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Optimization results for sphere maps

John P. D'Angelo

Abstract: We prove several optimization results for monomial and polynomial sphere maps. We formulate two open problems about the uniquely difficult situation in two dimensions.

Keywords: Optimization, polynomial sphere maps, linear programming.

1. Introduction

This paper considers an optimization problem motivated and introduced in [5]. Consider rational mappings f that send the unit sphere in $\mathbb{C}^n$ to the unit sphere in $\mathbb{C}^N$ for some N. Our first result is in the monomial case. Given the degree d, we assume that f includes the monomials $z_1^d, \ldots, z_n^d$. We then seek the minimum and maximum values of $\|f(\mathbf{1})\|^2$, where $\mathbf{1} = (1, 1, ..., 1)$, as a function of d and n. Basic results about this problem were obtained in [5] and additional results appear in [3]. A detailed account of this problem as part of the general theory of rational sphere maps will appear in the author's book [4]. We review the motivation for these questions in Section 2.

Finding the maximum value for monomial maps is elementary. In all source dimensions n and degrees d, the maximum value is n^d. This value is achieved by the d-fold tensor product map $H_d(z) = z^{\otimes d}$. It is well-known that all homogeneous polynomial sphere maps of degree d with linearly independent components are unitarily equivalent to H_d. See for example [2]. In Theorem 5.1 we prove that this map also gives the maximum value of $\|p(\mathbf{1})\|^2$ in the general polynomial case, where the proof is slightly more subtle.

In Section 6 we note some facts about the rational case. For rational maps, dimension 1 differs in the following obvious way; the point $\mathbf{1}$ lies outside the closed unit ball in dimensions at least 2, but lies on the circle in dimension 1. Hence, in one dimension, $\|f(\mathbf{1})\|^2 = 1$ for every *rational* sphere map, and the optimization problem is not interesting. In dimensions at least 2, for each

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non-zero degree d, there is a *rational* sphere map of degree d with a singularity at $\mathbf{1}$. Hence the *maximum* value of $\|f(\mathbf{1})\|^2$ does not exist. Furthermore the noncompactness of the automorphism group of the unit ball also has consequences for the minimum.

In the monomial case, finding the minimum in source dimension 2 is extremely difficult. The problem has been coded up to degree 201 and various results are known, but it is perhaps impossible to write down a general formula for the minimizer. Section 4 contains some complicated formulas and two open problems.

In this paper we prove Theorem 1.1, a result that will also appear in the author's book [4]. In source dimension 3 and higher (the result also holds in source dimension 1, where it is trivial), there is a unique explicit monomial mapping realizing this minimum, and the value of the minimum is the simple formula $n + n(n-1)(d-1)$.

Corollaries 3.1 and 3.2 of Theorem 1.1 give asymptotic results about the *minimum* as the degree tends to infinity. Theorem 5.1 and its Corollary 5.1 study the *maximum* in the more general polynomial case.

It is natural to ask why the case of dimension 2 is both so different and so difficult. Roughly speaking the reason is the following. For $n \geq 2$, consider the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$ as a CR manifold. Intersecting the sphere with a hyperplane defined by setting a variable equal to 0 gives the unit sphere in one less dimension. If $n = 2$, then we get the unit circle in $\mathbb{C}$, which is not a CR manifold. In source dimensions 3 or more, however, the intersection is a CR manifold. The extra structure then matters in solving a crucial system of equations.

In order to state Theorem 1.1, we introduce the following set. Let $\mathbf{S}(n,d)$ denote the collection of polynomials p in n real variables $\mathbf{x} = (x_1, ..., x_n)$ such that

- $p : \mathbb{R}^n \to \mathbb{R}$ is of degree d.
- p has non-negative coefficients.
- $p(\mathbf{x}) = 1$ on the hyperplane given by $s(\mathbf{x}) = \sum_{j=1}^{n} x_j = 1$.
- For each j, p contains the monomial x_j^d with coefficient 1.

This set of polynomials is closed in the topology determined by taking limits of coefficients, and because the coefficients are non-negative, it is also bounded. Evaluation at a point is continuous, and hence the minimum $m(n,d)$ and maximum $M(n,d)$ values of $p(\mathbf{1})$ for $p \in \mathbf{S}(n,d)$ are achieved.

Let f be a polynomial mapping such that $f(S^{2n-1}) \subseteq S^{2N-1}$. We call f a *polynomial sphere map*. When the components of f are single monomials we call f a *monomial sphere map*. Let f be a monomial sphere map and put

$p(\mathbf{x}) = \|f(z)\|^2$. Then p depends only upon the variables $\mathbf{x} = (|z_1|^2, ..., |z_n|^2)$, and furthermore,

$$p(\mathbf{x}) = 1 \quad \text{on} \quad \sum x_j = 1.$$

The numbers $m(n, d)$ and $M(n, d)$ therefore give the minimum and maximum of $\|f(\mathbf{1})\|^2$ for monomial sphere maps sending S^{2n-1} into S^{2N-1}. It is natural to ask how these values depend on the source dimension and the degree.

