of solutions to the GKZ hypergeometric system for our double covers via mirror symmetry. We will mainly follow the exposition in [15] and [5]. In what follows, let $X = X_\Sigma$ be as in §2.3.

4.1. A series solution to GKZ systems

We continuously assume the case $\beta = \begin{bmatrix} \mathbf{0} & -1/2 & \ldots & -1/2 \end{bmatrix}^\intercal \in \mathbb{Q}^{n+r}$. Let $\alpha \in \mathbb{C}^{p+r}$ such that $A_{\text{ext}}(\alpha) = \beta$. An obvious choice of α is $\alpha = (\alpha_{i,j})$ with $\alpha_{i,j} = 0$ for $j \neq 0$ and $\alpha_{i,0} = -1/2$ for $i = 1, \ldots, r$ (regarded as a column vector). A formal power series solution to the GKZ system $\mathcal{M}(A_{\mathrm{ext}}, \beta)$ is given by

$$\sum_{\ell \in L_{\mathrm{ext}}} \frac{1}{\prod_{i=1}^{r} \prod_{j=0}^{n_i} \Gamma(\ell_{i,j} + \alpha_{i,j} + 1)} x^{\ell+\alpha}. \tag{13}$$

Notice that in the present case the formal power series (13) is non-zero and will be convergent around the origin if we choose the charge vectors appropriately. However, in order to obtain an "integral" series, a renormalization is needed. Following the treatment in [15], we multiply the series (13) by an overall constant factor $\prod_{i=1}^{r} \Gamma(1+\alpha_{i,0})$. Manipulating the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ $(z \notin \mathbb{Z})$, we can rewrite the product $\prod_{i=1}^{r} \Gamma(1 + \alpha_{i,0}) \cdot (13)$ into the following form.

Definition 4.1 (The Γ-series, cf. [15, Equation (3.5)])**.** Let

$$\Phi^{\alpha}(x) := \sum_{\ell \in L_{\mathrm{ext}}} \frac{\prod_{i=1}^{r} \Gamma(-\ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^{r} \Gamma(-\alpha_{i,0}) \prod_{i=1}^{r} \prod_{j=1}^{n_i} \Gamma(\ell_{i,j} + \alpha_{i,j} + 1)} (-1)^{\sum_i \ell_{i,0}} x^{\ell+\alpha}. \tag{14}$$

Remark 4.2. We can multiply (14) by an overall factor $\prod_{i=1}^{r} \prod_{j=1}^{n_i+1} \Gamma(\alpha_{i,j}+1)$ to get the usual product form. It was pointed out in [12] that the Gamma function is crucial in order to get an integral, symplectic basis of the period integrals, although the period integrals obtained from the product form and the Gamma form are the same up to a Gamma factor.

4.2. A cohomology-valued series associated with the holomorphic period

Put $D_{i,0} = -\sum_{j=1}^{n_i} D_{i,j}$ for all $1 \le i \le r$. For each $\ell \in L_{\mathrm{ext}}$, we define

$$\mathcal{O}_{\ell}^{\alpha} := \frac{\prod_{i=1}^{r} (-1)^{\ell_{i,0}} \Gamma(-D_{i,0} - \ell_{i,0} - \alpha_{i,0})}{\prod_{i=1}^{r} \Gamma(-\alpha_{i,0}) \prod_{i=1}^{r} \prod_{j=1}^{n_i} \Gamma(D_{i,j} + \ell_{i,j} + \alpha_{i,j} + 1)}. \tag{15}$$

The quantity is understood as follows. The function $1/\Gamma(z)$ is an entire function on the complex plane. For $j \neq 0$, we can expand

$$\frac{1}{\Gamma(z + \ell_{i,j} + \alpha_{i,j} + 1)}$$

into a power series in z around $1/\Gamma(\ell_{i,j} + \alpha_{i,j} + 1)$; namely

$$\frac{1}{\Gamma(z + \ell_{i,j} + \alpha_{i,j} + 1)} = 1/\Gamma(\ell_{i,j} + \alpha_{i,j} + 1) + a_1 z + a_2 z^2 + \cdots .$$

