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# Sums of CR and projective dual CR functions

David E. Barrett* and Dusty E. Grundmeier

*To Joseph J. Kohn*

**Abstract:** A smooth, strongly $\mathbb{C}$-convex, real hypersurface $S$ in $\mathbb{CP}^n$ admits a projective dual CR structure in addition to the standard CR structure. Given a smooth function $u$ on $S$, we provide characterizations for when $u$ can be decomposed as a sum of a CR function and a dual CR function. Following work of Lee on pluriharmonic boundary values, we provide a characterization using differential forms. We further provide a characterization using tangential vector fields in the style of Audibert and Bedford.

**Keywords:** CR functions, pluriharmonic, projective duality.

## 1. Introduction

A smooth real hypersurface $S$ in complex projective space $\mathbb{CP}^n$ is *strongly* $\mathbb{C}$-*convex* if it is locally projectively equivalent to a strongly convex hypersurface. (Such $S$ are automatically strongly pseudoconvex. See [Bar, §5] for equivalent characterizations. We do not automatically assume $S$ to be compact.)

For $p \in S$ we let $H_pS = T_pS \cap JT_pS$, the maximal complex subspace of $T_pS$. (Here $J\colon T_p\mathbb{CP}^n \to T_p\mathbb{CP}^n$ is the complex structure tensor.)

In addition to the standard CR structure, $S$ admits a *projective dual* CR structure: if

(1.1) no complex tangent hyperplane for $S$ passes through the origin

this may be defined as the unique CR structure for which the functions

$$w_j(z) = \frac{\frac{\partial\rho}{\partial z_j}}{z_1\frac{\partial\rho}{\partial z_1} + \cdots + z_n\frac{\partial\rho}{\partial z_n}} \qquad (j = 1, \ldots, n) \tag{1.2}$$

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are CR, where $\rho$ is a defining function for $S$ (so $\rho$ is $\mathbb{R}$-valued with $\Omega = \{z\colon \rho(z) < 0\}$ and $d\rho \neq 0$ along $bD$). (Note that the values of the $w_j$ along $S$ will not depend on the choice of $\rho$.) The structure defined by this condition is projectively-invariant; along with a localization argument it follows that this construction induces a projectively-invariant CR structure on all of $S$ even when (1.1) fails. (See [Bar, §6], [BG, §3] and [BE, §4] for more detail.)

The two CR structures share the same maximal complex subspaces ([Bar, §6], [APS, §2.5]).

Given a smooth function $u$ on $S$, the goal of the current paper is to characterize whether $u$ can be decomposed as the sum of a CR function and a dual CR function. The projective decomposition problem is a natural analogue of the problem of attempting to decompose a function as a sum of a CR and conjugate-CR function, that is, of characterizing traces of pluriharmonic functions. We prove characterizations in terms of tangential vector fields (see section 3 for precise definitions of the vector fields $X$ and $T$ and section 5 for precise definitions of $X_{jk}$ and $\widetilde{T}_{jk\ell}$).

**Theorem A.** *For $S \subset \mathbb{C}^n$ $(n = 2)$ strongly $\mathbb{C}$-convex and simply-connected, the following conditions on smooth $u\colon S \to \mathbb{C}$ are equivalent:*

(1.3a) *$u$ decomposes as a sum $f + g$ where $f$ is CR and $g$ is dual-CR;*
(1.3b) $XXTu = 0 = TTXu$.

This result extends the main projective decomposition theorem of [BG] to non-circular hypersurfaces.

In higher dimensions, we give a second order vector field condition. We need to introduce the following additional condition:

$$(\star) \qquad z_j w_j + z_k w_k \neq 0 \text{ for all } j, k.$$

(In particular, all $z_j$ and $w_j$ are non-zero.)

**Theorem B.** *For $S \subset \mathbb{C}^n$ $(n > 2)$ strongly $\mathbb{C}$-convex and simply-connected and satisfying $(\star)$ the following conditions on smooth $u\colon S \to \mathbb{C}$ are equivalent.*

(1.4a) *$u$ decomposes as a sum $f + g$ where $f$ is CR and $g$ is dual-CR;*
(1.4b) *for all distinct $j, k, \ell$ we have*

$$X_{jk}\widetilde{T}_{jk\ell}u = 0. \tag{1.5}$$

The condition $(\star)$ allows for the relatively straightforward statement of (1.5), but when it fails we will see in Proposition 38 that it can be repaired (at least locally) by a linear change of variable, leading to a slightly less elegant version of (1.5).

The paper is organized as follows. In §2, we adapt Lee's characterization of CR pluriharmonic functions from [Lee] to the projective decomposition problem. In §3, we give a vector field characterization in two dimensions, and we prove Theorem A. In §4, we provide an alternate construction for the vector fields in Theorem A. In §5, we set up a vector field characterization in 3 or more dimensions, and we prove Theorem B. Finally in §6, we conclude by reviewing the pluriharmonic boundary value problem. In particular, we show the results on the sphere are remarkably similar to our projective decomposition results.

## 2. Operators $d'$ and $d''$

For $S$ as above define an *$H$-form of degree $k$* on $S$ to be a smoothly-varying $\mathbb{C}$-valued alternating $k$-tensor on each $H_pS$.

**Proposition 1.** *For smooth $u\colon S \to \mathbb{C}$ there are uniquely-determined degree one $H$-forms $d'u$ and $d''u$ satisfying*

- $du\big|_H = d'u + d''u$;
- *$d'u$ is $\mathbb{C}$-linear with respect to the standard CR structure on $S$;*
- *$d''u$ is $\mathbb{C}$-linear with respect to the projective dual CR structure on $S$.*

*The operators $d'$ and $d''$ are linear.*

**Lemma 2.** *The standard complex structure tensor $J\colon H_pS \to H_pS$ and the corresponding projective dual tensor $J^*\colon H_pS \to H_pS$ satisfy* $\ker(J^* - J) = \{0\}$.

*Proof of Lemma 2.* Working locally we may assume after possible application of a projective automorphism that (1.1) holds so that we have a local diffeomorphism [Bar, Thm. 16]

$$\begin{aligned} \mathcal{D}_S\colon S &\to \mathbb{C}^n \\ (z_1, \ldots, z_n) &\mapsto (w_1(z), \ldots, w_n(z)) \end{aligned}$$

with

$$J^* = \left(\mathcal{D}'(p)\right)^{-1} \circ J \circ \mathcal{D}'(p).$$

Quoting from [Bar, §6.4] we may choose projective transformations $\chi_1, \chi_2$ so that

$$\chi_1(0) = p$$

$$\chi_2\left(\mathcal{D}_S(p)\right) = 0$$
$$T_0\left(\chi_1^{-1}(S)\right) = \mathbb{C}^{n-1} \times \mathbb{R}$$

and

$$\left(\chi_2 \circ \mathcal{D}_S \circ \chi_1\right)'(0)\colon \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ u \end{pmatrix} \mapsto \begin{pmatrix} 2i\beta_1 z_1 + 2i\alpha_1 \overline{z}_1 \\ \vdots \\ 2i\beta_{n-1} z_{n-1} + 2i\alpha_{n-1}\overline{z}_{n-1} \\ -u \end{pmatrix}$$

with $0 \le \beta_j < \alpha_j$ (in fact $\alpha_j^2 - \beta_j^2 = 1/4$).