We remark that $\|f(z)\|^2 \leq 1$ for z in the closed unit ball; when $n \geq 2$ we are evaluating f at a point *outside* the sphere. The motivational section clarifies why we do so.

Theorem 1.1. *Suppose $p \in \mathbf{S}(n, d)$. For all source dimensions n and degrees d,*

1. *$M(n, d) = n^d$.*
2. *$m(n, d) \leq n + n(n-1)(d-1)$.*
3. *For $n \neq 2$, equality holds in (2).*
4. *The unique p for which equality holds in dimensions not 2 is given by*

$$p(\mathbf{x}) = \sum_{j=1}^{n} x_j^d + \sum_{j=1}^{n} (\sum_{j \neq k} x_j)(\sum_{l=1}^{d-1} x_k^l).$$

In Section 4, for various degree d, we list the polynomial for which $m(2, d)$ is achieved. We also recall results from [3] in the two dimensional case. The level of complexity is in stark contrast to the situation in dimensions not equal to 2. We pose two open problems in this section.

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2. Motivation

Let V, W be vector spaces and assume $T : V \to W$ is a linear map. If w is in the range of T, then each solution u to $Tu = w$ can be written $u = \mathbf{u}_0 + \mathbf{v}$, where $\mathbf{u_0}$ is a particular solution and $\mathbf{v}$ is an arbitrary element of the null-space of T. The linear system is called *underdetermined* when the null-space is non-trivial. Additional information is required to obtain a unique solution. One common approach is to work in L^2 and find the solution that minimizes the L^2 norm. When T is the Cauchy-Riemann operator $\overline{\partial}$ on a smoothly bounded pseudoconvex domain, the resulting solution is called the Kohn or canonical solution. It was believed for a long time that this solution would have optimal regularity properties, but this statement does not hold in general. See [1].

It is tempting to wonder whether there might be some related minimization problem that does yield optimal regularity properties. In this paper, however, we are concerned only with finite dimensional linear systems.

Consider a linear system $T(u) = w$ where $T : \mathbb{R}^n \to \mathbb{R}^N$. When solving the system, one can seek to minimize norms such as the L^1 or L^2 norms. Assume that the standard bases are used. In **compressed sensing**, one seeks a solution u for which the fewest number of components are non-zero. Such a solution is called *sparse*, and sparse solutions are particularly important in signal processing.

The L^1-norm of a solution $u = (u_1, ..., u_n)$ is given by $\|u\|_1 = \sum_{j=1}^n |u_j|$. A famous theorem of Donoho (See [7]) states for most large linear systems that a generic solution with minimum L^1 norm is also a sparse solution. Least squares solutions (those with minimal L^2 norm) turn out to be less useful in applications.

We next make the connection to monomial sphere maps. References [3], [4], [5], [6], [8], [9], [10] for example have made extensive use of studying monomial sphere maps by way of the linear system we next describe. Given a fixed source dimension and degree, a monomial sphere map can be regarded as the solution of a certain linear system. Both the L^1 norm and the number of terms in a sparse solution have useful interpretations in CR geometry. The starting point is homogenization. First note that there is a unique homogeneous polynomial of degree d that is 1 on the hyperplane given by $s(\mathbf{x}) = \sum x_j = 1$, namely s^d.

Next let $f : \mathbb{C}^n \to \mathbb{C}^N$ be a monomial map for which $f(S^{2n-1}) \subseteq S^{2N-1}$. If the components of f, in multi-index notation, are $C_\alpha z^\alpha$, then we have

$$\sum_\alpha |C_\alpha|^2 \mathbf{x}^\alpha = 1 \quad on \quad s(\mathbf{x}) = 1. \tag{1}$$

Homogenizing this equation leads to an equation that holds everywhere:

$$\sum_{|\alpha|=0}^{d} |C_\alpha|^2 \mathbf{x}^\alpha s(\mathbf{x})^{d-|\alpha|} = s(\mathbf{x})^d \tag{2}$$

Fix a degree d. Equate coefficients in (2) to obtain an underdetermined linear system for the unknown constants $|C_\alpha|^2$; this system determines all monomial sphere maps of degree at most d. Assume that $C_\alpha \neq 0$ for some α with $|\alpha| = d$, in order to guarantee that the solution is of degree precisely equal to d. Finding the minimum possible target dimension for the corresponding monomial map is the same as finding a solution for which the number of non-vanishing components is minimal. Thus the solution is sparse.

The number of unknowns in the linear system is $\binom{n+d}{d}$, the number of independent polynomials of degree at most d in n variables. The number of equations is $\binom{n+d-1}{d}$, the number of independent *homogeneous* polynomials of degree d in n variables. For example, when $n = 2$, there are $1+2+...+d+1 = \frac{(d+1)(d+2)}{2}$ unknowns and $d + 1$ equations.

