Pure and Applied Mathematics Quarterly
Volume 18, Number 2, 583–597, 2022

Basic estimates for the generalized ∂-complex

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Dedicated to J.J. Kohn on his 90th birthday

Abstract: We study certain densely defined unbounded operators on the Segal-Bargmann space, related to the annihilation and creation operators of quantum mechanics. We consider the corresponding D-complex and study properties of the complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, where D is a differential operator of polynomial type, in particular we discuss the corresponding basic estimates, where we express a commutator term as a sum of squared norms.

Keywords: ∂-complex, Segal-Bargmann space, sum of squared norms.

1. Introduction

We consider the classical Segal-Bargmann space

$$A^2(\mathbb{C}^n, e^{-|z|^2}) = \{u : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ entire} : \int_{\mathbb{C}^n} |u(z)|^2 e^{-|z|^2}\, d\lambda(z) < \infty\}$$

with inner product

$$(u, v) = \int_{\mathbb{C}^n} u(z)\, \overline{v(z)}\, e^{-|z|^2}\, d\lambda(z).$$

For $0 \leq p \leq n$ let $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ denote the space of $(p, 0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$. We define

$$\text{dom}(\partial) = \{f \in A^2(\mathbb{C}^n, e^{-|z|^2}) : \frac{\partial f}{\partial z_j} \in A^2(\mathbb{C}^n, e^{-|z|^2}), j = 1, \dots, n\}.$$

Received March 12, 2021.
2010 Mathematics Subject Classification: Primary 30H20, 32A36, 32W50; secondary 47B38.
*The author is supported by the Austrian Science Fund (FWF) project P28154.

The operator $\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j$ is a densely defined closed operator

$$\partial : A^2(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

The adjoint operator

$$\partial^* : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow A^2(\mathbb{C}^n, e^{-|z|^2})$$

is given by $\partial^* g = \sum_{j=1}^n z_j g_j$, where $g = \sum_{j=1}^n g_j\, dz_j \in \mathrm{dom}(\partial^*)$ and

$$\mathrm{dom}(\partial^*) = \{g \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) : \sum_{j=1}^n z_j g_j \in A^2(\mathbb{C}^n, e^{-|z|^2})\}.$$

Hence one has

$$(\partial f, g) = (\sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j, \sum_{j=1}^n g_j\, dz_j) = \sum_{j=1}^n (\frac{\partial f}{\partial z_j}, g_j) = (f, \sum_{j=1}^n z_j g_j) = (f, \partial^* g).$$

This kind of duality is used to describe the annihilation and creation operators in quantum mechanics [3], it is used by D.G. Quillen to represent Hermitian forms as sums of squares [13] and by H. Render in the real analytic setting to investigate sets of uniqueness for polyharmonic functions [14]. In [7] and [8] the ∂-operator on weighted Bergman spaces on Hermitian manifolds is investigated, a similar duality appears for instance on the unit ball endowed with Bergman-Kähler metric.

If one replaces the single derivative with respect to z_j by a differential operator of the form $p_j(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})$, where p_j is a complex polynomial on $\mathbb{C}^n$ (we write $p_j(u)$ for $p_j(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})u$), the duality relation is now of the form

$$(p_j(u), v) = (u, p_j^* v),$$

where $p_j^*(z_1, \dots, z_n)$ is the polynomial p_j with complex conjugate coefficients, taken as multiplication operator. Newman and Shapiro [11], [12] use this duality relation in their analysis of Fischer spaces of entire functions.

We generalize the ∂-operator by setting

$$Du = \sum_{j=1}^n p_j(u)\, dz_j, \tag{1}$$

where $u \in A^2(\mathbb{C}^n, e^{-|z|^2})$, see [4].

