

On the Kähler-Ricci flow on Fano manifolds*

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Abstract: A short proof of the convergence of the Kähler-Ricci flow on Fano manifolds admitting a Kähler-Einstein metric or a Kähler-Ricci soliton is given, using a variety of recent techniques.

Keywords: Kähler-Ricci flow, Fano manifolds.

1. Introduction

In 2002, G. Perelman announced that he could show the convergence of the Kähler-Ricci flow on Fano manifolds admitting a Kähler-Einstein metric. This led to many attempts to supply a detailed proof, including [18, 19, 20] and [2]. Perhaps the most transparent proof is that of Collins and Székelyhidi [2], which incorporated ideas from [19] and a general version allowing a twisting. The understanding of the Kähler-Ricci flow has progressed considerably since that time, and many new ideas and techniques have been introduced. The purpose of this brief note is to point out that, if one combines the ideas and techniques of Darvas-Rubinstein [3], Boucksom-Eyssidieux-Guedj [1], Collins-Székelyhidi [2], Kolodziej [6], and Phong-Song-Sturm-Weinkove [9, 10, 11], a relatively short proof can now be written down. Thus we shall prove the following:

Theorem 1. Let (M, g) be a compact Kähler manifold with $c_1(M) > 0$. If g is Kähler-Einstein, then the Kähler-Ricci flow with any initial metric in $c_1(M)$ will converge smoothly and exponentially fast to a Kähler-Einstein metric. More generally, if g is a Kähler-Ricci soliton with vector field X , then the Kähler-Ricci flow with any initial metric in $c_1(M)$ which is invariant under $\text{Im } X$ will converge smoothly and exponentially fast to a Kähler-Ricci soliton, after modified by the one-parameter group generated by X .

Remark. Compared to the works [20, 2], our theorem provides a more precise convergence property of the Kähler-Ricci flow when there is a Kähler-Ricci

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soliton, namely, the gauge transformation is simply given by the group generated by X . Finally we remark that the theorem still holds when the initial metric is not necessarily invariant under $\text{Im } X$, by the recent work of Dervan-Székelyhidi [4].

2. Proof of the Theorem

It may be instructive to compare the proof below to that of [18] which treats the simpler case of Fano manifolds admitting a Kähler-Einstein metric and no non-trivial holomorphic vector fields. There one develops a relative version of Kolodziej’s capacity theory [6] to get a one-sided bound on the potential. Then, combining the Moser-Trudinger [16, 8] inequality with a Harnack inequality, one gets the full C^0 bound. The higher order estimates are obtained by applying parabolic versions of Yau’s estimates [21].

The present proof also begins with the Moser-Trudinger inequality, which is now available for Kähler-Einstein or Kähler-Ricci soliton manifold with holomorphic vector fields [3]. But we obtain C^α estimates for the potentials directly from recent arguments introduced in [5] using [1, 6]. Higher order estimates and convergence can be obtained now more efficiently by combining techniques from [13, 11].

We begin by introducing the notation. Let (M, ω_0) be a compact Kähler manifold with $c_1(M) > 0$ and X be a holomorphic vector field whose imaginary part $\text{Im } X$ is a Killing vector field with respect to ω_0 . Write \mathcal{K}_X for the space of Kähler metrics in $c_1(M)$ that are invariant under $\text{Im } X$. We write

$$\mathcal{P}_X(M, \omega_0) = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_0 + i\partial\bar{\partial}\varphi \in \mathcal{K}_X, (\text{Im } X)(\varphi) = 0\}$$

to be the space of Kähler potentials in \mathcal{K}_X . Given $\omega \in \mathcal{K}_X$, we define the Hamiltonian $\theta_{X,\omega}$ as the real-valued function satisfying

$$\iota_X\omega = i\bar{\partial}\theta_{X,\omega}, \quad \int_M e^{\theta_{X,\omega}}\omega^n = \int_M \omega^n = V.$$

We define the Ricci potential $f = f_\omega$ by

$$(2.1) \quad -\text{Ric}(\omega) + \omega = i\partial\bar{\partial}f, \quad \int_M e^{-f}\omega^n = V.$$

We write

$$u_{X,\omega} = f_\omega + \theta_{X,\omega}$$

to be the *modified Ricci potential*, which satisfies

$$(2.2) \quad -\text{Ric}(\omega) + \omega + L_X\omega = i\partial\bar{\partial}u_{X,\omega},$$

where L_X denote the Lie derivative in the direction of X . It is known ([17, 11]) that if $\mathcal{K}_X \ni \omega = \omega_0 + i\partial\bar{\partial}\varphi$ then $\theta_{X,\omega} = \theta_{X,\omega_0} + X\varphi$ and $\|\theta_{X,\omega}\|_{C^0} \leq C(\omega_0, X)$ is uniformly bounded.

