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On the adjoint action of the group of symplectic diffeomorphisms

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Abstract: We study the action of Hamiltonian diffeomorphisms of a compact symplectic manifold (X, ω) on $C^\infty(X)$ and on functions $C^\infty(X) \to \mathbb{R}$. We describe various properties of invariant convex functions on $C^\infty(X)$. Among other things we show that continuous convex functions $C^\infty(X) \to \mathbb{R}$ that are invariant under the action are automatically invariant under so called strict rearrangements and they are continuous in the sup norm topology of $C^\infty(X)$; but this is not generally true if the convexity condition is dropped.

1. Introduction

Consider a connected, compact, symplectic manifold (X, ω), without boundary, of dimension $2n$. According to Omori [O], symplectic self–diffeomorphisms of X form a Fréchet–Lie group $\mathrm{Symp}(\omega)$, with Lie algebra the space $\mathfrak{v}(\omega)$ of smooth vector fields on X that are locally Hamiltonian. In this paper we will be interested in the action of $\mathrm{Symp}(\omega)$, by pull back, on the Fréchet space $C^\infty(X)$ of smooth real functions

$$\mathrm{Symp}(\omega) \times C^\infty(X) \ni (g, \xi) \mapsto \xi \circ g^{-1} \in C^\infty(X), \tag{1.1}$$

and in functions $C^\infty(X) \to \mathbb{R}$ that (1.1) leaves invariant. This action is no adjoint action, but it is close to one. The adjoint action Ad_g of $g \in \mathrm{Symp}(\omega)$ is, rather, push forward by g^{-1} of vector fields in $\mathfrak{v}(\omega)$. The subspace $\mathrm{ham}(\omega) \subset \mathfrak{v}(\omega)$ of globally Hamiltonian vector fields, those that are symplectic gradients $\mathrm{sgrad}\,\xi$ of some $\xi \in C^\infty(X)$, is invariant under Ad_g, and (1.1) induces via the projection $\xi \mapsto \mathrm{sgrad}\,\xi$ the restriction of the adjoint action to $\mathrm{ham}(\omega)$.

Other diffeomorphism groups of X also act on $C^\infty(X)$ by pull back. Our focus will be on the subgroup $\mathrm{Ham}(\omega) \subset \mathrm{Symp}(\omega)$ of Hamiltonian diffeomorphisms. Hamiltonian diffeomorphisms are the time 1 maps of time dependent Hamiltonian vector fields $\mathrm{sgrad}\,\xi_t$, $\xi_t \in C^\infty(X)$. Continuous norms—and

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also seminorms—on the Fréchet space $C^\infty(X)$, invariant under $\mathrm{Ham}(\omega)$, are of potential interest in symplectic geometry because they give rise to bi–invariant metrics on $\mathrm{Ham}(\omega)$, and have been investigated in the past. An obvious norm is $\|\xi\|_\infty = \max_X |\xi|$. That it gives rise to a genuine metric on $\mathrm{Ham}(\omega)$ was proved first by Hofer and Viterbo in $\mathbb{R}^{2n}$, and in general by Lalonde and McDuff; see also Polterovich's book, [Ho, LM, P, V]. Work by Ostrover–Wagner, Han, and Buhovsky–Ostrover [BO, Ha, OW] gave the following. Let (X,μ) and (Y,ν) be measure spaces. We say that measurable functions $\xi : X \to [-\infty,\infty]$, $\eta : Y \to [-\infty,\infty]$ are equidistributed, or strict rearrangements of each other, if

$$\mu(\xi^{-1}B) = \nu(\eta^{-1}B) \quad \text{for all Borel sets } B \subset [-\infty,\infty].$$

When $\mu(X) = \nu(Y) < \infty$, this is equivalent to $\mu\{x \in X : \xi(x) > t\} = \nu\{y \in Y : \eta(y) > t\}$ for all $t \in \mathbb{R}$. We have to use the qualifier 'strict', since the notion of rearrangement in harmonic analysis and Banach space theory typically refers to the relation $\mu\{x \in X : |\xi(x)| > t\} = \nu\{y \in Y : |\eta(y)| > t\}$. Back to our symplectic manifold (X,ω), we write μ for the measure on X defined by ω^n; the action (1.1) clearly sends functions on (X,μ) to their strict rearrangements.

Theorem 1.1 ([BO, H, OW])**.** *If $\| \ \|$ is a $\mathrm{Ham}(\omega)$ invariant continuous seminorm on the Fréchet space $C^\infty(X)$, then $\|\xi\| = \|\eta\|$ whenever $\xi,\eta \in C^\infty(X)$ are equidistributed. These seminorms satisfy $\| \ \| \le c\| \ \|_\infty$ with some $c \in (0,\infty)$. Unless $\| \ \|$ and $\| \ \|_\infty$ are equivalent, the pseudodistance on $\mathrm{Ham}(\omega)$ induced by $\| \ \|$ is identically 0.*

One of our goals in this paper is to offer a simpler proof to the first two statements, in fact in a slightly greater generality:

Theorem 1.2. *Suppose $p : C^\infty(X) \to \mathbb{R}$ is a continuous, convex function that is invariant under the action of $\mathrm{Ham}(\omega)$. Then p is continuous in the topology of $C^\infty(X)$ induced by $\| \ \|_\infty$, and is invariant under strict rearrangements: $p(\xi) = p(\eta)$ whenever ξ,η are equidistributed.*

The point is not the modest gain in generality, which can easily be achieved once Theorem 1.1 is known (for example along the lines of the proof of Theorem 4.1 below). Rather, it is the simplification of the proof. This is how the two proofs compare. [OW] first proved that any $\mathrm{Ham}(\omega)$ invariant seminorm $\| \ \| \le c\| \ \|_\infty$ is invariant under volume preserving diffeomorphisms. Han in [Ha] subsequently strengthened this to invariance under strict rearrangements. All this is obtained as a consequence of a lemma of Katok

[K, Section 3]. The final step is in [BO], that takes an arbitrary continuous $\mathrm{Ham}(\omega)$ invariant seminorm $\| \ \|$ on $C^\infty(X)$, and proves by an involved argument that $\| \ \| \le c\| \ \|_\infty$.

We obtain the simplification by restructuring the proof. First we prove that p in Theorem 1.2 is a limit point of the set of $\mathrm{Ham}(\omega)$ invariant functions $q : C^\infty(X) \to \mathbb{R}$ that are continuous in the L^1 topology on $C^\infty(X)$. This depends on studying linear forms on $C^\infty(X)$, i.e., distributions, and regularizing them using the action of $\mathrm{Ham}(\omega)$. Katok's lemma now gives that the functions q are invariant under strict rearrangements, whence so must be their limit point p. Another application of Katok's lemma, combined with real analysis type arguments then gives the continuity of p with respect to $\| \ \|_\infty$.

Continuity of p with respect to $\| \ \|_\infty$ in Theorem 1.2 is essentially an upper estimate of p. We will also prove a lower estimate:

Theorem 1.3. *Let $p\colon C^\infty(X) \to \mathbb{R}$ be $\mathrm{Ham}(\omega)$ invariant, convex, and continuous. Then either*

(i) $p(\xi) = p_1(\int_X \xi\omega^n)$, where $p_1\colon \mathbb{R} \to \mathbb{R}$ is convex; or

(ii) there are $a \in \mathbb{R}$, $b \in (0,\infty)$ such that

$$p(\xi) \ge a + b\int_X |\xi|\omega^n \quad \text{if} \quad \begin{cases} \int_X \xi\omega^n = 0, & \text{or} \\ \int_X \xi\omega^n \ge 0 \ \text{and} \ \lim_{\mathbb{R}\ni\lambda\to\infty} p(\lambda) = \infty, & \text{or} \\ \int_X \xi\omega^n \le 0 \ \text{and} \ \lim_{\mathbb{R}\ni\lambda\to-\infty} p(\lambda) = \infty. \end{cases}$$

If p is positively homogeneous ($p(c\xi) = cp(\xi)$ for positive constants c), then $a = 0$.