If $\chi_1'(0) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ 0 \end{pmatrix} \in \ker(J^* - J)$ then

$$\left(\chi_2 \circ \mathcal{D}_S \circ \chi_1\right)'(0) \cdot \begin{pmatrix} iz_1 \\ \vdots \\ iz_{n-1} \\ 0 \end{pmatrix} = i\left(\chi_2 \circ \mathcal{D}_S \circ \chi_1\right)'(0) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ 0 \end{pmatrix}$$

(since the $\chi_j$ are holomorphic) and so we must have $-2\beta_j z_j + 2\alpha_j z_j = -2\beta_j z_j - 2\alpha_j z_j$, hence $4\alpha_j z_j = 0$ and $z_j = 0$ for $j = 1, \ldots, n-1$. □

Note that the argument above also yields the following.

**Addendum 3.** *$\mathcal{D}'(p)$ is not $\mathbb{C}$-linear on any complex line in $H_pS$.*

From dimension considerations Lemma 2 has the following consequence.

**Corollary 4.** *The map $J - J^*\colon H_pS \to H_pS$ is surjective.*

*Proof of Proposition 1.* Let $H_p^*S$ denote the real dual of $H_pS$. We claim that for any $\omega$ in $H_p^*S \otimes \mathbb{C}$ there are unique $\omega_1, \omega_2 \in H_p^*S$ so that $\omega$ is the sum of the $J$-linear $\omega' \overset{\text{def}}{=} \omega_1 - i\omega_1 \circ J$ and the $J^*$-linear $\omega'' \overset{\text{def}}{=} \omega_2 - i\omega_2 \circ J^*$. We then set $d'u = (du\big|_H)'$, $d''u = (du\big|_H)''$.

To prove the claim we show that the map

$$H_p^*S \times H_p^*S \to H_p^*S \otimes \mathbb{C}$$
$$(\omega_1, \omega_2) \mapsto (\omega_1 + \omega_2) + i\left(-\omega_1 \circ J - \omega_2 \circ J^*\right)$$

is bijective. By dimension considerations it suffices to show that the map is injective. But a pair $(\omega_1, \omega_2)$ in the kernel must satisfy $\omega_1 = -\omega_2$, $\omega_1 \circ J = -\omega_2 \circ J^* = \omega_1 \circ J^*$, hence $\omega_1 \circ (J - J^*) = 0$. From Corollary 4 we now conclude that $\omega_1 = 0 = \omega_2$. $\square$

We note for future reference that $u$ is CR if and only if $du\big|_H = d'u$; this is equivalent in turn to the condition $d''u = 0$. Similarly, $u$ is dual CR if and only if $d'u = 0$.

**Remark 5.** Later on we will make use of a corresponding decomposition of $H_pS \otimes \mathbb{C}$: we claim that any vector $V \in H_pS \otimes \mathbb{C}$ decomposes uniquely as $V' + V''$, where $V' = V_1 + iJ^*V_1$ and $V'' = V_2 + iJV_2$ for real $V_1, V_2 \in H_pS$; equivalently, we claim that the map

$$\begin{aligned} H_pS \times H_pS &\to H_pS \otimes \mathbb{C} \\ (V_1, V_2) &\mapsto (V_1 + V_2) + i\,(J^*V_1 + JV_2) \end{aligned}$$

is bijective. By dimension considerations it suffices to show that the map is injective. But a pair $(V_1, V_2)$ in the kernel must satisfy $V_1 = -V_2, J^*V_1 = -JV_2 = JV_1$, forcing $V_1 = 0 = V_2$ by Lemma 2.

Setting

$$\begin{aligned} H'_pS &= \{V + iJ^*V : V \in H_pS\} \\ H''_pS &= \{V + iJV : V \in H_pS\} \end{aligned}$$

we have shown that

$$H_pS \otimes \mathbb{C} = H'_pS \oplus H''_pS.$$

A complex vector field on $S$ with values in $HS \otimes \mathbb{C}$ is in fact $H'$-valued if and only if it is annihilated by $J^*$-linear 1-forms; it is $H''$-valued if and only if it is annihilated by $J$-linear 1-forms. $\Diamond$

Moving forward, we will also need linear operators $\widetilde{d'}$ and $\widetilde{d''}$ mapping functions on $S$ to 1-forms on $S$ and satisfying $\widetilde{d'}u\big|_H = d'u, \widetilde{d''}u\big|_H = d''u$. The operators $\widetilde{d'}$ and $\widetilde{d''}$ are not uniquely determined but explicit choices of such operators are offered in (3.9) and (5.8) below and these also yield formulas for $d'$ and $d''$.

Let $u$ be a smooth function on $S$. We pose the question of whether $u$ may be decomposed as the sum of a CR function and a dual CR function. This problem – previously examined in [BG] – is a natural analogue of the classical problem of characterizing functions decomposable (at least locally) as the sum of a CR function and a conjugate-CR function (equivalently, of characterizing traces of pluriharmonic functions). Results on the latter problem are reviewed in §6 below.

**Theorem 6.** *If $S$ is strongly $\mathbb{C}$-convex and simply-connected and $\theta$ is a $\mathbb{C}$-valued contact form on $S$ (that is, a non-vanishing complex 1-form with $\theta\big|_H \equiv 0$) then the following conditions on smooth $u\colon S \to \mathbb{C}$ are equivalent:*

(2.1a) *$u$ decomposes as a sum $f+g$ where $f$ is CR and $g$ is dual-CR;*
(2.1b) *there is a scalar function $\lambda$ so that $\widetilde{d'}u + \lambda\theta$ is closed.*

Note that if $S$ fails to be simply-connected the result will still hold locally.

An explicit choice of contact form $\theta$ is offered in (3.4) (and again in (5.5)) below. (That choice is not $\mathbb{R}$-valued.)

Theorem 6 and its proof are adapted from [Lee, Lemma 3.1].

*Proof.* If $u$ decomposes as a sum $f+g$ where $f$ is CR and $g$ is dual-CR then from the discussion following the proof of Proposition 1 we have $d'u\big|_H = d'f\big|_H = df\big|_H$, hence $\widetilde{d'}u = df - \lambda\theta$ for smooth scalar $\lambda$ and so (2.1b) holds.