A Kähler metric ω_{KS} is a Kähler-Ricci soliton associated with X if $u_{X,\omega_{KS}} = 0$, i.e.

$$\text{Ric}(\omega_{KS}) = \omega_{KS} + L_X\omega_{KS}.$$

The *modified Mabuchi K-energy* $\mu_{X,\omega_0} : \mathcal{P}_X(M, \omega_0) \rightarrow \mathbb{R}$ is defined by the variation ([17])

$$\delta\mu_{X,\omega_0}(\varphi) = -\frac{1}{V} \int_M \delta\varphi(R - n - \nabla_j X^j - X(u_{X,\omega}))e^{\theta_{X,\omega}}\omega^n, \quad \mu_{X,\omega_0}(0) = 0$$

where $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ in the integral.

We will consider the following *normalized Kähler-Ricci flow* ([11], [17])

$$(2.3) \quad \frac{\partial\omega}{\partial t} = -\text{Ric}(\omega) + \omega, \quad \omega(0) = \omega_0,$$

where $\omega_0 \in \mathcal{K}_X$ is a fixed Kähler metric in $c_1(M)$. It is well-known that the solution $\omega(t)$ exists and lies in \mathcal{K}_X for all $t \in [0, \infty)$.

We recall Perelman’s estimates along the flow (2.3).

Lemma 1 ([13]). There exists a constant $C = C(M, \omega_0) > 0$ independent of t such that

$$\|f_\omega\|_{C^0} + \|\nabla_\omega f_\omega\|_{C^0} + \|\Delta_\omega f_\omega\|_{C^0} \leq C$$

where $\omega = \omega(t)$ is the Kähler metric along the flow (2.3).

Let $Aut(M)$ be the automorphism group of M and $\mathfrak{aut}(M)$ be the Lie algebra, i.e. the space of all real vector fields X such that $\mathcal{L}_X J = 0$, where J is the complex structure on M , and denote

$$\mathfrak{aut}_X(M) = \{Y \in \mathfrak{aut}(M) \mid L_X Y = [X, Y] = 0\}.$$

Write $G = Aut_X(M)$ to be the connected Lie group whose Lie algebra is $\mathfrak{aut}_X(M)$. It follows that for any $\sigma \in G$, $d\sigma(X) = X$.

For any $\sigma \in G$, we define $\sigma \cdot 0$ to be the Kähler potential of $\sigma^*\omega_0 = \omega_0 + i\partial\bar{\partial}\sigma \cdot 0$ (suitably normalized). Furthermore for any $\varphi \in \mathcal{P}_X(M, \omega_0)$, define

$$\sigma \cdot \varphi = \varphi \circ \sigma + \sigma \cdot 0$$

which is the Kähler potential of $\sigma^*\omega$, where $\omega = \omega_0 + i\partial\bar{\partial}\varphi$.

We recall the following Moser-Trudinger inequality from [3]. It is the extension to the case of Kähler-Ricci soliton manifolds with holomorphic vector fields of the sharp version for Kähler-Einstein manifolds with $\text{aut}(M) = 0$ in [8] of the inequality originally proved in [16]:

Lemma 2 (Theorem 8.1 in [3]). If M admits a Kähler-Ricci soliton metric associated to the holomorphic vector field X , then there exist positive constants C and D depending only on ω_0 and X such that

$$\mu_{X, \omega_0}(\varphi) \geq C\mathcal{J}_G(G\varphi) - D, \quad \forall \varphi \in \mathcal{P}_X(M, \omega_0),$$

where $\mathcal{J}_G(G\varphi) = \inf_{\sigma \in G} J_{\omega_0}(\sigma \cdot \varphi)$ and J_{ω_0} is the Aubin-Yau J -functional with the reference metric ω_0 .