For us, the L^1 norm is the value of a function at a certain point, and the number of terms in a sparse solution is the minimum target dimension for the monomial sphere map. Corollary 3.2 of Theorem 1.1 yields an asymptotic relationship between these problems as the degree tends to infinity. See [5] and [3] for details, based upon the degree bounds from [6] and [10].

For a polynomial p to lie in $\mathbf{S}(n, d)$, it must correspond to the squared norm of a monomial sphere map of degree d where an additional assumption holds. The assumption that p contains the specific monomials x_j^d with coefficient 1 has two significant effects. First, it forces $\mathbf{S}(n, d)$ to be compact, and hence the minimum and maximum values are attained. Second, it decreases the number of unknowns.

We interpret these statements in two real dimensions. Imagine a polynomial $p(x, y)$ of degree d with non-negative coefficients that equals 1 on the line segment $(t, 1 - t)$ for $0 \leq t \leq 1$. What is the smallest possible value of $p(1, 1)$? The infimum value is 1, but this value is not achieved. For $0 < \epsilon \leq 1$, the polynomial $(1 - \epsilon) + \epsilon(x + y)^d$ satisfies the first three conditions in the definition of $\mathbf{S}(2, d)$ but not the fourth. As ϵ tends to 0, the value at $(1, 1)$ tends to 1. The difficulty is that the set of polynomials satisfying the first three conditions is not closed. Including the fourth condition makes the set closed and allows us to ask a seemingly elementary but in fact extremely difficult question. Assume that $p(x, y)$ is a polynomial of degree d with non-negative coefficients, and that $p(x, 1 - x) = 1$ for all x. Suppose in addition that $p(x, 0) = x^d$ and $p(0, y) = y^d$. What is the minimum value of $p(1, 1)$? See Section 4 and especially [3] for formulas of staggering complexity.

3. Proof of Theorem 1.1

We need the following preliminaries. We write $\cong$ to denote equality on the hyperplane given by $s(\mathbf{x}) = 1$. Given a polynomial $p(x)$, we may symmetrize it by considering

$$\mathbf{Sym}(p) = \frac{1}{n!} \sum_{\gamma} (p \circ \gamma),$$

its average over the permutations γ of the coordinates. Since $\mathbf{1}$ is symmetric, it follows that $\mathbf{Sym}(p)(\mathbf{1}) = p(\mathbf{1})$. We note the following explicit example when $n = 3$, and the variables are (x, y, z):

$$6\ \mathbf{Sym}(x^k y) = x^k(y+z) + y^k(x+z) + z^k(x+y). \tag{3}$$

If $p \in \mathbf{S}(n.d)$ is a polynomial for which $p(\mathbf{1})$ is minimal, then $\mathbf{Sym}(p)$ also minimizes the value at 1. Hence we may assume from the start that a minimizer is symmetric.

First it is easy to show that $M(n,d) = n^d$ for all n, d. Note $s^d \in \mathbf{S}(n,d)$ and that $s(\mathbf{1})^d = n^d$. Assume $p \in \mathbf{S}(n,d)$. Put $p(\mathbf{x}) = \sum |c_\alpha|^2 \mathbf{x}^\alpha$. Then $p(\mathbf{x}) = 1$ when $s(\mathbf{x}) = 1$. Homogenizing gives

$$(s(\mathbf{x}))^d = \sum |c_\alpha|^2 \mathbf{x}^\alpha (s(\mathbf{x}))^{d-|\alpha|}$$

and hence

$$n^d = s(\mathbf{1})^d = \sum |c_\alpha|^2 (n)^{d-|\alpha|} \geq \sum |c_\alpha|^2 = p(\mathbf{1}).$$

In Theorem 5.1 we extend this result to the general polynomial case, where the proof is more subtle. We also show in Section 5 that no such result holds in the rational case. We return to the monomial setting.

Next, to show that $m(n,d) \leq n + n(n-1)(d-1)$ holds, it suffices to find a polynomial p for which $p(\mathbf{1}) = n + n(n-1)(d-1)$. That polynomial is given by

$$p(\mathbf{x}) = \sum_{j=1}^{n} x_j^d + \sum_{j=1}^{n} (\sum_{j \neq k} x_j)(\sum_{l=1}^{d-1} x_k^l). \tag{4}$$

The coefficients of p are non-negative, it contains the terms x_j^d and its value at $\mathbf{1}$ is $n + n(n-1)(d-1)$. It remains to show that $p(\mathbf{x}) = 1$ on $s(\mathbf{x}) = 1$. The proof is a computation with the finite geometric series:

$$\begin{aligned} p(\mathbf{x}) &= \sum_{j=1}^{n} x_j^d + \sum_{j=1}^{n} (\sum_{j \neq k} x_j)(\sum_{l=1}^{d-1} x_k^l) \cong \sum_{j=1}^{n} x_j^d + \sum_{k=1}^{n} (1 - x_k)(\sum_{l=1}^{d-1} x_k^l) \\ &= \sum_{j=1}^{n} x_j^d + \sum_{k=1}^{n} (x_k - x_k^d) = \sum_{j=1}^{n} x_k \cong 1. \end{aligned} \tag{5}$$

When $n = 1$ the only polynomial in $\mathbf{S}(1,d)$ is x^d and the conclusion holds. Assume now that $n \geq 3$. We must show that the polynomial in (4) achieves the minimum. We divide the proof into three steps.