Operating on $(p,0)$-forms we define

$$Du = \sum_{|J|=p}{}' \sum_{k=1}^{n} p_k(u_J)\, dz_k \wedge dz_J, \tag{2}$$

where $u = \sum_{|J|=p}{}' u_J\, dz_J$ is a $(p,0)$-form with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$, here $J = (j_1, \ldots, j_p)$ is a multiindex and $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and the summation is taken only over increasing multiindices. We get again densely defined closed operators and observe that $D^2 = 0$ and that we have

$$(Du, v) = (u, D^*v), \tag{3}$$

where $u \in \mathrm{dom}(D) = \{u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : Du \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2})\}$ and

$$D^*v = \sum_{|K|=p-1}{}' \sum_{j=1}^{n} p_j^* v_{jK}\, dz_K$$

for $v = \sum_{|J|=p}{}' v_J\, dz_J$.

Replacing ∂ by D one gets a corresponding complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, for which one can use duality and the machinery of the $\bar{\partial}$-Neumann operator ([9], [10]) in order to prove existence and boundedness of the inverse to $\tilde{\Box}_D$ and to find the canonical solutions to the inhomogeneous equations $Du = \alpha$ and $D^*v = \beta$. In addition, studying the spectrum of $\tilde{\Box}$ for ∂, one gets estimates for the canonical solutions, which are not attainable by standard methods, see [7], section 5 and [8], section 4.

In the $\bar{\partial}$-Neumann problem the underlying Hilbert space is $L^2(\Omega)$ and the $\bar{\partial}$-operator is defined in the sense of distributions in order to become a densely defined unbounded operator on $L^2(\Omega)$ with closed graph. The adjoint operator $\bar{\partial}^*$ is again a differential operator. In our setting, the underlying Hilbert space is $A^2(\mathbb{C}^n, e^{-|z|^2})$, the operator D is a densely defined unbounded operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$ with closed graph, the adjoint operator D^* is now a multiplication operator. This phenomenon is used to describe the commutator equation $[A, A'] = I$ of quantum mechanics on an appropriate Hilbert space, see [3], Chapter 1. The abstract theory of unbounded operators on Hilbert space is identical in both cases.

In our setting, the corresponding D-complex has the form

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{D^*}{\overset{D}{\rightleftarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{D^*}{\overset{D}{\rightleftarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

Similar to the classical $\overline{\partial}$-complex (see [4]) we consider the generalized box operator $\tilde{\Box}_{D,p} := D^*D + DD^*$ as a densely defined self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ with

$$\mathrm{dom}(\tilde{\Box}_{D,p}) = \{f \in \mathrm{dom}(D) \cap \mathrm{dom}(D^*) : Df \in \mathrm{dom}(D^*) \text{ and } D^*f \in \mathrm{dom}(D)\}.$$

see [5] for more details.

The $(p,0)$-forms with polynomial components are dense in $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. In addition, the $(p,0)$-forms with polynomial components are dense in $\mathrm{dom}(D) \cap \mathrm{dom}(D^*)$ endowed with the graph norm

$$u \mapsto (\|u\|^2 + \|Du\|^2 + \|D^*u\|^2)^{1/2}.$$

See [5] and [6] for the details.

In [5] it is shown that the basic estimate

$$\|u\|^2 \leq C(\|Du\|^2 + \|D^*u\|^2), \tag{4}$$

for any $u \in \mathrm{dom}(D) \cap \mathrm{dom}(D^*)$ implies that $\tilde{\Box}_{D,1}$ has a bounded inverse. The basic estimate can easily be shown for the ∂-complex (see [5]), whereas for the more general D-complex one has major difficulties. From [5] we know that for the basic estimates it suffices to show that there exists a constant $C > 0$ such that

$$\|u\|^2 \leq C \sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, u_k), \tag{5}$$

for any $(1,0)$-form $u = \sum_{j=1}^n u_j dz_j$ with polynomial components.

In this paper we will show that inequality (5) can be expressed by coerciveness of a corresponding densely defined Hermitian form on $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \times A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$.

In the last part we concentrate on $n = 2$ exhibiting some classes of homogeneous polynomials of arbitrary degree such that (5) holds.

2. Hermitian forms

We take the right hand side of (5) to define a Hermitian form.

Let

$$H(u, v) = \sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, v_k), \tag{6}$$

where u and v are $(1,0)$-forms with polynomial components.