In particular, if p is a norm, then it dominates L^1 norm, something that [OW] also found (cf. Proposition 6.1 there and its proof).

Above we have insisted on the difference between rearrangements and strict rearrangements. Nevertheless, Theorem 1.3 implies that in our setting the difference between the two is minimal. The notion of rearrangement invariant Banach spaces in the next theorem is defined in [BS], see also the Appendix, section 9; or section 7.

Theorem 1.4. *Given a $\mathrm{Ham}(\omega)$ invariant continuous norm p on $C^\infty(X)$, there is a rearrangement invariant Banach function space on X whose norm, restricted to $C^\infty(X)$, is equivalent to p.*

A natural question is whether Theorem 1.2 holds for all continuous $\mathrm{Ham}(\omega)$ invariant functions p, independently of convexity. It does not:

Theorem 1.5. *If* $\dim X \geq 4$*, there is a smooth* $\mathrm{Ham}(\omega)$ *invariant function* $p : C^\infty(X) \to \mathbb{R}$ *that is not invariant under volume preserving diffeomorphisms* $X \to X$.

The last statement of Theorem 1.1 suggests that, after all, the only invariant norm on $C^\infty(X)$ that is of interest for symplectic geometry, is Hofer's norm $\| \ \|_\infty$. However, all invariant norms are of interest for Kähler geometry. The groups $\mathrm{Symp}(\omega)$ and $\mathrm{Ham}(\omega)$ can be regarded as symmetric spaces. When (X, ω) is Kähler, Donaldson, Mabuchi, and Semmes proposed that the infinite dimensional manifold $\mathcal{H}_\omega$ of relative Kähler potentials, endowed with a natural connection on its tangent bundle, should be viewed as the dual symmetric space, at least in a formal sense; see [Do, M, S1, S2]. $\mathrm{Ham}(\omega)$ invariant norms on $C^\infty(X)$ induce Finsler metrics on $\mathcal{H}_\omega$ that are invariant under parallel transport, and, perhaps surprisingly, all these Finsler metrics induce genuine metrics on $\mathcal{H}_\omega$. Mabuchi was the first to study such a metric, associated with L^2-norm $\|\xi\| = (\int_X |\xi|^2\omega^n)^{1/2}$; more recently, Darvas in [Da] introduced various Orlicz norms on $C^\infty(X)$ and the induced metrics on $\mathcal{H}_\omega$. Generalizing Darvas's norms and metrics, in [L] we study general $\mathrm{Ham}(\omega)$ invariant Lagrangians and the associated action on $\mathcal{H}_\omega$, and most results here are motivated by the needs of that paper.

Acknowledgement. My referee indicated places in the first version of this paper that risked to be confusing, one outright incorrect. In addition to clarifying and correcting, here I have replaced my original proof of Lemma 7.2c by the simpler proof that s/he suggested.

2. Reduction to linear forms

In this section (X, ω) can be any $2n$ dimensional symplectic manifold, not necessarily compact. The space of compactly supported smooth functions on X will be denoted $\mathcal{D}(X)$, with its usual locally convex inductive limit topology. Its dual is $\mathcal{D}'(X)$, the space of distributions. The group $\mathrm{Ham}_0(\omega)$, time 1 maps of compactly supported Hamiltonian flows, acts on $\mathcal{D}(X)$ by pull back and on $\mathcal{D}'(X)$ by push forward. We denote the pairing between $\mathcal{D}'(X)$ and $\mathcal{D}(X)$ by $\langle\ ,\ \rangle$. The locally convex topology of $\mathcal{D}'(X)$ is generated by the seminorms $\|f\|'_\xi = |\langle f, \xi\rangle|$ with $\xi \in \mathcal{D}(X)$. Integration against any smooth $2n$–form defines a distribution. Such distributions will be called smooth. If $h \in \mathcal{D}'(X)$, we denote by $\mathrm{conv}(h)$ the closed convex hull of the $\mathrm{Ham}_0(\omega)$ orbit of h.

The main result of this section is

Lemma 2.1. *Suppose* $p : \mathcal{D}(X) \to \mathbb{R}$ *is a* $\mathrm{Ham}_0(\omega)$ *invariant, continuous, convex function. There is a family* $\mathcal{A} \subset \mathbb{R} \times C^\infty(X)$ *such that*

$$(2.1) \qquad p(\xi) = \sup\Big\{a + \int_X f\xi\omega^n : (a,f) \in \mathcal{A}\Big\}, \qquad \text{for all } \ \xi \in \mathcal{D}(X).$$

If p *is positively homogeneous as well* ($p(c\xi) = cp(\xi)$ *for* $0 < c < \infty$), *then* $\mathcal{A}$ *can be chosen in* $\{0\} \times C^\infty(X)$.

For the proof we need certain regularization maps $\mathcal{D}'(X) \to \mathcal{D}'(X)$. Let $U \subset\subset X$ be open, and assume that on a neighborhood of $\overline{U}$ there are local coordinates x_ν in which ω takes the form $\sum_1^n dx_\nu \wedge dx_{n+\nu}$. Let $C \subset X \setminus \overline{U}$ be compact. Fix $\varphi_\nu \in \mathcal{D}(X)$, $\nu = 1, \dots, 2n$, vanishing on a neighborhood of C, such that $\varphi_\nu = x_\nu$ in a neighborhood of $\overline{U}$. Let $g_\nu^\tau, \tau \in \mathbb{R}$, denote the Hamiltonian flow of φ_ν for $\nu \le n$ and of $-\varphi_\nu$ for $\nu > n$; i.e., the flow of the vector fields $\pm\mathrm{sgrad}\,\varphi_\nu$. If $t = (t_1, \dots, t_{2n}) \in \mathbb{R}^{2n}$, put

$$g^t = g_1^{t_1} \circ g_2^{t_2} \circ \cdots \circ g_{2n}^{t_{2n}}.$$

Near C we have $g^t = \mathrm{id}$; on U, for small t, $g^t(x) = x - t$. Let furthermore $\chi \in \mathcal{D}(\mathbb{R}^{2n})$ be nonnegative, $\int_{\mathbb{R}^{2n}} \chi(t) dt_1 \dots dt_{2n} = 1$. For $\lambda \in [1, \infty)$ define operators $R_\lambda : \mathcal{D}'(X) \to \mathcal{D}'(X)$ by

$$R_\lambda h = \lambda^{2n} \int_{\mathbb{R}^{2n}} \chi(\lambda t)(g_*^t h) dt_1 \dots dt_{2n} \in \mathrm{conv}(h), \quad h \in \mathcal{D}'(X).$$

Standard properties of convolutions imply

Lemma 2.2. $\lim_{\lambda\to\infty} R_\lambda h = h$ *for* $h \in \mathcal{D}'(X)$. *If the support of* χ *is sufficiently close to 0, then* $R_\lambda h \in$ *conv*(h) *is smooth on* U *and* $R_\lambda h = h$ *on a neighborhood of* C. *Furthermore, if* $V \subset\subset W \subset X$ *are open, and* h *is smooth on* W, *then* $R_\lambda h$ *is smooth on* V *for sufficiently large* λ.

Lemma 2.3. *For any* $h \in \mathcal{D}'(X)$, *smooth distributions are dense in conv*(h).

