Pure and Applied Mathematics Quarterly
Volume 18, Number 2, 617–637, 2022

q-effectiveness for holomorphic subelliptic multipliers

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Dedicated to Professor J.J. Kohn on the occasion of his birthday

Abstract: We provide a solution to the effectiveness problem in Kohn's algorithm for generating holomorphic subelliptic multipliers for $(0, q)$ forms for arbitrary q. As application, we obtain subelliptic estimates for $(0, q)$ forms with effectively controlled order $\varepsilon > 0$ (the Sobolev exponent) for domains given by sums of squares of holomorphic functions (J.J. Kohn called them "special domains" in [K79]). These domains are of particular interest due to their relation with complex and algebraic geometry. Our methods include *triangular resolutions* introduced by the authors in [KZ20].

1. Introduction

In his celebrated paper [K79], J.J. Kohn invented a purely algebraic approach to subelliptic estimates to the $\bar\partial$ problem, based on *generating multiplier ideals* that, quoting Y.-T. Siu [S17], "measure location and extent of failure of subelliptic estimates":

Definition 1.1. Let $\Omega \subset \mathbb{C}^n$ be a domain and $p \in \partial\Omega$ a boundary point.

1. [K79, Definition 1.11] A *subelliptic estimate* of order $\varepsilon > 0$ for $(0, q)$ forms is said to hold at p if there exist an open neighborhood U of p and $C > 0$ satisfying

$$\|u\|_\varepsilon^2 \le C(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \|u\|^2)$$

for all $(0, q)$ forms u with compact support in $U \cap \overline{\Omega}$ which belong to the domain of the adjoint $\bar\partial^*$. Here $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are respectively the tangential Sobolev norm of the (fractional) order ε and the standard L^2 norm on Ω.

Received May 6, 2021.
*This work was supported by the Institute for Basic Science (IBS-R032-D1-2021-a00).

2. [K79, Definition 4.2] A *subelliptic multiplier* of order $\varepsilon > 0$ at p for $(0, q)$ forms, called here briefly a "*q-multiplier*", is a germ f of a smooth function at p, for which there is a representative in a neighborhood U of p, also denoted by f and $C > 0$ satisfying

$$\|fu\|_\varepsilon^2 \le C(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \|u\|^2)$$

for all $(0, q)$ forms u as in the subelliptic estimate.

In particular, a subelliptic estimate of order $\varepsilon > 0$ holds at p if and only if $f = 1$ is a *q-multiplier* of order $\varepsilon > 0$ at p. Although multipliers are *defined* in terms of the above *a priori estimate*, J.J. Kohn discovered in [K79] purely algebraic procedures generating multipliers starting from the defining equation of $\partial\Omega$. Based on these procedures, J.J. Kohn proved for bounded domains with *real-analytic* boundary of finite D'Angelo type, that the trivial multiplier $f = 1$ can be generated by a finite sequence of these procedures. [K79, Theorem 1.19]

On the other hand, the more general case of *smooth boundary* remains open (also formulated by Y.-T. Siu [S17, §2]):

Conjecture 1.2 (Kohn's conjecture)**.** *For a bounded pseudoconvex domain with smooth boundary of finite type in* $\mathbb{C}^n$*, the trivial multiplier* $f = 1$ *can be generated by a finite sequence of Kohn's procedures.*

The stronger *Effective Kohn's conjecture* with the additional control of the order ε remains open even for real-analytic boundaries:

Conjecture 1.3 (Effective Kohn's conjecture)**.** *Kohn's conjecture holds under the same assumptions with an additional effective estimate of the* order of subellipticity *of the multiplier* $f = 1$ *as function of the finite type and the dimension* n*.*

The effective conjecture is known for $n = 2$, see [K79, §8], where it is based on fundamental results by Hörmander [Ho65] and Rothschild-Stein [RS76]. In higher dimension, the situation is much less understood, in fact, examples of [He08] (§1.1 in the preprint version) and [CD10, Proposition 4.4] in dimension 3 illustrate a lack of such control, see also [S17, §4.1] for a detailed explanation of this important phenomenon.

When $n > 2$, only the case $q = 1$ has been previously considered. To tackle the effectiveness, Siu [S10, S17] introduced algebraic geometric techniques to obtain the effectiveness in the important case of *special domains* (see Definition 1.7 below) of finite type in dimension 3, with further indications of how to proceed in the more general cases of special domains in higher dimension, and outlining a program to treat the more general real-analytic and smooth cases. A different effective procedure in Kohn's algorithm was given by D'Angelo [D95] and Catlin-D'Angelo [CD10, Section 5] for special domains given by so-called *triangular systems* of holomorphic functions. In [N14] A.C. Nicoara proposed a construction for the termination of the Kohn algorithm in the real-analytic case with an indication of the ingredients needed for the effectivity. More recently, the authors of this article established another effective procedure for special domains in dimension $n = 3$ [KZ18] and arbitrary n in [KZ20] (Y.-T. Siu also told us about his unpublished proof in this case).