Conversely, if (2.1b) holds we may write $\widetilde{d'}u + \lambda\theta = df$; it follows that $d'u = df\big|_H = d'f\big|_H$ and hence that $f$ is CR and that $g \stackrel{\text{def}}{=} u - f$ satisfies $d'g = 0$ and thus is dual CR. □

Note that (2.1b) implies that

$$d\widetilde{d'}u\big|_H = -\lambda\, d\theta\big|_H. \tag{2.2}$$

Note that the strong pseudoconvexity of $S$ guarantees that $d\theta\big|_H$ is nowhere-vanishing.

$\boxed{\mathbf{n = 2}}$ We may define

$$\lambda = -\frac{d\widetilde{d'}u\big|_H}{d\theta\big|_H}, \tag{2.3}$$

then check whether or not this works. See Theorem A for the result of this approach.

$\boxed{\mathbf{n > 2}}$ In higher dimension we have the following result.

**Theorem 7.** *For $S$, $\theta$ and $u$ as in Theorem 6 the following are equivalent:*

(2.4a) *there is a smooth scalar function $\lambda$ so that (2.2) holds;*

(2.4b) *$u$ may be decomposed as the sum of a CR function and a dual CR function.*

*Proof.* The discussion above shows that (2.4b) implies (2.4a).

Suppose on the other hand that the necessary condition (2.4a) holds.

Then the restriction of $d(\widetilde{d}'u + \lambda\theta) = d\widetilde{d}'u + \lambda\, d\theta + d\lambda \wedge \theta$ to $H$ vanishes identically. In view of the dimension condition and the non-degeneracy of $H$, Lemma 3.2 from [Lee] tells us that a closed 2-form whose restriction to $H$ vanishes must vanish identically, hence in particular $d(\widetilde{d}'u+\lambda\theta) = 0$ and thus (2.1b) holds. Theorem 6 now furnishes the desired decomposition. □

## 3. The projective decomposition problem for $n = 2$

We make the standing assumption that $S$ is a strongly $\mathbb{C}$-convex hypersurface satisfying (1.1). (Note that (1.1) holds automatically if $S$ is a compact hypersurface enclosing 0 [APS, §2.5].)

We define $w_1(z)$ and $w_2(z)$ as in (1.2).

**Lemma 8.** *We have*

$$z_1 w_1 + z_2 w_2 = 1 \ \textit{on } S \tag{3.1a}$$

$$w_1\, dz_1 + w_2\, dz_2 + z_1\, dw_1 + z_2\, dw_2 = 0 \ \textit{as 1-forms on } S \tag{3.1b}$$

$$w_1\, dz_1 + w_2\, dz_2 = 0 \ \textit{as forms on } H \tag{3.1c}$$

$$z_1\, dw_1 + z_2\, dw_2 = 0 \ \textit{as forms on } H \tag{3.1d}$$

*Proof.* Equation (3.1a) is immediate; (3.1b) follows from differentiation of (3.1a). Equation (3.1c) follows from the fact that $\partial\rho$ vanishes along $H_pS$; then (3.1d) follows by combining (3.1b) with (3.1c). □

**Lemma 9.** *At each point of $S$ at least one of*

$$dz_1 \wedge dz_2 \wedge dw_1$$
$$dz_1 \wedge dz_2 \wedge dw_2$$

*is non-zero as a 3-form on $S$*

*Proof.* From Addendum 3 we see that at least one of the $dw_j$ fails to be $\mathbb{C}$-linear; the claim follows immediately. □

**Lemma 10.** *The intersection of $S$ with $\{z_j = 0\}$ has no relative interior.*

*Proof.* This follows from the strong pseudoconvexity of $S$. □

**Proposition 11.** *There are uniquely-defined tangential vector fields $X, T$ on $S$ satisfying*

$$\begin{array}{llll} Xz_1 = 0 & Xz_2 = 0 & Xw_1 = z_2 & Xw_2 = -z_1 \\ Tz_1 = w_2 & Tz_2 = -w_1 & Tw_1 = 0 & Tw_2 = 0. \end{array} \tag{3.2}$$

*$T$ and $X$ take values in $H'$ and $H''$, respectively.*

*Proof.* Consider $p \in S$ and suppose that $dz_1 \wedge dz_2 \wedge dw_1 \neq 0$ at $p$; then linear independence guarantees the existence and uniqueness of $X$ and $T$ in a neighborhood of $p$ satisfying all conditions above other than $Xw_2 = -z_1$ and $Tw_2 = 0$. If $z_2(p) \neq 0$ then the remaining equations follow from differentiation of $z_1w_1 + z_2w_2 = 1$; if $z_2(p) = 0$ then by Lemma 10 the remaining equations still must hold at many points near to $p$, hence by passing to the limit they must also hold at $p$.

A similar argument holds if $dz_1 \wedge dz_2 \wedge dw_2 \neq 0$ at $p$; uniqueness guarantees that the local solutions patch together to form a global solution. □

Let $\Upsilon = [X, T]$. (This corresponds to $iR$ in the notation from [BG].)

**Proposition 12.** *We have*

$$\begin{array}{ll} \Upsilon z_1 = -z_1 & \Upsilon z_2 = -z_2 \\ \Upsilon w_1 = w_1 & \Upsilon w_2 = w_2. \end{array} \tag{3.3}$$

*Proof.* These follow directly from (3.2). □

**Proposition 13.** *We have*

$$[\Upsilon, X] = -2X \qquad [\Upsilon, T] = 2T.$$

*Proof.* By Lemma 9 it suffices to use (3.2) and (3.3) to test both sides against $z_j, w_j$. □

Now let

$$\begin{aligned} \eta' &= z_2\,dz_1 - z_1\,dz_2 \\ \eta'' &= w_2\,dw_1 - w_1\,dw_2 \\ \theta &= -w_1\,dz_1 - w_2\,dz_2 = z_1\,dw_1 + z_2\,dw_2. \end{aligned} \tag{3.4}$$

(The equivalence of the two descriptions of $\theta$ follows from (3.1b).)

**Lemma 14.** *We have*

$$du = (Tu)\,\eta' + (Xu)\,\eta'' + (\Upsilon u)\,\theta. \tag{3.5}$$

*Proof.* Direct computation using (3.4) reveals that

$$\begin{aligned} dz_1 &= w_2\eta' - z_1\theta \\ dz_2 &= -w_1\,\eta' - z_2\theta \\ dw_1 &= z_2\eta'' + w_1\theta \\ dw_2 &= -z_1\eta'' + w_2\theta; \end{aligned} \tag{3.6}$$

using (3.2) it follows that that (3.5) holds for $u = z_1, z_2, w_1$ or $w_2$.