Then for a divisor class $D \in \mathrm{H}^2(X, \mathbb{Z})$, we define

$$\frac{1}{\Gamma(D + \ell_{i,j} + \alpha_{i,j} + 1)} = \mathbf{1}/\Gamma(\ell_{i,j} + \alpha_{i,j} + 1) + a_1 D + a_2 D^2 + \cdots ,$$

where $\mathbf{1} \in \mathrm{H}^0(X, \mathbb{Z})$ is the Poincaré dual of the fundamental class. This is an honest element in $\mathrm{H}^\bullet(X, \mathbb{C})$ since D is nilpotent. For $j = 0$, we consider the *deformed* coefficient

$$\frac{\Gamma(-z - \ell_{i,0} - \alpha_{i,0})}{\Gamma(-\alpha_{i,0})}$$

and expand it into a power series in z around $z = 0$; namely

$$\frac{\Gamma(-z - \ell_{i,0} - \alpha_{i,0})}{\Gamma(-\alpha_{i,0})} = \frac{\Gamma(-\ell_{i,0} - \alpha_{i,0})}{\Gamma(-\alpha_{i,0})} + a_1 z + a_2 z^2 + \cdots .$$

For any divisor class $D \in \mathrm{H}^2(X, \mathbb{Z})$, we define

$$\frac{\Gamma(-D - \ell_{i,0} - \alpha_{i,0})}{\Gamma(-\alpha_{i,0})} = \frac{\Gamma(-\ell_{i,0} - \alpha_{i,0}) \cdot \mathbf{1}}{\Gamma(-\alpha_{i,0})} + a_1 D + a_2 D^2 + \cdots .$$

Consequently, $\mathcal{O}_\ell^\alpha$ is a *well-defined* element in $\mathrm{H}^\bullet(X, \mathbb{C})$.

Remark 4.3. Note that $1/\Gamma(w + D)$ is *divisible* by D if $w \in \mathbb{Z}_{\leq 0}$.

The following lemma follows from the multiplicative property of the Gamma function.

Lemma 4.1. *Let $w \in \mathbb{C}$. Then for any $D \in \mathrm{H}^\bullet(X, \mathbb{Z})$,*

$$\frac{(w + D)}{\Gamma(1 + w + D)} = \frac{1}{\Gamma(w + D)}.$$

Proof. Fix $w \in \mathbb{C}$, we have $\Gamma(1 + w + z) = (w + z)\Gamma(w + z)$ as a function in z. Therefore,

$$\frac{(w + z)}{\Gamma(1 + w + z)} = \frac{1}{\Gamma(w + z)}. \qquad \square$$

We now define the cohomology-valued series. Recall that we have an isomorphy $L_{\mathrm{ext}} \simeq L$ between the lattice relation of A_{ext} and that of A. The Mori cone $\overline{\mathrm{NE}}(X)$ can thus be regarded as a cone in L_{ext} which is also denoted by $\overline{\mathrm{NE}}(X)$.

Definition 4.4 (Cf. [15, 5])**.** We define the cohomology-valued B series to be

$$B_X^\alpha(x) := \left(\sum_{\ell\in\overline{\mathrm{NE}}(X)\cap L_{\mathrm{ext}}} \mathcal{O}_\ell^\alpha x^{\ell+\alpha}\right) \exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right). \tag{16}$$

B_X^α is regarded as an element in $\mathbb{C}[\![x_{i,j}]\!]\otimes_{\mathbb{C}}\mathrm{H}^\bullet(X,\mathbb{C})$.

Proposition 4.2. *We have $\mathcal{O}_\ell^\alpha = 0$ for $\ell\in L_{\mathrm{ext}}\setminus\overline{\mathrm{NE}}(X)$.*

Proof. Let $\ell=(\ell_{i,j})\in L_{\mathrm{ext}}\setminus\overline{\mathrm{NE}}(X)$. We claim that there exists a primitive collection $\mathcal{P}\subset\{(i,j)\in J\colon \ell_{i,j}<0\}$. Assuming the claim, we see that

$$\prod_{(i,j)\in\mathcal{P}} D_{i,j}$$

appears in the numerator of $\mathcal{O}_\ell^\alpha$ and hence $\mathcal{O}_\ell^\alpha=0$ in $\mathrm{H}^\bullet(X,\mathbb{C})$.

To prove the claim, we choose an ample divisor B with $B.\ell<0$. B corresponds to a term order on $\mathbb{C}[y_{i,j}]$, the homogeneous coordinate ring of X. Write $\ell=\ell^+-\ell^-$ as before. Then $B.(\ell^+-\ell^-)<0$ and y^{ℓ^-} will be the leading term of $y^{\ell^+}-y^{\ell^-}$ with respect to B. Hence y^{ℓ^-} is contained in the Stanley–Reisner ideal of X.