Proof. It will suffice to prove that given a finite $\Xi \subset \mathcal{D}(X)$ and $\varepsilon > 0$, there is a smooth $h' \in \mathrm{conv}(h)$ such that $|\langle h' - h, \xi\rangle| \le \varepsilon$ for all $\xi \in \Xi$. To show this latter, for each $z \in X$ construct an open neighborhood $V(z) \subset\subset X$ so that in a neighborhood of $\overline{V(z)}$ we can write $\omega = \sum dx_\nu \wedge dx_{n+\nu}$ in suitable local coordinates. Select a locally finite cover $V(z_1), V(z_2), \dots$ of X. Thus the $V(z_j)$ form a finite or infinite cover depending on whether X is compact or not. For each j we can find $U_j \supset\supset V(z_j)$ such that $\{U_j\}_j$ is still locally finite, and $\omega = \sum dx_\nu \wedge dx_{n+\nu}$ is still valid in some neighborhood of $\overline{U_j}$. Fix furthermore open sets V_j^i, $i \in \mathbb{N}$, such that

$$U_j = V_j^1 \supset\supset V_j^2 \supset\supset \cdots \supset V(z_j),$$

and compact sets $C_j \subset X \setminus \bigcup_{k>j} \overline{U}_k$, $C_0 = \emptyset$, such that $C_{j-1} \subset \operatorname{int} C_j$ and $\bigcup_j C_j = X$. We let $h_0 = h$ and construct $h_j \in \operatorname{conv}(h)$ so that for $j \geq 1$

$$\begin{aligned}
&|\langle h_j - h, \xi\rangle| < \varepsilon \quad \text{if } \xi \in \Xi;\\
&h_j|V_1^j \cup \cdots \cup V_j^j \quad \text{is smooth;}\\
&h_j = h_{j-1} \quad \text{on } \operatorname{int} C_{j-1}.
\end{aligned}$$

Assuming we already have h_{j-1}, we apply Lemma 2.2 with $U = U_j$, $C = C_{j-1}$, $V = V_1^j \cup \cdots \cup V_{j-1}^j$, and $W = V_1^{j-1} \cup \cdots \cup V_{j-1}^{j-1}$. If λ is sufficiently large, then $h_j = R_\lambda h_{j-1}$ will do as the next distribution. Note that h_j is smooth over $V \cup U_j \supset V_1^j \cup \cdots \cup V_{j-1}^j \cup V_j^j$.

Thus $h_j = h_{j+1} = \ldots$ on $\operatorname{int} C_j$ and $h_j|V(z_1) \cup \cdots \cup V(z_j)$ is smooth. If X is compact, we take h' to be the last h_j; otherwise, we take $h' = \lim_{j\to\infty} h_j$.

Proof of Lemma 2.1. By an affine function we mean a function $\mathcal{D}(X) \to \mathbb{R}$ of the form const + linear. Clearly, if an affine function is bounded above on a symmetric neighborhood of $0 \in \mathcal{D}(X)$, it is bounded below as well, hence continuous.

Let $\mathcal{B}$ denote a collection of affine functions $\beta : \mathcal{D}(X) \to \mathbb{R}$ such that $\beta \leq p$. Thus $\beta \in \mathcal{B}$ can be written

$$\beta(\xi) = a + \langle h, \xi\rangle, \qquad \text{with } a \in \mathbb{R}, \quad h \in \mathcal{D}'(X). \tag{2.2}$$

The Banach–Hahn separation theorem gives that $p = \sup_{\beta\in\mathcal{B}} \beta$ with a suitable choice of $\mathcal{B}$. If p is positively homogeneous, another version of the Banach–Hahn theorem, see e.g. [Sc, p.317-319], gives that $\mathcal{B}$ can be taken to consists of linear forms, i.e. all a will be 0.

By the invariance of p, if β in (2.2) is in $\mathcal{B}$, then for any $g \in \operatorname{Ham}(\omega)$

$$(g_*\beta)(\xi) = a + \langle g_*h, \xi\rangle = a + \langle h, g^*\xi\rangle \leq p(\xi).$$

This means that all $g_*\beta$ can be adjoined to $\mathcal{B}$, and in fact we can arrange that all $\beta' = a + \langle h', \cdot\rangle$ are in $\mathcal{B}$ whenever $\beta = a + \langle h, \cdot\rangle \in \mathcal{B}$ and $h' \in \operatorname{conv}(h)$. Therefore, if we take all $\beta \in \mathcal{B}$ of form (2.2) with smooth h and write h as $f\omega^n$, the family $\mathcal{A}$ of pairs (a, f) thus obtained will do according to Lemma 2.3.

3. Proof of the second part of Theorem 1.2

This was the second part:

Theorem 3.1. *Let (X, ω) be a connected, compact, symplectic manifold. Any continuous, convex, and* $\mathrm{Ham}(\omega)$ *invariant function $p : C^\infty(X) \to \mathbb{R}$ is strict rearrangement invariant: $p(\xi) = p(\eta)$ if ξ, η are equidistributed.*

As before, μ denotes the Borel measure on X that the form ω^n determines. In our integrals below we will often omit $d\mu$ and write $\int_E f$ for $\int_E f\, d\mu$; and when $E = X$, we will even omit X and write $\int f$ for $\int_X f\, d\mu$. In the same spirit, we write $L^q(X)$ for $L^q(X, \mu)$.

We need the following result, an equivalent of Katok's Basic Lemma, valid for noncompact (but connected) X as well:

Lemma 3.2. *If $\xi, \eta \in L^1(X)$ are equidistributed, then there is a sequence of $g_k \in \mathrm{Ham}_0(\omega)$ such that*

$$\lim_{k\to\infty} \int_X |\xi - \eta \circ g_k| d\mu = 0.$$

Proof. (Essentially as in [OW], [Ha, Proposition 1.12].) Given $\varepsilon > 0$, we will find $g \in \mathrm{Ham}_0(\omega)$ such that $\int |\xi - \eta \circ g| < 5\varepsilon$. Assume first $\mu(X) < \infty$.

The measures $|\xi| d\mu$, $|\eta| d\mu$ are absolutely continuous with respect to $d\mu$, hence there is a $\delta > 0$ such that $\int_E |\xi|, \int_E |\eta| < \varepsilon$ if $\mu(E) < \delta$. Construct disjoint intervals $J_1, \dots, J_N \subset \mathbb{R}$ of length $< \varepsilon/\mu(X)$ so that $\mu(X \setminus \bigcup_i \xi^{-1} J_i) < \delta/2$, and choose compact sets $K_i \subset \xi^{-1} J_i$ so that also

$$\mu(X \setminus \bigcup_i K_i) < \delta/2. \tag{3.1}$$

By equidistribution $\mu(\eta^{-1} J_i) = \mu(\xi^{-1} J_i)$, hence there are compact $L_i \subset \eta^{-1} J_i$ such that $\mu(L_i) = \mu(K_i)$. The K_i are disjoint among themselves and so are the L_i. In this situation Katok's Basic Lemma [K, Section 3] provides a $g \in \mathrm{Ham}_0(\omega)$ such that

$$\mu(K_i \setminus g^{-1} L_i) < \delta/(2N), \quad i = 1, \dots, N. \tag{3.2}$$

If $x \in K_i \cap g^{-1} L_i$ then $\xi(x), \eta(gx) \in J_i$ and so $|\xi(x) - \eta(gx)| < \varepsilon/\mu(X)$. Conversely, $|\xi(x) - \eta(gx)| \geq \varepsilon/\mu(X)$ can happen only if

$$x \in E, \quad \text{where } E = (X \setminus \bigcup_i K_i) \cup \bigcup_i (K_i \setminus g^{-1} L_i).$$

By (3.1), (3.2) $\mu(E) < \delta$, whence $\mu(gE) < \delta$ and

$$\int |\xi - \eta \circ g| = \int_{X \setminus E} |\xi - \eta \circ g| + \int_E |\xi - \eta \circ g| < \varepsilon + \int_E |\xi| + \int_{gE} |\eta| < 3\varepsilon.$$

This takes care of X of finite measure. In general, choose an $a > 0$ so that the super–level sets $Y_1 = \{|\xi| \geq a\}$ and $Y_2 = \{|\eta| \geq a\}$ satisfy $\int_{X\setminus Y_1} |\xi| = \int_{X\setminus Y_2} |\eta| < \varepsilon$. Then $\mu(Y_1) = \mu(Y_2) < \infty$. The functions

$$\xi' = \begin{cases} \xi & \text{on } Y_1 \\ 0 & \text{on } X \setminus Y_1 \end{cases} \qquad \text{and} \qquad \eta' = \begin{cases} \eta & \text{on } Y_2 \\ 0 & \text{on } X \setminus Y_2 \end{cases}$$

are also equidistributed. Construct a connected open $X' \subset X$ of finite measure containing $Y_1 \cup Y_2$. By what we have proved so far, there is a $g \in \mathrm{Ham}_0(\omega|X')$ such that $\int_{X'} |\xi' - \eta' \circ g| < 3\varepsilon$. Extend g to all of X by identity on $X \setminus X'$. Denoting this extension also by g, we have

$$\int |\xi - \eta \circ g| \leq \int |\xi' - \eta' \circ g| + \int |\xi - \xi'| + \int |\eta - \eta'| < 3\varepsilon + \varepsilon + \varepsilon = 5\varepsilon.$$

To finish the proof, we let $\varepsilon = 1/k$ and $g = g_k$, $k \in \mathbb{N}$, and obtain the sequence sought.

