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Vanishing theorem of Kohn-Rossi cohomology class and rigidity of Sasakian space form

Der-Chen Chang*, Shu-Cheng Chang†, Ting-Jung Kuo†, and Chien Lin

Dedicated to Professor J.J. Kohn on his 90th birthday

Abstract: In this note, under the positivity assumption of the pseudohermitian curvature, we derive the existence theorem for pseudo-Einstein contact forms and rigidity theorems for Sasakian space forms in a closed spherical strictly pseudoconvex CR 3-manifold of the nonnegative CR Paneitz operator with a kernel consisting of the CR pluriharmonic functions and the CR Q-curvature is pluriharmonic.

Keywords: Pseudo-Einstein, CR pluriharmonic operator, CR Paneitz operator, CR Q-curvature, Tangential Cauchy-Riemann equation, Sasakian manifold, CR rigidity theorem, Sasakian space form.

1. Introduction

A Riemannian manifold is Einstein if the Ricci curvature tensor is function-proportional to its Riemannian metric. For dimension greater than 2, it is equivalent to the constant-proportional case. In contrast to the Riemannian geometry situation, there is an analog notion that a strictly pseudoconvex CR $(2n+1)$-manifold is pseudo-Einstein if the pseudohermitian Ricci curvature tensor is function-proportional to its Levi metric. The pseudo-Einstein condition is less rigid than the Einstein condition in Riemannian geometry. Indeed, in the case $n \geq 2$, the CR contracted Bianchi identity does not imply

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the pseudohermitian scalar curvature R to be a constant due to the presence of pseudohermitian torsion

$$R_{\alpha\overline{\beta},\beta} = R_\alpha - i(n-1)A_{\alpha\beta,\overline{\beta}}.$$

Any contact form on a closed strictly pseudoconvex 3-manifold is actually pseudo-Einstein since the pseudohermitian Ricci tensor has only one component $R_{1\overline{1}}$.

In [Lee88], J. Lee showed that the obstruction to the existence of a pseudo-Einstein contact form θ is that its first Chern class $c_1(T_{1,0}M)$ vanishes. Indeed, for a closed strictly pseudoconvex $(2n+1)$-manifold (M, J, θ) with $c_1(T_{1,0}M) = 0$ and $n \geq 2$, he proved that M admits a globally defined pseudo-Einstein contact form if either M admits a contact form θ with nonnegative pseudohermitian Ricci curvature tensor or the vanishing pseudohermitian torsion. However, his method couldn't be applied to the case $n = 1$ directly.

So it is natural to focus on the existence theorem of pseudo-Einstein contact forms for $n = 1$. Of course, we must find another appropriate definition for the pseudo-Einstein contact form. In fact, by Lemma 2.2 below, it is reasonable to view

$$W_1 \doteqdot (R_{,1} - iA_{11,\overline{1}}) = 0$$

as the pseudo-Einstein contact form in a closed strictly pseudoconvex CR 3-manifold (M, J, θ).

Before we start to work on the existence of pseudo-Einstein contact forms in a closed strictly pseudoconvex CR 3-manifold, we make the following observations in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, J) with a choice of pseudohermitian contact form θ.

(i) For $n \geq 2$: Assume that the pseudohermitian Ricci curvature is positive, it is well-known ([K64], [Lee88]) that we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation

$$\overline{\partial}_b \varphi = \eta \tag{1.1a}$$

for any $\overline{\partial}_b$-closed $(0,1)$-form η. That is to say that

$$H^{0,1}_{\overline{\partial}_b}(M) = 0.$$

(ii) For $n = 1$: We consider a closed strictly pseudoconvex CR 3-manifold (M, θ) with $c_1(T_{1,0}M) = 0$. There is a pure imaginary 1-form

$$\sigma = \sigma_{\overline{1}}\theta^{\overline{1}} - \sigma_1\theta^1 + i\sigma_0\theta \tag{1.2}$$

such that

$$d\omega_1^1 = d\sigma.$$

Kohn's result (Lemma 3.2 below) implies that there is a complex function

$$\varphi = u + iv \in C_{\mathbb{C}}^{\infty}(M)$$

and $\gamma = \gamma_{\bar{1}}\theta^{\bar{1}} \in \Omega^{0,1}(M) \cap \ker(\Box_b)$ such that

$$\overline{\partial}_b \varphi = \sigma_{\bar{1}}\theta^{\bar{1}} - \gamma_{\bar{1}}\theta^{\bar{1}} \tag{1.3}$$

with

$$\overline{\partial}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = 0.$$

Here $\Box_b = 2\left(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b\right)$ is the Kohn-Rossi Laplacian. Thus it is natural to ask when we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation (i.e. $\gamma = 0$)

$$\overline{\partial}_b \varphi = \sigma_{\bar{1}}\theta^{\bar{1}}. \tag{1.4}$$

In this paper, we focus on the existence of pseudo-Einstein contact forms as in Corollary 1.1, Theorem 1.2 and Theorem 1.4, an upper bound eigenvalue estimate for the CR Paneitz operator as in Theorem 1.3, Corollary 1.3 and its applications to the CR rigidity theorem for Sasakian space forms as in Corollary 1.3 and Corollary 1.4 in a closed spherical, strictly pseudoconvex CR 3-manifold.

We first state one of the main theorems as follows:

Theorem 1.1. *If (M, J, θ) is a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Then*

(i) $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form if and only if f satisfies the third-order partial differential equation

$$P_1 f = i\left(A_{11}\gamma_{\bar{1}} - \gamma_{1,0}\right). \tag{1.5}$$

Here P_1 is a third-order CR pluriharmonic operator

$$P_1 f = f_{\bar{1}11} + iA_{11}f_{\bar{1}}.$$

(ii) In particular, $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form for a CR-pluriharmonic function f if and only if the equality holds:

$$\left(A_{11}\gamma_{\bar{1}} - \gamma_{1,0}\right) = 0. \tag{1.6}$$

As a consequence, we are able to show that one of the existence theorems for the pseudo-Einstein contact form in this paper.

Corollary 1.1. *Let (M, J, θ) be a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Then M admits a globally defined pseudo-Einstein contact form $e^{\frac{(f+2u)}{3}}\theta$ for any CR-pluriharmonic function f if the pseudohermitian torsion is vanishing (Sasakian). More precisely, we have*

$$\gamma_{1,0} = 0.$$

Note that we do not know whether $\gamma = 0$ holds in the situation as in Corollary 1.1. However, by deriving the Bochner-type estimate as in the Lemma 3.4, we can conclude

$$\gamma = 0$$

under certain pseudohermitian geometric assumptions and obtain the solvability of the inhomogeneous tangential Cauchy-Riemann equation (1.4):

Theorem 1.2. *Let (M, J, θ) be a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$ and nonnegative CR Paneitz operator P_0. Assume that the pseudohermitian curvature is $\frac{1}{2}$-positive*

$$R(x) > |A_{11}|(x)$$

for all $x \in M$. Then $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form for any CR-pluriharmonic function f if and only if the inhomogeneous tangential Cauchy-Riemann equation (1.4) is solvable.