From Lemma 9 we see that this implies the general case. ☐

**Lemma 15.** *We have*

$$\begin{array}{ccc} \theta(X) = 0 & \eta'(X) = 0 & \eta''(X) = 1 \\ \theta(T) = 0 & \eta'(T) = 1 & \eta''(T) = 0 \\ \theta(\Upsilon) = 1 & \eta'(\Upsilon) = 0 & \eta''(\Upsilon) = 0. \end{array} \tag{3.7}$$

Note that from $\theta(X) = 0 = \theta(T)$ we see that $\theta$ is a contact form on $S$; as we quote below from §2 above we will use this choice of contact form. Note also that (3.7) shows that $\eta', \eta''$ and $\theta$ are linearly independent at each point of $S$.

*Proof.* From (3.4) and (3.2) we have

$$\begin{aligned} \theta(X) &= -w_1(Xz_1) - w_2(Xz_2) = 0 \\ \theta(T) &= z_1(Tw_1) - z_2(Tw_2) = 0 \\ \theta(\Upsilon) &= -w_1(\Upsilon z_1) - w_2(\Upsilon z_2) = 1. \end{aligned}$$

Similar computations serve to verify the remaining entries. ☐

From (3.4) and (3.6) we find that

$$\begin{aligned} d\eta' &= -2\,dz_1 \wedge dz_2 = 2\,\eta' \wedge \theta \\ d\eta'' &= -2\,dw_1 \wedge dw_2 = -2\,\eta'' \wedge \theta \\ d\theta &= dz_1 \wedge dw_1 + dz_2 \wedge dw_2 = \eta' \wedge \eta''. \end{aligned} \tag{3.8}$$

Returning to the discussion from §2 we now set

$$\begin{aligned} \widetilde{d'}u &= (Tu)\,\eta' \\ \widetilde{d''}u &= (Xu)\,\eta'' \\ d^0u &= (\Upsilon u)\,\theta \end{aligned} \tag{3.9}$$

so that

$$d = \widetilde{d'} + \widetilde{d''} + d^0$$

and

$$\begin{aligned} \widetilde{d'}u\big|_H &= d'u \\ \widetilde{d''}u\big|_H &= d''u \\ d^0u\big|_H &= 0. \end{aligned}$$

In Proposition 16 and Theorem A below, the strongly $\mathbb{C}$-convex hypersurface $S \subset \mathbb{C}^2$ is assumed to be simply-connected. (In the case of compact $S$, the simple-connectivity holds automatically since $S$ will be diffeomorphic to the sphere $S^3$ – one way to show this is to extend the result in [Sem, §5] using the results of [Lem].)

**Proposition 16.** *If the equivalent conditions of Theorem 6 hold (with the above choice of $\theta$) then $\lambda = \Upsilon f = XTf = XTu$.*

*Proof.* From the proof of Theorem 6 we have

$$\begin{aligned} \widetilde{d'}u + \lambda\theta &= \widetilde{d'}f + \widetilde{d''}f + d^0f \\ &= \widetilde{d'}f + (\Upsilon f)\,\theta; \end{aligned}$$

matching terms we find that $\lambda = \Upsilon f = XTf = XTu$ as claimed. □

We need to better understand the condition that $\widetilde{d'u} + (XTu)\,\theta = (Tu)\,\eta' + (XTu)\,\theta$ is closed, i.e.,

$$\begin{aligned} 0 &= d(Tu)\wedge\eta' + (Tu)\cdot d\eta' + d(XTu)\wedge\theta + (XTu)\cdot d\theta \\ &= ((TTu)\,\eta' + (XTu)\,\eta'' + (\Upsilon Tu)\,\theta)\wedge\eta' + 2(Tu)\,\eta'\wedge\theta \\ &\qquad + ((TXTu)\,\eta' + (XXTu)\,\eta'' + (\Upsilon XTu)\,\theta)\wedge\theta - (XTu)\,\eta''\wedge\eta' \\ &= (-\Upsilon Tu + 2Tu + TXTu)\;\eta'\wedge\theta + (XXTu)\;\eta''\wedge\theta \\ &= (2Tu - [\Upsilon, T]u + TTXu)\;\eta'\wedge\theta + (XXTu)\;\eta''\wedge\theta \\ &= (TTXu)\;\eta'\wedge\theta + (XXTu)\;\eta''\wedge\theta. \end{aligned}$$

We have proved the following.

**Theorem A.** *For $S \subset \mathbb{C}^n$ $(n = 2)$ strongly $\mathbb{C}$-convex and simply-connected, the following conditions on smooth $u \colon S \to \mathbb{C}$ are equivalent:*

(3.10a) *$u$ decomposes as a sum $f + g$ where $f$ is CR and $g$ is dual-CR;*
(3.10b) $XXTu = 0 = TTXu$.

## 4. Alternate construction of $X$, $T$ and $\Upsilon$

In this section we set out an alternate approach to the development of the vector fields $X, T, \Upsilon$.

Let $\mathcal{I}' = \{(z_1, z_2, w_1, w_2) \in \mathbb{C}^4 \mid z_1 w_1 + z_2 w_2 = 1\}$. (See Remark 21 below.)

The holomorphic vector fields

$$\mathcal{X} = z_2 \frac{\partial}{\partial w_1} - z_1 \frac{\partial}{\partial w_2}$$
$$\mathcal{T} = w_2 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial z_2}$$
$$\mathcal{Y} = -z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}.$$

on $\mathbb{C}^4$ are tangent to $\mathcal{I}'$. We have

$$\begin{aligned} [\mathcal{X}, \mathcal{T}] &= \mathcal{Y} \\ [\mathcal{Y}, \mathcal{X}] &= -2\mathcal{X} \\ [\mathcal{Y}, \mathcal{T}] &= 2\mathcal{T}. \end{aligned} \tag{4.1}$$

Consider the diffeomorphism

$$\begin{aligned} &\mathcal{D}^\sharp_S \,:\, S \to \Gamma_S \subset \mathcal{I}' \\ &(z_1, z_2) \mapsto (z_1, z_2, w_1(z), w_2(z))\,. \end{aligned}$$

with $\Gamma_S \overset{\text{def}}{=} \mathcal{D}^\sharp_S(S)$. Using Addendum 3 we see that $\Gamma_S$ is a totally real 3-manifold inside the complex 3-manifold $\mathcal{I}'$.