Proof of Theorem 3.1. Consider $\mathcal{A} \subset \mathbb{R} \times C^\infty(X)$ of Lemma 2.1:

$$p(\xi) = \sup \Big\{ a + \int f\xi : (a, f) \in \mathcal{A} \Big\}.$$

Suppose $\xi, \eta \in C^\infty(X)$ are equidistributed, and let g_k be as in Lemma 3.2. With any $(a, f) \in \mathcal{A}$

$$p(\eta) = p(\eta \circ g_k) \geq a + \int (\eta \circ g_k) f \to a + \int f\xi \quad \text{as } k \to \infty.$$

Taking sup over $(a, f) \in \mathcal{A}$, $p(\eta) \geq p(\xi)$ follows, and in fact $p(\xi) = p(\eta)$ by symmetry.

4. Proof of the first part of Theorem 1.2

This is what the first part says:

Theorem 4.1. *If (X, ω) is a connected compact symplectic manifold, any continuous, convex,* $\mathrm{Ham}(\omega)$ *invariant function $p : C^\infty(X) \to \mathbb{R}$ is continuous in the sup norm topology on $C^\infty(X)$.*

We will use the following standard fact:

Lemma 4.2. *Let V be a locally convex topological vector space over $\mathbb{R}$. If $p : V \to \mathbb{R}$ is convex and bounded above on some open $U \subset V$, then it is continuous on U.*

Proof. We can assume U is convex. Say, we want to prove continuity at $0 \in U$. Let $s = \sup_U p < \infty$. With $0 < \lambda < 1$ and $v \in (\lambda U) \cap (-\lambda U)$ convexity implies

$$\left.\begin{array}{l} p(v) - p(0) \le \lambda(p(v/\lambda) - p(0)) \le \lambda(s - p(0)) \\ p(0) - p(v) \le \lambda(p(-v/\lambda) - p(0)) \le \lambda(s - p(0)) \end{array}\right\} \to 0$$

when $\lambda \to 0$, as needed.

The key to the proof of Theorem 4.1 is the following.

Lemma 4.3. *Let $\mathcal{F} \subset L^1(X)$ be a* $\mathrm{Ham}(\omega)$ *invariant family of functions. If for every $\xi \in C^\infty(X)$*

$$\sup_{f \in \mathcal{F}} \int_X f\xi \, d\mu < \infty, \tag{4.1}$$

then $\sup_{f \in \mathcal{F}} \int_X |f| \, d\mu < \infty$.

This is not hard to show and will suffice to prove Theorem 4.1; but later we will need a more precise statement, whose proof is just a little more involved. Let $\xi^+ = \max(\xi, 0)$ and $\xi^- = \max(-\xi, 0)$ denote the positive and negative parts of functions $\xi \colon X \to \mathbb{R}$. If $E \subset X$ has positive measure, write $⨍_E \xi$ for the average $\int_E \xi / \mu(E)$ of an integrable function. If $\mu(E) = 0$, we let $⨍_E \xi = 0$.

Lemma 4.4. *Let $f \in L^1(X)$, $\xi \in L^\infty(X)$, and $S, T \subset X$ be of equal measure. If $\xi \ge 0$ on T and $\xi \le 0$ on $X \setminus T$, then*

$$\sup\Big\{ \int_X (f \circ g)\xi \colon g \in \mathrm{Ham}(\omega) \Big\} \ge ⨍_S f \int \xi^+ - ⨍_{X \setminus S} f \int \xi^- . \tag{4.2}$$

First we show how this implies Lemma 4.3.

Proof of Lemma 4.3. We can assume $\mu(X) = 1$. Let $M(\xi)$ denote the left hand side of (4.1). Fix a nonnegative $\xi \in C^\infty(X)$ that is not identically

0, but $T' = \{\xi > 0\}$ has measure $\le 1/2$. Let $f \in \mathcal{F}$. Suppose first that $S = \{f \ge 0\}$ has measure $\ge 1/2$, and choose $T \supset T'$ so that $\mu(S) = \mu(T)$. By Lemma 4.4 $M(\xi) \ge ⨍_S f \int \xi$, hence

$$\int f^+ \le M(\xi) \Big/ \int \xi.$$

If, instead of S, $\{f \le 0\}$ has measure $\ge 1/2$, Lemma 4.4 implies in the same way that $\int f^- \le M(-\xi)/\int \xi$. Since $\max\big(M(1), M(-1)\big) \ge |\int f| = |\int f^+ - \int f^-|$, in both cases we obtain a bound for $\int |f| = \int f^+ + \int f^-$, as claimed.

Given $f \in L^1(X)$, we will write $\mathrm{conv}_1(f)$ for the closure, in the $L^1(X)$ topology, of the convex hull of the orbit of f under $\mathrm{Ham}(\omega)$. In light of Lemma 3.2 this is the same as the closed convex hull of all strict rearrangements of f. To prove Lemma 4.4 we need the following.

Lemma 4.5. *If $f \in L^1(X)$ and $E \subset X$ has positive measure, then the function*

$$f' = \begin{cases} ⨍_E f & \text{on } E \\ f & \text{on } X \setminus E \end{cases}$$

is in $\mathrm{conv}_1(f)$.

Proof. If two functions $f, h \in L^1(X)$ are at L^1 distance $\le \varepsilon$, then their $\mathrm{Ham}(\omega)$ orbits are at Hausdorff distance $\le \varepsilon$, and so are therefore $\mathrm{conv}_1(f)$ and $\mathrm{conv}_1(h)$. Hence, given E, if the lemma holds for a sequence $f = f_k$, $k = 1, 2, \ldots$, and $f_k \to f_0$ in L^1, then the lemma will hold for f_0 as well.

Now suppose that E is the disjoint union of E_j, $j = 1, \ldots, m$, of equal measure, and $f = c_j$ is constant on each E_j. If σ is a permutation of $1, \ldots, m$, define $f_\sigma \in L^1(X)$ by

$$f_\sigma = c_{\sigma(j)} \quad \text{on } E_j, \qquad f_\sigma = f \quad \text{on } X \setminus E.$$

As a strict rearrangement of f, by Lemma 3.2 f_σ is in the closure of the $\mathrm{Ham}(\omega)$ orbit of f. Therefore

$$f' = \sum_\sigma f_\sigma / m!$$

is indeed in $\mathrm{conv}_1(f)$. Since any $f \in L^1(X)$ is the limit of functions of the above type, the claim follows.