The reader is referred to [D95, DK99, S01, S02, K04, S05, S07, Ch06, S09, CD10, S10, S17, Fa20] for more extensive details on subelliptic multipliers and Siu's accounts [S07, S09, S17] on their broad role and relation with other multipliers arising in complex and algebraic geometry. See also [K00, K02, K04, K05, Ce08, St08, CS09, Ba15, BPZ15, CZ17, S17] for multipliers in more general settings.

1.1. Main results

The goal of this note is to provide a solution to the effectiveness problem in Kohn's algorithm for holomorphic subelliptic multipliers for $(0, q)$ *forms for arbitrary* $1 \le q \le n$. We first recall *Kohn's multiplier generating procedures* for holomorphic multipliers [K79, §7] that can be described algebraically starting with an abstract *initial set* of germs:

Definition 1.4 (Holomorphic Kohn's procedures)**.** For an arbitrary initial subset S in the set $\mathcal{O}_{n,p}$ of holomorphic function germs in $\mathbb{C}^n$ at a point $p \in \mathbb{C}^n$ and integer $1 \le q \le n$, the *holomorphic Kohn's (q-)procedures* consist of:

(P1) for $0 < \varepsilon \le 1/2$ and $f_1, \dots, f_{n-q+1}$ either in S or multipliers of order $\ge \varepsilon$, it follows that the partial Jacobian $(n-q+1) \times (n-q+1)$ minors

$$\frac{\partial(f_1, \dots, f_{n-q+1})}{\partial(z_{j_1}, \dots, z_{j_{n-q+1}})}, \quad 1 \le j_1 < \dots < j_{n-q+1} \le n,$$

are multipliers of order $\ge \varepsilon/2$;

(P2) for $0 < \varepsilon < 1$, $k, r \in \mathbb{N}_{\ge 1}$, $f_1, \dots, f_k$ multipliers or order $\ge \varepsilon$, and g a holomorphic function (germ) with $g^r \in (f_1, \dots, f_k)$, it follows that g is a multiplier of order $\ge \varepsilon/r$.

Rather than directly using the finite type, we control the order of subellpticity of q-multipliers in terms of the *q-multiplicity* defined as follows:

Definition 1.5. The *q-multiplicity* $\mathrm{mult}_q(I)$ of an ideal $I \subset \mathcal{O}_{n,p}$ in the ring $\mathcal{O}_{n,p}$ of germs at p of holomorphic functions in $\mathbb{C}^n$ is the minimum of the dimension of the quotient space

$$\mathrm{mult}_q(I) := \min \dim \mathcal{O}_{n,p}/(I + (L_1, \ldots, L_{q-1})),$$

where the minimum is taken over all choice of $(q-1)$ affine linear functions $L_1, \ldots, L_{q-1}$ vanishing at p. By the *q-multiplicity* of a subset $S \subset \mathcal{O}_{n,p}$ we mean the q-multiplicity of the ideal generated by the set $\{f - f(p) : f \in S\}$.

Note that q-multiplicity is in fact a biholomorphic invariant (§2.1). We formulate our first result purely in terms of Kohn's procedures (P1) and (P2):

Theorem 1.6. *For every number $\nu > 1$ and initial subset $S \subset \mathcal{O}_{n,p}$ of finite q-multiplicity $\leq \nu$, there exists an effectively computable sequence $f_1, \ldots, f_m \in \mathcal{O}_{n,p}$, where $f_m = 1$ and each f_j is either in S or is obtained by applying to S or multipliers from $\{f_1, \ldots, f_{j-1}\}$ one of the Kohn's procedures (P1) or (P2). Furthermore, the number of steps and the root orders in (P2) are effectively bounded by functions depending only on (n, q, ν).*

As the first application, we obtain the effectiveness for the so-called special domains [K79, §7], [S17, §2.8]:

Definition 1.7. A *special domain* in $\mathbb{C}^{n+1}$ is one defined locally near each boundary point p by

$$\mathsf{Re}\,(z_{n+1}) + \sum_{j=1}^{N} |F_j(z_1, \ldots, z_n)|^2 < 0 \tag{1}$$

where $F_1, \ldots, F_N$ are holomorphic functions in a neighborhood of p. By the *q-multiplicity* of domain (1) at p we mean the q-multiplicity of the set $S = \{F_1, \ldots, F_N\}$.

As an immediate consequence of Kohn's theory and Theorem 1.6 applied to S as in Definition 1.7, we obtain:

Corollary 1.8. *There exists a positive function $\varepsilon\colon \mathbb{N}_{>0} \times \mathbb{N}_{>0} \times \mathbb{N}_{>0} \to \mathbb{R}_{>0}$ such that for integers $\nu, n, q \in \mathbb{N}_{>0}$ and any domain (1) of finite q-multiplicity $\leq \nu$ at a boundary point p, a subelliptic estimate for $(0, q)$ forms holds at p with effectively bounded order of subellipticity $\geq \varepsilon(n, q, \nu)$.*

Remark 1.9. Since the q-multiplicity of (1) is $\leq (T/2)^{n-q+1}$ where T is the D'Angelo *q-finite type* of (1) at p by a result of D'Angelo [D82, Theorem 2.7], an effective bound in terms of the type can be obtained by substituting $(T/2)^{n-q+1}$ for ν in Theorem 1.8. See also [BS92, BHR96, FIK96, FLZ14, MM17, D17, Fa19, Fa20, HY19, Z19] for relations of the finite type with other invariants.