We observe that, for a strictly pseudoconvex 3-manifold (M^3, J, θ), we have the invariance property for the CR pluriharmonic operator P_1 and CR Paneitz operator P_0. It is to say that, for rescaled contact form $\widetilde{\theta} = e^{2g}\theta$, we have

$$\widetilde{P}_1 = e^{-3g}P_1 \quad \text{and} \quad \widetilde{P}_0 = e^{-4g}P_0. \tag{1.7}$$

Then the nonnegativity of CR Paneitz operator P_0 is CR conformal invariant ([H93]). Since the CR Paneitz operator P_0 is nonnegative ([CCC07]) if the pseudohermitian torsion is vanishing, it follows from Theorem 1.2 and Corollary 1.1 that

Corollary 1.2. *Let (M, J, θ) be a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Assume that the manifold is Sasakian and the Tanaka-Webster scalar curvature is positive. Then we have the solvability of the inhomogeneous tangential Cauchy-Riemann equation (1.4). That is to say that the Kohn–Rossi cohomology class of $\sigma_{\bar{1}}\theta^{\bar{1}}$ is vanishing.*

When the torsion is nonvanishing, with the help of the notion of C_0-*convexity*, we have the eigenvalue estimate for the CR Paneitz operator P_0 in terms of the CR Q-curvature.

Theorem 1.3. *Let (M, J, θ) be a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$ and the nonnegative CR Paneitz operator P_0 has kernel consisting of the CR pluriharmonic functions. Assume that the pseudohermitian curvature is $\frac{1}{2}$-positive*

$$R(x) > |A_{11}|(x)$$

and $A_{11,\bar{1}}(x) = 0$ for all $x \in M$. If $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form for any CR-pluriharmonic function f, then one can derive the upper bound estimate for the first eigenvalue of the CR Paneitz operator P_0

$$\Lambda^2 \int_M (u^{\perp})^2 d\mu \leq \int_M (Q^{\perp})^2 d\mu, \tag{1.8}$$

where the decomposition $Q = Q_{\ker} + Q^{\perp}$ and $u = u_{\ker} + u^{\perp}$ is with respect to the CR Paneitz operator P_0. Here Λ is the positive constant as in (2.4).

For a closed strictly pseudoconvex CR 3-manifold of vanishing pseudohermitian torsion (Sasakian), we have

$$\ker P_1 = \ker P_0. \tag{1.9}$$

In general, we only have $\ker P_1 \subsetneq \ker P_0$. Then combining Theorem 1.3, Corollary 1.2 and (3.2), we have the following CR rigidity theorem ([T69]) in a Sasakian manifold due to the eigenvalue estimate of the CR Paneitz operator (1.8).

Corollary 1.3. *Let (M, J, θ) be a closed, strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Assume that the manifold is Sasakian and the Tanaka-Webster scalar curvature is positive, then*

$$\Lambda^2 \int_M (u^{\perp})^2 d\mu \leq \int_M (Q^{\perp})^2 d\mu.$$

In additional, if the CR Q-curvature is pluriharmonic (i.e. $Q^{\perp} = 0$*), then* (M, J, θ) *is the Sasakian space form with the positive constant Tanaka-Webster scalar curvature and vanishing torsion.*

We observe that any compact simply connect Sasakian 3-manifold with the positive Tanaka-Webster scalar curvature is diffeomorphic to the sphere $\mathbf{S}^3$. This result was proved by Belgun ([B00]) as a part of the classification of three dimensional Sasakian manifolds. Also we refer to [HS16] for higher dimensional Sasakian manifolds.

Finally, if we do not assume the torsion is vanishing (non-Sasakian), we can derive another existence theorem for the pseudo-Einstein contact form with the stronger condition.

Theorem 1.4. *Let* (M, J, θ) *be a closed strictly pseudoconvex CR* 3*-manifold with* $c_1(T_{1,0}M) = 0$ *and the nonnegative CR Paneitz operator* P_0 *has kernel consisting of the CR pluriharmonic functions. Assume that the pseudohermitian curvature is* $\frac{1}{2}$*-positive*

$$R(x) > |A_{11}|(x)$$

and $A_{11,\bar{1}}(x) = 0$ *for all* $x \in M$*. If the CR Q-curvature is pluriharmonic, then*

$$\gamma = 0.$$

Hence $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ *is a pseudo-Einstein contact form for any CR-pluriharmonic function* f*.*

As a consequence of Theorem 1.4, we have the another CR rigidity theorem for Sasakian space forms ([T69]) in a spherical CR 3-manifold.

Corollary 1.4. *Let* (M, J, θ) *be a closed spherical strictly pseudoconvex CR* 3*-manifold of* $c_1(T_{1,0}M) = 0$ *and the pluriharmonic CR Q-curvature. Assume that the CR Paneitz operator* P_0 *is nonnegative with kernel consisting of the CR pluriharmonic functions and the pseudohermitian curvature is* $\frac{1}{2}$*-positive*

$$R(x) > |A_{11}|(x)$$

with $A_{11,\bar{1}}(x) = 0$ *for all* $x \in M$*. Then* (M, J, θ) *is the Sasakian space form.*

We conclude the introduction with the outline of the paper. In Section 2, we derive some preliminary results and indicate the geometry and topology of CR 3-manifolds with the positivity of pseudohermitian curvature. In Section 3, we prove main Theorems such as Theorem 1.3 and Corollary 1.4. In Appendix, we survey basic notions in the pseudohermitian (strictly pseudoconvex CR) geometry.

2. Preliminaries

In this section, we will derive some necessary ingredients for the proof of main results. In particular, we define the positivity of pseudohermitian curvature and indicate the geometry and topology of strictly pseudoconvex CR 3-manifolds.

Definition 2.1. *We say that a strictly pseudoconvex CR 3-manifolds is C_0-convex if the pseudohermitian curvature is C_0-positive, for a positive constant C_0. That is*

$$(R - C_0 Tor)(X, X) \quad = \quad Rx^1x^{\bar{1}} - 2C_0 \mathrm{Re}[i(A_{\bar{1}\bar{1}}x^{\bar{1}}x^{\bar{1}})] > 0 \tag{2.1}$$

for any $X = x^1 Z_1 \in T_{1,0}(M)$.