Proof of Lemma 4.4. Write χ_A for the characteristic function of a set A. By Lemma 3.2 there is a sequence $g_k \in \mathrm{Ham}(\omega)$ such that $\chi_S \circ g_k \to \chi_T$ in L^1. Two applications of Lemma 4.5 give that

$$f' = \begin{cases} \fint_S f & \text{on } S \\ \fint_{X\setminus S} f & \text{on } X \setminus S \end{cases} \quad \text{and so} \quad f'' = \lim_k f' \circ g_k = \begin{cases} \fint_S f & \text{on } T \\ \fint_{X\setminus S} f & \text{on } X \setminus T \end{cases}$$

are in $\mathrm{conv}_1(f)$. Lemma 4.4 follows, since the left hand side in (4.2) is

$$\geq \int f''\xi = \fint_S f \int_T \xi + \fint_{X\setminus S} f \int_{X\setminus T} \xi = \fint_S f \int \xi^+ - \fint_{X\setminus S} f \int \xi^-.$$

Proof of Theorem 4.1. If a function is continuous in the sup norm topology, we will say it is $\| \ \|_\infty$–continuous, and use similar terminology for other topological notions. First assume that p of the theorem is positively homogeneous as well. By Lemma 2.1 there is a family $\mathcal{F} \subset L^1(X)$ such that

$$p(\xi) = \sup\Big\{ \int f\xi : f \in \mathcal{F} \Big\}. \tag{4.3}$$

If we replace $\mathcal{F}$ by its $\mathrm{Ham}(\omega)$ orbit, the supremum in (4.3) will not change, for

$$\int (f \circ g)\xi = \int (\xi \circ g^{-1}) f \leq p(\xi \circ g^{-1}) = p(\xi) \qquad \text{if } f \in \mathcal{F},\ g \in \mathrm{Ham}(\omega).$$

Therefore we may assume that the family $\mathcal{F}$ in (4.3) is already invariant under $\mathrm{Ham}(\omega)$. Hence Lemma 4.3 gives $\sup_{\mathcal{F}} \int |f| < \infty$. This implies p is bounded on $\| \ \|_\infty$–bounded subsets of $C^\infty(X)$, and by Lemma 4.2 it is $\| \ \|_\infty$–continuous.

For general p, pick a number $c > p(0)$ and consider the Minkowski functional q of the convex set $\{p < c\}$ (see e.g. [Sc, pp. 315-317]),

$$q(\xi) = \inf\{\lambda \in (0,\infty) : p(\xi/\lambda) < c\} \in [0,\infty).$$

This is a convex, positively homogeneous, strict rearrangement invariant function, that is continuous—because locally bounded—in the topology of $C^\infty(X)$. By what we have already proved, it is $\| \ \|_\infty$–continuous, in particular, the set $U_c = \{q < 1\} \supset \{p < c\}$ is $\| \ \|_\infty$–open. If $\xi \in U_c$ then $p(\xi/\lambda) < c$ with some $\lambda < 1$. Also $p(0) < c$. As ξ is a point on the segment connecting 0, ξ/λ, convexity implies $p(\xi) < c$. Thus p is bounded above on the $\| \ \|_\infty$–open set U_c, and by Lemma 4.2 it is continuous there. The theorem follows since $\bigcup_c U_c = C^\infty(X)$.

5. Extending convex functions

The above ideas can be developed to prove that p can be extended to $C(X)$ and, under an additional assumption, to the Banach space $B(X)$ of bounded Borel functions, with the supremum norm. (Thus $L^\infty(X)$ is a quotient of $B(X)$, but $B(X)$ is more natural to use in our setting.) In this section, X is compact and connected.

Definition 5.1. *If $V \subset B(X)$ is a vector subspace, we say that a function $p\colon V \to \mathbb{R}$ is strongly continuous if $p(\xi_k)$ is convergent whenever $\xi_k \in V$ is an almost everywhere convergent sequence of uniformly bounded functions.*

The limit $\lim p(\xi_k)$ depends only on $\lim \xi_k = \xi$, since two such sequences can be combined into one sequence, converging to ξ.

Theorem 5.2. *Any continuous, convex,* $\mathrm{Ham}(\omega)$ *invariant $p\colon C^\infty(X) \to \mathbb{R}$ has a unique continuous extension to $C(X)$; this extension is convex and* $\mathrm{Ham}(\omega)$ *(hence strict rearrangement) invariant. If p is strongly continuous, then it has a unique strongly continuous extension $q : B(X) \to \mathbb{R}$. This extension is convex, and invariant under strict rearrangements.*

Since $C^\infty(X)$ is dense in $C(X)$, and p is known to be continuous in supremum norm, for the first part of Theorem 5.2 one only needs to prove that a continuous extension exists. This is a special case of the following:

Lemma 5.3. *Let W be a locally convex topological vector space over $\mathbb{R}$, $V \subset W$ a dense subspace. Any continuous, convex $p : V \to \mathbb{R}$ can be extended to a continuous $q : W \to \mathbb{R}$.*

Proof. First we show that any $w \in W$ has a convex neighborhood U such that p is bounded on $V \cap U$. By continuity, there certainly is a symmetric, convex neighborhood $U_0 \subset W$ of 0 such that p is bounded on $V \cap 4U_0$. Now $w + 2U_0$ is a neighborhood of w, and if $v_1 \in V$ is sufficiently close to w, then $U = v_1 + 2U_0$ is also. For any $v \in V \cap U$ convexity implies

$$2p(v) \le p(2v_1) + p\big(2(v - v_1)\big).$$

Since $v - v_1 \in 2U_0$, the right hand side is bounded as v varies in $V \cap U$. Thus p is bounded above on $V \cap U$. But then $p(v) + p(2v_1 - v) \ge 2p(v_1)$ gives that p is also bounded below. Set $s = \sup_U |p|$.

We let $U' = v_1 + U_0$ and show that p is uniformly continuous on $V \cap U'$. For suppose $\lambda \in (0, \infty)$. If $u, v \in V \cap U'$ and $v - u \in U_0/\lambda$, then $v + \lambda(v - u) \in$

$v_1 + U_0 + U_0 = U$, hence by convexity

$$p(v) - p(u) \le \frac{p\big(v + \lambda(v-u)\big) - p(u)}{1+\lambda} \le \frac{2s}{1+\lambda}.$$

Since the roles of u, v are symmetric, this indeed proves locally uniform continuity; which in turn implies continuous extension.

Proof of Theorem 5.2. We have already seen that the first half of the theorem follows from Lemma 5.3. As to the uniqueness of extension to $B(X)$, we note that Lusin's theorem implies that any $\xi \in B(X)$ is the a.e. limit of a uniformly bounded sequence of continuous, hence also of smooth functions ξ_k. Therefore at ξ the extension of p must take the value $\lim_k p(\xi_k)$, so it is unique. What remains is to construct the required extension q.

If $\xi \in B(X)$, predictably we let $q(x) = \lim_k p(\xi_k)$, where the uniformly bounded sequence $\xi_k \in C^\infty(X)$ converges to ξ a.e., cf. Definition 5.1. As we saw, this is independent of the choice of the sequence ξ_k. Clearly $p = q$ on $C^\infty(X)$. If uniformly bounded $\eta_k \in C^\infty(X)$ converge to $\eta \in B(X)$ a.e., and $\lambda \in [0, 1]$, then

$$\begin{aligned} q\big(\lambda\xi + (1-\lambda)\eta\big) &= \lim_k p\big(\lambda\xi_k + (1-\lambda)\eta_k\big) \\ &\le \lim_k \lambda p(\xi_k) + (1-\lambda)p(\eta_k) = \lambda q(\xi) + (1-\lambda)q(\eta), \end{aligned}$$

i.e., q is convex. It is also strongly continuous. For this it suffices to show that if uniformly bounded $\xi_k \in B(X)$ converge to ξ a.e., then a subsequence of $q(\xi_k)$ tends to $q(\xi)$. By dominated convergence,

$$\lim_k \int |\xi_k - \xi| = 0. \tag{5.1}$$

Let each ξ_k be the a.e. limit of a uniformly bounded sequence $\xi_k^i \in C^\infty(X)$, as $i \to \infty$. We can arrange that the double sequence ξ_k^i is also uniformly bounded. Thus $\lim_{i\to\infty} p(\xi_k^i) = q(\xi_k)$. For each k choose $i = i_k$ so that $\eta_k = \xi_k^i$ satisfies

$$|p(\eta_k) - q(\xi_k)| < 1/k, \qquad \int |\eta_k - \xi_k| < 1/k. \tag{5.2}$$

In view of (5.1) $\lim_k \int |\eta_k - \xi| = 0$, so a subsequence $\eta_{k(j)}$ converges to ξ a.e. Hence, by (5.2)

$$q(\xi) = \lim_j p(\eta_{k(j)}) = \lim_j q(\xi_{k(j)}),$$

as needed.

