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Symmetry algebras of polynomial models in complex dimension three.

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Dedicated to Professor J.J. Kohn on the occasion of his 90th birthday

Abstract: We consider the Lie algebra of infinitesimal CR automorphisms of a real hypersurface at a point of Levi degeneracy. As a main result, we give a complete classification of symmetry algebras of dimension at least six for polynomial models of finite Catlin multitype in $\mathbb{C}^3$. As a consequence, this also provides understanding of "exotic" higher order symmetries, which violate 2-jet determination.
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1. Introduction

The Lie algebra of infinitesimal CR automorphisms $\mathrm{aut}(M, p)$ is a fundamental local holomorphic invariant of a real hypersurface $M \subseteq \mathbb{C}^N$ at a point $p \in M$. For suitably invariantly defined polynomial models, this algebra is in one to one correspondence with the kernel of the generalized Chern-Moser operator, as defined in [12]. A classification of such algebras is thus an essential step before addressing the local equivalence problem by the normal form approach ([5, 15, 16, 17]).

Since the work of Poincaré ([19]), the study of invariants and symmetries of Levi nondegenerate hypersurfaces played fundamental role in the development of CR geometry. In his pioneering work [9], J. J. Kohn started to investigate the case of Levi degenerate manifolds – weakly pseudoconvex boundaries of domains. He introduced an integer valued higher order invariant at points where the Levi form vanishes – the type of the point, measuring the maximal order of contact between complex curves and the given manifold ([6, 9]).

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The subsequent works of Kohn, D'Angelo, Catlin and others proved that the existence, or nonexistence, of complex varieties in the boundary is reflected in a fundamental way in the analytic properties of the $\bar{\partial}$ operator. By Kohn's ideas, complex algebraic varieties in $\mathbb{C}^n$ can be realized as subsets of smooth weakly pseudoconvex CR manifolds in $\mathbb{C}^{n+1}$ and the invariants of these smooth manifolds thus carry information about the original varieties.

Given the importance of weakly pseudoconvex manifolds in complex analysis, it became an important problem to find a local biholomorphic classification of such manifolds.

As a serious obstruction, one cannot hope to extend the differential geometric approach of Cartan, Chern, Tanaka ([3, 5, 20]), since the structure is not uniform anymore, the rank of the Levi form could change from point to point.

The first complete normal form in the Levi degenerate case was obtained in [15], which applies the normal form approach to the class of hypersurfaces of finite type in $\mathbb{C}^2$. Combining this result with a convergence result of Baouendi-Ebenfelt-Rothschild [1] solves the Poincaré local equivalence problem for this class.

Later, Kolář, Meylan and Zaitsev ([12]) generalized the Chern-Moser theory to the Levi degenerate case in arbitrary dimension. Instead of quadratic models from the Chern-Moser case, it permits considering general, invariantly defined, polynomial models.

An essential inevitable step in extending the Chern-Moser theory in the Levi degenerate setting is to classify the appropriate polynomial models according to the form of the Lie algebra of infinitesimal automorphisms. The normal form construction then has to be carried out for each of the possible cases separately.

In dimension two, such a classification is rather simple. There are three different cases, two exceptional ones and one generic. The two exceptional ones are the circular model $\{\mathsf{Im}\, w = |z|^{2k}\}$ with a 4-dimensional symmetry algebra, and the tubular model $\{\mathsf{Im}\, w = (\mathsf{Re}\, z)^k\}$, which has a three dimensional symmetry algebra. All other models fall into the generic case, which admits a two dimensional algebra of infinitesimal symmetries.

Our aim in this paper is to establish such a classification in complex dimension three. We completely classify polynomial models of finite Catlin multitype according to the type of their symmetry algebra of dimension at least six.

The crucial starting point is the structure result obtained in [12], which shows that hypersurfaces of finite Catlin multitype provide a natural class of manifolds for which a generalization of the Chern-Moser operator is well defined. Using this operator, it is proven that the Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \mathrm{aut}(M_P, 0)$ admits the weighted grading given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\eta \in E} \mathfrak{g}_\eta \oplus \mathfrak{g}_1,$$

where E is the set of integer combinations of the multitype weights, which lie between zero and one (see Section 2 for notation and definitions). As a consequence, it is proven that the automorphisms of M at p are uniquely determined by their *weighted 2-jets*.

A new phenomenon here is the existence of a component that we shall call $\mathfrak{g}_c$ of nonlinear vector fields with coefficients independent of w, which has no analog in the nondegenerate case. Our first result gives a description of manifolds with nontrivial $\mathfrak{g}_c$ in our setting. Note that in complex dimension three, the dimension of $\mathfrak{g}_c$ is at most one ([14]).

In the following, we will assume that M_P is a holomorphically nondegenerate model of finite Catlin multitype.