Then

$$H : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \times A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow \mathbb{C}$$

is a densely defined Hermitian form. This follows from

$$\begin{aligned} H(v,u) &= \sum_{j,k=1}^n ([p_k, p_j^*]v_j, u_k) \\ &= \sum_{j,k=1}^n \Big[(p_j^* v_j, p_k^* u_k) - (p_k v_j, p_j u_k)\Big] \\ &= \sum_{j,k=1}^n \Big[(p_k^* u_k, p_j^* v_j) - (p_j u_k, p_k v_j)\Big]^- \\ &= \sum_{j,k=1}^n \Big[(p_j^* u_j, p_k^* v_k) - (p_k u_j, p_j v_k)\Big]^- \\ &= H(u,v)^- \end{aligned}$$

Condition (5) can be written in the form

$$H(u,u) \geq \frac{1}{C}\|u\|^2, \tag{7}$$

which means that the Hermitian form H is lower semibounded, and as $C > 0$ even that H is coercive.

We just mention that associated with the form H there is an operator T_H defined by $T_H u := w_u$ for $u \in \operatorname{dom}(T_H)$, where $\operatorname{dom}(T_H) = \{u \in \operatorname{dom}(H) :$ there exists $w_u \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$ such that $H(u,v) = (w_u, v)$ for $v \in \operatorname{dom}(H)\}$, where (w_u, v) denotes the inner product in $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$. Since $\operatorname{dom}(H)$ is dense, the $(1,0)$-form is uniquely determined by u, and the operator T_H is well defined and linear. By definition, $\operatorname{dom}(T_H) \subseteq \operatorname{dom}(H)$, and

$$H(u,v) = (T_H u, v) \quad \text{for} \quad u \in \operatorname{dom}(T_H) \quad \text{and } v \in \operatorname{dom}(H).$$

In our case, it's easy to give an explicit expression for

$$T_H : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

We have

$$T_H u = \sum_{j=1}^n [p_1, p_j^*] u_j \, dz_1 + \sum_{j=1}^n [p_2, p_j^*] u_j \, dz_2 + \cdots + \sum_{j=1}^n [p_n, p_j^*] u_j \, dz_n.$$

See [15] for the general properties of Hermitian forms and the corresponding operators.

3. Commutator terms as a sum of squared norms

In order to handle the expression

$$\sum_{j,k=1}^n ([p_k, p_j^*] u_j, u_k)$$

we use the following operator theoretic method. Let A_j and $B_j, j = 1, \dots, n$ be operators satisfying

$$[A_j, A_k] = [B_j, B_k] = [A_j, B_k] = 0, j \neq k$$

and

$$[A_j, B_j] = I, j = 1, \dots, n.$$

Let P and Q be polynomials of n variables and write $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$.

The assumptions are satisfied, if one takes $A_j = \frac{\partial}{\partial z_j}$ and $B_j = z_j$ the multiplication operator. The inspiration for this comes from quantum mechanics, where the annihilation operator A_j can be represented by the differentiation with respect to z_j on $A^2(\mathbb{C}^n, e^{-|z|^2})$ and its adjoint, the creation operator B_j, by the multiplication by z_j.

Then

$$Q(A)P(B) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} P^{(\alpha)}(B) Q^{(\alpha)}(A), \tag{8}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multiindices and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$, see [13], [16].

Applying (8) we get

$$([p_k, p_j^*] u_j, u_k) = \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} (p_j^{(\alpha)*} p_k^{(\alpha)} u_j, u_k), \tag{9}$$

see [6], and try to express

$$\sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, u_k)$$

as a sum of squared norms. See also [1] and [2] for the term squared norms in a different context.

In the following we concentrate on polynomials of two complex variables to demonstrate the method of squared norms, and to show for what kinds of polynomials it is applicable.