Finally, to show that q is invariant under strict rearrangements, consider equidistributed $\xi, \eta \in B(X)$. By Lemma 3.2 there are $g_k \in \mathrm{Ham}(\omega)$ such that $\int |\eta - \xi \circ g_k| \to 0$ as $k \to \infty$. Choose uniformly bounded $\xi_k \in C^\infty(X)$ converging to ξ a.e. In particular, $\lim_k \int |\xi_k - \xi| = 0$. Then

$$\lim_k \int |\xi_k \circ g_k - \eta| \le \limsup_k \int |(\xi_k - \xi) \circ g_k| + \limsup_k \int |\xi \circ g_k - \eta| = 0.$$

Again, this means that a subsequence of $\xi_k \circ g_k$ converges a.e. to η, whence

$$q(\xi) = \lim_k p(\xi_k) = \lim_k p(\xi_k \circ g_k) = q(\eta),$$

which proves that q is indeed invariant under strict rerrangements.

Here is the last theorem in this section.

Theorem 5.4. *If a strict rearrangement invariant convex $p : B(X) \to \mathbb{R}$ is strongly continuous, then it is Lipschitz continuous on bounded sets.*

Lemma 5.5. *There is a continuous $\theta : X \to [0, \mu(X)]$ that is smooth away from the preimage of finitely many $t \in [0, \mu(X)]$, and that preserves measure (the target is endowed with Lebesgue measure).*

Proof. If $\zeta \in C^\infty(X)$ is a Morse function, its reverse distribution function

$$\lambda(t) = \mu(\zeta < t), \qquad t \in [\min \zeta, \max \zeta],$$

is continuous, strictly increasing, and smooth away from the set C of critical values of ζ. It is a homeomorphism $[\min \zeta, \max \zeta] \to [0, \mu(X)]$, and a diffeomorphism away from C. The function $\theta = \lambda \circ \zeta$ will therefore do, as

$$\mu(\theta < s) = \mu(\zeta < \lambda^{-1}(s)) = \lambda(\lambda^{-1}(s)) = s, \qquad s \in [0, \mu(X)].$$

We will need the notion of decreasing rearrangement of a measurable $\xi : X \to \mathbb{R}$. It is the decreasing, say, upper semicontinuous function $\xi^\star : (0, \mu(X)] \to \mathbb{R}$ that is equidistributed with ξ. Thus $\mu(s \le \xi \le t)$ is equal to the length of the maximal interval on which $s \le \xi^\star \le t$. In particular,

$$\mu(\xi \ge \xi^\star(s)) \ge s \ge \mu(\xi > \xi^\star(s)). \tag{5.3}$$

The upper semicontinuity requirement translates to left continuity of the decreasing function $\xi^\star$, which differs from the more usual convention of right continuity, but the difference is inconsequential. We can extend $\xi^\star$ to 0 to be continuous there. Obviously, with θ of Lemma 5.5 ξ and $\xi^\star \circ \theta$ are equidistributed.

Lemma 5.6. *If $\xi \in C(X)$, then $\xi^\star$ is continuous.*

Proof. Since $\xi^\star$ is always u.s.c., i.e., left continuous, all we need to show is that $\alpha = \lim_{t\to s^+} \xi^\star(t)$ cannot be less than $\beta = \xi^\star(s)$, $s \in (0, \mu(X)]$. If it were, then $\xi^{-1}(\alpha, \beta) \subset X$ would be a nonempty open subset, of positive measure. But with $t > s$

$$\mu(\alpha < \xi < \beta) = \mu(\xi > \alpha) - \mu(\xi \geq \beta) \leq \mu(\xi > \xi^\star(t)) - \mu(\xi \geq \xi^\star(s)) \leq t - s,$$

see (5.3). Letting $t \to s^+$ gives a contradiction.

Proof of Theorem 5.4. Let θ be as in Lemma 5.5. We start by showing that p is bounded on bounded sets. Otherwise there would be a bounded sequence $\xi_k \in B(X)$ such that $|p(\xi_k)| \to \infty$. The decreasing rearrangements $\xi_k^\star$ are uniformly bounded, hence by Helly's theorem contain a pointwise convergent subsequence. But along that subsequence $\xi_k^\star \circ \theta$ converges pointwise and therefore by strong continuity

$$p(\xi_k) = p(\xi_k^\star \circ \theta)$$

also converges, a contradiction.

Now boundedness on bounded sets implies Lipschitz continuity on bounded sets. For suppose $\xi \neq \eta$ have norm $\leq R$, and let ρ be the unit vector in the direction of $\xi - \eta$. With $M = \sup_{||\zeta||_\infty \leq R+1} |p(\zeta)|$, by convexity

$$\frac{p(\xi) - p(\eta)}{||\xi - \eta||_\infty} \leq \frac{p(\xi + \rho) - p(\eta)}{||\xi + \rho - \eta||_\infty} \leq 2M.$$

The roles of ξ, η being symmetric, we obtain Lipschitz continuity.

6. Proof of Theorem 1.3

To simplify notation, we will assume $\mu(X) = 1$. By Lemma 2.1 a $\mathrm{Ham}(\omega)$ invariant convex, continuous, $p\colon C^\infty(X) \to \mathbb{R}$ can be written

$$p(\xi) = \sup\Big\{a + \int f\xi : (a, f) \in \mathcal{A}\Big\} \tag{6.1}$$

with a family $\mathcal{A} \subset \mathbb{R} \times C^\infty(X)$, that can be chosen convex and invariant under $\mathrm{Ham}(\omega)$. The possible behaviors of p described in Theorem 1.3 are determined by whether all functions f that occur in $\mathcal{A}$ are constant or not.

If in $\mathcal{A}$ only constant functions occur, then (6.1) gives $p(\xi) = p(\int \xi)$ (viewing $\int \xi$ itself as a constant function on X). Henceforward we will assume

$\mathcal{A}$ contains a pair (a, f) with a nonconstant function f. According to (ii) of Theorem 1.3, we must estimate $p(\xi)$ from below with the L^1 norm of $\xi \in C^\infty(X)$. We do this in a somewhat greater generality, of relevance in the next section.

Lemma 6.1. *Suppose* $\mathcal{A} \subset \mathbb{R}\times L^1(X)$ *is convex and invariant under* $\mathrm{Ham}(\omega)$*. For* $\xi \in L^\infty(X)$ *let* $q(\xi) = \sup_{(a,f)\in\mathcal{A}} a + \int f\xi$*. If* $\mathcal{A}$ *contains a pair* (a, f) *with* f *nonconstant, then there are* $a_0 \in \mathbb{R}$ *and* $b \in (0, \infty)$ *such that*

$$q(\xi) \geq a_0 + b\int|\xi| \quad \text{if} \quad \begin{cases} \int \xi = 0, & \text{or} \\ \int \xi \geq 0 \quad \text{and} \quad \lim_{\mathbb{R}\ni\lambda\to\infty} q(\lambda) > q(0), & \text{or} \\ \int \xi \leq 0 \quad \text{and} \quad \lim_{\mathbb{R}\ni\lambda\to-\infty} q(\lambda) > q(0). \end{cases}$$

If $\mathcal{A} \subset \{0\} \times L^1(X)$*, then* a_0 *can be chosen* 0*.*

Proof. It will suffice to prove when $\mathcal{A} \subset \mathbb{R} \times L^\infty(X)$ for the following reason. If we enlarge $\mathcal{A}$ of the lemma to contain all pairs $(a, \lim_{j\to\infty} f_j)$—limit in L^1—with $(a, f_j) \in \mathcal{A}$, q will not change. Once so enlarged, we can replace in $\mathcal{A}$ each (a, f) by pairs (a, f_k), $k \in \mathbb{N}$,

$$f_k(x) = \begin{cases} \fint_{|f|>k} f & \text{if } |f(x)| > k \\ f(x) & \text{if } |f(x)| \leq k. \end{cases}$$

By Lemma 4.5 $(a, f_k) \in \mathcal{A}$, and clearly $f_k \to f$ in L^1. Hence q will not change if we make all these modifications to $\mathcal{A}$; but now the new $\mathcal{A}$ will be in $\mathbb{R} \times L^\infty(X)$.