Theorem 1.1. *Let* $\dim \mathfrak{g}_c = 1$ *and* $\dim \mathfrak{g} \geq 6$*. Then* M_P *is biholomorphically equivalent to the polynomial model*

$$\mathsf{Im}\, w = |z_1|^{2k} |z_2|^{2l} (\mathsf{Re}\, z_1^\alpha z_2^\beta)^m$$

for some nonnegative integers k, l, m *and a pair of integers* (α, β)*, such that* $\alpha \geq -1, \beta \geq -1$.

Further we give a complete classification according to dimension up to dimension 6. It is known ([14]) that there exist three kinds of models with

$$\dim \mathfrak{g} \geq 9.$$

The Levi nondegenerate models, hyperquadrics of the two possible signatures, possess a 15-dimensional symmetry algebra. Dimension 10 is realized for the model

$$\mathsf{Im}\, w = \mathsf{Re}\, z_1 \bar{z}_2^l, \tag{1}$$

and dimension 9 is realized for

$$\mathsf{Im}\, w = |z_1|^2 \pm |z_2|^{2l} \tag{2}$$

for some $l > 1$. Moreover, it was proved in [14] that there is a “secondary” gap in dimension eight. The following result describes all models with a seven dimensional symmetry algebra.

Theorem 1.2. *Let* $\dim \mathfrak{g} = 7$. *Then either* M_P *is biholomorphic to*

$$\mathsf{Im}\, w = |z_2|^{2l} (\mathsf{Re}\, z_1)^m \tag{3}$$

for some positive integers l *and* m, *or to*

$$\mathsf{Im}\, w = (|z_1|^2 \pm |z_1|^2)^m$$

for some integer $m > 1$.

For polynomial models with symmetry algebras of dimension six we obtain three additional types of exceptional models.

Theorem 1.3. *Let* $\dim \mathfrak{g} = 6$. *Then* M_P *is biholomorphic to one of the following models:*

$$\mathsf{Im}\, w = |z_1|^{2k} |z_2|^{2l} (\mathsf{Re}\, z_1^\alpha z_2^\beta)^m$$

for some nonnegative integers m, k, l *and a pair of integers* (α, β) *such that* $\alpha \geq -1, \beta \geq -1$, *different from those which lead to* (3) *or* (1),

$$\mathsf{Im}\, w = (z_1 \bar{z}_2 + \bar{z}_1 z_2)^\alpha |z_2|^{2k} \tag{4}$$

for positive integers α *and* k, *or*

$$\mathsf{Im}\, w = (\mathsf{Re}\, z_1)^2 + (\mathsf{Re}\, z_2)^p \tag{5}$$

for an integer $p > 2$.

Let us remark that if the dimension of the symmetry algebra is five, then moduli (parameters) start to appear. A complete description of this case is left open. Let us also remark that very little is known about symmetries of hypersurfaces of infinite multitype ([7, 10, 18]).

The structure of the paper is as follows. In Section 2 we recall some needed definitions. In Section 3 we consider models with nontrivial $\mathfrak{g}_c$. In Section 4 models with nontrivial $\mathfrak{g}_0^{Nil}$ are described. In Section 5 we treat the remaining cases of models with nontrivial $\mathfrak{g}_1$. Section 6 deals with models, which admit none of these three types of symmetries. Section 7 proves Theorem 1.1–1.3.

2. Preliminaries

In this section we introduce notation and recall briefly some needed definitions and results (for more details, see e.g. [12]).

Consider a smooth real hypersurface $M \subseteq \mathbb{C}^3$ and let $p \in M$ be a point of finite type $m \geq 2$ (in the sense of Kohn and Bloom-Graham, [2], [9]).

Let (z, w) be local holomorphic coordinates centered at p, where $z = (z_1, z_2)$, $z_j = x_j + iy_j$, $j = 1, 2$, and $w = u + iv$. We assume that the hyperplane $\{v = 0\}$ is tangent to M at p, so M is described near p as the graph of a uniquely determined real valued function

$$v = \Psi(z_1, z_2, \bar{z}_1, \bar{z}_2, u), \tag{6}$$

with $d\Psi(0) = 0$. We can assume that (see e.g. [2])

$$\Psi(z_1, z_2, \bar{z}_1, \bar{z}_2, u) = P_m(z, \bar{z}) + o(u, |z|^m), \tag{7}$$

where $P_m(z, \bar{z})$ is a nonzero homogeneous polynomial of degree m without pluriharmonic terms.

The type is the first nontrivial local invariant of M. A more refined invariant, which captures the behaviour of the defining function in all tangential directions was introduced by Catlin in [4].

The Catlin multitype is defined in terms of rational weights associated to the variables w, z_1, z_2. Initially, the complex normal variable w, and its components u and v are assigned weight one. The complex tangential variables (z_1, z_2) are treated in the following way.