Before giving the proof of Theorem 1.3, we explain why we introduce the notion of C_0-convexity as follows:

Lemma 2.1 ([CaCC20]). *Let M be a closed strictly pseudoconvex CR 3-manifold. For any nonnegative constant C_0, C_0-convexity is equivalent to the curvature-torsion pinching condition*

$$R(x) > 2C_0|A_{11}|(x)$$

for all $x \in M$. Moreover, if M is C_0-positive with $C_0 \geq \frac{1}{2}$, then M admits a Riemannian metric of positive scalar curvature. In particular, it is the case if

$$R(x) > |A_{11}|(x).$$

Definition 2.2 ([Lee88]). *(i) A contact form θ on a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, θ) is said to be pseudo-Einstein for $n \geq 2$ if the pseudohermitian Ricci tensor $R_{\alpha\overline{\beta}}$ is proportional to the Levi form $h_{\alpha\overline{\beta}}$, i.e.,*

$$R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}},$$

where $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ is the Tanaka-Webster scalar curvature of (J, θ).

(ii) (Lemma 2.2) Note that any contact form on a closed strictly pseudoconvex 3-manifold is actually pseudo-Einstein (since the pseudohermitian Ricci tensor has only one component $R_{1\bar{1}}$). Then we define a contact form θ on a closed strictly pseudoconvex CR 3-manifold (M, θ) to be pseudo-Einstein if the following tensor is vanishing

$$W_1 \doteqdot \left(R_1 - iA_{11,\bar{1}} \right) = 0.$$

(iii) We define the first Chern class $c_1(T_{1,0}M) \in H^2(M, \mathbf{R})$ *for the holomorphic tangent bundle* $T^{1,0}M$ *by*

$$\begin{aligned} c_1(T^{1,0}M) &= \frac{i}{2\pi}[d\omega_\alpha{}^\alpha] \\ &= \frac{i}{2\pi}[R_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + A_{\alpha\mu,\overline{\alpha}}\theta^\mu \wedge \theta - A_{\overline{\alpha}\overline{\mu},\alpha}\theta^{\overline{\mu}} \wedge \theta]. \end{aligned}$$

(iv) Note that any pseudo-Einstein manifold (M^{2n+1}, θ)*, the first Chern class* $c_1(T_{1,0}M)$ *of* $T_{1,0}(M)$ *is vanishing ([Lee88]).*

Next let us recall the equivalent definitions of the pseudo-Einsteinian $(2n+1)$-manifold for $n \geq 2$ and $n = 1$ as well.

Lemma 2.2 ([Lee88], [CCKL19])**.** *(i) If* (M, J, θ) *is a strictly pseudoconvex CR* $(2n+1)$*-manifold for* $n \geq 2$*, then the following propositions are all equivalent:*

$$\begin{array}{ll} (1) & R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}}, \\ (2) & \left(\omega_\alpha^\alpha + \frac{i}{n} R\theta\right) \text{ is closed,} \\ (3) & W_\alpha \doteqdot \left(R_{,\alpha} - inA_{\alpha\beta,}{}^{\beta}\right) = 0. \end{array}$$

As for $n = 1$*, we still have the equivalence between* (2) *and* (3)*.*

(ii) By the equivalence of (2) *and* (3)*, we see the first Chern class* $c_1(T_{1,0}M)$ *is vanishing if* (M, J, θ) *is a pseudo-Einsteinian* 3*-manifold.*

Proof. The equivalence of (1) and (2) could be found in [Lee88] for $n \geq 2$. The proof of (2) $\Longleftrightarrow$ (3) for $n \geq 2$ is the same with $n = 1$. So, for simplification, we just give the prove the equivalence of (2) and (3) for $n = 1$.

Because

$$d\omega_1^1 = R\theta^1 \wedge \theta^{\overline{1}} + A_{11,\overline{1}}\theta^1 \wedge \theta - A_{\overline{1}\overline{1},1}\theta^{\overline{1}} \wedge \theta,$$

we have

$$\begin{array}{rcl} d\left(\omega_1^1 + iR\theta\right) & = & d\omega_1^1 + i\left(R_{,1}\theta^1 + R_{,\overline{1}}\theta^{\overline{1}}\right) \wedge \theta - R\theta^1 \wedge \theta^{\overline{1}} \\ & = & i\left[\left(R_{,1} - iA_{11,\overline{1}}\right)\theta^1 + \left(R_{,\overline{1}} + iA_{\overline{1}\overline{1},1}\right)\theta^{\overline{1}}\right] \wedge \theta \end{array}.$$

Hence

$$d\left(\omega_1^1 + iR\theta\right) = 0 \Longleftrightarrow R_{,1} - iA_{11,\overline{1}} = 0.$$

□

We recall some useful notations.

Definition 2.3 ([Lee88]). *(i) Let (M, J, θ) be a three-dimensional strictly pseudoconvex CR manifold. We define*

$$P\varphi = (P_1\varphi)\theta^1,$$

which is an operator that characterizes CR-pluriharmonic functions. Here $P_1\varphi = \varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1$ and $\overline{P}\varphi = (\overline{P}_1)\theta^{\bar{1}}$, the conjugate of P. The CR Paneitz operator P_0 is defined by

$$P_0\varphi = \left(\delta_b(P\varphi) + \bar{\delta}_b(\overline{P}\varphi)\right) \tag{2.2}$$

where δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_{1,}{}^1$ and, similarly, $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1},}{}^{\bar{1}}$. We observe that

$$\int_M \langle P\varphi + \overline{P}\varphi, d_b\varphi\rangle_{L^*_\theta}\, d\mu = -\int_M P_0\varphi \cdot \varphi\, d\mu \tag{2.3}$$

with $d\mu = \theta \wedge d\theta$. One can check that P_0 is self-adjoint, that is, $\langle P_0\varphi, \psi\rangle = \langle \varphi, P_0\psi\rangle$ for all smooth functions φ and ψ. For the details about these operators, the reader can make references to [GL88], [H93], [Lee88], [GG05] and [FH03].

(ii) On a complete pseudohermitian 3-manifold (M, J, θ), we call the Paneitz operator P_0 with respect to (J, θ) essentially positive if there exists a constant $\Lambda > 0$ such that

$$\int_M P_0\varphi \cdot \varphi d\mu \geq \Lambda \int_M \varphi^2 d\mu \tag{2.4}$$

for all real smooth functions $\varphi \in (\ker P_0)^\perp$ (i.e. perpendicular to the kernel of P_0 in the L^2 norm with respect to the volume form $d\mu = \theta \wedge d\theta$). We say that P_0 is nonnegative if

$$\int_M P_0\varphi \cdot \varphi d\mu \geq 0$$

for all real smooth functions φ.

Remark 2.1. 1. The notions of Paneitz operator P_0 and Q-curvature were initially introduced on a Riemannian manifold, and were considered as a kind of generalization of Laplacian and Gaussian curvature on a two-dimensional manifold, respectively ([H93]).

2. The kernel of the CR Paneitz operator P_0 is infinite dimensional, containing all CR-pluriharmonic functions.

3. Let (M, J, θ) be a closed strictly pseudoconvex 3-manifold with vanishing pseudohermitian torsion. Then the corresponding CR Paneitz operator P_0 is essentially positive ([CCC07]).