Next fix $(a, f) \in \mathcal{A}$ with f nonconstant. If $\alpha \in (0, 1]$ let

$$s_\alpha = s_\alpha(f) = \sup_{\mu(E)=\alpha} \fint_E f, \qquad i_\alpha = i_\alpha(f) = \inf_{\mu(E)=\alpha} \fint_E f, \tag{6.2}$$

and let $s_0 = \operatorname{ess\,sup} f$, $i_0 = \operatorname{ess\,inf} f$. For every $\alpha > 0$ there is an $S = S_\alpha \subset X$ of measure α for which $\fint_S f = s_\alpha$. Indeed, consider

$$u = \inf \{t \in \mathbb{R} \colon \mu\{f > t\} \leq \alpha\}.$$

Since $\mu\{f > u\} \leq \alpha \leq \mu\{f \geq u\}$, any set S of measure α sandwiched between $\{f > u\}$ and $\{f \geq u\}$ will provide the sup in (6.2). Similarly, $S' = X \setminus S$, of measure $1 - \alpha$, satisfies $i_{1-\alpha} = \fint_{S'} f$. This implies that $s_\alpha > i_{1-\alpha}$. From the absolute continuity of $fd\mu$ with respect to $d\mu$ we deduce that s_α, i_α are continuous functions of $\alpha > 0$; continuity trivially holds at $\alpha = 0$ as well. Hence

(6.3)
$$2c = 2c(f) = \min_{0\le\alpha\le1}(s_\alpha - i_{1-\alpha}) > 0, \quad 2m = 2m(f) = \max_{0\le\alpha\le1}|s_\alpha| + |i_{1-\alpha}| < \infty.$$

Consider a $\xi \in L^\infty(X)$ and let $T = \{\xi \ge 0\}$. With $\alpha = \mu(T)$ and $S = S_\alpha$ as above, Lemma 4.4 implies

$$(6.4)\quad q(\xi) \ge a + s_\alpha \int \xi^+ - i_{1-\alpha}\int \xi^- = a + \frac{s_\alpha - i_{1-\alpha}}{2}\int|\xi| + \frac{s_\alpha + i_{1-\alpha}}{2}\int \xi$$

(even if $\alpha = 0$). When $\int \xi = 0$, by (6.3) we obtain $q(\xi) \ge a + c\int|\xi|$.

Next suppose that $\lim_{\lambda\to\infty} q(\lambda) > q(0)$. There are $\lambda > 0$ and $(a_1, f_1) \in \mathcal{A}$ with $a_1 + \int f_1\lambda > q(0) \ge a_1$; hence $\int f_1 > 0$. Because $\mathcal{A}$ is convex, we can arrange that our fixed $(a, f) \in \mathcal{A}$ already satisfies $\int f > 0$. Let $b = s_1c/(s_1+m)$. We will show that if $\int \xi \ge 0$, then $q(\xi) \ge a+b\int|\xi|$. Note that the constant function $f' = \int f$ is in $\mathrm{conv}_1(f)$ according to Lemma 4.5, and (a, f') is in $\mathcal{A}$. Hence $q(\xi) \ge a + \int f'\xi = a + s_1\int\xi$. By (6.4) $q(\xi) \ge a + c\int|\xi| - m\int\xi$. Combining these two we can eliminate $\int\xi$ and obtain

$$mq(\xi) + s_1q(\xi) \ge (m + s_1)a + s_1c\int|\xi|,$$

as needed. Finally, if $\lim_{\lambda\to-\infty} q(\lambda) > q(0)$, we choose $(a, f) \in \mathcal{A}$ such that f is nonconstant and $\int f < 0$. Letting $b = c(f)|s_1(f)|/(|s_1(f)| + m(f))$ we can similarly prove $q(\xi) \ge a + b\int|\xi|$ whenever $\int\xi \le 0$. This completes the proof of the lemma, and also of the theorem.

7. Proof of Theorem 1.4

This was the theorem:

Theorem 7.1. *Given a* $\mathrm{Ham}(\omega)$ *invariant continuous norm* p *on* $C^\infty(X)$*, there is a rearrangement invariant Banach function space on* X *whose norm, restricted to* $C^\infty(X)$*, is equivalent to* p*.*

We will get to the notion of rearrangement invariant Banach spaces shortly (or see the Appendix), but first we formulate a few auxiliary results that we will need. Let us say that two functions $\phi, \psi : X \to \mathbb{R}$ are similarly ordered if $(\phi(x) - \phi(y))(\psi(x) - \psi(y)) \ge 0$ for all $x, y \in X$. Put it differently, $\phi(x) > \phi(y)$ should imply $\psi(x) \ge \psi(y)$. In spite of what the language may suggest, this is not an equivalence relation (all functions are similarly ordered as a constant).

However, it is true that if ϕ and ψ are similarly ordered, and $U : \mathbb{R} \to \mathbb{R}$ is increasing, then ϕ and $U \circ \psi$ are also similarly ordered.

We will write $\phi \sim \psi$ for measurable functions $X \to \mathbb{R}$ if they are equidistributed. The following lemma in one form or another is known and, like Lemmas 7.3, 7.4, 7.5, holds in any finite measure space (X, μ) without atoms.

Lemma 7.2. *Let $\phi_0 \in L^1(X)$ be bounded below and $\psi_0 \in L^\infty(X)$.*

(a) $\sup_{\phi\sim\phi_0} \int \phi\psi_0 = \sup_{\psi\sim\psi_0} \int \phi_0\psi$.

(b) The suprema in (a) are attained, by ϕ and ψ that are similarly ordered as ψ_0 and ϕ_0.

(c) $\int \phi\psi$ is independent of the choice of $\phi \sim \phi_0$, $\psi \sim \psi_0$, as long as ϕ, ψ are similarly ordered.

Proof. (b) That the suprema are attained, at least when $\phi_0, \psi_0 \geq 0$, is proved in [BS, Chapter 2, Theorems 2.2 and 2.6]. The general result follows upon adding a constant to the functions. The proof in [BS, pp. 49-50], say, for the first supremum in (a), proceeds by first considering a simple ϕ_0 (i.e., one that takes only finitely many values), and representing the maximizing ϕ by an explicit formula, then passing to a limit. The formula shows that ϕ and ψ_0 are similarly ordered when ϕ_0 is simple; but similar ordering is preserved under pointwise limits, and must hold in general.

(c) (Borrowed from the referee's report.) We can assume that ϕ_0 and ψ_0 are similarly ordered. We will show that the vector valued functions (ϕ, ψ) and (ϕ_0, ψ_0) are equidistributed, in the sense that for any Borel set $S \subset \mathbb{R}^2$

$$\mu\{(\phi,\psi) \in S\} = \mu\{(\phi_0,\psi_0) \in S\} \quad \text{or, equivalently,} \tag{7.1}$$

$$\mu\{\phi > a, \psi > b\} = \mu\{\phi_0 > a, \psi_0 > b\} \quad \text{for all } a, b \in \mathbb{R}. \tag{7.2}$$

Indeed, given a, b, let $A = \{\phi > a\}$, $B = \{\psi > b\}$, $A_0 = \{\phi_0 > a\}$, $B_0 = \{\psi_0 > b\}$. The point is that one of $A \setminus B$ and $B \setminus A$ must be empty, since if x were in the former and y in the latter, then $\phi(x) > a \geq \phi(y)$ but $\psi(x) \leq b < \psi(y)$ would contradict the similar order. It follows that $A \subset B$ or $B \subset A$, and

$$\mu(A \cap B) = \min\big(\mu(A), \mu(B)\big).$$

Similarly,

$$\mu(A_0 \cap B_0) = \min\big(\mu(A_0), \mu(B_0)\big) = \min\big(\mu(A), \mu(B)\big) = \mu(A \cap B),$$

which is the same as (7.2). Therefore (7.1) also holds, whence $\phi\psi, \phi_0\psi_0$ are equidistributed, and have the same integral.