Theorem 3.1. *Let $p_1(z_1, z_2)$ and $p_2(z_1, z_2)$ be polynomials of degree 2 with real coefficients. Suppose that for $b_1 \neq 0$ and $a_2 \neq 0$ the polynomials have the form*

$$p_1(z_1, z_2) = b_1 z_1 z_2 + d_1 z_1 + e_1 z_2 \text{ and}$$
$$p_2(z_1, z_2) = a_2 z_1^2 + a_2 z_2^2 + \frac{2e_1 a_2}{b_1} z_1 + \frac{2d_1 a_2}{b_1} z_2,$$

where d_1 and e_1 are arbitrary real numbers, or for $a_1 \neq 0$ and $c_2 \neq 0$ the polynomials have the form

$$p_1(z_1, z_2) = a_1 z_1^2 + d_1 z_1 \text{ and } p_2(z_1, z_2) = c_2 z_2^2 + e_2 z_2,$$

where d_1 and e_2 are arbitrary real numbers. Then

$$\|u\|^2 \leq C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k), \tag{10}$$

for any $(1,0)$-form $u = \sum_{j=1}^{2} u_j dz_j$ with polynomial components.

Proof. We denote the first derivative with respect to z_1 by (10), and with respect to z_2 by (01). It is easily seen that our polynomials p_1 and p_2 satisfy

$$p_2^{(01)*} p_1^{(01)} = p_1^{(10)*} p_2^{(10)} \quad , \quad p_2^{(10)*} p_1^{(10)} = p_1^{(01)*} p_2^{(01)}. \tag{11}$$

So we can use the technique in the proof of Theorem 3.2 of [6] to compute in the following way: If $p_1(z_1, z_2) = a_1 z_1^2 + d_1 z_1$ and $p_2(z_1, z_2) = c_2 z_2^2 + e_2 z_2$, then

$$\sum_{j,k=1}^{2}([p_j,p_k^*]u_k,u_j) = 2a_1^2\|u_1\|^2+2c_2^2\|u_2\|^2 + \|d_1u_1+2a_1\frac{\partial u_1}{\partial z_1}\|^2 + \|e_2u_2+2c_2\frac{\partial u_2}{\partial z_2}\|^2.$$

If $p_1(z_1,z_2)=bz_1z_2+dz_1+ez_2$ and $p_2(z_1,z_2)=az_1^2+az_2^2+fz_1+gz_2$ with $bg=2ad$ and $bf=2ae$, then

$$\sum_{j,k=1}^{2}([p_j,p_k^*]u_k,u_j) = b^2\|u_1\|^2+4a^2\|u_2\|^2 + \|du_1+fu_2+2a\frac{\partial u_2}{\partial z_1}+b\frac{\partial u_1}{\partial z_2}\|^2 + \|eu_1+gu_2+2a\frac{\partial u_2}{\partial z_2}+b\frac{\partial u_1}{\partial z_1}\|^2.$$

□

Finally we determine homogeneous polynomials p_1 and p_2 of degree K such that (5) holds.

Let $a,b\in\mathbb{R}, a,b\neq 0$. We consider the following homogeneous polynomials of degree K: if K is even we set

$$p_1(z_1,z_2)^* = a\sum_{\ell=0}^{K/2}\binom{K}{2\ell}z_1^{K-2\ell}z_2^{2\ell}, \tag{12}$$

and

$$p_2(z_1,z_2)^* = b\sum_{\ell=0}^{(K-2)/2}\binom{K}{2\ell+1}z_1^{K-2\ell-1}z_2^{2\ell+1}, \tag{13}$$

if K is odd we set

$$p_1(z_1,z_2)^* = a\sum_{\ell=0}^{(K-1)/2}\binom{K}{2\ell}z_1^{K-2\ell}z_2^{2\ell}, \tag{14}$$

and

(15) $$p_2(z_1,z_2)^* = b \sum_{\ell=0}^{(K-1)/2} \binom{K}{2\ell+1} z_1^{K-2\ell-1} z_2^{2\ell+1}.$$

In addition we have

$$bp_1(z_1,z_2)^* + ap_2(z_1,z_2)^* = ab(z_1+z_2)^K,$$

for all cases.

The derivatives of p_1 and p_2 with respect to z_1 or z_2 yield homogeneous polynomials of less degree, but of exactly the same type, for instance, we get for K being even

$$p_1^{(10)*} = Ka \sum_{\ell=0}^{(K-2)/2} \binom{K-1}{2\ell} z_1^{K-2\ell-1} z_2^{2\ell},$$

which corresponds to (14).