Finally, we define the CR Q-curvature in a pseudohermitian 3-manifold by

$$Q := -\mathrm{Re}(R_{,1} - iA_{11,\bar{1}})_{\bar{1}} = -\mathrm{Re}(R_{,1\bar{1}} - iA_{11,\bar{1}\bar{1}}). \tag{2.5}$$

Then

$$Q = -\frac{1}{2}[\Delta_b R - i(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})].$$

Now for $\theta = e^{2\gamma}\theta_0$, under this conformal change, it is known that we have the following transformation laws ([H93]):

$$Q = e^{-4\gamma}(Q_0 + \frac{3}{4}P_0\gamma) \tag{2.6}$$

and

$$W_1 := (R_{,1} - iA_{11,\bar{1}}) = e^{-3\gamma}[\overset{0}{R}_{,1} - i\overset{0}{A}_{11,\bar{1}} - 6\overset{0}{P}_1\gamma], \tag{2.7}$$

Finally, we recall that

Definition 2.4. *We call a CR structure J spherical if Cartan curvature tensor Q_{11} vanishes identically. Here*

$$Q_{11} = \frac{1}{6}R_{11} + \frac{i}{2}RA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1}.$$

Note that (M, J, θ) is called a spherical CR 3-manifold if J is a spherical structure. We observe that the spherical structure is CR invariant and a closed spherical CR 3-manifold (M, J, θ) is locally CR equivalent to the standard CR 3-sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$. In additional, if M is simply connected, then (M, J, θ) is the standard CR 3-sphere.

3. Proofs of main theorems

In this section, we prove the main theorems. We start from the groundwork for Theorem 1.1.

Lemma 3.1. *If (M, J, θ) is a closed strictly pseudoconvex 3-manifold with $c_1(T_{1,0}M) = 0$, then there is a pure imaginary 1-form*

$$\sigma = \sigma_{\overline{1}}\theta^{\overline{1}} - \sigma_1\theta^1 + i\sigma_0\theta$$

with $d\omega_1^1 = d\sigma$ such that

$$\begin{cases} R = R_{1\overline{1}} = \sigma_{\overline{1},1} + \sigma_{1,\overline{1}} - \sigma_0 \\ A_{11,\overline{1}} = \sigma_{1,0} + i\sigma_{0,1} - A_{11}\sigma_{\overline{1}} \end{cases} . \tag{3.1}$$

Proof. Because

$$c_1(T_{1,0}M) = -\frac{1}{2\pi i}\left[d\omega_1^1\right] = 0,$$

we know there is a pure imaginary 1-form

$$\sigma = \sigma_{\overline{1}}\theta^{\overline{1}} - \sigma_1\theta^1 + i\sigma_0\theta$$

such that

$$d\omega_1^1 = d\sigma.$$

By the structure equation

$$\begin{cases} d\theta = i\theta^1 \wedge \theta^{\overline{1}} \\ d\theta^1 = A_{\overline{1}\overline{1}}\theta \wedge \theta^1 \end{cases} ,$$

we have

$$\begin{aligned} d\sigma &= \left(\sigma_{\overline{1},1}\theta^1 + \sigma_{\overline{1},0}\theta\right) \wedge \theta^{\overline{1}} + \sigma_{\overline{1}}d\theta^{\overline{1}} - \left(\sigma_{1,\overline{1}}\theta^{\overline{1}} + \sigma_{1,0}\theta\right) \wedge \theta^1 - \\ &\quad \sigma_1 d\theta^1 + i\left(\sigma_{0,1}\theta^1 + \sigma_{0,\overline{1}}\theta^{\overline{1}}\right) \wedge \theta + i\sigma_0 d\theta \\ &= \left(\sigma_{\overline{1},1} + \sigma_{1,\overline{1}} - \sigma_0\right)\theta^1 \wedge \theta^{\overline{1}} - \left(\sigma_{1,0} + i\sigma_{0,1} - \sigma_{\overline{1}}A_{11}\right)\theta \wedge \theta^1 + \\ &\quad \left(\sigma_{\overline{1},0} - i\sigma_{0,\overline{1}} - \sigma_1 A_{\overline{1}\overline{1}}\right)\theta \wedge \theta^{\overline{1}}. \end{aligned} .$$

Due to

$$d\sigma = d\omega_1^1 = R_{1\overline{1}}\theta^1 \wedge \theta^{\overline{1}} + A_{11,\overline{1}}\theta^1 \wedge \theta - A_{\overline{1}\overline{1},1}\theta^{\overline{1}} \wedge \theta,$$

we derive

$$\begin{cases} R = R_{1\overline{1}} = \sigma_{\overline{1},1} + \sigma_{1,\overline{1}} - \sigma_0 \\ A_{11,\overline{1}} = \sigma_{1,0} + i\sigma_{0,1} - A_{11}\sigma_{\overline{1}} \end{cases} .$$

∎

We would need need the J.J. Kohn's Hodge theory for the $\overline{\partial}_b$ complex (see [K64]):

Lemma 3.2. *If (M, J, θ) is a closed strictly pseudoconvex CR $(2n+1)$-manifold and $\eta \in \Omega^{0,1}(M)$, a smooth $(0,1)$-form on M with*

$$\overline{\partial}_b \eta = 0,$$

then there are a smooth complex function $\varphi \in C^{\infty}_{\mathbb{C}}(M)$ and a smooth $(0,1)$-form $\gamma \in \Omega^{0,1}(M)$ such that

$$\left(\eta - \overline{\partial}_b \varphi\right) = \gamma \in \ker(\Box_b),$$

where $\Box_b = 2\left(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b\right)$ is the Kohn-Rossi Laplacian.

Subsequently, we deduce the expression for W_1. We denote $\gamma_1 := \overline{\gamma_{\overline{1}}}$.

Lemma 3.3. *If (M, J, θ) is a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$, then there are $u \in C^{\infty}_{\mathbb{R}}(M)$ and $\gamma = \gamma_{\overline{1}}\theta^{\overline{1}} \in \Omega^{0,1}(M)$ with*

$$\gamma_{\overline{1},1} = \gamma_{1,\overline{1}} = 0$$

such that

$$W_1 = 2P_1 u + i\left(A_{11}\gamma_{\overline{1}} - \gamma_{1,0}\right). \tag{3.2}$$

Proof. By choosing $\eta = \sigma_{\overline{1}}\theta^{\overline{1}}$ as in Lemma 3.2, where σ is chosen from Lemma 3.1, there are

$$\varphi = u + iv \in C^{\infty}_{\mathbb{C}}(M)$$

and

$$\gamma = \gamma_{\overline{1}}\theta^{\overline{1}} \in \Omega^{0,1}(M) \cap \ker(\Box_b)$$

such that

$$\sigma_{\overline{1}} = \varphi_{\overline{1}} + \gamma_{\overline{1}} \tag{3.3}$$