1.2. Triangular resolutions and effective meta-procedures

In this section we introduce our main tools. Recall that the crucial lack of effectiveness in (P2) (see Definition 1.4) is due to the fact that the order of subellipticity of the generated multiplier depends on the root order that can be arbitrarily large in general.

To quantify this phenomenon, we call a procedure *effective* if the order of the new multiplier can be effectively estimated in terms of a quantity associated to the data that we call *a complexity*. We don't seek complexities of individual multipliers but rather of their *finite tuples and tuples of their ideals, or more precisely, their filtrations*. More specifically, we shall use the notion of triangular resolution that we defined in the earlier paper [KZ20]:

Definition 1.10 ([KZ20])**.** A *triangular resolution of length* $k \geq 1$ and multi-order $(\mu_1, \dots, \mu_k) \in \mathbb{N}^k$ of a pair $(\Gamma, \mathcal{I})$, where $\Gamma\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic map germ and $I_1 \subset \dots \subset I_k \subset \mathcal{O}_{n,0}$ a filtration $\mathcal{I}$ of ideals, is a system of holomorphic function germs $(h_1, \dots, h_k)$ satisfying

$$h_j = h_j(w_j, \dots, w_n), \quad h_j \circ \Gamma \in I_j, \quad \operatorname{ord}_{w_j} h_j = \mu_j, \quad 1 \leq j \leq k.$$

Our proof of the results from previous section is based on the following effective meta-procedures involving triangular resolutions:

Theorem 1.11. *For integers $n \geq 1$, $1 \leq q \leq n$, $0 \leq k \leq n-q$ and $\mu \geq 1$, the following hold:*

(MP1) (Selection of a partial Jacobian). For any collections of germs

$$f = (f_1, \dots, f_k) \in (\mathcal{O}_{n,0})^k, \quad \psi = (\psi_{k+1}, \dots, \psi_{n-q+1}) \in (\mathcal{O}_{n,0})^{n-k-q+1},$$
$$\operatorname{mult}_q(f, \psi) \leq \mu,$$

there exist linear changes of the coordinates $z \in \mathbb{C}^n$ and of the components of ψ in $\mathbb{C}^{n-k-q+1}$ such that for the partial Jacobian determinant

$$J := \frac{\partial(\psi_{k+1}, \dots, \psi_{n-q+1})}{\partial(z_{k+1}, \dots, z_{n-q+1})},$$

the q-multiplicity

$$\mathrm{mult}_{\,q}(f, J, \psi_{k+2}, \ldots, \psi_{n-q+1})$$

is effectively bounded by a function depending only on (n, q, μ).

(MP2) *(Selection of a triangular resolution). For any collections of germs*

$$f = (f_1, \ldots, f_k) \in (\mathcal{O}_{n,0})^k, \quad \psi = (\psi_1, \ldots, \psi_{n-q+1}) \in (\mathcal{O}_{n,0})^{n-q+1},$$

with

$$\mathrm{mult}_{\,q}(f_1, \ldots, f_j, \psi_{j+1}, \ldots, \psi_{n-q+1}) \leq \mu \quad \text{for all } \ 0 \leq j \leq k,$$

there exist a germ of a holomorphic map

$$\Gamma_\psi := (\psi, L_{n-q+2}, \ldots, L_n) \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), L_j \text{ are linear functions},$$

such that

$$\mathrm{mult}_{\,q}(\psi) = \mathrm{mult}\,(\Gamma_\psi)$$

and a triangular resolution $h = (h_1, \ldots, h_k)$ *of* $(\Gamma_\psi, \mathcal{I})$, *where* $\mathcal{I}$ *is the filtration*

$$(f_1) \subset (f_1, f_2) \ldots \subset (f_1, \ldots, f_k),$$

such that orders $\mathrm{ord}_{\,w_j} h_j$ *are effectively bounded by functions depending only on* (n, q, μ).

(MP3) *(Jacobian extension in a triangular resolution). For any*

$$\Gamma = (\phi, \psi, z_{n-q+2}, \ldots, z_n), \quad (\phi, \psi) \in (\mathcal{O}_{n,0})^k \times (\mathcal{O}_{n,0})^{n-k-q+1}$$

and filtration $\mathcal{I}$ *of ideals* $I_1 \subset \ldots \subset I_{k+1} \subset \mathcal{O}_{n,0}$ *satisfying*

$$I_{k+1} \subset I_k + (J),$$

where J *is the Jacobian determinant of* Γ, *let* $h = (h_1, \ldots, h_{k+1})$ *be a triangular resolution with*

$$\mathrm{ord}_{\,z_j} h_j \leq \mu, \quad 1 \leq j \leq k.$$

Then $h_{k+1} \circ \Gamma$ *can be obtained by holomorphic Kohn's procedures (P1) and (P2) starting with the initial set consisting of components of* ψ *and the ideal* I_k, *where the number of procedures and the root order in (P2) are effectively bounded by a function depending only on* (n, q, μ).