Step 1 is to show (for $n \geq 2$) that a polynomial achieving $m(n,d)$ can be written in the form

$$\sum_{j=1}^{n} x_j^d + n! \sum_{k=1}^{d-1} c[k]\mathbf{Sym}(x_1^k x_2) \tag{6}$$

for non-negative numbers $c[k]$.

Step 2 is to show (for $n \neq 2$) that there is a unique collection of numbers $c[k]$ for which the expression in (6) equals 1 on the hyperplane. Step 2 fails in source dimension 2, as we show in Section 4.

Conclusion (4) of Theorem 1.1 is then easy. The polynomial in (4) satisfies all our conditions, and by uniqueness it is the minimizer. Note that $c[k] = 1$ for all k and we obtain property (4) from Theorem 1.1.

We begin Step 1. The argument in source dimension 3 relies on computations done in source dimension 2. It is relatively easy to reduce the case of source dimension $n \geq 3$ to that of $n = 3$. For $n \geq 3$, we need both conclusions of the following result when $n = 2$.

Proposition 3.1. *Suppose $p(x,y)$ is a symmetric element of $\mathbf{S}(2,d)$, written for some symmetric polynomial $h(x,y)$ with non-negative coefficients in the form*

$$x^d + y^d + Axy + \sum_{k=2}^{d-1} c[k](x^k y + xy^k) + (xy)^2 h(x,y). \tag{6}$$

Then $A + \sum_{k=2}^{d-1} c[k] = d$. If also $p(1,1)$ is minimal, then $h(x,y) = 0$.

Proof. We homogenize (6) to get, for some homogeneous polynomial H,

$$x^d + y^d + Axy(x+y)^{d-2} + \sum_{k=2}^{d-1} c[k](x^k y + xy^k)(x+y)^{d-k-1} + (xy)^2 H(x,y). \tag{7}$$

The polynomial in (7) is homogeneous and equals 1 on $x + y = 1$; therefore it must be $(x+y)^d$. Therefore the coefficient of $x^{d-1}y$ on the left-hand side of (7) must equal d. But this coefficient is precisely equal to

$$A + \sum_{k=2}^{d-1} c[k].$$

The first statement follows.

Next we substitute the formula $\sum c[k] = d - A$ in (6) to obtain

$$p(1,1) = 2 + A + 2\sum_{k=2}^{d-1} c[k] + h(1,1)$$
$$= 2 + A + 2(d-A) + h(1,1) = 2d + 2 - A + h(1,1).$$

We minimize $p(1,1)$ by choosing $h(1,1)$ to vanish and choosing A as large as possible. Since h has non-negative coefficients, we set it equal to 0 and the conclusion follows. Note that we have not determined how to maximize A. We only know that we wish to choose the $c[k]$ such that

- $c[k] \geq 0$ for each k,
- $A + \sum_{k=2}^{d-1} c[k] = d$,
- A is as large as possible for p to satisfy the linear system.

In [3] the bound $A \leq 4 - \frac{2}{[\frac{d+1}{2}]}$ is proved, but determining the actual largest value for a given degree seems impossibly difficult. (Here [ ] denotes the floor function.) $\square$

Proposition 3.2. *Assume that $n = 3$ and $p \in \mathbf{S}(3,d)$ is symmetric. Group terms as follows:*

(8)

$$p(x,y,z) = x^d+y^d+z^d+A(xy+xz+yz)+\sum_{j=2}^{d-1} c[j](x^j(y+z)+y^j(x+z)+z^j(x+y))$$
$$+(xyz)g(x,y,z) + ((xy)^2 + (xz)^2 + (yz)^2)h(x,y,z),$$

where g and h are symmetric. Suppose that $p(1,1,1)$ is minimal. Then g and h vanish identically.

Proof. Note that $p(x,y,0)$ satisfies the hypotheses of Proposition 3.1. Therefore $A + \sum_{j=2}^{d-1} c[j] = d$. It follows that

$$p(1,1,1) = 3 + 3A + 6\sum_{j=2}^{d-1} c[k] + g(1,1,1) + 3h(1,1,1)$$

$$= 3+3A+6(d-A)+g(1,1,1)+3h(1,1,1) = 3+6d-3A+g(1,1,1)+h(1,1,1).$$

Note that $g(1,1,1)$ and $h(1,1,1)$ are non-negative, as each polynomial has non-negative coefficients. If either of these does not vanish identically, we can decrease this expression by multiplying g and h by parameters $t_1, t_2 \in [0,1)$.

The linear system will have a solution for all such t_1, t_2. Thus we minimize the value at $\mathbf{1}$ by choosing g and h to vanish and then choosing A as large as possible. ☐

The next result yields step 2 in the proof of Theorem 1.1.