By a weight we understand a pair of nonnegative rational numbers $\Lambda = (\lambda_1, \lambda_2)$, where $0 \leq \lambda_j \leq \frac{1}{2}$, and $\lambda_1 \geq \lambda_2$. Let $\Lambda = (\lambda_1, \lambda_2)$ be a weight, and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ be multiindices. The weighted degree κ of a monomial

$$q(z, \bar{z}, u) = c_{\alpha\beta l} z^\alpha \bar{z}^\beta u^l, \; l \in \mathbb{N},$$

is then

$$\kappa := l + \sum_{i=1}^{2} (\alpha_i + \beta_i)\lambda_i.$$

A polynomial $Q(z, \bar{z}, u)$ is weighted homogeneous of weighted degree κ if it is a sum of monomials of weighted degree κ.

Analogously, a holomorphic vector field with polynomial coefficients of the form

$$\sum_{j=1}^{2} f^j(z, w)\partial_{z_j} + g(z, w)\partial_w \tag{8}$$

is weighted homogeneous of weighted degree κ, provided $g(z,w)$ is weighted homogeneous of degree $\kappa+1$, and each $f^j(z,w)$ is weighted homogeneous of degree $\kappa+\lambda_j$.

For a weight Λ, the weighted length of a multiindex $\alpha=(\alpha_1,\alpha_2)$ is defined by

$$|\alpha|_\Lambda := \lambda_1\alpha_1+\lambda_2\alpha_2.$$

Similarly, if $\alpha=(\alpha_1,\alpha_2)$ and $\hat{\alpha}=(\hat{\alpha}_1,\hat{\alpha}_2)$ are two multiindices, the weighted length of the pair $(\alpha,\hat{\alpha})$ is

$$|(\alpha,\hat{\alpha})|_\Lambda := \lambda_1(\alpha_1+\hat{\alpha}_1)+\lambda_2(\alpha_2+\hat{\alpha}_2).$$

Definition 2.1. A weight Λ will be called distinguished for M if there exist local holomorphic coordinates (z,w) in which the defining equation of M takes form

$$v = P_C(z,\bar{z}) + o_\Lambda(1), \tag{9}$$

where $P_C(z,\bar{z})$ is a nonzero Λ – homogeneous polynomial of weighted degree 1 without pluriharmonic terms, and $o_\Lambda(1)$ denotes a smooth function whose derivatives of weighted order less than or equal to one vanish.

As we see from (7), there always exist distinguished weights. In (7), we can take $\Lambda=\left(\frac{1}{m},\frac{1}{m}\right)$. In the following we shall consider the standard lexicographic order on the set of pairs. We recall the following definition (see [4, 11]).

Definition 2.2. Let $\Lambda_M=(\mu_1,\mu_2)$ be the infimum of all possible distinguished weights Λ with respect to the lexicographic order. The multitype of M at p is defined to be the pair

$$(m_1,m_2),$$

where

$$m_j = \begin{cases} \frac{1}{\mu_j} & \text{if } \ \mu_j\neq 0 \\ \infty & \text{if } \ \mu_j=0. \end{cases}$$

If both m_j are finite, we say that M is of *finite multitype at p*. Since the definition of multitype includes all distinguished weights, the infimum is clearly a biholomorphic invariant.

Coordinates corresponding to the multitype weight Λ_M, in which the local description of M has form (9), with P_C being Λ_M-homogeneous, are called multitype coordinates.

Definition 2.3. Let M be given by (9). We define a model hypersurface M_P associated to M at p by

$$M_P = \{(z, w) \in \mathbb{C}^3 \mid v = P_C(z, \bar{z})\}. \tag{10}$$

We will write for simplicity $P = P_C$, when there is no danger of confusion. Let $w = u + iv$ and let W be the vector field of weighted degree -1 given by

$$W = \partial_w = \partial_u - i\partial_v.$$

Then we have $\mathsf{Re}\, W(\mathsf{Im}\,(w) - P(z, \bar{z})) = 0$, hence W is a symmetry.

Recall that, as was proved in [12], the Lie algebra $\mathfrak{g}$ of infinitesimal automorphisms $\mathrm{aut}(M_P, 0)$ at $0 \in M_P \subset \mathbb{C}^3$ admits a weighted decomposition, which we rewrite now as

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^{2} \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_1, \tag{11}$$

where the vector fields in $\mathfrak{g}_c$ commute with W, the non-zero vector fields in $\mathfrak{g}_n$ do not commute with W and their weights are between 0 and 1. In particular, $W = \partial_w$ is contained in $\mathfrak{g}_{-1}$, which has real dimension one (for more details, see [12], [14]).