Note that

$$\Box_b \gamma = 0 \Longrightarrow \overline{\partial}_b^* \gamma = 0 \Longrightarrow \gamma_{\overline{1},1} = 0 \tag{3.4}$$

and

$$\sigma_1 = \left(\overline{\varphi}\right)_1 + \gamma_1. \tag{3.5}$$

Thus

$$\begin{array}{rcl}\sigma_{1,\bar{1}1} & = & (\overline{\varphi})_{,1\bar{1}1}+\gamma_{1,\bar{1}1} \text{ by } (3.5) \\ & = & (\overline{\varphi})_{,1\bar{1}1} \text{ by } (3.4) \\ & = & (\overline{\varphi})_{,\bar{1}11}+i(\overline{\varphi})_{,01} \text{ by } (A.5) \\ & = & (\overline{\varphi})_{,\bar{1}11}+i\left[(\overline{\varphi})_{,10}+A_{11}(\overline{\varphi})_{,\bar{1}}\right] \text{ by } (A.5)\end{array}$$

and

$$\sigma_{\bar{1},11}=\varphi_{,\bar{1}11} \text{ from } (3.3) \text{ and } (3.4)$$

imply

$$\begin{array}{rcl}W_1 & = & R_{,1}-iA_{11,\bar{1}} \\ & = & \sigma_{\bar{1},11}+\sigma_{1,\bar{1}1}-i\sigma_{1,0}+iA_{11}\sigma_{\bar{1}} \text{ by } (3.1) \\ & = & \varphi_{,\bar{1}11}+(\overline{\varphi})_{,\bar{1}11}+iA_{11}(\overline{\varphi})_{,\bar{1}}-i\gamma_{1,0}+iA_{11}\left(\varphi_{\bar{1}}+\gamma_{\bar{1}}\right) . \\ & = & 2\left(u_{,\bar{1}11}+iA_{11}u_{\bar{1}}\right)+i\left(A_{11}\gamma_{\bar{1}}-\gamma_{1,0}\right) \\ & = & 2P_1u+i\left(A_{11}\gamma_{\bar{1}}-\gamma_{1,0}\right)\end{array}$$

This completes the proof. □

Now we are ready to give the proof of Theorem 1.1:

Proof. (Proof of Theorem 1.1) Set

$$\widetilde{\theta}=e^{2\lambda}\theta.$$

By the transformation law (refer to Lemma 5.4 in [H93] or Lemma 3.1 in [CW18]), we know

$$\widetilde{W_1}=e^{-3\lambda}\left(W_1-6P_1\lambda\right), \tag{3.6}$$

where the notation with "tilde" means such quantity corresponds to the new contact form $\widetilde{\theta}$. With the help of (3.6) and Lemma 3.3, we have

$$\widetilde{W_1}=0$$

if and only if

$$W_1=6P_1\lambda$$

if and only if

$$6P_1\lambda=2P_1u+i\left(A_{11}\gamma_{\bar{1}}-\gamma_{1,0}\right).$$

That is to say

$$P_1 f = i\left(A_{11}\gamma_{\overline{1}} - \gamma_{1,0}\right)$$

for

$$f = (6\lambda - 2u)\,.$$

□

Remark 3.1. From (1.5) and

$$\gamma_{1,0\overline{1}} = \gamma_{1,\overline{1}0} + A_{\overline{1}\overline{1}}\gamma_{1,1} + A_{\overline{1}\overline{1},1}\gamma_1 = A_{\overline{1}\overline{1}}\gamma_{1,1} + A_{\overline{1}\overline{1},1}\gamma_1,$$

we could deduce f satisfies the fourth-order partial differential equation

$$P_0 f = 2i\left[\left(A_{11}\gamma_{\overline{1}}\right)_{,\overline{1}} - \left(A_{\overline{1}\overline{1}}\gamma_1\right)_{,1}\right] \tag{3.7}$$

where P_0 is the CR Paneitz operator (see section 2). This suggests us there is an obstruction to the existence of pseudo-Einstein contact form pertaining to the CR Paneitz operator. See Theorem 1.2 below for more details.

As for the proof of the case of vanishing pseudohermitian torsion:

Proof. (Proof of Corollary 1.1)
Setting $A_{11} = 0$ in (1.6), by Theorem 1.1, it suffices to show that

$$\gamma_{1,0} = 0$$

in order to have a globally defined pseudo-Einstein contact form $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$.
Note that, from (3.2) and $A_{11} = 0$,

$$R_{,1} = 2u_{\overline{1}11} - i\gamma_{1,0}. \tag{3.8}$$

Utilizing integration by parts, it follows from (3.8) and $\gamma_{\overline{1},1} = 0$ that

$$\begin{array}{rcl} 0 & \leq & \int_M \left|\gamma_{1,0}\right|^2 d\mu \\ & = & -\int_M \gamma_1\gamma_{\overline{1},00} d\mu \\ & = & -\int_M \gamma_1\left(-iR_{,\overline{1}} + 2iu_{,1\overline{1}\overline{1}}\right)_{,0} d\mu \\ & = & i\int_M \gamma_1\left(R_{,0} - 2u_{,1\overline{1}0}\right)_{,\overline{1}} d\mu \\ & = & -i\int_M \gamma_{1,\overline{1}}\left(R_{,0} - 2u_{,1\overline{1}0}\right) d\mu \\ & = & 0. \end{array}.$$

The third equality comes from ($A.5$) and $A_{11}=0$. Then

$$\gamma_{1,0}=0.$$

□

Before giving the proof of Theorem 1.2, we need the following Bochner-type equality.

Lemma 3.4. *Let (M,J,θ) be a closed strictly pseudoconvex CR 3-manifold and $\widetilde{\theta}=e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form. Then we have*

$$\int_M (2R-Tor)(\gamma,\gamma)\,d\mu+2\int_M |\gamma_{1,1}|^2\,d\mu+\frac{1}{2}\int_M (P_0 f)\,f d\mu=0. \tag{3.9}$$

Proof. From Theorem 1.1 and the commutation formula, it follows that

$$\widetilde{\theta}=e^{\frac{(f+2u)}{3}}\theta$$

is a pseudo-Einstein contact form if and only if

$$P_1 f=iA_{11}\gamma_{\overline{1}}+R\gamma_1-\gamma_{1,1\overline{1}}. \tag{3.10}$$

By the fact that $\gamma_{1,\overline{1}}=0$, it's easy to see

$$\int_M (P_1 f)\,\gamma_{\overline{1}}d\mu=\int_M (f_{\overline{1}11}+iA_{11}f_{\overline{1}})\,\gamma_{\overline{1}}d\mu=i\int_M A_{11}f_{\overline{1}}\gamma_{\overline{1}}d\mu.$$

Then, substituting (3.10) into the last equality and adding its conjugation, we have

$$-\int_M Tor\,(d_b f,\gamma)\,d\mu=\int_M (2R-Tor)\,(\gamma,\gamma)\,d\mu+2\int_M |\gamma_{1,1}|^2\,d\mu. \tag{3.11}$$