Lemma 3.2. *Let p_1 and p_2 be like in (12) and (13), or like in (14) and (15). Then*

(16) $$p_2^{(01)*} p_1^{(01)} = p_1^{(10)*} p_2^{(10)} \quad , \quad p_2^{(10)*} p_1^{(10)} = p_1^{(01)*} p_2^{(01)}.$$

Proof. We have only to prove the first equality of (16), the second will then follow by interchanging the roles of p_1 and p_2. First we consider (12) and (13):

$$p_2^{(01)*} = Kb \sum_{\ell=0}^{(K-2)/2} \binom{K-1}{2\ell} z_1^{K-2\ell-1} z_2^{2\ell},$$

$$p_1^{(01)} = Ka \sum_{\ell=0}^{(K-2)/2} \binom{K-1}{2\ell+1} \frac{\partial^{K-1}}{\partial z_1^{K-2\ell-2} \partial z_2^{2\ell+1}};$$

and we have

$$p_1^{(10)*} = Ka \sum_{\ell=0}^{(K-2)/2} \binom{K-1}{2\ell} z_1^{K-2\ell-1} z_2^{2\ell},$$

$$p_2^{(10)} = Kb \sum_{\ell=0}^{(K-2)/2} \binom{K-1}{2\ell+1} \frac{\partial^{K-1}}{\partial z_1^{K-2\ell-2} \partial z_2^{2\ell+1}}.$$

Comparing the coefficients of the differential operators $p_2^{(01)*} p_1^{(01)}$ and $p_1^{(10)*}$ $p_2^{(10)}$ yields the desired result.

Next we take (14) and (15):

$$p_2^{(01)*} = Kb \sum_{\ell=0}^{(K-1)/2} \binom{K-1}{2\ell} z_1^{K-2\ell-1} z_2^{2\ell},$$

$$p_1^{(01)} = Ka \sum_{\ell=0}^{(K-3)/2} \binom{K-1}{2\ell+1} \frac{\partial^{K-1}}{\partial z_1^{K-2\ell-2} \partial z_2^{2\ell+1}};$$

and we have

$$p_1^{(10)^*} = Ka \sum_{\ell=0}^{(K-1)/2} \binom{K-1}{2\ell} z_1^{K-2\ell-1} z_2^{2\ell},$$

$$p_2^{(10)} = Kb \sum_{\ell=0}^{(K-3)/2} \binom{K-1}{2\ell+1} \frac{\partial^{K-1}}{\partial z_1^{K-2\ell-2} \partial z_2^{2\ell+1}}.$$

Finally, compare again the coefficients of the differential operators $p_2^{(01)*} p_1^{(01)}$ and $p_1^{(10)*} p_2^{(10)}$ to see that they coincide. □

Theorem 3.3. *Let p_1, p_2 be homogeneous polynomials of degree K as in (12), (13) or (14), (15). Then there exists a constant $C > 0$ such that*

$$\|u\|^2 \leq C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k), \tag{17}$$

for any $(1,0)$-form $u = \sum_{j=1}^2 u_j dz_j$ with polynomial components.

Proof. We will express the right hand side of (17) as a sum of squared norms. We use

$$([p_k, p_j^*]u_j, u_k) = \sum_{|\alpha|\geq 1} \frac{1}{\alpha!} (p_j^{(\alpha)*} p_k^{(\alpha)} u_j, u_k),$$

see [6]. By Lemma 3.2 we have

$$p_2^{(01)*} p_1^{(01)} = p_1^{(10)*} p_2^{(10)} \quad , \quad p_2^{(10)*} p_1^{(10)} = p_1^{(01)*} p_2^{(01)},$$

and hence

$$\begin{aligned} \sum_{j,k=1}^{2} [(p_j^{(10)*} p_k^{(10)} u_j, u_k) \quad &+ \quad (p_j^{(01)*} p_k^{(01)} u_j, u_k)] \\ &= \quad \|p_1^{(10)} u_1 + p_2^{(10)} u_2\|^2 + \|p_1^{(01)} u_1 + p_2^{(01)} u_2\|^2. \end{aligned}$$