Let E be the *weighted Euler field*, given by

$$E = w\partial_w + \sum_{j=1}^{2} \mu_j z_j \partial_{z_j}. \tag{12}$$

Then it is immediate that

$$\mathsf{Re}\, E(\mathsf{Im}\, w - P(z, \bar{z})) = 0, \tag{13}$$

which implies $E \in \mathfrak{g}_0$ and hence $\dim \mathfrak{g}_0 \geq 1$ for an arbitrary model M_P.

3. Exotic nonlinear symmetries

In this section we consider in detail the component $\mathfrak{g}_c$ consisting of vector fields with nonlinear coefficients, which are independent of the variable w. Let us recall that such symmetries do not exist in the Levi nondegenerate case.

We recall some definitions and results from [13].

Definition 3.1. Let Y be a weighted homogeneous holomorphic vector field. A pair of finite sequences of holomorphic weighted homogeneous polynomials $\{U^1, \dots, U^n\}$ and $\{V^1, \dots, V^n\}$ is called a symmetric pair of $Y-$chains if

$$Y(U^n) = 0, \quad Y(U^j) = c_j U^{j+1}, \quad j = 1, \dots, n-1, \tag{14}$$

$$Y(V^n) = 0, \quad Y(V^j) = d_j V^{j+1}, \quad j = 1, \dots, n-1, \tag{15}$$

where c_j, d_j are non zero complex constants, which satisfy

$$c_j = -\bar{d}_{n-j}. \tag{16}$$

If the two sequences are identical we say that $\{U^1, \dots, U^n\}$ is a symmetric Y-chain.

The following theorem shows that in general the elements of $\mathfrak{g}_c$ arise from symmetric pairs of chains (see [13]).

Theorem 3.2. *Let M_P be a holomorphically nondegenerate hypersurface given by (10), which admits a nontrivial $Y \in \mathfrak{g}_c$. Then P_C can be decomposed in the following way*

$$P_C = \sum_{j=1}^{M} T_j, \tag{17}$$

where each T_j is given by

$$T_j = \mathsf{Re}\,(\sum_{k=1}^{N_j} U_j^k \overline{V_j^{N_j-k+1}}), \tag{18}$$

where $\{U_j^1, \dots, U_j^{N_j}\}$ and $\{V_j^1, \dots, V_j^{N_j}\}$ are a symmetric pair of $Y-$ chains.

Conversely, if Y and P_C satisfy (14) – (18), then $Y \in \mathfrak{g}_c$.

The simplest example of this situation is given by

$$\mathsf{Im}\, w = \mathsf{Re}\, z_1 \bar{z}_2^l, \tag{19}$$

where $U_2 = z_2^l$ and $U_1 = iz_1$. In this case, $Y = iz_2^l \partial_{z_1}$ and $\dim \mathfrak{g} = 10$.

Definition 3.3. If P_C satisfies (14) – (18), the associated hypersurface M_P is called a chain hypersurface.

Lemma 3.4. *Let* $\dim \mathfrak{g}_c = 1$ *and* $\dim \mathfrak{g} \geq 6$*. Then* M_P *is biholomorphically equivalent to*

$$\mathsf{Im}\, w = |z_1|^{2k}|z_2|^{2l}(\mathsf{Re}\, z_1^{\alpha} z_2^{\beta})^m \tag{20}$$

for some nonnegative integers k, l, m *and a pair of integers* (α, β) *such that* $\alpha \geq -1, \beta \geq -1$*. Moreover, if* $k = 0, \alpha = 1, \beta = 0$*, or* $l = 0, \alpha = 0, \beta = 1$*, then*

$$\dim \mathfrak{g} = 7.$$

In all other cases of holomorphically nondegenerate polynomial models defined by (20)*, except for* (19)*,* $\dim \mathfrak{g} = 6$.

Proof. We start by analyzing the structure of $\mathfrak{g}_0$ under the assumptions of the lemma. We will show that $\mathfrak{g}_0$ has real dimension three. Let $Z \in \mathfrak{g}_0$ be a rotation and $Y \in \mathfrak{g}_1$ be a nonzero vector field. By Theorem 4.7 in [12], Y has the form up to a real multiple,

$$Y = \sum_{j=1}^{2} \phi_j(z) w \partial_{z_j} + \frac{1}{2} w^2 \partial_w, \tag{21}$$

where the first term gives a complex reproducing field, i.e.,

$$2 \sum_{j=1}^{2} \phi_j(z) P_{z_j}(z, \bar{z}) = P(z, \bar{z}). \tag{22}$$

It follows that the Jordan normal form of the vector field $\sum_{j=1}^{2} \phi_j(z) \partial_{z_j}$ is diagonal with real coefficients. Hence we consider multitype coordinates such that

$$Y = (\lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2})\, w + \frac{1}{2} w^2 \partial_w.$$

We will show that in such coordinates, Z is diagonal. Let $X \in \mathfrak{g}_c$ be a nonzero vector field. We obtain $[X, Y] = 0$, since otherwise the commutator is of weight bigger than one. As a consequence, (λ_1, λ_2) is linearly independent with the multitype weights (μ_1, μ_2), by the grading property of the Euler field. If $\mu_1 \neq \mu_2$, then any rotation is diagonal. Hence we may assume that $\mu_1 = \mu_2$, which gives $\lambda_1 \neq \lambda_2$. The commutator of Z and Y has to be a real multiple of Y. By computing directly the commutator, we obtain that Z is diagonal.