**Proposition 17.** *We have*

$$\left(\mathcal{D}^\sharp_S\right)_* X = \left(\mathcal{X} + \phi\overline{\mathcal{X}} + \alpha\overline{\mathcal{T}}\right)\big|_{\Gamma_S} \tag{4.2a}$$

$$\left(\mathcal{D}^\sharp_S\right)_* T = \left(\mathcal{T} + \beta\overline{\mathcal{X}} + \psi\overline{\mathcal{T}}\right)\big|_{\Gamma_S} \tag{4.2b}$$

$$\left(\mathcal{D}_S^\sharp\right)_* \Upsilon = \left(\mathcal{Y} + (X\beta - T\phi)\overline{\mathcal{X}} + (X\psi - T\alpha)\overline{\mathcal{T}} + (\phi\psi - \alpha\beta)\overline{\mathcal{Y}}\right)|_{\Gamma_S} \tag{4.2c}$$

*for certain smooth functions $\alpha$, $\beta$, $\psi$, $\phi$.*

*Proof.* Quoting (4.1) from [BG] (but correcting typos in the last two entries) there are smooth functions $\alpha$, $\beta$, $\psi$, $\phi$ satisfying

$$\begin{aligned} X\overline{z}_1 &= \alpha\overline{w}_2 & X\overline{z}_2 &= -\alpha\overline{w}_1 \\ X\overline{w}_1 &= \phi\overline{z}_2 & X\overline{w}_2 &= -\phi\overline{z}_1 \\ T\overline{z}_1 &= \psi\overline{w}_2 & T\overline{z}_2 &= -\psi\overline{w}_1 \\ T\overline{w}_1 &= \beta\overline{z}_2 & T\overline{w}_2 &= -\beta\overline{z}_1. \end{aligned}$$

The first two lines of (4.2) follow from applying both sides to the functions $z_j, \overline{z}_j, w_j, \overline{w}_j$: for (4.2a) application of either side to

$$z_1, z_2, \overline{z}_1, \overline{z}_2, w_1, w_2, \overline{w}_1, \overline{w}_2$$

leads to

$$0, 0, \alpha\overline{w}_2, -\alpha\overline{w}_1, w_1, w_2, \phi\overline{z}_2, -\phi\overline{z}_1,$$

respectively, while a similar computation verifies (4.2b). The remaining line (4.2c) now follows from a bracket computation using the previous results. □

Any vector field $V$ (with values in $T\mathcal{I}'$) defined on $\Gamma_S$ may be written uniquely as $V^{\mathsf{tang}} + V^{\mathsf{normal}}$, where $V^{\mathsf{tang}}$ and $JV^{\mathsf{normal}}$ are tangent to $\Gamma_S$.

**Proposition 18.** *We have*

$$\begin{aligned} \left(\mathcal{D}_S^\sharp\right)_* X &= 2\left(\mathcal{X}|_{\Gamma_S}\right)^{\mathsf{tang}} \\ \left(\mathcal{D}_S^\sharp\right)_* T &= 2\left(\mathcal{T}|_{\Gamma_S}\right)^{\mathsf{tang}} \\ \left(\mathcal{D}_S^\sharp\right)_* \Upsilon &= 2\left(\mathcal{Y}|_{\Gamma_S}\right)^{\mathsf{tang}}. \end{aligned}$$

Thus $X$ is the unique vector field on $S$ pushing forward to twice the tangential part of $\mathcal{X}$ – that is, $X = 2(\mathcal{D}_S^\sharp)_*^{-1}((\mathcal{X}|_{\Gamma_S})^{\mathsf{tang}})$ – and similarly for $T$ and $\Upsilon$.

We will prove Proposition 18 as a consequence of a related result using type considerations. Recall that any vector field $V$ on a subset of $\mathcal{I}'$ decomposes uniquely as $V^{(1,0)} + V^{(0,1)}$ with

$$V^{(1,0)} = \frac{1}{2}(V - iJV)$$

$$V^{(0,1)} = \frac{1}{2}\,(V + iJV)\,.$$

Holomorphic vector fields are of type (1,0).

**Proposition 19.** *We have*

$$\begin{aligned}
\left(\left(\mathcal{D}_S^\sharp\right)_* X\right)^{(1,0)} &= \mathcal{X}|_{\Gamma_S}\\
\left(\left(\mathcal{D}_S^\sharp\right)_* T\right)^{(1,0)} &= \mathcal{T}|_{\Gamma_S}\\
\left(\left(\mathcal{D}_S^\sharp\right)_* \Upsilon\right)^{(1,0)} &= \mathcal{Y}|_{\Gamma_S}.
\end{aligned}$$

*Proof.* These follow directly from Proposition 17. □

**Lemma 20.** *If $V$ is a vector field tangent to $\Gamma_S$ then $V = 2(V^{(1,0)})^{\mathsf{tang}}$.*

*Proof.* This follows from $2V^{(1,0)} = V - iJV$. □

*Proof of Proposition 18.* Apply Lemma 20 to the results of Proposition 19. □

**Remark 21.** We may identify $\mathcal{I}'$ with an open subset of the full incidence manifold

$$\mathcal{I} \overset{\text{def}}{=} \{((z_0 : z_1 : z_2), (w_0 : w_1 : w_2)) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid z_0w_0 + z_1w_1 + z_2w_2 = 0\}$$

important in projective duality theory (as discussed in [APS, §3.2]) via the map

$$\begin{aligned}
\mathcal{I}' &\to \mathcal{I} \setminus \{z_0w_0 = 0\}\\
(z_1, z_2, w_1, w_2) &\mapsto ((i : z_1 : z_2), (i : w_1 : w_2))\,.
\end{aligned}$$

We may also identify $\mathcal{I}'$ with $SL(2,\mathbb{C})$ via $(z_1, z_2, w_1, w_2) \mapsto \left(\begin{smallmatrix} z_1 & -w_2 \\ z_2 & w_1 \end{smallmatrix}\right)$. Then the flows $\exp(t\,\mathrm{Re}\,\mathcal{X})$, $\exp(t\,\mathrm{Re}\,\mathcal{T})$ and $\exp(t\,\mathrm{Re}\,\mathcal{Y})$ correspond to right-multiplication by $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{smallmatrix}\right)$ respectively.

## 5. The projective decomposition problem for $n > 2$

Again we make the standing assumption that $S$ is a strongly $\mathbb{C}$-convex hypersurface satisfying (1.1).

We define $w_k(z)$ as in (1.2).

**Lemma 22.** *We have*

$$z_1 w_1 + \cdots + z_n w_n = 1 \text{ on } S \tag{5.1a}$$

$$w_1\,dz_1 + \cdots + w_n\,dz_n + z_1\,dw_1 + \cdots + z_n\,dw_n = 0 \text{ as 1-forms on } S \tag{5.1b}$$

$$w_1\,dz_1 + \cdots + w_n\,dz_n = 0 \text{ as forms on } H \tag{5.1c}$$

$$z_1\,dw_1 + \cdots + z_n\,dw_n = 0 \text{ as forms on } H. \tag{5.1d}$$

*Proof.* Like Lemma 8. □

**Proposition 23.** *For $1 \le j, k \le n$, $j \ne k$, there are uniquely-determined tangential vector fields $X_{jk}, T_{jk}$ on $S$ satisfying*

$$\begin{aligned} &X_{jk} z_\ell = 0 && T_{jk} w_\ell = 0 \\ &X_{jk} w_\ell = \begin{cases} z_k & \ell = j \\ -z_j & \ell = k \\ 0 & \text{otherwise} \end{cases} && T_{jk} z_\ell = \begin{cases} w_k & \ell = j \\ -w_j & \ell = k \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.2}$$