In order to get the corresponding equations for the derivatives of order 2, we start with with the first order derivatives of p_1 and p_2 and observe that they are of the same type as the original polynomials, just of one degree lower. We have for the derivatives of order 2 that

$$\sum_{|\alpha|=2} \frac{1}{\alpha!}\,(p_j^{(\alpha)*}p_k^{(\alpha)}u_j,u_k) = \frac{1}{2}(p_j^{(20)*}p_k^{(20)}u_j,u_k) + (p_j^{(11)*}p_k^{(11)}u_j,u_k) + \frac{1}{2}(p_j^{(02)*}p_k^{(02)}u_j,u_k),$$

so we get

$$\frac{1}{2}\sum_{j,k=1}^{2}[(p_j^{(20)*}p_k^{(20)}u_j,u_k) + (p_j^{(11)*}p_k^{(11)}u_j,u_k)] = \frac{1}{2}\|p_1^{(20)}u_1+p_2^{(20)}u_2\|^2 + \frac{1}{2}\|p_1^{(11)}u_1+p_2^{(11)}u_2\|^2$$

and

$$\frac{1}{2}\sum_{j,k=1}^{2}[(p_j^{(11)*}p_k^{(11)}u_j,u_k) + (p_j^{(02)*}p_k^{(02)}u_j,u_k)] = \frac{1}{2}\|p_1^{(11)}u_1+p_2^{(11)}u_2\|^2 + \frac{1}{2}\|p_1^{(02)}u_1+p_2^{(02)}u_2\|^2.$$

For the derivatives of order m we have the following types of derivatives together with the corresponding factor in formula (8):

$$\frac{1}{m!}\,(m,0);\ \frac{1}{(m-1)!1!}\,(m-1,1);\ \frac{1}{(m-2)!2!}\,(m-2,2);\ldots$$

$$\ldots\frac{1}{1!(m-1)!}\,(1,m-1);\ \frac{1}{m!}\,(0,m).$$

We take the factor $\frac{1}{m!}$ for the derivatives of type $(m,0)$ and $(m-1,1)$:

$$\frac{1}{m!}\sum_{j,k=1}^{2}[(p_j^{(m,0)*}p_k^{(m,0)}u_j,u_k) + (p_j^{(m-1,1)*}p_k^{(m-1,1)}u_j,u_k)]$$

$$= \frac{1}{m!}\|p_1^{(m,0)}u_1+p_2^{(m,0)}u_2\|^2 + \frac{1}{m!}\|p_1^{(m-1,1)}u_1+p_2^{(m-1,1)}u_2\|^2.$$

So, for the type $(m-1,1)$, the factor $\frac{1}{(m-1)!} - \frac{1}{m!} = \frac{m-1}{m!}$ is left. We continue with what was left for type $(m-1,1)$ and get

$$\frac{m-1}{m!} \sum_{j,k=1}^{2} [(p_j^{(m-1,1)*} p_k^{(m-1,1)} u_j, u_k) + (p_j^{(m-2,2)*} p_k^{(m-2,2)} u_j, u_k)]$$
$$= \frac{m-1}{m!} \|p_1^{(m-1,1)} u_1 + p_2^{(m-1,1)} u_2\|^2 + \frac{m-1}{m!} \|p_1^{(m-2,2)} u_1 + p_2^{(m-2,2)} u_2\|^2.$$

Now the factor $\frac{1}{(m-2)!2!} - \frac{m-1}{m!} = \frac{(m-1)(m-2)}{m!2!}$ is left for the derivatives of type $(m-2,2)$. So after $\ell - 1$ steps, the factor

$$\frac{(m-1)(m-2)\dots(m-\ell+1)}{m!(\ell-1)!} \tag{18}$$

is left. Therefore we obtain for the factor in the next step

$$\frac{1}{(m-\ell)!\ell!} - \frac{(m-1)(m-2)\dots(m-\ell+1)}{m!(\ell-1)!} = \frac{(m-1)(m-2)\dots(m-\ell)}{m!\ell!},$$

which is of the same type as (18) for the derivatives of type $(m-\ell-1, \ell+1)$.