Using the fact that $\alpha_{i,j}=0$ for all i and $j\neq 0$, we see that $\mathcal{O}_\ell^\alpha$ is divisible by $D_{i,j}$ for those (i,j) such that $\ell_{i,j}<0$ and hence it is divisible by $\prod_{(i,j)\in\mathcal{P}}D_{i,j}$ for some primitive collection $\mathcal{P}$ of X. This establishes the claim. ∎

This proposition allows us to rewrite

$$B_X^\alpha(x) = \left(\sum_{\ell\in L_{\mathrm{ext}}} \mathcal{O}_\ell^\alpha x^{\ell+\alpha}\right) \exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right).$$

Proposition 4.3 (Cf. [5, Proposition 2.17])**.** *We regard $B_X^\alpha(x)$ as an element in $\mathbb{C}[\![x_{i,j}]\!]\otimes_{\mathbb{C}}\mathrm{H}^\bullet(X,\mathbb{C})$. For any $h\in\mathrm{H}^\bullet(X,\mathbb{C})^\vee$, the pairing $\langle B_X^\alpha(x),h\rangle\in\mathbb{C}[\![x_{i,j}]\!]$ is annihilated by $\mathcal{M}(A_{\mathrm{ext}},\beta)$.*

Proof. For simplicity, we drop the subscripts α and X in $B_X^\alpha(x)$. For each variable $x_{i,j}$, we have

(17)

$$\begin{aligned}\frac{\partial B(x)}{\partial x_{i,j}} &= \sum_{\ell\in L_{\mathrm{ext}}}\left(\frac{\ell_{i,j}+\alpha_{i,j}+D_{i,j}}{x_{i,j}}\right)\mathcal{O}^{\alpha}_{\ell}x^{\ell+\alpha}\exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right)\\ &= \left(\frac{1}{x_{i,j}}\right)\sum_{\ell\in L_{\mathrm{ext}}}(\ell_{i,j}+\alpha_{i,j}+D_{i,j})\mathcal{O}^{\alpha}_{\ell}x^{\ell+\alpha}\exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right).\end{aligned}$$

Hence the series $\langle B^{\alpha}_{X}(x),h\rangle$ is annihilated by the Euler operators. Now we examine the box operators. We write

$$(17) = \left(\frac{1}{x_{i,j}}\right)\sum_{\ell\in L_{\mathrm{ext}}}\mathcal{O}^{\alpha}_{\ell-\mathrm{e}_{i,j}}x^{\ell+\alpha}\exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right), \tag{18}$$

where $\{\mathrm{e}_{i,j}\colon (i,j)\in J\}$ is the standard basis of $\mathbb{Z}^{p+r}$. Here we extend the definition of $\mathcal{O}^{\alpha}_{\xi}$ to any element $\xi\in\mathbb{Z}^{p+r}$ by (15).

For $l=l^{+}-l^{-}\in L_{\mathrm{ext}}$, we have

(19)

$$\prod_{l_{i,j}>0}\left(\frac{\partial}{\partial x_{i,j}}\right)^{l_{i,j}}B(x) = \sum_{\ell\in L_{\mathrm{ext}}}\mathcal{O}^{\alpha}_{\ell-l^{+}}x^{\ell+\alpha-l^{+}}\exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right).$$

Similarly, we have

(20)

$$\prod_{l_{i,j}<0}\left(\frac{\partial}{\partial x_{i,j}}\right)^{-l_{i,j}}B(x) = \sum_{\ell\in L_{\mathrm{ext}}}\mathcal{O}^{\alpha}_{\ell-l^{-}}x^{\ell+\alpha-l^{-}}\exp\left(\sum_{i=1}^{r}\sum_{j=0}^{n_i}(\log x_{i,j})D_{i,j}\right).$$

Note that the ranges of the summations appeared on the right hand side of (19) and (20) are the same. Indeed, for any $\ell\in L_{\mathrm{ext}}$ and $l\in L_{\mathrm{ext}}$, there exists $\ell'\in L_{\mathrm{ext}}$ such that $\ell-l^{+}=\ell'-l^{-}$ since $l=l^{+}-l^{-}\in L_{\mathrm{ext}}$. This implies that $\Box_{l}B(x)=0$. $\square$

Corollary 4.4. *Assume X is smooth as before. When $h\in\mathrm{H}^{\bullet}(X,\mathbb{C})^{\vee}$ runs through a basis of $\mathrm{H}^{\bullet}(X,\mathbb{C})^{\vee}$, the series $\langle B^{\alpha}_{X}(x),h\rangle$ give a complete set of solution to $\mathcal{M}(A_{\mathrm{ext}},\beta)$.*