*The $T_{jk}$ and $X_{jk}$ take values in $H'$ and $H''$, respectively.*

For $k = j$, we set $X_{jj} = 0 = T_{jj}$.
Note the relation

$$z_j X_{k\ell} + z_k X_{\ell j} + z_\ell X_{jk} = 0. \tag{5.3}$$

**Proposition 24.** *There is a uniquely-determined tangential vector field $\Upsilon$ on $S$ satisfying*

$$\begin{aligned} \Upsilon z_1 &= -z_1 & \cdots \quad && \Upsilon z_n &= -z_n \\ \Upsilon w_1 &= w_1 & \cdots \quad && \Upsilon w_n &= w_n. \end{aligned} \tag{5.4}$$

*Proofs of Propositions 23 and 24.* These are similar to the proof of Proposition 11. □

**Proposition 25.** *We have*

$$\begin{aligned} [\Upsilon, X_{jk}] &= -2X_{jk} \\ [\Upsilon, T_{jk}] &= 2T_{jk}. \end{aligned}$$

*Proof.* Similar to the proof of Proposition 13. □

**Proposition 26.** *For $p \in S$ we have*

$$H'_p = \mathrm{Span}\,\{T_{jk} \colon 1 \le j, k \le n\}$$
$$H''_p = \mathrm{Span}\,\{X_{jk} \colon 1 \le j, k \le n\}\,.$$

*Proof.* First note that $\dim_{\mathbb{C}} H'_p = n - 1 = \dim_{\mathbb{C}} H''_p$; then note that after possibly reordering the coordinates we may assume that $z_1 w_1 \neq 0$ at $p$ and thus that $T_{12}, \ldots, T_{1n}$ are linearly independent in $H'_p$ while $X_{12}, \ldots, X_{1n}$ are linearly independent in $H''_p$. □

Now let

$$\begin{aligned} \eta'_{jk} &= z_k\, dz_j - z_j\, dz_k \\ \eta''_{jk} &= w_k\, dw_j - w_j\, dw_k \\ \theta &= -w_1\, dz_1 - \cdots - w_n\, dz_n = z_1\, dw_1 + \cdots + z_n\, dw_n. \end{aligned} \tag{5.5}$$

Note that

$$d\theta = dz_1 \wedge dw_1 + \cdots + dz_n \wedge dw_n \tag{5.6}$$

and that $d\theta\big|_H$ is non-degenerate.

**Lemma 27.** *We have*

$$du = \frac{1}{2} \sum_{j,k} (T_{jk} u)\ \eta'_{jk} + \frac{1}{2} \sum_{j,k} (X_{jk} u)\ \eta''_{jk} + (\Upsilon u)\ \theta. \tag{5.7}$$

*Proof.* Check that the result holds for $u = z_j$ or $w_j$, then apply adapted version of Lemma 9. □

Following (3.9) we set

$$\begin{aligned} \widetilde{d'} u &= \frac{1}{2} \sum_{j,k} (T_{jk} u)\ \eta'_{jk} \\ \widetilde{d''} u &= \frac{1}{2} \sum_{j,k} (X_{jk} u)\ \eta''_{jk} \\ d^0 u &= (\Upsilon u)\, \theta \end{aligned} \tag{5.8}$$

so that again

$$d = \widetilde{d'} + \widetilde{d''} + d^0.$$

Let $\omega$ be an $H$-form of degree two. We will say that $\omega$ has

- type (2,0) if $\omega$ is $J$-bilinear;
- type (0,2) if $\omega$ is $J^*$-bilinear;
- type (1,1) if it can be written as a finite sum of wedge products of $J$-linear $H$-forms of degree one with $J^*$-linear $H$-forms of degree one.

From standard arguments we obtain the following.

**Proposition 28.** *Every $H$-form $\omega$ of degree two decomposes uniquely as a sum $\omega^{(2,0)} + \omega^{(0,2)} + \omega^{(1,1)}$ of forms of specified type.*

Recall the decomposition of $H_pS \otimes \mathbb{C}$ from Remark 5.

**Lemma 29.** *If an $H$-form $\omega$ has type (1,1) then*

$$\omega\left(V' + V'', W' + W''\right) = \omega\left(V', W''\right) - \omega\left(W', V''\right).$$

*Proof.* Using the definition of a form of type (1,1) and the last sentence of Remark 5 we have $\omega\left(V', W'\right) = 0 = \omega\left(V'', W''\right)$; the claim follows. ☐

**Proposition 30.** $d\widetilde{d'}u\big|_H = \frac{1}{4}\sum_{j,k,\ell,m}\left(X_{\ell m}T_{jk}u\right)\eta''_{\ell m}\wedge\eta'_{jk}$*; in particular,* $d\widetilde{d'}u\big|_H$ *has type (1,1).*

*Proof.* We first show that $d\widetilde{d'}u\big|_H$ has type (1,1).

Note first that $d\eta'_{jk}\big|_H$ has type (2,0), $d\eta''_{jk}\big|_H$ has type (0,2) and $d\theta\big|_H$ has type (1,1).

From direct inspection we now find that

$$\left(d\widetilde{d'}u\big|_H\right)^{(0,2)} = 0$$
$$\left(d\widetilde{d''}u\big|_H\right)^{(2,0)} = 0$$
$$\left(dd^0u\big|_H\right)^{(2,0)} = 0.$$

It suffices now to show that $(d\widetilde{d'}u\big|_H)^{(2,0)} = 0$; this follows from taking (2,0)-components in

$$\begin{aligned} 0 &= ddu\big|_H \\ &= d\widetilde{d'}u\big|_H + d\widetilde{d''}u\big|_H + dd^0u\big|_H. \end{aligned}$$

Ignoring the cancelling (2,0)-terms now find that

$$\begin{aligned} d\widetilde{d'}u\big|_H &= \frac{1}{2}\sum_{j,k}\widetilde{d''}\left(T_{jk}u\right)\,\eta'_{jk}\big|_H \\ &= \frac{1}{4}\sum_{j,k,\ell,m}\left(X_{\ell m}T_{jk}u\right)\eta''_{\ell m}\wedge\eta'_{jk}\big|_H. \end{aligned}$$

□

For conciseness we now fix $p \in S$ and set

$$\nu_p = d\widetilde{d'}u(p)\big|_H. \tag{5.9}$$

**Lemma 31.** *The condition (2.2) holds (at $p$) if and only if there is a scalar $\lambda$ satisfying*

$$\nu_p\left(T_{jk}, X_{\ell m}\right) = \lambda \cdot d\theta\left(T_{jk}, X_{\ell m}\right)$$

*for all $j, k, \ell, m$.*

*Proof.* This follows from Proposition 30 along with Lemma 29 and Proposition 26. □