The proof for each of the statements in (MP1), (MP2) and (MP3) will be provided respectively in §3 and Propositions 4.1 and 5.1. All three meta-procedures will be subsequently used one after another in §6 to prove the following explicit description of q-multipliers arising from our algorithm:

Corollary 1.12. *For integers $n, q, \nu \geq 1$, initial system $\psi_0 = (\psi_{0,1}, \dots, \psi_{0,n-q+1})$ of q-multiplicity $\leq \nu$, and $1 \leq k \leq n-q+1$, there exist:*

1. *holomorphic coordinates $(z_1, \dots, z_n)$ chosen among linear combinations of any given holomorphic coordinate system;*
2. *systems $\psi_k = (\psi_{k,k+1}, \dots, \psi_{k,n-q+1})$ chosen among generic linear combination of ψ_0, and associated maps*

$$\Gamma_k(z) := (z_1, \dots, z_k, \psi_{k,k+1}(z), \dots, \psi_{k,n-q+1}(z), z_{n-q+2}, \dots, z_n);$$

3. *systems of multipliers $f_k = (f_{k,1}, \dots, f_{k,k})$ obtained via effective meta-procedures applied to (ψ_{k-1}, f_{k-1}) (where f_0 is empty);*
4. *integer functions $\nu_{k,j}(n, q, \nu) > 0$ and decompositions of the form $f_{k,j} = Q_{k,j} \circ \Gamma_k$, $j = 1, \dots, k$, where each $Q_{k,j} = Q_{k,j}(w_j, \dots, w_n)$ is a holomorphic function depending only on the last $n-j+1$ coordinates with* $\operatorname{ord}_{w_j} Q_{k,j} \leq \nu_{k,j}(n, q, \nu)$;
5. *positive functions $\varepsilon_{k,j}(n, q, \nu) > 0$ such that the order of subellipticity of each $f_{k,j}$ for $(0, q)$ forms is $\geq \varepsilon_{k,j}(n, q, \nu)$.*

Using Corollary 1.12 for the largest k, we prove Theorem 1.6 by applying the meta-procedure (MP3) from Theorem 1.11 one last time:

Proof of Theorem 1.6. Taking $k = n - q + 1$ in Corollary 1.12, we find

$$\Gamma_k(z) = (z_1, \dots, z_k, z_{k+1}, \dots, z_n) = z,$$

whose Jacobian determinant $J = 1$. Hence, using $f_{k,j} = Q_{k,j} \circ \Gamma_k$ provided by Corollary 1.12, we can apply (MP3) from Theorem 1.11 to

$$\Gamma = \Gamma_k, \quad I_j = (f_{k,1}, \dots, f_{k,j}) \text{ for } j \leq k, \quad I_{k+1} = 1 \subset I_k + (J),$$

and

$$h_j = Q_j \text{ for } j \leq k, \quad h_{k+1} = 1,$$

to conclude that $h_{k+1} \circ \Gamma = 1$ is a q-multiplier, completing the proof. □

2. Preliminaries

2.1. Multiplicity and degree

Denote by $\mathcal{O} = \mathcal{O}_{n,p}$ the ring of germs at a point p of holomorphic functions in $\mathbb{C}^n$. Since our considerations are for germs at a fixed point, we shall assume $p = 0$ unless specified otherwise.

Recall that an ideal $\mathcal{I} \subset \mathcal{O}$ is of *finite type* if $\dim \mathcal{O}/\mathcal{I} < \infty$, or equivalently the (germ at 0 of the) zero variety $\mathcal{V}(\mathcal{I})$ is zero-dimensional at 0. In the latter case, the classical *algebraic intersection multiplicity* of $\mathcal{I}$ (see e.g. [Fu84, §1.6, §2.4]) is defined as

$$\operatorname{mult} \mathcal{I} := \dim \mathcal{O}/\mathcal{I}. \tag{2}$$

Similarly, for a germ of holomorphic map $\psi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, we have $\operatorname{mult} \psi := \operatorname{mult} (\psi)$, where (ψ) is the ideal generated by the components of ψ, and the quotient $\mathcal{O}/(\psi)$ is the *local algebra of* ψ (see e.g. [AGV85]). More generally (cf. [D93, §2.4]), for every integer $0 < q \leq n$, define the q-*multiplicity* by

$$\operatorname{mult}_q \mathcal{I} := \min \dim \mathcal{O}/(\mathcal{I} + (L_1, \dots, L_{q-1})), \tag{3}$$

where the minimum is taken over sets of $q - 1$ linear functions L_j on $\mathbb{C}^n$. The same minimum is achieved when L_j are germs of holomorphic functions with linearly independent differentials, as can be easily shown by a change of coordinates linearizing the functions. In particular, the q-multiplicity of an ideal is a *biholomorphic invariant.* In a similar vein, given a collection $\phi = (\phi_1, \dots, \phi_{n-d}) \in \mathcal{O}_n$ of $n - d$ function germs vanishing at 0, we write

$$\operatorname{mult} (\phi) = \operatorname{mult} (\phi_1, \dots, \phi_{n-d}) := \min \dim \mathcal{O}/(\phi_1, \dots, \phi_{n-d}, L_1, \dots, L_d), \tag{4}$$

where L_j are as above. That is, we will adopt the following convention:

Convention. For every $1 \leq k \leq n$ and a k-tuple of holomorphic function germs $\phi_1, \dots, \phi_k$, their multiplicity $\operatorname{mult} (\phi_1, \dots, \phi_k)$ is always assumed to be the $(n - k + 1)$-multiplicity, i.e. with $(n - k)$ generic linear functions added to the ideal.

Further recall that the *degree* $\deg(\psi)$ of a germ (also called "index" in [AGV85]) of a finite holomorphic map $\psi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is the minimum m such that ψ restricts to a ramified m-sheeted covering between neighborhoods of 0 in $\mathbb{C}^n$. Both integers are known to coincide (see e.g. [ELT77, AGV85, D93]):

Theorem 2.1 ([AGV85, §4.3])**.** *Let* $\psi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ *be germ of finite holomorphic map. Then*

$$\operatorname{mult}(\psi)=\deg\psi.$$

2.2. Basic properties of multiplicity

The proofs of the following lemmas can be found in [S10] and [KZ20].

Lemma 2.2 (Semicontinuity of multiplicity)**.** *Let* $\psi_t\colon(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$ *be a continuous family of germs of holomorphic maps, in the sense that all coefficients of the power series expansion of* ψ_t *depend continuously on* $t\in\mathbb{R}^m$*. Then* $\operatorname{mult}(\psi_t)$ *is upper semicontinuous in* t*.*

In the following we keep using the notation (4).

Corollary 2.3. *For every germs*

$$(f,g)\colon(\mathbb{C}^{n+m},0)\to(\mathbb{C}^n,0)\times(\mathbb{C}^m,0),$$

we have

$$\operatorname{mult} f\le\operatorname{mult}(f,g).$$

Definition 2.4 (D'Angelo, [D82])**.** Let $S\subset\mathcal{O}_{n,0}$ be a subset of germs of holomorphic functions.

1. The D'Angelo 1-type of S is

   $$\Delta^1(S):=\sup_{\gamma}\inf_{f\in S}\frac{\operatorname{ord} f\circ\gamma}{\operatorname{ord}\gamma},$$

   where ord denotes the vanishing order, and the supremum is taken over all nonzero germs of holomorphic maps $\gamma\colon(\mathbb{C},0)\to(\mathbb{C}^n,0)$.
2. the D'Angelo q-type of S for $q\ge 1$ is

   $$\Delta^q(S):=\inf_{L}\ \Delta^1(S\cup L),$$

   where the infimum is taken over all sets L of $(q-1)$ complex linear functions.

Let Ω be a domain defined locally by

$$\mathsf{Re}\,(z_{n+1})+\sum_{j=1}^{N}|F_j(z_1,\dots,z_n)|^2<0,\tag{5}$$

where $F_1, \dots, F_N$ are holomorphic functions in a neighborhood of 0. By *q-type* $\Delta^q(\Omega)$ of (5) at 0 we mean twice the q-type of $F_1, \dots, F_N$. Let p be the smallest integer such that

$$|z|^p \leq A \sum_j |F_j| + |L(z)|$$

holds for some linear map $L : \mathbb{C}^n \to \mathbb{C}^{q-1}$. By following the argument in (I.2) of [S10], we can show that

$$2p = \Delta^q(\Omega) \leq \operatorname{mult}_q \mathcal{I}(F_1, \dots, F_N) =: s.$$

Furthermore, for the smallest integer r satisfying

$$\mathfrak{m}^r \subset \mathcal{I}(F_1, \dots, F_N, L_1, \dots, L_{q-1})$$

for some linear functions $L_1, \dots, L_{q-1}$, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{n,0}$, we obtain

$$r \leq s \leq (n + r - 1)!/n!(r - 1)!.$$

An important ingredient is the following consequence from Siu's lemma on selection of linear combinations of holomorphic functions for effective multiplicity [S10, (III.3)] combined with effective comparison of the invariants of holomorphic map germs [S10, (I.3-4)] (see also [D93, §2.2]):

Lemma 2.5 (q-type version of Siu's lemma on effective mixed multiplicity)**.** *Let $0 \leq j \leq n - q$ and $f_1, \dots, f_j, F_1, \dots, F_N$ be holomorphic function germs in $\mathcal{O}_{n,0}$ such that*

$$\mu := \operatorname{mult}(f_1, \dots, f_j) < \infty, \quad \nu := \operatorname{mult}_q(F_1, \dots, F_N) < \infty.$$

Then

$$\operatorname{mult}(f_1, \dots, f_j, G_{j+1}, \dots, G_\ell) \leq \mu\nu^{\ell - j}, \quad \ell \leq n - q + 1$$

holds for generic linear combinations $G_{j+1}, \dots, G_\ell$ of F_k's.