The following lemma, which is a version of estimates in [5], is the key starting point:

Lemma 3. Let $\{t_j\}$ be an arbitrary sequence of times, and denote $\omega_j = \omega(t_j) = \omega_0 + i\partial\bar{\partial}\varphi_j$. Then there exists a uniform constant $C > 0$ and $\sigma_j \in G$ with

$$(2.4) \quad \|\psi_j\|_{C^\alpha(M, \omega_0)} \leq C$$

if we set $\tilde{\omega}_j = \sigma_j^*\omega_j \in \mathcal{K}_X$, and $\tilde{\omega}_j = \omega_0 + i\partial\bar{\partial}\psi_j$, $\sup_M \psi_j = 0$. In particular,

$$(2.5) \quad C^{-1}\omega_0^n \leq \tilde{\omega}_j^n \leq C\omega_0^n.$$

Proof of Lemma 3. From Lemma 2, for each j there exists a $\sigma_j \in G$ such that $\mathcal{J}_G(G\varphi_j) \geq J_{\omega_0}(\sigma_j \cdot \varphi_j) - 1$ and $J_{\omega_0}(\sigma_j \cdot \varphi_j) \leq C^{-1}\mu_{X, \omega_0}(\varphi_j) + C^{-1}D + 1 \leq C(\omega_0, X)$, since the modified Mabuchi K -energy μ_{X, ω_0} is non-increasing along the flow. It follows by a straightforward calculation that $f_{\tilde{\omega}_j} = \sigma_j^*f_{\omega_j}$ so $f_{\tilde{\omega}_j}$ is also uniformly bounded by Lemma 1. From the equation (2.1), the metric $\tilde{\omega}_j$ satisfies the complex Monge-Ampère equation

$$(2.6) \quad \tilde{\omega}_j^n = (\omega_0 + i\partial\bar{\partial}\psi_j)^n = e^{-(\psi_j - \sup_M \psi_j) + f_{\tilde{\omega}_j} - f_{\omega_0} + c_j} \omega_0^n,$$

where c_j is a normalizing constant so that both sides have the same integral over M . It follows from the normalizing condition $\frac{1}{V} \int_M e^{-f_{\tilde{\omega}_j}} \tilde{\omega}_j^n = 1$ and equation (2.6) that

$$V = \int_M e^{-(\psi_j - \sup_M \psi_j) - f_{\omega_0} + c_j} \omega_0^n \geq e^{c_j} \int_M e^{-f_{\omega_0}} \omega_0^n = e^{c_j} V \quad \Rightarrow \quad c_j \leq 0.$$

So

$$(2.7) \quad (\omega_0 + i\partial\bar{\partial}\psi_j)^n \leq C e^{-(\psi_j - \sup_M \psi_j)} \omega_0^n = C e^{-\tilde{\psi}_j} \omega_0^n$$

where $C = C(\omega_0, X) > 0$ depends on the bound on $f_{\tilde{\omega}_j}$ and f_{ω_0} . $\tilde{\psi}_j = \psi_j - \sup_M \psi_j$ belongs to a set S_A for some $A = A(\omega_0, X) > 0$, which is

$$S_A := \{\psi \in \mathcal{P}_X(M, \omega_0) \subset \mathcal{E}_X^1(M, \omega_0) \mid \sup_M \psi = 0 \text{ and } J_{\omega_0}(\psi) \leq A\}.$$

S_A is compact under the weak $L^1(M, \omega_0^n)$ -topology in $\mathcal{E}_X^1(M, \omega_0)$, and each $\psi \in S_A$ has zero Lelong numbers at any point $x \in M$. This implies, as in [1], that for any $p > 0$, there exists a constant $C_p = C(\omega_0, A, p) > 0$ such that

$$(2.8) \quad \int_M e^{-p\psi} \omega_0^n \leq C_p, \quad \forall \psi \in S_A.$$

Combining equations (2.7) and (2.8) and Kolodziej’s theorem ([6]), for $p > 1$, there exist an $\alpha = \alpha(n, p) \in (0, 1)$ and $C = C(\omega_0, A, p) > 0$ such that

$$(2.9) \quad \|\tilde{\psi}\|_{C^\alpha(M, \omega_0)} \leq C.$$

The estimate (2.5) follows next from the equation (2.6). The lemma is proved.