Theorem 3.1. *For each positive integer d, there is a unique polynomial of degree d of the form (9) that equals 1 on the hyperplane $x+y+z=1$.*
(9)

$$p(x,y,z) = x^d+y^d+z^d+A(xy+xz+yz)+\sum_{j=2}^{d-1} c[j](x^j(y+z)+y^j(x+z)+z^j(x+y))$$

Suppose $d \geq 2$ and p satisfies (9): Then $A = 2$ and $c[j] = 1$ for each j.

Proof. The result is trivial when $d = 1$. First suppose $d = 2$. Then (9) becomes $x^2+y^2+z^2+A(xy+xz+yz)$, which equals 1 on $x+y+z=1$ only if $A = 2$. Next suppose that $d \geq 3$. We have seen from (4) that these values of A and $c[j]$ give a solution. It therefore suffices to show that the linear system has a unique solution. For $j \geq 2$, let T_j denote six times the symmetrization of $x^j y$. By formula (3),

$$T_j(x,y,z) = x^j(y+z) + y^j(x+z) + z^j(x+y).$$

Notice that $T_j(1,1,-2) = -1-1+(-2)^j 2 \neq 0$ for $j \neq 0$.

We homogenize (9) and set the result equal to $(x+y+z)^d$. We get a formula that holds for all (x,y,z). The same holds if we use the values 2 for the coefficient of the quadratic term and 1 for all the other coefficients.

Subtracting the two resulting formulas gives an identity of the form
(10)

$$0 = (A-2)(xy+xz+yz)(x+y+z)^{d-2}+\sum_{j=2}^{d-1}(c[j]-1))T_j(x,y,z)(x+y+z)^{d-1-j}$$

First set $(x,y,z) = (1,1,-2)$ in (10). All terms divisible by $(x+y+z)$ vanish, and the only term remaining is when $j = d-1$. We get

$$0 = (c[d-1]-1)\, T_{d-1}(1,1,-2)$$

and hence $c[d-1]-1 = 0$. Thus $c[d-1] = 1$. Plug this value into (10) and we get the same formula except now the sum goes only up to $d-2$. Divide both sides of the result by $(x+y+z)$ to get an equation that holds for all (x,y,z). The only term not now divisible by $(x+y+z)$ is the term with $j = d-2$.

Again substitute $(1,1,-2)$ for (x,y,z) to conclude that $c[d-2]-1=0$. By the method of descent we get $c[j]=1$ for all j with $2 \leq j \leq d-1$. Plugging into (10) then yields $A=2$ as well, and hence the solution is unique. $\square$

Next suppose that the source dimension n is at least 4 and that p is an arbitrary symmetric element of $\mathbf{S}(n.d)$. Setting all but three of the variables equal to 0 puts us into the situations of Proposition 3.2 and Theorem 3.1. It thus follows that p is a minimizer if and only if p satisfies (4).

The proof of Theorem 1.1 is complete. The following corollary follows instantly in dimensions not equal to 2, and also holds in dimension 2.

Corollary 3.1. *For all dimensions n, the following asymptotic result holds:*

$$\lim_{d\to\infty} \frac{m(n,d)}{d} = n(n-1).$$

Proof. For $n \neq 2$, the result follows from the explicit formula for $m(n,d)$ in Theorem 1.1. When $n=2$, we need an appropriate bound on $m(n,d)$ from below. Let λ be the coefficient of the xy term in a minimizer. It is elementary to show that $2 \leq \lambda < 4$ holds for all degrees at least 2. In fact λ increases a a function of d, but we do not need to know this monotonicity. For example, the lower bound in (*) is proved for all dimensions in [4] and the upper bound comes from Theorem 1.1:

$$(*) \qquad n + n(n-1)(d - \frac{\lambda}{2}) \leq m(2,d) \leq n + n(n-1)(d-1).$$

In (*), for general n, λ denotes the coefficient of the quadratic term, but the lower bound is relevant only when $n=2$, because otherwise equality holds with the right-hand side. In any case, since λ is bounded as a function of d, dividing by d in (*) and letting d tend to infinity gives $n(n-1)$. $\square$

The next corollary relies on the well-known degree bounds for monomial sphere maps with target dimension N. For $n=1$ of course there is no such bound, as z^d maps the circle to the circle and d can be arbitrarily large. For $n=2$, the sharp bound is $d \leq 2N-3$, and for $n \geq 3$ the sharp bound is $d \leq \frac{N-1}{n-1}$. See [6] and [10]. Let $\mathcal{N}(n,d)$ denote the minimum target dimension satisfying these degree bounds. Thus $N(n,d) = 1 + d(n-1)$ when $n \geq 3$ and $N(2,d)$ is the ceiling of $\frac{d+3}{2}$.

Corollary 3.2. *Let $m(n,d)$ denote the minimum value of $p(\mathbf{1})$ for $p \in$* $\mathbf{S}(n,d)$. *Let $\mathcal{N}(n,d)$ denote the minimum target dimension of a monomial* *sphere map of degree d with source dimension n. If $n \neq 2$, then*

$$\lim_{d\to\infty} \frac{m(n,d)}{\mathcal{N}(n,d)} = n.$$

If $n = 2$, the limit is 4.