We shall also need the following lemma proved in [KZ20, §3]:

Lemma 2.6 (Effective Nullstellensatz, [KZ20])**.** *Let $\phi_1, \dots, \phi_k, f \in \mathcal{O}_{n,0}$ satisfy*

$$\mu := \operatorname{mult}(\phi_1, \dots, \phi_k) < \infty, \quad f \in \sqrt{(\phi_1, \dots, \phi_k)}.$$

Then

$$f^{n\mu} \in (\phi_1, \dots, \phi_k).$$

3. Multiplicity estimates for Jacobian determinants

The meta-procedure (MP1) in Theorem 1.11 is the special case of the following proposition where one can put $d = n - q - k$ and $\psi = (\psi_{k+1}, \dots, \psi_{n-q+1}, z_{n-q+2}, \dots, z_n)$ assuming that

$$\begin{aligned} &\operatorname{mult}_q(f_1, \dots, f_k, \psi_{k+1}, \dots, \psi_{n-q+1}) \\ &= \operatorname{mult}(f_1, \dots, f_k, \psi_{k+1}, \dots, \psi_{n-q+1}, z_{n-q+2}, \dots, z_n) < \infty \end{aligned}$$

after a suitable linear coordinate change of $(z_1, \dots, z_n)$. The proof of the proposition is given in [KZ20]. In what follows, we use the convention that $\operatorname{mult}(f) = 1$ if f has 0 components.

Proposition 3.1 ([KZ20])**.** *Let $(f, \psi)\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d} \times \mathbb{C}^d, 0)$, $1 \le d \le n$, be a holomorphic map germ satisfying*

$$\nu := \operatorname{mult}(f) < \infty, \quad \mu := \operatorname{mult}(f, \psi) < \infty.$$

Then after a linear change of coordinates $(z_1, \dots, z_n)$ and another linear coordinate change in $\mathbb{C}^d$, the partial Jacobian determinant

$$J := \frac{\partial(\psi_1, \dots, \psi_d)}{\partial(z_1, \dots, z_d)} \tag{6}$$

satisfies

$$\operatorname{mult}(f, J) \le d\nu\mu, \quad \operatorname{mult}(f, J, \psi_2, \dots, \psi_d) \le d\nu\mu^d, \tag{7}$$

where ψ_j is the j-th component of ψ in the new coordinates.

4. Existence of effective triangular resolutions

The following is a more precise version of the meta-procedure (MP2) in Theorem 1.11. We shall denote by $\operatorname{ord}_{w_j} h$ the vanishing order at 0 of $h(0, \dots, 0, w_j, 0, \dots, 0)$ (where all variables are zero except w_j).

Proposition 4.1. *Let $1 \le k \le n - q + 1$, $f_1, \dots, f_k, \phi_1, \dots, \phi_{n-q+1} \in \mathcal{O}_{n,0}$, satisfy*

$$\mu_j := \operatorname{mult}_q(f_1, \dots, f_j, \phi_{j+1}, \dots, \phi_{n-q+1}) < \infty, \quad 1 \le j \le k.$$

Let $\mathcal{I}$ be the filtration of ideals

$$I_j := (f_1, \dots, f_j),\ 1 \le j \le k.$$

Then there exist a germ of a holomorphic map

$$\Gamma_\phi = (\phi_1, \dots, \phi_{n-q+1}, L_{n-q+2}, \dots, L_n)$$

and a triangular resolution $h = (h_1, \dots, h_k)$ of $(\Gamma_\phi, \mathcal{I})$ such that

$$\operatorname{ord}_{w_j} h_j \le n \cdot \mu_j \cdot \operatorname{mult}(f_1, \dots, f_j), \quad 1 \le j \le k. \tag{8}$$

Furthermore, each $h_j(w_j, \dots, w_n)$ can be chosen as Weierstrass polynomial in w_j.

Proof. Since

$$\operatorname{mult}(\psi_1, \dots, \psi_k) = \operatorname{mult}(\psi_1, \dots, \psi_k, L_{k+1}, \dots, L_n),$$

for generic choice of $n-k$ linear functions L_j, we can choose a set of linear functions $L_{n-q+2}, \dots, L_n$ such that

$$\operatorname{mult}(f_1, \dots, f_j, \phi_{j+1}, \dots, \phi_{n-q+1}, L_{n-q+2}, \dots, L_n) = \mu_j, \quad \text{for all } j. \tag{9}$$