**Proposition 32.** *Suppose that*

- *$T$ is a vector field taking values in $H'$;*
- *$X$ is a vector field taking values in $H''$;*
- *$d\theta(T, X) \equiv 0$.*

*Then $\nu_p\left(T, X\right) = -XTu(p)$.*

**Lemma 33.** *We can write*

$$u = C + f_1 + g_2 + \sum_{j=3}^{N} f_j g_j + E \tag{5.10}$$

*with*

- *all $f_j$ are CR;*
- *all $f_j(p) = 0$;*
- *all $g_j$ are dual CR;*
- *all $g_j(p) = 0$;*
- *all second derivatives of $E$ vanish at $p$.*

*Proof.* The computations from the proof of Lemma 2 show that

$$\text{Span}\left\{dz_j(p), dw_k(p)\right\} = T_pS \otimes \mathbb{C}.$$

A "Taylor polynomial"-type argument now serves to prove the Lemma. $\square$

*Proof of Proposition 32.* Invoking the decomposition from Lemma 33 we note first that

$$\widetilde{d'}C = 0 = -XT(C)(p) \tag{5.11}$$

and

$$d\widetilde{d'}g_2 = 0 = -XTg_2(p). \tag{5.12}$$

Next we note that

$$\widetilde{d'}f_1 = df_1 - d^0 f_j = df_1 - (\Upsilon f_1)\,\theta$$

(since $\widetilde{d''}f_1 = 0$) and thus

$$d\widetilde{d'}f_1(T, X) = (\Upsilon f_1)(p) \cdot d\theta(T, X) = 0 \tag{5.13}$$

(using $\theta(T) = 0 = \theta(X)$).

For the general term $f_j g_j$ we have

$$\begin{aligned}
\widetilde{d'}\left(f_j g_j\right) &= \widetilde{d'}\left(f_j\right) g_j \\
&= \left(df_j - d^0 f_j\right) g_j \\
&= g_j\, df_j - g_j\left(\Upsilon f_j\right)\theta
\end{aligned}$$

and so

$$\begin{aligned}
d\widetilde{d'}\left(f_j g_j\right) &= -df_j \wedge dg_j - d\left(g_j\left(\Upsilon f_j\right)\right) \wedge \theta - g_j\left(\Upsilon f_j\right)\, d\theta \\
d\widetilde{d'}\left(f_j g_j\right)\big|_H &= -df_j \wedge dg_j\big|_H - g_j\left(\Upsilon f_j\right)\, d\theta\big|_H \\
d\widetilde{d'}\left(f_j g_j\right)(T, X) &= -\left(df_j \wedge dg_j\right)(T, X) = -Tf_j \cdot Xg_j
\end{aligned}$$

thus

$$d\widetilde{d'}\left(f_j g_j\right)(T, X)(p) = -Tf_j(p) \cdot Xg_j(p) = -XT(f_j g_j)(p). \tag{5.14}$$

Adding (5.11), (5.13), (5.12) and (5.14) and recalling (5.9) and (5.10) we obtain

$$\nu_p\,(T,X) = -XTu(p).$$

$\square$

Let

$$\begin{aligned}\widetilde{X}_{jk\ell} &= w_j X_{\ell j} + w_k X_{\ell k}\\ \widetilde{T}_{jk\ell} &= z_j T_{\ell j} + z_k T_{\ell k}.\end{aligned}$$

The $\widetilde{X}_{jk\ell}$ and $\widetilde{T}_{jk\ell}$ take values in $H''$ and $H'$, respectively with

(5.15)
$$\widetilde{X}_{jk\ell} z_m = 0 \qquad\qquad \widetilde{T}_{jk\ell} w_m = 0$$

$$\widetilde{X}_{jk\ell} w_m = \begin{cases} -z_j w_\ell & m=j\\ -z_k w_\ell & m=k\\ z_j w_j + z_k w_k & m=\ell\\ 0 & otherwise\end{cases} \qquad \widetilde{T}_{jk\ell} z_m = \begin{cases} -z_\ell w_j & m=j\\ -z_\ell w_k & m=k\\ z_j w_j + z_k w_k & m=\ell\\ 0 & otherwise.\end{cases}$$

**Lemma 34.** *For $j,k,\ell$ distinct we have $d\theta(\widetilde{T}_{jk\ell}, X_{jk}) = 0 = d\theta(T_{jk}, \widetilde{X}_{jk\ell})$.*

*Proof.* Using (5.6) we find that $d\theta(\widetilde{T}_{jk\ell}, X_{jk}) = -w_\ell z_j z_k + w_\ell z_j z_k = 0$ and $d\theta(T_{jk}, \widetilde{X}_{jk\ell}) = -z_\ell w_j w_k + z_\ell w_j w_k = 0$. $\square$

For $X \in H'_p S$ we set

$$\begin{aligned}X^{\perp_{d\theta}} &= \left\{T \in H''_p S\colon d\theta(T,X)(p) = 0\right\}\\ X^{\perp_{\nu_p}} &= \left\{T \in H''_p S\colon \nu_p(T,X) = 0\right\}.\end{aligned}$$

**Lemma 35.**
1. *If $z_j \neq 0$ the vectors $\{X_{jk}(p)\colon k \neq j\}$ form a basis for $H''_p$.*
2. *If ($\star$) holds and $k \neq j$ the vectors $\{\widetilde{T}_{jk\ell}(p)\colon \ell \notin \{j,k\}\}$ form a basis of $X_{jk}^{\perp_{d\theta}}(p)$.*

*Proof.* In each case we have the right number of linearly independent vectors in the indicated space. $\square$

The following result is based on Theorem 3 in [Aud].

**Theorem B.** *For $S \subset \mathbb{C}^n$ $(n > 2)$ strongly $\mathbb{C}$-convex and simply-connected and satisfying $(\star)$ the following conditions on smooth $u \colon S \to \mathbb{C}$ are equivalent.*

(5.16a) *$u$ decomposes as a sum $f + g$ where $f$ is CR and $g$ is dual-CR;*
(5.16b) *for all distinct $j, k, \ell$ we have*

$$X_{jk}\widetilde{T}_{jk\ell}u = 0. \tag{5.17}$$

*Proof.* If (5.16a) holds then using (2.2) along with Lemma 34 and Proposition 32 we have

$$\begin{aligned} X_{jk}\widetilde{T}_{jk\ell}u(p) &= \nu_p\left(\widetilde{T}_{jk\ell}, X_{jk}\right) \\ &= \lambda \cdot d\theta\left(\widetilde{T}_{jk\ell}, X_{jk}\right)(p) \\ &= 0 \end{aligned}$$

for distinct $j, k, \ell$.