On the other hand, the equality (3.10) and the commutation formulas enable us to get

$$\begin{aligned}
\textstyle\int_M (P_1 f)\,f_{\overline{1}}d\mu &= \textstyle\int_M (iA_{11}\gamma_{\overline{1}}+R\gamma_1)\,f_{\overline{1}}d\mu-\int_M \gamma_1 f_{\overline{1}\overline{1}1}d\mu\\
&= \textstyle\int_M (iA_{11}\gamma_{\overline{1}}+R\gamma_1)\,f_{\overline{1}}d\mu\\
&\quad \textstyle+\int_M \gamma_1\,(-f_{\overline{1}1\overline{1}}+if_{\overline{1}0}-R\gamma_1 f_{\overline{1}})\,d\mu\\
&= \textstyle\int_M iA_{11}\gamma_{\overline{1}}f_{\overline{1}}d\mu+\int_M \gamma_1\,(-f_{\overline{1}1\overline{1}}+if_{\overline{1}0})\,d\mu\\
&= \textstyle\int_M iA_{11}\gamma_{\overline{1}}f_{\overline{1}}d\mu+\int_M \gamma_1\,(-f_{\overline{1}1\overline{1}}+if_{0\overline{1}}-iA_{\overline{1}\overline{1}}f_1)\,d\mu\\
&= \textstyle i\int_M (A_{11}\gamma_{\overline{1}}f_{\overline{1}}-A_{\overline{1}\overline{1}}\gamma_1 f_1)\,d\mu\\
&= \textstyle-\int_M Tor\,(d_b f,\gamma)\,d\mu.
\end{aligned}$$

By the definition of the CR Paneitz operator, we obtain

$$\int_M (P_0 f) f d\mu = -\int_M ((P_1 f) f_{\bar{1}} + (P_{\bar{1}} f) f_1) d\mu = 2\int_M Tor(d_b f, \gamma) d\mu \tag{3.12}$$

Therefore, it follows from the equalities (3.11) and (3.12) that

$$\int_M (2R - Tor)(\gamma,\gamma) d\mu + 2\int_M |\gamma_{1,1}|^2 d\mu + \frac{1}{2}\int_M (P_0 f) f d\mu = 0.$$

Then we are done. □

Such equality enables us to prove Theorem 1.2 as follows:

Proof. (Proof of Theorem 1.2) From the equality (3.9) and the hypotheses, it is clear that if $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form, then

$$\gamma = 0.$$

Hence we can solve the inhomogeneous tangential Cauchy-Riemann equation

$$\overline{\partial}_b \varphi = \sigma_{\bar{1}} \theta^{\bar{1}}$$

by Lemma 3.2. Note that this implicitly implies f is CR-pluriharmonic. So the sufficient part is completed.

As for the necessary part, it's obvious from Theorem 1.1. □

Before to go further, we need the following key lemma.

Lemma 3.5. *Let (M, J, θ) be a closed strictly pseudoconvex CR 3-manifold with $c_1(T_{1,0}M) = 0$. Then, with the notations as above, the following equality holds*

$$\begin{array}{l} \int_M \left(R - \frac{1}{2}Tor - \frac{1}{2}Tor'\right)(\gamma,\gamma) d\mu + \int_M |\gamma_{1,1}|^2 d\mu + \\ \int_M Q u d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu = 0. \end{array} \tag{3.13}$$

Here $Tor'(\gamma,\gamma) := -i\left(A_{11,\bar{1}}\gamma_{\bar{1}} - A_{\bar{1}\bar{1},1}\gamma_1\right)$.

Proof. From the equality (3.2), we are able to get

$$\begin{aligned} \left(R_{,1} - iA_{11,\bar{1}}\right)\gamma_{\bar{1}} &= W_1 \gamma_{\bar{1}} \\ &= 2\left(u_{\bar{1}11} + iA_{11}u_{\bar{1}}\right)\gamma_{\bar{1}} + iA_{11}\gamma_{\bar{1}}\gamma_{\bar{1}} - i\gamma_{1,0}\gamma_{\bar{1}} \\ &= 2\left(u_{\bar{1}11} + iA_{11}u_{\bar{1}}\right)\gamma_{\bar{1}} + iA_{11}\gamma_{\bar{1}}\gamma_{\bar{1}} - \left(\gamma_{1,1\bar{1}} - R\gamma_1\right)\gamma_{\bar{1}}. \end{aligned}$$

Taking the integration over M of both sides and its conjugation, we have, by the fact that $\gamma_{1,\overline{1}}=0$,

$$\begin{array}{l} i\int_M \left(A_{11,\overline{1}}\gamma_{\overline{1}}-A_{\overline{11},1}\gamma_1\right)d\mu+\int_M\left(2R-Tor\right)(\gamma,\gamma)\,d\mu+ \\ 2\int_M\left|\gamma_{1,1}\right|^2d\mu-2\int_M Tor\left(d_bu,\gamma\right)d\mu=0. \end{array}$$

That is

$$\int_M\left(R-\frac{1}{2}Tor-\frac{1}{2}Tor'\right)(\gamma,\gamma)\,d\mu+\int_M\left|\gamma_{1,1}\right|^2d\mu-\int_M Tor\left(d_bu,\gamma\right)d\mu=0. \tag{3.14}$$

On the other hand, it follows from the equality (3.2) that

$$\left(R,_1-iA_{11},_{\overline{1}}\right)u_{\overline{1}}=W_1u_{\overline{1}}=\left[2P_1u+i\left(A_{11}\gamma_{\overline{1}}-\gamma_{1,0}\right)\right]u_{\overline{1}}. \tag{3.15}$$

By the fact that $\gamma_{1,\overline{1}}=0$, we see that

$$\begin{array}{rcl} \int_M\gamma_{1,0}u_{\overline{1}}d\mu &=& \int_M\gamma_1u_{\overline{1}0}d\mu \\ &=& -\int_M\gamma_1\left(u_{0\overline{1}}-A_{\overline{11}}u_1\right)d\mu \\ &=& \int_M A_{\overline{11}}u_1\gamma_1d\mu. \end{array} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{array}{l} \quad 2\int_M Qud\mu+2\int_M\left(P_0u\right)ud\mu \\ =i\int_M\left[\left(A_{11}u_{\overline{1}}\gamma_{\overline{1}}-A_{\overline{11}}u_1\gamma_1\right)-conj\right]d\mu \\ =-2\int_M Tor\left(d_bu,\gamma\right)d\mu. \end{array}$$

Thus by (3.14)

$$\begin{array}{l} \int_M\left(R-\frac{1}{2}Tor-\frac{1}{2}Tor'\right)(\gamma,\gamma)\,d\mu+\int_M\left|\gamma_{1,1}\right|^2d\mu+ \\ \int_M Qud\mu+\int_M\left(P_0u^{\perp}\right)u^{\perp}d\mu=0. \end{array}$$