Let

$$\Gamma_\phi := (\phi_1, \dots, \phi_{n-q+1}, L_{n-q+2}, \dots, L_n).$$

Consider the coordinate projections

$$\pi_j(w_1, \dots, w_n) = (w_j, \dots, w_n) \in \mathbb{C}^{n-j+1}, \quad 1 \le j \le k,$$

and let

$$W_j := \mathcal{V}(f_1, \dots, f_j), \quad \widetilde{W}_j := (\pi_j \circ \Gamma_\phi)(W_j) \subset \mathbb{C}^{n-j+1}, \quad 1 \le j \le k,$$

where $\mathcal{V}$ is the zero variety. Then W_j is of codimension $\ge k$ in $\mathbb{C}^n$. In fact, counting preimages and using (9), we conclude that $\widetilde{W}_j \subset \mathbb{C}^{n-j+1}$ is a proper subvariety of codimension 1 and

$$\pi_{j+1}|_{\widetilde{W}_j} : \widetilde{W}_j \to \mathbb{C}^{n-j}$$

is a finite holomorphic map germ of degree $\le \mu_j$. Then there exist Weierstrass polynomials $Q_j(w_j, \dots, w_n)$, $j = 1, \dots, k$, satisfying

$$Q_j = w_j^{\nu_j} + \sum_{\ell<\nu_j} b_{j,\ell}(w_{j+1}, \dots, w_n) w_j^{\ell}, \quad Q_j|_{\widetilde{W}_j} = 0, \quad \nu_j = \operatorname{ord}_{w_j} Q_j \le \mu_j.$$

Furthermore, Lemma 2.6 implies

$$h_j \circ \Gamma_\phi \in (f_1, \dots, f_j), \quad h_j := Q_j^{\lambda_j},$$

for suitable $\lambda_j \in \mathbb{N}_{\ge 1}$ satisfying

$$\lambda_j \le n \cdot \operatorname{mult}(f_1, \dots, f_j).$$

Then $(h_1, \dots, h_k)$ is a triangular resolution satisfying (8) as desired. □

5. Effective Kohn's procedures for triangular resolutions

The following is a more precise version of the meta-procedure (MP3) in Theorem 1.11:

Proposition 5.1. *Let $1 \le k \le n-q$ and $(Q_1 \circ \Gamma, \dots, Q_{k+1} \circ \Gamma)$ be a triangular resolution of $(\Gamma, \mathcal{I})$, where $\Gamma\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic map germ of the form*

$$\Gamma = (\psi_1, \dots, \psi_{n-q+1}, z_{n-q+2}, \dots, z_n), \tag{10}$$

and $\mathcal{I}$ a filtration of ideals $I_1 \subset \dots \subset I_{k+1} \subset \mathcal{O}_{n,0}$. Assume

$$\mu_j = \operatorname{ord}_{w_j} Q_j < \infty, \quad 1 \le j \le k, \tag{11}$$

and

$$I_{k+1} \subset I_k + (J), \tag{12}$$

where J is the Jacobian determinant of Γ.

Then $Q_{k+1} \circ \Gamma$ can be obtained by applying holomorphic Kohn's procedures (P1) and (P2) to (Γ, I_k) and each procedure (P1) and (P2) is applied $\mu_1 \cdots \mu_k$ number of times with the root order in (P2) being $\le k+1$. In particular, if I_k consists of multipliers of order $\ge \varepsilon$, then $Q_{k+1} \circ \Gamma$ is a multiplier of order $\ge (2k+2)^{-\mu_1 \cdots \mu_k} \varepsilon$.

Proof. Since $\mu_j < \infty$ for $j \leq k$, multiplying by invertible holomorphic functions, we may assume that

$$Q_j = w_j^{\mu_j} + \sum_{\ell<\mu_j} b_{j,\ell}(w_{j+1}, \ldots, w_n) w_j^{\ell}, \quad 1 \leq j \leq k,$$

are Weierstrass polynomials satisfying

$$f_j := Q_j \circ \Gamma \in I_j.$$

In addition, (12) implies

$$f_{k+1} := Q_{k+1} \circ \Gamma \in I_k + (J). \tag{13}$$

For simplicity of notation, we use the remaining indices to denote the coordinate functions in (10), i.e.

$$\Gamma = (\psi_1, \ldots, \psi_{n-q+1}, z_{n-q+2}, \ldots, z_n) = (\psi_1, \ldots, \psi_{n-q+1}, \psi_{n-q+2}, \ldots, \psi_n).$$

Since

$$(f_1, \ldots, f_k, \psi_{k+1}, \ldots, \psi_n) = \Phi \circ \Gamma,$$

where

$$\Phi(w) := (Q_1(w), \ldots, Q_k(w), w_{k+1}, \ldots, w_n), \tag{14}$$

we obtain the Jacobian determinants

$$\frac{\partial(f_1, \ldots, f_k, \psi_{k+1}, \ldots, \psi_{n-q+1})}{\partial(z_1, \ldots, z_{n-q+1})} = \frac{\partial(f_1, \ldots, f_k, \psi_{k+1}, \ldots, \psi_n)}{\partial(z_1, \ldots, z_n)} =: J_{(1,\ldots,1)}.$$

For $L = (\ell_1, \ldots, \ell_k) \in \mathbb{N}^k$, define $A_L \in \mathcal{O}_{n,0}$ by

$$A_L(w) := \partial_{w_1}^{\ell_1} Q_1(w) \cdots \partial_{w_k}^{\ell_k} Q_k(w).$$

Then the Jacobian factors as

$$J_{(1,\ldots,1)} = (A_{(1,\ldots,1)} \circ \Gamma) J$$

and hence by (13),

$$(A_{(1,\ldots,1)} \circ \Gamma) f_{k+1} \in I_k + (J_{(1,\ldots,1)})$$

is obtained by applying Kohn's procedure (P1) to I_k.