From now on we use the above coordinates, in which all rotations are diagonal. By the reproducing property, there is a real rotation with coefficients given by $\lambda_1 - \mu_1$ and $\lambda_2 - \mu_2$. In addition, there is an imaginary rotation with coefficients $i\lambda_1, i\lambda_2$.

On the other hand, if there exist two linearly independent imaginary rotations, writing P as

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = \sum_{|\alpha,\hat{\alpha}|_{\Lambda_M}=1} C_{\alpha.\hat{\alpha}} z^{\alpha} \bar{z}^{\hat{\alpha}}$$

we have $\alpha_1 = \hat{\alpha}_1$, $\alpha_2 = \hat{\alpha}_2$ whenever $A_{\alpha.\hat{\alpha}} \neq 0$. From the real rotation and weighted homogeneity we obtain a unique solution for $\alpha_1, \alpha_2, \hat{\alpha}_1, \hat{\alpha}_2$. That contradicts holomorphic nondegeneracy of M_P. Hence there are two linearly independent rotations, and therefore $\dim \mathfrak{g}_0 = 3$.

Next, consider again the nonzero vector field $X \in \mathfrak{g}_c$. The commutator with Y has to vanish, since otherwise its weight exceeds one. It follows that X is of weight zero with respect to the weights λ_1, λ_2.

If $\lambda_1 = \lambda_2$, we obtain a contradiction with nonlinearity of X. Without any loss of generality, we can assume that $\lambda_1 < \lambda_2$. First assume that $\lambda_1 > 0$. Then the coefficient of ∂_{z_2} has to vanish, and the coefficient of ∂_{z_1} has to be a power of z_2, hence

$$X = i z_1^m \partial_{z_2}, \tag{23}$$

where $m\lambda_1 - \lambda_2 = 0$.

Now let $\lambda_1 \leq 0$. Then we obtain

$$X = i z_1^{\alpha} z_2^{\beta} (\sigma_1 z_1 \partial_{z_1} + \sigma_2 z_2 \partial_{z_2}), \tag{24}$$

where $\lambda_1 \alpha + \lambda_2 \beta = 0$. Note that (23) is a special case of (24), for $\beta = -1$ and $\sigma_1 = 0$.

We compute the general form of chains corresponding to such vector fields. Let us first assume that the hypersurface is defined by one chain, and let n denote the lenght of that chain. We obtain

$$U_n = z_1^p z_2^q$$

where $\sigma_1 p + \sigma_2 q = 0$, and

$$U_j = \frac{1}{(n-j)!} z_1^{p-(n-j)\alpha} z_2^{q-(n-j)\beta}.$$

We verify directly that this gives the model of the form

$$\mathsf{Im}\, w = |z_1|^{2k} |z_1|^{2l} (\mathsf{Re}\, z_1^{\alpha} z_2^{\beta})^m. \tag{25}$$

We also verify that

$$X = iz_1^\alpha z_2^\beta(\sigma_1 z_1 \partial_{z_1} + \sigma_2 z_2 \partial_{z_2})$$

is a symmetry of (25), provided that

$$\sigma_1 k + \sigma_2 l + m(\sigma_1\alpha + \sigma_2\beta) = 0. \tag{26}$$

Indeed, $X(P)$ is equal to

$$iz_1^\alpha z_2^\beta |z_1|^{2k}|z_1|^{2l}(\mathsf{Re}\, z_1^\alpha z_2^\beta)^{m-1}\left[(\sigma_1 k + \sigma_2 l)(\mathsf{Re}\, z_1^\alpha z_2^\beta) + \frac{m}{2}(\sigma_1\alpha + \sigma_2\beta)z_1^\alpha z_2^\beta\right]. \tag{27}$$

Further

$$\mathsf{Re}\, X(P) = \mathsf{Re}\left[iz_1^\alpha z_2^\beta((\sigma_1 k + \sigma_2 l)(\mathsf{Re}\, z_1^\alpha z_2^\beta) + \frac{m}{2}(\sigma_1\alpha + \sigma_2\beta)(z_1^\alpha z_2^\beta))\right]. \tag{28}$$

Hence

$$\mathsf{Re}\, X(P) = \mathsf{Re}\, iz_1^\alpha z_2^\beta \frac{\sigma_1 k + \sigma_2 l}{2} \bar{z}_1^\alpha \bar{z}_2^\beta = 0. \tag{29}$$

Now we verify that if $k = 0, \alpha = 1, \beta = 0$, or $l = 0, \alpha = 0, \beta = 1$, then M admits an additional tubular symmetry. Hence $\dim \mathfrak{g} = 7$. In all other cases we obtain $\dim \mathfrak{g} = 6$.