Proof. It is clear that all the coefficients are linearly independent. On one hand, for a general x, we know that the solution space to $\mathcal{M}(A_{\mathrm{ext}},\beta)$ has dimensional $\mathrm{vol}_{r+n}(A_{\mathrm{ext}})$, where vol_{r+n} denotes the normalized volume in $\mathbb{R}^{n+r}$. On the other hand, by [13, Proposition 1.2],

$$\mathrm{vol}_{r+n}(A_{\mathrm{ext}}) = \chi(X) = \dim \mathrm{H}^{\bullet}(X, \mathbb{C})$$

since X is a smooth toric variety. $\square$

Remark 4.5. For odd n, from the proof of [13, Theorem 2.2], we know $\chi(Y^{\vee}) = \chi(X^{\vee}) - \chi(X)$. Also from [loc. cit., Theorem 2.1], we have

$$\dim \mathrm{H}^{p,q}(Y^{\vee}, \mathbb{C}) = \dim \mathrm{H}^{p,q}(X^{\vee}, \mathbb{C}) \text{ for } p + q \neq n.$$

Since $X^{\vee}$ is also a smooth toric variety, it follows that $\dim \mathrm{H}^{n}(Y^{\vee}, \mathbb{C}) = \chi(X) = \mathrm{vol}_{r+n}(A_{\mathrm{ext}})$. Together with Remark 2.3, it suggests that the affine periods are all the solutions to the GKZ system.

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References

[1] Victor V. Batyrev. Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori. *Duke Math. J.*, **69**(2):349–409, 1993. MR1203231

[2] Victor V. Batyrev. Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. *Journal of Algebraic Geometry*, **3**:493–545, 1994. MR1269718

[3] Victor V. Batyrev and Lev A. Borisov. On Calabi–Yau complete intersections in toric varieties. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 39–65. de Gruyter, Berlin, 1996. MR1463173

[4] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes. *Comm. Math. Phys.*, **165**(2):311–427, 1994. MR1301851

[5] Lev A. Borisov and R. Paul Horja. Mellin–Barnes integrals as Fourier–Mukai transforms. *Advances in Mathematics*, **207**(2):876–927, 2006. MR2271990

[6] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, **359**(1):21–74, 1991. MR1115626

[7] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky. Generalized Euler integrals and A-hypergeometric functions. *Advances in Mathematics*, **84**(2):255–271, 1990. MR1080980

[8] I. M. Gel'fand, A. V. Zelevinskiĭ, and M. M. Kapranov. Hypergeometric functions and toric varieties. *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya*, **23**(2):12–26, 1989. MR1011353

[9] A. Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, *Progr. Math.*, 160, pages 141–175. Birkhäuser Boston, Boston, MA, 1998. MR1653024

[10] B. R. Greene and M. R. Plesser. Duality in Calabi–Yau moduli space. *Nuclear Physics B*, **338**:15–37, 1990. MR1059831

[11] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau. Mirror symmetry, mirror map and applications to complete intersection Calabi–Yau spaces. *Nuclear Phys. B*, **433**(3):501–552, 1995. MR1319280

[12] Shinobu Hosono. Local mirror symmetry and type IIA monodromy of Calabi–Yau manifolds. *Advances in Theoretical and Mathematical Physics*, **4**(2):335–376, 2000. MR1838444

[13] Shinobu Hosono, Tsung-Ju Lee, Bong H. Lian, and Shing-Tung Yau. Mirror symmetry for double cover Calabi–Yau varieties. Available at arXiv:2003.07148v1. March 2020.

[14] Shinobu Hosono, Bong H. Lian, Hiromichi Takagi, and Shing-Tung Yau. K3 surfaces from configurations of six lines in $\mathbb{P}^2$ and mirror symmetry I. *Communications in Number Theory and Physics*, **14**(4):739–783, 2020. MR4164174

[15] Shinobu Hosono, Bong H. Lian, and Shing-Tung Yau. GKZ-generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces. *Communications in Mathematical Physics*, **182**(3):535–577, 1996. MR1461942

[16] Shinobu Hosono, Bong H. Lian, and Shing-Tung Yau. Maximal degeneracy points of GKZ systems. *Journal of the American Mathematical Society*, **10**(2):427–443, April 1997. MR1423031

[17] Shinobu Hosono, Bong H. Lian, and Shing-Tung Yau. K3 surfaces from configurations of six lines in $\mathbb{P}^2$ and mirror symmetry II – λ_{K3}-functions. *International Mathematics Research Notices*, 11 2019. rnz259. MR4307688

[18] Bernd Sturmfels. Gröbner bases of toric varieties. *Tohoku Math. J. (2)*, **43**(2):249–261, 1991. MR1104431

[19] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996. MR1363949

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