Corollary 3.2 gives an asymptotic result comparing the L^1 norm of a particular solution to a certain linear system with the sparsest solution, the one with with fewest non-zero components.

4. The case $n = 2$

Recall the basic problem. A polynomial $p(x, y)$ satisfies the following properties:

- p is of degree d
- p has non-negative coefficients.
- $p(x, 1 - x) = 1$ for all x
- $p(x, 0) = x^d$ and $p(0, y) = y^d$.

What is the minimum possible value of $p(1, 1)$?

We begin by providing the solution for degrees up to 11. Recall that $m(2, d)$ denotes the minimum. This list is taken from [3] and will appear also in [4]. Notice that passing from odd degree to the next even degree is easy, but passing from even degree to the next odd degree is quite subtle.

$$m(2,1) = 2 \quad \text{for} \quad x + y$$

$$m(2,2) = 4 \quad \text{for} \quad x^2 + y^2 + 2xy$$

$$m(2,3) = 5 \quad \text{for} \quad x^3 + y^3 + 3xy$$

$$m(2,4) = 7 \quad \text{for} \quad x^4 + y^4 + xy(3 + x^2 + y^2)$$

$$m(2,5) = \frac{26}{3} \quad \text{for} \quad x^5 + y^5 + xy\left(\frac{10}{3} + \frac{5}{3}(x^3 + y^3)\right)$$

$$m(2,6) = \frac{32}{3} \quad \text{for} \quad x^6 + y^6 + xy\left(\frac{10}{3} + \frac{5}{3}(x^3 + y^3) + x^4 + y^4\right)$$

$$m(2,7) = \frac{25}{2} \quad \text{for} \quad x^7 + y^7 + xy\left(\frac{7}{2} + \frac{7}{2}(x^4 + y^4)\right)$$

$$m(2,8) = \frac{29}{2} \quad \text{for} \quad x^8 + y^8 + xy\left(\frac{7}{2} + \frac{7}{2}(x^4 + y^4) + x^6 + y^6\right)$$

$$m(2,9) = 577/35 \quad \text{for}$$
$$x^9 + y^9 + xy\left(\frac{123}{35} + 3(x^4+y^4) + \frac{6}{5}(x^5+y^5) + \frac{9}{7}(x^7+y^7)\right)$$
$$m(2,10) = \frac{647}{35} \quad \text{for}$$
$$x^{10} + y^{10} + xy\left(\frac{123}{35} + 3(x^4+y^4) + \frac{6}{5}(x^5+y^5) + \frac{9}{7}(x^7+y^7) + x^9 + y^9\right)$$
$$m(2,11) = \frac{573}{28} \quad \text{for}$$
$$x^{11} + y^{11} + xy\left(\frac{99}{28} + \frac{33}{14}(x^4+y^4) + \frac{33}{14}(x^5+y^5) + \frac{55}{28}(x^8+y^8) + \frac{11}{14}(x^9+y^9)\right).$$

Let λ_d denote the coefficient of the xy term in the minimizer of degree d. It always holds (See [3] or [4]) that $\lambda_d + m(2,d) = 2d+2$ and that λ_d is a monotone sequence bounded above by 4. We may thus rephrase the basic question as follows. Consider a polynomial with non-negative coefficients of the form

$$p(x,y) = x^d + y^d + Axy + \sum_{j=2}^{d-1} c[k](x^k y + xy^k).$$

Suppose $p(x,y) = 1$ on $x+y=1$. What is the maximum possible value of A as a function of d? Equivalently, what is the minimum value of $p(1,1)$?

We challenge the reader to find a general pattern! To convince the reader of the difficulty, λ_{35} (the coefficient in degree 35) is the following fraction:

$$\frac{17966598676264183}{4976648507631528}.$$

The value of this fraction to ten decimal places is 3.6101803551.

We pose two open problems in source dimension 2.

Open problem. Let λ_d denote the coefficient of xy in a polynomial realizing $m(2,d)$. It is known that λ_d is a monotone sequence bounded above by 4. Find $\lim_{d\to\infty}\lambda_d$. In degree 201, the value of λ_d is approximately 3.626016659.

Open problem. Given d, determine those k for which the coefficient $c[k]$ in a minimizer is non-zero. Call this set $NZ(d)$. This problem is perhaps too difficult, but the following might hold. Consider a very large odd degree d. Let $c[k]$ denote the coefficient of $(x^k y + xy^k)$ in a minimizer. Then, for small k, $k \in NZ(z)$ only for k congruent to 5 or 6 modulo 6. Thus these k come in consecutive pairs determined by a congruence. The pattern must eventually break down, because the cardinality of $NZ(d)$ should be approximately half the degree, and hence this pattern gives too few. We illustrate when $d = 121$. Here is the set $NZ(121)$:

5 6 11 12 17 18 23 24 29 30 35 36 40 41 46 47 51 52 57 58 62 63

67 68 71 72 76 77 80 81 84 85 88 89 92 93 95 96 98 99 101

102 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120

The first twelve coefficients follow this pattern, but then it breaks down. The lists in degrees 123 and 125 begin the same way; the first 25 such k are the same. The data suggest stabilization as the degree grows. The first 62 values in $NZ(197)$ and $NZ(201)$ are the same, and most of the remaining values are the same. Can one prove something precise along these lines as d tends to infinity?