□

Proof. (proof of Theorem 1.3 and Corollary 1.3) If we assume that

$$\ker P_1=\ker P_0.$$

Then we also have

$$\begin{array}{rcl} 0 &=& \int_M\left(R-\frac{1}{2}Tor-\frac{1}{2}Tor'\right)(\gamma,\gamma)\,d\mu+\int_M\left|\gamma_{1,1}\right|^2d\mu+ \\ && \int_M Qu^{\perp}d\mu+\int_M\left(P_0u^{\perp}\right)u^{\perp}d\mu. \end{array} \tag{3.17}$$

Here we have used the fact that P_0 is self-adjoint and

$$\int_M Tor\,(d_b u_{\ker}, \gamma)\, d\mu = 0.$$

Now if $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$ is a pseudo-Einstein contact form for any CR-pluriharmonic function f, it follows from (3.9) that

$$\gamma = 0$$

and then

$$0 = \int_M Qu^{\perp} d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu = \int_M Q^{\perp} u^{\perp} d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu.$$

By the Hölder's inequality and essentially positivity of the CR Paneitz operator, we have

$$\begin{array}{ll} & \int_M Q^{\perp} u^{\perp} d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu \\ \geq & \Lambda \int_M (u^{\perp})^2 d\mu - (\int_M (Q^{\perp})^2 d\mu)^{\frac{1}{2}})(\int_M (u^{\perp})^2 d\mu)^{\frac{1}{2}} \\ \geq & [\Lambda(\int_M (u^{\perp})^2 d\mu)^{\frac{1}{2}} - (\int_M (Q^{\perp})^2 d\mu)^{\frac{1}{2}}](\int_M (u^{\perp})^2 d\mu)^{\frac{1}{2}} \end{array}$$

and then

$$0 \geq \Lambda(\int_M (u^{\perp})^2 d\mu)^{\frac{1}{2}} - (\int_M (Q^{\perp})^2 d\mu)^{\frac{1}{2}}.$$

Hence

$$\int_M (Q^{\perp})^2 d\mu \geq \Lambda^2 \int_M (u^{\perp})^2 d\mu.$$

Furthermore, if the CR Q-curvature is pluriharmonic (i.e. $Q^{\perp} = 0$), then

$$u^{\perp} = 0$$

and by (3.2)

$$W_1 = 0.$$

Hence θ is also a globally defined pseudo-Einstein contact form. Moreover, if the pseudohermitian torsion is vanishing, then (M, J, θ) is the Sasakian space form. □

Proof. (proof of Theorem 1.4 and Corollary 1.4) As before

$$\begin{array}{rcl} \int_M Qu^{\perp} d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu & = & \int_M Q^{\perp} u^{\perp} d\mu + \int_M \left(P_0 u^{\perp}\right) u^{\perp} d\mu \\ & \geq & \Lambda \int_M (u^{\perp})^2 d\mu \\ & \geq & 0 \end{array}$$

if

$$Q^{\perp} = 0. \tag{3.18}$$

It follows from (3.17) that

$$0 \geq \int_M \left(R - \frac{1}{2}Tor - \frac{1}{2}Tor' \right) (\gamma, \gamma)\, d\mu + \int_M |\gamma_{1,1}|^2\, d\mu, \tag{3.19}$$

if (3.18) holds. Hence

$$\gamma = 0$$

if the pseudohermitian curvature is $\frac{1}{2}$-*positive* and $A_{11,\bar{1}} = 0$. It follows from Theorem 1.2 that M admits a globally defined pseudo-Einstein contact form $\widetilde{\theta} = e^{\frac{(f+2u)}{3}}\theta$.

Furthermore, if the CR Q-curvature is pluriharmonic (i.e. $Q^{\perp} = 0$), then

$$u^{\perp} = 0$$

and by (3.2),

$$W_1 = 0.$$

Hence θ is also a globally defined pseudo-Einstein contact form and R is a positive constant.

Now if (M, J, θ) is spherical and pseudo-Einstein, we have

$$W_1 = R_{,1} - iA_{11,\bar{1}} = 0$$

and

$$iR_{,11} = 3RA_{11} + 6iA_{11,0} - 4A_{11,\bar{1}1}.$$

By cancelling $R_{,11}$, one derives

$$3RA_{11} + 6iA_{11,0} - 3A_{11,\bar{1}1} = 0.$$

On the other hand, it follows from the commutation relation ([Lee88]) that

$$A_{11,1\bar{1}} - A_{11,\bar{1}1} = iA_{11,0} + 2RA_{11},$$

we obtain

$$-3RA_{11} + 2A_{11,1\bar{1}} - 3A_{11,\bar{1}1} = 0$$

and then

$$-2\int_M |A_{11,1}|^2\, d\mu + 3\int_M \left|A_{11,\bar{1}}\right|^2 d\mu = 3\int_M R\,|A_{11}|^2\, d\mu.$$

Hence

$$-2\int_M |A_{11,1}|^2\, d\mu = 3\int_M R\,|A_{11}|^2\, d\mu.$$

Moreover, since R is a positive constant, then

$$A_{11} = 0.$$

It follows that (M, J, θ) is the Sasakian space form with positive constant Tanaka-Webster scalar curvature and vanishing pseudohermitian torsion. □

Appendix A. Appendix section

In this appendix, we introduce some basic notions from pseudohermitian geometry as in [Lee88].

Definition A.1. *Let M be a smooth manifold and $\xi \subset TM$ a subbundle. A **CR structure** on ξ consists of an endomorphism $J : \xi \to \xi$ with $J^2 = -id$ such that the following integrability condition holds.*

1. *If $X, Y \in \xi$, then so is $[JX, Y] + [X, JY]$.*
2. *$J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.*

The CR structure J can be extended to $\xi \otimes \mathbb{C}$, which we can then decompose into the direct sum of eigenspaces of J. The eigenvalues of J are i and $-i$, and the corresponding eigenspaces will be denoted by $T^{1,0}$ and $T^{0,1}$, respectively. The integrability condition can then be reformulated as

$$X, Y \in T^{1,0} \text{ implies } [X, Y] \in T^{1,0}.$$

Now consider a closed $2n+1$-manifold M with a cooriented contact structure $\xi = \ker\theta$. This means that $\theta \wedge d\theta^n \neq 0$. The **Reeb vector field** of θ is the vector field T uniquely determined by the equations

$$\theta(T) = 1, \quad \text{and} \quad d\theta(T, \cdot) = 0. \tag{A.1}$$

A **pseudohermitian manifold** is a triple (M^{2n+1}, θ, J), where θ is a contact form on M and J is a CR structure on $\ker\theta$. The **Levi form** $\langle\ ,\ \rangle$ is the Hermitian form on $T^{1,0}$ defined by

$$H(Z, W) = \langle Z, W\rangle = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle.$$

We can extend this Hermitian form $\langle\ ,\ \rangle$ to $T^{0,1}$ by defining $\left\langle \overline{Z}, \overline{W} \right\rangle = \overline{\langle Z, W\rangle}$ for all $Z, W \in T^{1,0}$. Furthermore, the Levi form naturally induces a Hermitian form on the dual bundle of $T^{1,0}$, and hence on all induced tensor bundles.