If the chain hypersurface is given by a pair of chains, or is composed from several such chains, or pairs of chains, it is immediate to verify that $\mathfrak{g}_1$ is trivial and also there is no real rotation. It follows that in this case $\dim \mathfrak{g} \le 5$. That proves the statement of the lemma. □

4. Nilpotent rotations

Let X be an infinitesimal CR automorphism in $\mathfrak{g}_0$. By results of [12], X is a linear vector field in suitable multitype coordinates. Its Jordan normal form can be decomposed into $X^{Re} + X^{Im} + X^{Nil}$, where X^{Re} is the real diagonal of the Jordan normal form, X^{Im} is the imaginary diagonal and X^{Nil} is the nilpotent part. As was proved in [12], each of the components itseft is a symmetry.

In this section we will consider models which admit a nilpotent rotation.

Lemma 4.1. *Let M_P admit a nilpotent rotation and* $\dim \mathfrak{g} \geq 6$. *Then M_P is equivalent to*

$$(z_1\bar{z}_2 + \bar{z}_1 z_2)^\alpha Q(z_2, \bar{z}_2), \tag{30}$$

where Q is a real valued homogeneous polynomial. Moreover,

- $\dim \mathfrak{g} = 7$ *if and only if Q is constant*
- $\dim \mathfrak{g} = 6$ *if and only if* $Q(z_2, \bar{z}_2) = |z_2|^{2k}$.

Proof. Let $X = z_2\partial_{z_1}$ be a nilpotent rotation in Jordan normal form. Since X is a symmetry of M_P, we have $\mu_1 = \mu_2$. Let M_P be given by

$$\mathsf{Im}\, w = P(z, \bar{z}), \tag{31}$$

where P is homogeneous of degree d. Consider the bihomogeneous expansion of P,

$$P(z, \bar{z}) = \sum_{m=0}^{d} P_m(z, \bar{z}) \tag{32}$$

and expand each of the polynomials P_m as

$$P_m(z, \bar{z}) = \sum_{j=0}^{m} \sum_{l=0}^{d-m} A^m_{jl} z_1^j z_2^{m-j} \bar{z}_1^l \bar{z}_2^{d-m-l}.$$

Hence

$$X(P_m) = \sum_{j=0}^{m} \sum_{l=0}^{d-m} j A^m_{jl} z_1^{j-1} z_2^{m-j+1} \bar{z}_1^l \bar{z}_2^{k-m-l}.$$

We have $P_m = \bar{P}_{k-m}$. Hence from $\mathsf{Re}\, X(P) = 0$ we obtain

$$X(P_m) = -\overline{X(\bar{P}_m)},$$

which leads to the recurence

$$j A^m_{jl} = -(l+1) A^m_{j-1,l+1}.$$

It follows that the coeffients A^m_{0j} are arbitrary, which gives the term $A^m_{0j} \bar{z}_1^j z_2^m \bar{z}_2^{d-m-j}$. Note that for $j > l$ the recurrence forces $A^m_{jl} = 0$. We verify that the solution is given by

$$\sum_{j=0}^{m}(z_1\bar{z}_2+\bar{z}_1 z_2)^j A_{0j}^m z_2^{m-j}\bar{z}_2^{d-m-j}. \tag{33}$$

P is then obtained by summation over m, which leads to

$$P(z,\bar{z})=\sum_{j=0}^{m}(z_1\bar{z}_2+\bar{z}_1 z_2)^j Q_{d-2j}(z_2,\bar{z}_2)$$

for some real valued polynomials Q_j of degree j. Further, if the sum contains more than one term, the resulting M_P does not admit a real rotation and has trivial $\mathfrak{g}_1$. It follows that $\dim \mathfrak{g} \leq 5$.

It remains to consider polynomials of the form (30). If Q is constant and $\alpha > 0$, $\mathfrak{g}_0$ is identical to the case of the hyperquadric with indefinite signature, hence $\dim \mathfrak{g}_0 = 5$. Since there are no tubular symmetries and $\mathfrak{g}_c$ is trivial, we obtain $\dim \mathfrak{g} = 7$.

If $Q(z,\bar{z}) = |z_2|^{2k}$, then all symmetries remain, except for $z_1\partial_{z_2}$ and we obtain $\dim \mathfrak{g} = 6$.