Now for $B = (B_1, \dots, B_k)$, where

$$B_j := (A_{L_j} \circ \Gamma) f_{k+1} = (A_{L_j} Q_{k+1}) \circ \Gamma$$

or

$$B_j := Q_j \circ \Gamma = f_j$$

with each B_j in both cases are obtained by applying Kohn's procedures, we obtain

$$\frac{\partial(B_1, \dots, B_k, \psi_{k+1}, \dots, \psi_{n-q+1})}{\partial(z_1, \dots, z_{n-q+1})} = \frac{\partial(B_1, \dots, B_k, \psi_{k+1}, \dots, \psi_n)}{\partial(z_1, \dots, z_n)} =: J_L.$$

In view of our assumption that each Q_j, $j \le k$, is a Weierstrass polynomial in w_j of degree μ_j, each top derivative $\partial_{w_j}^{\mu_j} Q_j$ is constant and hence B_j only depends on $(w_j, \dots, w_n)$. Then using factorization of the Jacobian determinant and the triangular property of B_j's, we obtain

$$J_L = c \Big(((\partial_{w_1}^{\ell_1} Q_1)^{m_1} \cdots (\partial_{w_k}^{\ell_k} Q_k)^{m_k} Q_{k+1}^{m_{k+1}}) \circ \Gamma \Big) J$$

for some constant $c \neq 0$ and integers m_j, $j = 1, \dots, k+1$ and hence by (13),

$$((A_L Q_{k+1}) \circ \Gamma)^{m_{k+1}+1} \in I_k + (J_L).$$

Then $(A_L Q_{k+1}) \circ \Gamma$ is obtained by the Kohn's procedure (P2) with root order $\le m_{k+1} + 1$, and by using the lexicographic order for $L = (\ell_1, \dots, \ell_k)$ as in the proof [KZ20], we can complete the proof. $\square$

6. Proof of Corollary 1.12

We will use the induction on k. For the case $k = 1$, take $\psi_0 = (\psi_{0,1}, \dots, \psi_{0,n-q+1})$ to be a (generic) linear combinations of F_j's such that $\operatorname{mult}_q(\psi_0)$ is effectively bounded and assume that

$$\operatorname{mult}_q(\psi_0) = \operatorname{mult}(\psi_0, z_{n-q+2}, \dots, z_n)$$

after a linear coordinate change of $\mathbb{C}^n$. Such ψ_0 exists by Lemma 2.5.

Now suppose that the statement of the corollary holds for $k - 1$. Applying Lemma 2.5 and (MP1), we obtain (generic) linear combinations $\psi_k =$

$(\psi_{k,k+1},\ldots,\psi_{k,n-q+1})$ of $\psi_{k-1,j}$'s such that $\operatorname{mult}_q(z_1,\ldots,z_k,\psi_k)$ and $\operatorname{mult}_q(f_{k-1},J,\psi_k)$ are effectively bounded, where

$$J := \frac{\partial(\psi_{k-1,k},\ldots,\psi_{k-1,n-q+1})}{\partial(z_k,\ldots,z_{n-q+1})}.$$

Next apply (MP2) for the map germ

$$\Gamma(z) := (z_1,\ldots,z_{k-1},\psi_{k-1}(z),z_{n-q+2},\ldots,z_n)$$

and the filtration of ideals

$$I_j := (f_{k-1,1},\ldots,f_{k-1,j}),\ 1\le j\le k-1,$$

to obtain a triangular resolution $(h_1,\ldots,h_{k-1})$ such that $\operatorname{ord}_{w_j} h_j$ is effectively bounded.

Finally, apply (MP3) for

$$(\phi,\psi) = (z_1,\ldots,z_{k-1},\psi_{k-1})$$

and a filtration $\mathcal{I}$ of ideals

$$\tilde I_j = (h_1\circ\Gamma,\ldots,h_j\circ\Gamma),\quad j=1,\ldots,k-1$$

and

$$\tilde I_k = \tilde I_{k-1} + (\tilde J) \subset \tilde I_{k-1} + (J),$$

where

$$\tilde J = \frac{\partial((h_1\circ\Gamma),\ldots,(h_{k-1}\circ\Gamma),\psi_{k-1,k},\ldots,\psi_{k-1,n-q+1})}{\partial(z_1,\ldots,z_{n-q+1})}.$$

Then we obtain a new set of multipliers

$$f_k = (f_{k,1},\ldots,f_{k,k})$$

given by the triangular resolution $(h_1,\ldots,h_k)$ of $(\Gamma,\mathcal{I})$ together with a set of premultipliers

$$\psi_k = (\psi_{k,k+1},\ldots,\psi_{k,n-q+1})$$

that satisfy the condition of the corollary, completing the proof.

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