We can give now the proof of the theorem. Let $\tilde{\omega}_j(s) := \sigma_j^* \omega(t_j + s)$ for $s \in [0, 3]$. Then $\tilde{\omega}_j(s)$ satisfies the *Kähler-Ricci flow* equation

$$(2.10) \quad \frac{\partial \tilde{\omega}_j(s)}{\partial s} = -\text{Ric}(\tilde{\omega}_j(s)) + \tilde{\omega}_j(s), \quad \tilde{\omega}_j(0) = \tilde{\omega}_j.$$

Let $\tilde{\omega}_j(s) = \omega_0 + i\partial\bar{\partial}\tilde{\psi}_j(s)$. The equation (2.10) is equivalent to the following complex Monge-Ampère equation for $\tilde{\psi}_j(s)$

$$(2.11) \quad \frac{\partial \tilde{\psi}_j(s)}{\partial s} = \log \frac{(\omega_0 + i\partial\bar{\partial}\tilde{\psi}_j(s))^n}{\omega_0^n} + \tilde{\psi}_j(s) + f_{\omega_0}, \quad \tilde{\psi}_j(0) = \tilde{\psi}_j,$$

where $\tilde{\psi}_j$ is the Kähler potential of the initial metric $\tilde{\omega}_j$ with $\sup_M \tilde{\psi}_j = 0$. Equations (2.5) and (2.9) imply that

$$\|\tilde{\psi}_j(0)\|_{C^0} + \left\| \frac{\partial}{\partial s} \Big|_{s=0} \tilde{\psi}_j(s) \right\|_{C^0} \leq C(\omega_0, X).$$

We recall the following parabolic version of Yau’s estimates for complex Monge-Ampère equations. □

Lemma 4 (Proposition 2.1 in [15]). Suppose $\varphi \in PSH(M, \omega_0)$ satisfies the parabolic Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + i\partial\bar{\partial}\varphi)^n}{\omega_0^n} + \varphi + f_0, \quad \varphi(0) = \varphi_0$$

for some smooth function $f_0 : M \rightarrow \mathbb{R}$. Then for any $k \in \mathbb{N}$, there exists a smooth function $C_k : (0, \infty) \rightarrow \mathbb{R}_+$ which depends only on $\|\varphi_0\|_{C^0}$ and $\|\dot{\varphi}|_{t=0}\|_{C^0}$ such that

$$\|\varphi(t)\|_{C^k(M, \omega_0)} \leq C_k(t), \quad \forall t \in (0, \infty).$$

We now apply Lemma 4 to conclude that for each $k \in \mathbb{N}$, there exists a constant $C_k = C(\omega_0, X) > 0$ such that

$$\|\tilde{\psi}_j(s)\|_{C^k(M, \omega_0)} \leq C_k, \quad \forall s \in [1, 2].$$

In particular, this implies the metric $\sigma_j^*(\omega(t_j + 1))$ has a uniformly bounded Kähler potential and its derivatives are also uniformly bounded.

Since $t_j > 0$ is arbitrarily chosen, we conclude that for any $t \geq 1$, there exists a $\sigma_t \in G$ such that

$$\sigma_t^*(\omega(t)) = \omega_0 + i\partial\bar{\partial}\psi_t, \quad \text{with } \|\psi_t\|_{C^k(M, \omega_0)} \leq C_k(\omega_0, X).$$

Let $\eta_t = \exp(tX)$ be the one-parameter group generated by X , and $\lambda(t)$ be the first nonzero eigenvalue of the $\bar{\partial}$ -operator associated to $\eta_t^*(\omega(t))$. Since $\lambda(t)$ is also the corresponding eigenvalue of $\sigma_t^*(\omega(t))$, which satisfies uniform estimates and hence forms a compact set, we conclude that

$$\inf_{t \in [0, \infty)} \lambda(t) > 0.$$

On the other hand, since M admits a Kähler-Ricci soliton, by [17] we have

$$\inf_{\varphi \in \mathcal{P}_X(M, \omega_0)} \mu_{X, \omega_0}(\varphi) > -\infty.$$

Applying Theorem 3 in [11] we conclude that $\hat{\omega}_t = \eta_t^*(\omega(t))$ converge exponentially fast to a Kähler-Ricci soliton metric, observing that $\eta_t^*(\omega(t))$ satisfies the *modified Kähler-Ricci flow* defined in [11].

The proof of the theorem is complete.

3. Additional remarks

We also observe that another proof of the convergence of the Kähler-Ricci flow can also be obtained using the ideas in [10, 11] and the recent result in [5]. For example, if M is Kähler-Einstein, then the K -energy is bounded from below, and by [5], the lowest strictly positive eigenvalue of the Laplacian on vector fields is uniformly bounded away from 0 along the Kähler-Ricci flow. By the main result of [10], the flow converges.

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