If Q is not circular, the imaginary rotation is not present and at the same time $\dim \mathfrak{g}_1 = 0$. Thus $\dim \mathfrak{g} \leq 5$. □

Lemma 4.2. *Let $\mathfrak{g}_0$ be non-abelian. Then either M_P admits a nilpotent rotation, or P is of the form*

$$P(z,\bar{z}) = (|z_1|^2+|z_2|^2)^l$$

for some integer $l \in \mathbb{N}$. As a consequence, in the latter case, $\dim \mathfrak{g}_0 = 5$.

Proof. If $\mu_1 \neq \mu_2$, then $\mathfrak{g}_0$ is obviously abelian. Hence we assume that $\mu_1 = \mu_2$ and that M_P does not admit any nilpotent rotation. Let the defining polynomial $P(z,\bar{z})$ be a homogeneous polynomial of degree d. Let us consider a real or imaginary rotation in normal form, $X = \sigma_1 z_1\partial_{z_1} + \sigma_2 z_2\partial_{z_2} \in \mathfrak{g}_0$ for some $\sigma_1, \sigma_2 \in \mathbb{C}$ which are both real or both purely imaginary and satisfy $\sigma_1 \neq \sigma_2$. Assume that

$$Y = az_1\partial_{z_1} + bz_1\partial_{z_2} + cz_2\partial_{z_1} + dz_2\partial_{z_2} \tag{34}$$

also belongs to $\mathfrak{g}_0$ for some $a,b,c,d \in \mathbb{C}$ with $|b|+|c| > 0$.

Let first X be real. We have

$$[X,Y] = (\sigma_1-\sigma_2)(bz_1\partial_{z_2} - cz_2\partial_{z_1}) \in \mathfrak{g}_0$$

and

$$[X,[X,Y]] = (\sigma_1 - \sigma_2)^2 (b z_1 \partial_{z_2} + c z_2 \partial_{z_1}) \in \mathfrak{g}_0.$$

Hence, by a suitable linear combination of these two commutators we obtain a nilpotent rotation in $\mathfrak{g}_0$, which is a contradiction.

We may further assume that M_P admits only imaginary rotations, and use an argument from [8]. Since the Jordan normal form of $[X,Y]$ only has purely imaginary part, the second-order equation $x^2 - (\sigma_1 - \sigma_2)^2 bc = 0$ has to have purely imaginary solution, and therefore $bc < 0$ and $c = -\bar{b}$ up to positive scalar. After re-scaling the z_2-axis with a positive scalar if necessarily, let $c = -\bar{b}$. Note that $z_2 \partial_{z_2}$ is stable under the re-scaling of z_2-axis and so is X. Then

$$[X,Y] = i(\sigma_1 - \sigma_2)(b z_1 \partial_{z_2} + \bar{b} z_2 \partial_{z_1})$$

and

$$[X,[X,Y]] = -(\sigma_1 - \sigma_2)^2 (b z_1 \partial_{z_2} - \bar{b} z_2 \partial_{z_1})$$

for some $b \in \mathbb{C}$. Hence, $Z_1 = z_1 \partial_{z_2} - z_2 \partial_{z_1} \in \mathfrak{g}_0$ and $Z_2 = i z_1 \partial_{z_2} + i z_2 \partial_{z_1} \in \mathfrak{g}_0$.

It follows that away from the origin, M_P admits three linearly independent vector fields, Y, Z_1, Z_2, which are also tangent to the spherical model

$$\mathsf{Im}\, w = \sum_{j=1}^{2} |z_j|^2. \tag{35}$$

It follows that the level sets of $P(z,\bar{z})$ are the same as those for the function $\sum_{j=1}^{2} |z_j|^2$. By homogeneity, we obtain

$$P(z,\bar{z}) = (\sum_{j=1}^{2} |z_j|^2)^l \tag{36}$$

for some l. □

5. Nonvanishing $\mathfrak{g}_1$

In this section, we will assume that $\dim \mathfrak{g}_1 = 1$, $\dim \mathfrak{g}_c = 0$ and $\mathfrak{g}_0$ is abelian. We will denote by G the element of $\mathfrak{g}_1$ whose z-part has the complex reproducing property.

Lemma 5.1. *Let* $\dim \mathfrak{g}_1 = 1$, $\dim \mathfrak{g}_c = 0$ *and* $\mathfrak{g}_0$ *be abelian. Then* $\dim \mathfrak{g} \leq 5$.

Proof. The argument splits into two cases. Either the complex reproducing field is the Euler field itself, in which case $[G, W] = E$, or it is a different field and $[G, W] \neq E$. We start by considering the latter case, $[G, W] \neq E$. We will show that P takes the form

$$P(z, \bar{z}) = |z_1|^{2k} |z_1|^{2l} Q(z_1^\gamma z_2^\delta, \bar{z}_1^\gamma \bar{z}_2^\delta) \tag{37}$$

for some real valued homogeneous polynomial $Q(\zeta, \bar{\zeta})$ and nonzero integer pair (γ, δ).