5. The maximum in the general polynomial case

Recall that $\mathbf{1}$ denotes the n-tuple of all 1's. Note also that $\|\mathbf{1}\|^2 = n$. Consider a polynomial sphere map of degree d that is not necessarily a monomial map. In this section we prove the following result:

Theorem 5.1. *Let $p : \mathbb{C}^n \to \mathbb{C}^N$ be a polynomial mapping of degree d. Assume that $p(S^{2n-1}) \subseteq S^{2N-1}$. Then $\|p(\mathbf{1})\|^2 \leq n^d$. Equality holds if p is homogeneous but can hold otherwise.*

Theorem 5.2 (from [2]) characterizes homogeneous polynomial sphere maps p in terms of the volume of the image of the unit ball under p. This volume is the integral of the determinant of the complex Hessian of $\|p\|^2$ over the source ball. When p is not homogeneous, Proposition 5.1 applies. The crucial point in measuring the volume of the image is that the partial tensor product operation from Proposition 5.1 increases the volume of the image while preserving the degree. Examples 5.1 and 5.2 show that the value at $\mathbf{1}$ does not characterize homogeneity.

Theorem 5.2. *Let $p : \mathbb{C}^n \to \mathbb{C}^N$ be a polynomial sphere map of degree d. For $n \geq 2$, the following are equivalent:*

- *p is homogeneous.*
- *The $2n$-dimensional volume of the image of the unit ball under p is $\frac{d^n \pi^n}{n!}$.*

We begin the proof of Theorem 5.1. The key idea considers the expansion of p into homogeneous parts:

$$p = \sum_{j=0}^{d} p_j,$$

where each p_j is a vector-valued homogeneous polynomial of degree j. By letting the circle act on the sphere, one can prove various identities relating the inner products $\langle p_j, p_k\rangle$. See [2] for detailed discussion of these identities and their applications. In the monomial case, these inner products vanish for $j \neq k$. One can regard Proposition 5.1 as the first step in a process of *orthogonal homogenization.* The proof of Theorem 5.1 requires only this proposition and not all the identities.

Proposition 5.1. *Suppose $p = p_\nu + ... + p_d$ is a polynomial sphere map with source dimension n and target dimension N. Suppose that ν, the order of vanishing at 0, is less than d, the degree. Then there is a proper subspace $V \subseteq \mathbb{C}^N$ such that*

- *The image of p_ν lies in V.*
- *The image of p_d lies in the orthogonal complement of V.*
- *Let π_V denote orthogonal projection onto V. Then the map*

$$Ep = (\pi_V p \otimes z) \oplus (1 - \pi_V)(p) \tag{11}$$

  is also a polynomial sphere map.
- *Ep is of degree d and its order of vanishing at 0 exceeds ν.*
- *For all $z \in \mathbb{C}^n$, we have*

$$\|Ep(z)\|^2 = \|p(z)\|^2 + (\|z\|^2 - 1)\|\pi_V p(z)\|^2. \tag{12}$$

Proof. Assume $\nu < d$. Replace z by $e^{i\theta} z$ in the identity (that holds on the sphere)

$$\|p_\nu(z) + ... + p_d(z)\|^2 = \|z\|^{2d}$$

to get, for each θ,

$$\|e^{i\nu\theta} p_\nu(z) + ... + e^{id\theta} p_d(z)\|^2 = \|z\|^{2d}.$$

The right-hand side is independent of θ. Expanding the inner product on the left-hand side yields a trig polynomial that thus must be constant. The only term of the form $e^{i(d-\nu)\theta}$ is the inner product $\langle p_d, p_\nu\rangle$, which must then vanish. Hence these polynomials map into orthogonal subspaces. Let V be the smallest subspace into which p_ν maps. The first two items follow. The third item follows from the Pythagorean theorem. Since we do not tensor on the orthogonal complement of V, the map Ep remains of degree d. Formula (12) follows from (11), after using $||p||^2 = ||\pi_V p||^2 + ||(1-\pi_V)p||^2$. □

Corollary 5.1. *Suppose $n \geq 2$ and that p is a polynomial sphere of degree d, but that p is* **not** *homogeneous. Assume that $(\pi_V p)(\mathbf{1}) \neq 0$. Then $||p(\mathbf{1})||^2 < n^d$.*