We now restrict ourselves to **strictly pseudoconvex** CR manifolds, or in other words, compatible complex structures J. This means that the Levi form induces a Hermitian metric $\langle \cdot, \cdot \rangle_{J,\theta}$ by

$$\langle V, U\rangle_{J,\theta} = d\theta(V, JU).$$

The associated norm is defined as usual: $|V|^2_{J,\theta} = \langle V, V\rangle_{J,\theta}$. It follows that H also gives rise to a Hermitian metric for $T^{1,0}$, and hence we obtain Hermitian metrics on all induced tensor bundles. By integrating this Hermitian metric over M with respect to the volume form $d\mu = \theta \wedge d\theta^n$, we get an L^2-inner product on the space of sections of each tensor bundle.

The **pseudohermitian connection** or **Tanaka-Webster connection** ([Ta75], [We78]) of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $\{Z_\alpha\}$ for $T^{1,0}$ by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = {\omega_{\bar\alpha}}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where ${\omega_\alpha}^\beta$ is the 1-form uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta \\ \tau_\alpha \wedge \theta^\alpha &= 0 \\ {\omega_\alpha}^\beta + {\omega_{\bar\beta}}^{\bar\alpha} &= 0. \end{aligned} \tag{A.2}$$

Here τ^α is called the **pseudohermitian torsion**, which we can also write as

$$\tau_\alpha = A_{\alpha\beta}\theta^\beta.$$

The components $A_{\alpha\beta}$ satisfy

$$A_{\alpha\beta} = A_{\beta\alpha}.$$

We often consider the **torsion tensor** given by

$$A_{J,\theta} = {A^\alpha}_{\bar\beta} Z_\alpha \otimes \theta^{\bar\beta} + {A^{\bar\alpha}}_\beta Z_{\bar\alpha} \otimes \theta^\beta.$$

We now consider the curvature of the Tanaka-Webster connection in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\beta}\}$. The second structure equation gives

$$\Omega_\beta{}^\alpha = \overline{\Omega_{\bar\beta}{}^{\bar\alpha}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha,$$
$$\Omega_0{}^\alpha = \Omega_\alpha{}^0 = \Omega_0{}^{\bar\beta} = \Omega_{\bar\beta}{}^0 = \Omega_0{}^0 = 0.$$

In [We78, Formulas 1.33 and 1.35], Webster showed that the curvature $\Omega_\beta{}^\alpha$ can be written as

(A.3)
$$\Omega_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

In addition, by [Lee88, (2.4)] the coefficients $W_\alpha{}^\beta{}_\rho$ are determined by the torsion,

$$W_\alpha{}^\beta{}_\rho = A_{\alpha\rho,}{}^\beta.$$

Contraction of (A.3) yields

$$\text{(A.4)} \qquad \begin{aligned} \Omega_\alpha{}^\alpha = d\omega_\alpha{}^\alpha &= R_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\alpha{}^\alpha{}_\rho\theta^\rho \wedge \theta - W_{\bar\alpha}{}^{\overline{\alpha}}{}_{\bar\rho}\theta^{\bar\rho} \wedge \theta \\ &= R_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + A_{\alpha\rho}{}^\alpha\theta^\rho \wedge \theta - A_{\bar\alpha\bar\rho}{}^{\bar\alpha}\theta^{\bar\rho} \wedge \theta \end{aligned}$$

We will denote components of covariant derivatives by indices preceded by a comma. For instance, we write $A_{\alpha\beta,\gamma}$. Here the indices $\{0, \alpha, \bar\beta\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\beta}\}$. For derivatives of a scalar function, we will often omit the comma. For example, $\varphi_\alpha = Z_\alpha\varphi$, $\varphi_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha\varphi - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma\varphi, \varphi_0 = T\varphi$ for a (smooth) function φ.

In particular, we define the followings for $n = 1$. For a real function φ, the subgradient ∇_b is defined by $\nabla_b\varphi \in \xi$ and $\langle Z, \nabla_b\varphi\rangle_{L_\theta} = d\varphi(Z)$ for all vector fields Z tangent to the contact plane. Locally $\nabla_b\varphi = \varphi_{\bar 1}Z_1 + \varphi_1 Z_{\bar 1}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2\varphi : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1},$$

by

$$(\nabla^H)^2\varphi(Z) = \nabla_Z\nabla_b\varphi.$$

Also

$$\Delta_b\varphi = Tr\left((\nabla^H)^2\varphi\right) = (\varphi_{1\bar 1} + \varphi_{\bar 1 1}).$$

For all $Z = x^1 Z_1 \in T_{1,0}$, we define

$$\begin{aligned} Ric(Z,Z) &= W x^1 x^{\bar{1}} = W|Z|^2_{L_\theta}, \\ Tor(Z,Z) &= 2Re\ iA_{\bar{1}\bar{1}} x^{\bar{1}} x^{\bar{1}}. \end{aligned}$$

We also need the following commutation relations ([Lee88]).

$$\begin{aligned} C_{I,01} - C_{I,10} &= C_{I,\bar{1}} A_{11} - k C_{I,} A_{11,\bar{1}}, \\ C_{I,0\bar{1}} - C_{I,\bar{1}0} &= C_{I,1} A_{\overline{11}} - k C_{I,} A_{\overline{11},1}, \\ C_{I,1\bar{1}} - C_{I,\bar{1}1} &= i C_{I,0} + k W C_I. \end{aligned} \tag{A.5}$$

Here C_I denotes a coefficient of a tensor with multi-index I consisting of only 1 and $\bar{1}$, and k is the number of 1's minus the number of $\bar{1}$'s in I.

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Der-Chen Chang
Department of Mathematics and Statistics
Georgetown University
Washington D. C. 20057
USA and
Graduate Institute of Business Administration
College of Management
Fu Jen Catholic University
New Taipei City, 24205
Taiwan, R.O.C.
E-mail: chang@georgetown.edu

Shu-Cheng Chang
Department of Mathematics and
Taida Institute for Mathematical Sciences (TIMS)
National Taiwan University
Taipei, 10617
Taiwan, R.O.C.
E-mail: scchang@math.ntu.edu.tw

Ting-Jung Kuo
Department of Mathematics
National Taiwan Normal University
Taipei, 11677
Taiwan
E-mail: tjkuo1215@ntnu.edu.tw

Chien Lin
Mathematical Science Research Center
Chongqing University of Technology
Banan District 400054
Chongqing P.R. China
E-mail: chienlin@cqut.edu.cn