Note that $Q(\zeta, \bar{\zeta}) = |\zeta|^{2k}$ leads to a holomorphically degenerate model. Also note that the integers γ, δ are allowed to assume negative values. The only restriction is that the resulting product is a holomorphic polynomial P.

We have $\dim \mathfrak{g}_0^{Re} = 1$. Since in suitable multitype coordinates $X = \lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2}$ gives a complex reproducing field, in such coordinates each monomial $z_1^a z_2^b$ appearing in the expansion of P satisfies $a\lambda_1 + b\lambda_2 = C$ for a fixed constant C.

We will assume that $\lambda_1 \lambda_2 \neq 0$. Let us denote by (a, b) the smallest pair in lexicographic order of nonnegative integers that solve this equation, and let (γ, δ) be the smallest pair in Euclidean norm of integers such that $\gamma\lambda_1 + \delta\lambda_2 = 0$ with $\gamma > 0$.

It follows that P is of the form (37). The case when one of the λ_j vanishes is completely analogous.

As in the proof of Lemma 3.4, we obtain that all rotations are diagonal in the coordinates which diagonalize the complex reproducing field determined by G. Combining this fact with the above form of P gives $\dim \mathfrak{g}_0 \leq 3$. Further, if M_P admits a tubular symmetry, then $\gamma = 0$, $\delta = 1$, or vice versa, and $Q(\zeta, \bar{\zeta}) = (\mathsf{Re}\, \zeta)^k$. It then follows by Lemma 3.4 that $\dim \mathfrak{g}_c = 1$, a contradiction. It follows that $\dim \mathfrak{g} \leq 5$.

It remains to consider the case $[G, W] = E$. We claim that in this case, $\dim \mathfrak{g} = 5$ if either

$$P(z, \bar{z}) = \sum_{\mu_1 k + \mu_2 l = \frac{1}{2}} B_k |z_1|^{2k} |z_2|^{2l} \tag{38}$$

for some real constatants B_k, or

$$P(z, \bar{z}) = \mathsf{Re} \sum_{\mu_1 j + \mu_2 l = \frac{1}{2}} A_j z_1^j z_2^l \bar{z}_1^{K-j} \bar{z}_2^{L-l},$$

where $\mu_1 K + \mu_2 L = 1$ for some complex constants A_j. In all other cases, $\dim \mathfrak{g} = 4$.

Indeed, the first form corresponds to the case $\dim \mathfrak{g}_0^{Im} = 2$. The second form corresponds to case $\dim \mathfrak{g}_0^{Im} = \dim \mathfrak{g}_0^{Re} = 1$. Using commutativity of $\mathfrak{g}_0$, it is immediate to verify that in all other cases M_P admits only a one dimensional space of imaginary rotations as a consequence of nontriviality of $\mathfrak{g}_1$. That proves the statement of the lemma. □

6. Remaining cases

In this section we will analyze the remaining cases. Thus we assume $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_c = \dim \mathfrak{g}_0^{Nil} = 0$. We start by describing models which admit two complex dependent tubular symmetries.

Lemma 6.1. *Let M_P admit two tubular symmetries, which are linearly independent at* 0 *over* $\mathbb{R}$*, but dependent over* $\mathbb{C}$*. Then in suitable modified multitype coordinates*

$$P(z, \bar{z}) = (\mathsf{Re}\, z_1)^2 + Q(z_2, \bar{z}_2). \tag{39}$$

If $Q(z_2, \bar{z}_2) = (\mathsf{Re}\, z_2)^l$, then $\dim \mathfrak{g} = 6$*. In all other cases,* $\dim \mathfrak{g} = 5$.

Proof. The proof follows from Lemma 5.1 in [14]. □

It remains to consider all possible combinations of tubular symmetries with real or imaginary rotations. Since under our assumption, $\mathfrak{g}_0$ is abelian, we can use modified multitype coordinates in which the rotations are diagonal and the tubular symmetry is straightened. It is immediate to verify that in this case $\dim \mathfrak{g} \leq 5$.

7. Proof of the main results

Theorem 1.1 now follows from Lemma 3.4. In order to prove Theorem 1.2, we first consider the case $\dim \mathfrak{g}_c \neq 0$ and use Lemma 3.4, which leads to the models (3). Then we apply Lemma 4.1 in the case of existence of a nilpotent rotation, and 4.2 in the case of non-abelian $\mathfrak{g}_0$, which together provide the second type of models. Finally, Lemmas 5.1 and 6.1 guarantee that there are no other models with $\dim \mathfrak{g} = 7$. The proof of Theorem 1.3 follows the same steps. In addition to the models described in Lemmas 4.1 and 4.2, Lemma 6.1 provides the third class of models with $\dim \mathfrak{g} = 6$.

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