If (2.1b) holds then fixing $j$ the above computation along with Lemma 35 yields $X_{jk}^{\perp_{\nu_p}} \supset X_{jk}^{\perp_{d\theta}}$ for $k \neq j$, thus there are $\lambda_k$ so that

$$X_{jk} \lrcorner\, \nu_p = \lambda_k X_{jk} \lrcorner\, d\theta.$$

For distinct $k_1, k_2$ not equal to $j$ we have

$$\begin{aligned} z_{k_2}X_{jk_1} \lrcorner\, \nu_p &= \lambda_{k_1} z_{k_2} X_{jk_1} \lrcorner\, d\theta \\ z_{k_1}X_{jk_2} \lrcorner\, \nu_p &= \lambda_{k_2} z_{k_1} X_{jk_2} \lrcorner\, d\theta; \end{aligned}$$

using (5.3) this yields

$$\begin{aligned} z_j X_{k_1k_2} \lrcorner\, \nu_p &= (z_{k_1}X_{jk_2} - z_{k_2}X_{jk_1}) \lrcorner\, \nu_p \\ &= (\lambda_{k_2} z_{k_1} X_{jk_2} - \lambda_{k_1} z_{k_2} X_{jk_1}) \lrcorner\, d\theta; \end{aligned}$$

but repeating the above argument there is also $\lambda^*$ with $z_j X_{k_1k_2} \lrcorner\, \nu_p = z_j \lambda^* X_{k_1k_2} \lrcorner\, d\theta$ and the non-degeneracy of $d\theta$ then yields

$$\lambda_{k_2} z_{k_1} X_{jk_2} - \lambda_{k_1} z_{k_2} X_{jk_1} = \lambda^* z_j X_{k_1k_2} = \lambda^* z_{k_1} X_{jk_2} - \lambda^* z_{k_2} X_{jk_1}.$$

From the independence of $X_{jk_1}, X_{jk_2}$ and the fact that $z_{k_1}$ and $z_{k_2}$ are non-zero we obtain $\lambda_{k_2} = \lambda^* = \lambda_{k_1}$.

By Lemma 31 (2.2) holds; by Theorem 7 we then have (5.16a). ☐

**Remark 36.** The proof of Theorem B shows that if $u$ satisfies (5.16a) then it also must satisfy

$$T_{jk}\widetilde{X}_{jk\ell}u = 0 \text{ for distinct } j, k, \ell. \tag{5.18}$$

In particular, (5.17) implies the companion condition (5.18). On the other hand, the equations $TTXu = 0$ and $XXTu = 0$ from Theorem A do not imply each other locally (see Example 21 in [BG]) but they do imply each other when $S$ is compact and circular (see Theorem B in [BG]). $\Diamond$

**Remark 37.**

(a) Recall that in higher dimensions we required an additional second order vector field condition, given in ($\star$). Failure of condition ($\star$) may be repaired (at least locally) by a linear change of coordinates as shown in the following proposition.

**Proposition 38.** *For $p \in S$ there is a linear transformation $T$ so that $T(S)$ satisfies ($\star$) at $T(p)$ (hence also in a neighborhood of $T(p)$).*

*Proof.* We set $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

If $M$ is an invertible square matrix then replacing $z$ by $Mz$ the transformation law from [Bar,§6] tells us that $w$ is replaced by ${}^tM^{-1}w$. From (3.1a) we know that $z$ and $w$ are non-zero; choosing $M$ from a Zariski-open dense set of matrices we may assume that all entries of the new vectors $z, w$ are non-zero.

With this in place we make a further change of variables, replacing $z$ by $\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ 0 & 1 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} z$ and $w$ by $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -a_2 & 1 & \cdots & 0 \\ & & \vdots & \\ -a_n & 0 & \cdots & 1 \end{pmatrix} w$.

We find that

$$\text{for } 1 < j \text{ the sum } z_1w_1 + z_jw_j \text{ is replaced by } z_1w_1 + z_jw_j + w_1 \sum_{k \notin \{1,j\}} a_k z_k$$

and that

$$\text{for } 1 < k < \ell \text{ the sum } z_kw_k + z_\ell w_\ell \text{ is replaced by } z_kw_k + z_\ell w_\ell - w_1(a_kz_k + a_\ell z_\ell).$$

For a Zariski-open dense set of $a_j$'s the transformed sums will all be non-zero. $\square$

(b) From Theorem B and Proposition 38 it follows easily that for any relatively compact open $U \subset S$ there are finitely many vector fields $\widehat{X}_1, \ldots, \widehat{X}_N$ with values in $H''$ and $\widehat{T}_1, \ldots, \widehat{T}_N$ with values in $H'$ so that decomposable functions on $U$ are characterized by the system

$$\widehat{X}_k \widehat{T}_k u = 0 \text{ for } k = 1 \ldots, N.$$

$\Diamond$

## 6. Pluriharmonic boundary values

One inspiration for the current paper comes from the problem of characterizing the boundary values of pluriharmonic functions. The pluriharmonic boundary value problem has a long history, and we refer the reader to the introduction of [BG] for an outline of the history. We briefly recall some key results.

Nirenberg observed that there is no second order system of differential operators which is tangential to the boundary of the ball in $\mathbb{C}^2$ that characterizes pluriharmonic boundary values (see [BG] for a discussion of this result). Bedford [Bed1] provided a system of third order operators that solved the global problem for the unit ball. In higher dimensions, Audibert [Aud] and Bedford [Bed] solved the global and local problems using second order systems. Bedford and Federbush [BeFe] extended these results to the case of embedded CR manifolds, and Lee extended the results to abstract CR manifolds.

Our results parallel the results on the sphere, and we briefly recall the local results on the sphere. Define the tangential vector fields

$$L_{jk} = z_j \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_j} \qquad \overline{L}_{jk} = \bar{z}_j \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial z_j}$$

for $1 \leq j, k \leq n$. Further let

$$\widetilde{L}_{jk\ell} = z_j \overline{L}_{\ell j} + z_k \overline{L}_{\ell k}.$$

**Theorem 39.** *[Aud] Suppose $S$ is a relatively open subset of $S^{2n-1}$, and $u$ is smooth on $S$.*

1. *$u$ extends to a pluriharmonic function on a one-sided neighborhood of $S$ if and only if*

$$L_{jk} L_{lm} \overline{L}_{rs} u = 0 = \overline{L}_{jk} \overline{L}_{lm} L_{rs} u$$

*for $1 \leq j, k, l, m, r, s \leq n$.*

*2. If $n > 2$, then $u$ extends to a pluriharmonic function on a one-sided neighborhood of $S$ if and only if*

$$L_{jk}\widetilde{L}_{jk\ell}u = 0$$

*for all distinct $j, k, \ell$.*

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David E. Barrett
Department of Mathematics
University of Michigan
Ann Arbor
MI 48109-1043
USA
E-mail: barrett@umich.edu

Dusty E. Grundmeier
Department of Mathematics
Harvard University
Cambridge
MA 02138-2901
USA
E-mail: deg@math.harvard.edu