Proof. Put $z = \mathbf{1}$ in (12). Since $||\mathbf{1}||^2 = n \geq 2$, and $(\pi_V p)(\mathbf{1}) \neq 0$, the term $(||z||^2 - 1)||\pi_V p(z)||^2$ is positive at $\mathbf{1}$ and hence $||p(\mathbf{1})||^2 < ||Ep(\mathbf{1})||^2$. Thus the operation of replacing p by Ep increases the value of the norm at $\mathbf{1}$. By Proposition 5.1, either Ep is homogeneous or we are in the same situation. Keep applying this partial tensor product process until we reach a homogeneous map. The only homogeneous polynomial sphere map of degree d has norm $||z||^{2d}$. We conclude for some polynomial mapping w that

$$||z||^{2d} = ||p(z)||^2 + ||w(z)||^2(||z||^2 - 1). \tag{13}$$

Since $||w||^2 \geq ||\pi_V p||^2$, evaluating at $\mathbf{1}$ yields the desired result. □

Example 5.1. The second conclusion of Theorem 5.1 fails when $n = 1$. For $c = \cos(\theta)$ and $s = \sin(\theta)$, and both non-zero, consider the polynomial sphere maps $z \to (cz^d, sz^k)$ for $k < d$. Then p is of degree d and $||p(1)||^2 = 1 = n^d$ but p is not homogeneous.

Equality can also occur in higher dimensions. In the next example, $p_\nu(\mathbf{1}) = 0$ and equality holds. In the notation of Proposition 5.1, $\pi_V p = (\frac{z-w}{\sqrt{2}}, 0, 0)$.

Example 5.2. The following polynomial sphere map p is of degree 2 with source dimension 2:

$$p(z, w) = \left(\frac{z-w}{\sqrt{2}}, z(\frac{z+w}{\sqrt{2}}), w(\frac{z+w}{\sqrt{2}})\right).$$

We have $||p(1,1)||^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4 = n^d$, but p is not homogeneous.

Remark 5.1. One can write down a formula for the polynomial map w in (13). Let E^k denote the iteration of k partial tensor product operations. Let π_k denote the orthogonal projection arising at the k-th step. Then

$$||w||^2 = \sum_{k=1}^{d-\nu} ||\pi_k E^{k-1} p||^2.$$

6. Rational sphere maps

We close the paper with several simple facts about the situation for rational sphere maps. First we note a trivial inequality:

$$|\sum_{j=1}^{n} z_j| \leq \sqrt{n}\|z\|. \tag{14}$$

Statement (14) follows from applying the Cauchy-Schwarz inequality to the vectors z and $\mathbf{1}$. As a consequence, in dimensions at least 2, the zero set of the polynomial $q(z) = 1 - \frac{\sum_j z_j}{n}$ lies outside the closed unit ball. There is thus a vector-valued polynomial map p such that $f = \frac{p}{q}$ maps the unit sphere in $\mathbb{C}^n$ to the unit sphere in $\mathbb{C}^N$ and $\frac{p}{q}$ is reduced to lowest terms. See [2] for a general statement; for this given q, one can easily construct p directly. In fact there is a p of degree 1. By taking tensor products, there is an example p of each degree $d \geq 1$. Since f has a singularity at $\mathbf{1}$, for each degree d there is no maximum value of $\|f(\mathbf{1})\|^2$. It follows that Theorem 5.1 fails for rational sphere maps that are not polynomials.

We next consider the values of $\|f(\mathbf{1})\|^2$ for rational sphere maps f formed from automorphisms. The automorphism group of the unit ball is not compact, which has consequences. For example, for a point a in the unit disk, and $s^2 = 1 - |a|^2$, consider the following automorphism for $n \geq 2$:

$$\phi_a(z) = \frac{(a - z_1, -sz_2, ... - sz_n)}{1 - z_1\overline{a}}.$$

One computes that

$$\|\phi_a(\mathbf{1})\|^2 = 1 + \frac{(1 - |a|^2)(n - 1)}{|1 - a|^2}$$

Let $0 > a > -1$ and let a tend to -1. Then $\|\phi_a(\mathbf{1})\|^2$ tends to 1. Since $\|\phi_a(z)\|^2 = 1$ on the sphere, the maximum principle forces $\|\phi_a(\mathbf{1})\|^2$ to be at least 1. Therefore the infimum value as a varies is 1, and this value is not achieved. The d-th tensor power of ϕ_a is a rational sphere map of degree d, and again,

$$\|(\phi_a^{\otimes d})(\mathbf{1})\|^2 = \|\phi_a(\mathbf{1})\|^{2d} = \left(1 + \frac{(1 - |a|^2)(n - 1)}{|1 - a|^2}\right)^d$$

tends to 1. Thus, for each d the infimum is 1, and this value is not attained. On the other hand, if $0 < a < 1$, and a tends to 1, then the limits are infinite, providing a second proof that the supremum is infinity.

These considerations help explain why the optimization problems are more interesting in the polynomial case. They also indicate why we introduced the fourth condition in the definition of the set $\mathbf{S}(n,d)$.

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John P. D'Angelo
Dept. of Mathematics
Univ. of Illinois
1409 W. Green St.
Urbana IL 61801
USA
E-mail: jpda@illinois.edu