In this way we can proceed until the derivatives of order K and observe that for $|\alpha| = K$ one of the constants $p_1^{(\alpha)}$ and $p_2^{(\alpha)}$ is zero and the other positive. So we get $c_1\|u_1\|^2$ and $c_2\|u_2\|^2$, for $c_1, c_2 > 0$; all other terms are squared norms. $\square$

Remark 3.4. **a)** *If there exists a real constant $C \neq 0$ such that $p_2(z_1,z_2) = Cp_1(z_1,z_2)$, then we set $[p_1, p_1^*] = A$ and get for the $(1,0)$-forms $u = u_1 dz_1 + u_2 dz_2$, where $u_2 = -\frac{u_1}{C}$:*

$$\sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k) = (Au_1, u_1) - (Au_1, u_1) - (Au_1, u_1) + (Au_1, u_1) = 0.$$

So, (17) does not hold in this case.

b) *Let $p_1(z_1,z_2) = az_1^2 + bz_1z_2 + cz_2^2$ and $p_(z_1,z_2) = dz_1^2 + ez_1z_2 + fz_2^2$ be two homogeneous polynomials with real coefficients and suppose that condition (16)*

$$p_2^{(01)*} p_1^{(01)} = p_1^{(10)*} p_2^{(10)} \quad , \quad p_2^{(10)*} p_1^{(10)} = p_1^{(01)*} p_2^{(01)}.$$

holds. The first equation gives

$$(ez_1 + 2fz_2)\left(b\frac{\partial}{\partial z_1} + 2c\frac{\partial}{\partial z_2}\right) = (2az_1 + bz_2)\left(2d\frac{\partial}{\partial z_1} + e\frac{\partial}{\partial z_2}\right),$$

which, by a comparison of the coefficients, yields

$$be = 4ad, ce = ae, bf = bd, 4cf = be.$$

The second equation of (16) *gives no further information. If* $e \neq 0$ *and* $b \neq 0$, *we obtain* $p_1(z_1, z_2) = az_1^2 + bz_1z_2 + az_2^2$ *and* $p_2(z_1, z_2) = dz_1^2 + ez_1z_2 + dz_2^2$, *where* $4ad = be$. *If* $b = 0$, *but* $e \neq 0$, *we get* $a = c$. *So, if we want* p_1 *to be non-trivial, we have to suppose that* $a \neq 0$ *and we get* $f = d = 0$. *Hence, in this case:* $p_1(z_1, z_2) = az_1^2 + az_2^2$ *and* $p_2(z_1, z_2) = ez_1z_2$. *So, if we consider*

$$q_1(z_1, z_2) := z_1^2 + z_2^2 \text{ and } q_2(z_1, z_2) := z_1z_2$$

as basis polynomials for the solution of (16), *we can express an arbitrary nontrivial solution of* (16) *in the form*

$$\begin{pmatrix} \frac{\beta\delta}{4} & \beta \\ 1 & \delta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{\beta\delta}{4} q_1 + \beta q_2 \\ q_1 + \delta q_2 \end{pmatrix},$$

where $\left(\begin{smallmatrix} \frac{\beta\delta}{4} & \beta \\ 1 & \delta \end{smallmatrix}\right)$ *is a real invertible matrix. A primitive function with respect to* z_1 *of the basis polynomial* q_1 *is* $\frac{z_1^3}{3} + z_1z_2^2$ *and a primitive function with respect to* z_1 *of the basis polynomial* q_2 *is* $\frac{z_1^2}{2}z_2 + \frac{z_2^3}{6}$, *which corresponds to the polynomials* $z_1^3 + 3z_1z_2^2$ *and* $3z_1^2z_2 + z_2^3$, *which we consider in Theorem 3.3. Continuing this procedure one finally gets the polynomials* (12), (13) *and* (14), (15). *In this way it was possible to guess what kind of homogeneous polynomials of degree* K *can be chosen such that the basic estimate holds.*

Acknowledgements

The author thanks the referee for the careful reading, several useful suggestions and for pointing out an error in the first version.

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