(a) now follows from (b) and (c).

Lemma 7.3. *If $\phi \in L^1(X)$ and $\psi \in L^\infty(X)$ are similarly ordered, then $\int \phi\psi \geq ⨍ \phi \int \psi$.*

Proof. This is Chebishev's integral inequality. See for the discrete version of the inequality—from which the lemma follows—p. 43 in [HLP], and also p. 168.

Lemma 7.4. *If $\phi_0, \psi \in L^\infty(X)$, then*

$$\sup_{\phi\sim\phi_0} \int |\phi|\psi \leq \sup_{\phi\sim\phi_0} \int \phi\psi + \sup_{\phi\sim\phi_0} \int (-\phi)\psi + ⨍ |\phi_0| \int \psi. \tag{7.3}$$

Proof. First we estimate $\int \phi^+\psi$. By Lemma 7.2 we can choose $\phi_1 \sim \phi_0$, similarly ordered as ψ, that realizes $\sup_{\phi\sim\phi_0} \int \phi\psi$. It follows that ϕ_1^+, a composition of ϕ_1 with an increasing function, is also similarly ordered as ψ. Using Lemma 7.2 once more we obtain

$$\sup_{\phi\sim\phi_0} \int \phi^+\psi = \int \phi_1^+\psi = \int \phi_1\psi + \int \phi_1^-\psi.$$

As $-\phi_1^-$ and ψ are similarly ordered, Lemma 7.3 gives $-\int \phi_1^-\psi \geq -⨍ \phi_1^- \int \psi$, and so

$$\sup_{\phi\sim\phi_0} \int \phi^+\psi \leq \int \phi_1\psi + ⨍ \phi_1^- \int \psi = \sup_{\phi\sim\phi_0} \int \phi\psi + ⨍ \phi_0^- \int \psi. \tag{7.4}$$

Replacing ϕ_0 with $-\phi_0$,

$$\sup_{\phi\sim\phi_0} \int \phi^-\psi \leq \sup_{\phi\sim\phi_0} \int (-\phi)\psi + ⨍ \phi_0^+ \int \psi, \tag{7.5}$$

and (7.3) follows by adding (7.4) and (7.5).

Lemma 7.5. *If $f_0, \xi \in L^\infty(X)$ then $\sup_{f\sim f_0} \int |f\xi| \leq 4 \sup_{f\sim f_0} |\int f\xi| + 3 ⨍ |f_0| \int |\xi|$.*

Proof. Let us start with a simple ξ. Lemma 7.4, with $\phi_0 = f_0$, $\psi = |\xi|$ gives

$$\sup_{f\sim f_0} \int |f\xi| \leq 2 \sup_{f\sim f_0} \Big| \int f|\xi| \Big| + ⨍ |f_0| \int |\xi|. \tag{7.6}$$

By Lemma 7.2

$$\sup_{f\sim f_0} \int f|\xi| = \sup_{\zeta\sim|\xi|} \int f_0\zeta. \tag{7.7}$$

Any $\zeta \sim |\xi|$ can be written as $\zeta = |\eta|$ with $\eta \sim \xi$. Indeed, suppose ξ takes distinct values $a_1, \dots, a_k$. If for some i there is no j with $a_i = -a_j$, we let $\eta \equiv a_i$ on the set $(\zeta = |a_i|)$. If for some i there is a (necessarily unique) j with $a_i = -a_j$, for each such pair we divide the set $(\zeta = |a_i| = |a_j|)$ in two parts, of measures $\mu(\xi = a_i)$, $\mu(\xi = a_j)$, and define $\eta \equiv a_i$ on the former, $\eta \equiv a_j$ on the latter.

Hence, applying Lemma 7.4 again, this time with $\phi_0 = \xi$, $\psi = f_0$, we obtain

$$\sup_{\zeta\sim|\xi|} \int f_0\zeta = \sup_{\eta\sim\xi} \int f_0|\eta| \le 2\sup_{\eta\sim\xi} \Big|\int f_0\eta\Big| + ⨍ f_0 \int |\xi|.$$

In light of (7.7) and Lemma 7.2 therefore

$$\sup_{f\sim f_0} \int f|\xi| \le 2 \sup_{f\sim f_0} \Big|\int f\xi\Big| + ⨍ |f_0| \int |\xi|.$$

Substituting this, and its counterpart with f_0 replaced by $-f_0$, into (7.6) gives the lemma, when ξ is simple. A general ξ can be uniformly approximated by simple functions ξ_m, and knowing the estimate for each ξ_m gives the estimate for ξ in the limit.

Proof of Theorem 7.1. By Lemma 2.1 $p(\xi) = \sup\{\int f\xi \colon f \in \mathcal{F}\}$ with a family $\mathcal{F} \subset L^\infty(X)$, that we can choose to be invariant under $\mathrm{Ham}(\omega)$. Because of Lemma 3.2 we can even choose it to be invariant under strict rearrangements. For any measurable $\zeta : X \to [-\infty, \infty]$ define

$$q(\zeta) = \sup\Big\{\int |f\zeta| \colon f \in \mathcal{F}\Big\} \in [0, \infty],$$

and let $B = \{\zeta \colon q(\zeta) < \infty\}$, $\| \ \| = q|B$. Some obvious properties of q are: it is positively homogeneous, $q(\eta + \zeta) \le q(\eta) + q(\zeta)$, and $|\eta| \le |\zeta|$ a.e. implies $q(\eta) \le q(\zeta)$. If $q(\zeta) = 0$ then $\zeta = 0$ a.e. on any set where some $f \in \mathcal{F}$ is nonzero; since $\mathcal{F}$ is invariant under strict rearrangements, this simply means $\zeta = 0$ a.e. By Lemma 4.3 $\sup_{f\in\mathcal{F}} \int |f| < \infty$, hence $L^\infty(X) \subset B$. Furthermore, q is invariant under all rearrangements, strict or not; this also implies by Lemma 6.1, with a suitable $b > 0$,

$$q(\zeta) \ge b \int |\zeta| \tag{7.8}$$

if $\zeta \in L^\infty(X)$.

Following Bennett–Sharpley's definition [BS, pp. 2, 3, 59], $(B, \| \enspace \|)$ is a rearrangement invariant Banach space if, in addition to the properties above, (7.8) holds for all measurable ζ, and

$$\lim_{k\to\infty} q(\zeta_k) = q(\zeta) \tag{7.9}$$

for every increasing sequence $\zeta_k \geq 0$ converging to ζ. We start with the latter. On the one hand, since q is monotone, the limit in (7.9) exists, and is $\leq q(\zeta)$. On the other, the monotone convergence theorem implies that with any $f \in \mathcal{F}$

$$\int |f\zeta| = \lim_{k\to\infty} \int |f\zeta_k| \leq \lim_{k\to\infty} q(\zeta_k).$$

Taking the sup over all $f \in \mathcal{F}$ we obtain $q(\zeta) \leq \lim_k q(\zeta_k)$, which proves (7.9). That (7.8) holds for all measurable ζ now follows because $|\zeta|$ is the limit of an increasing sequence of functions in $L^\infty(X)$.

It remains to verify that p and $\| \enspace \|$ are equivalent on $C^\infty(X)$. Clearly $p \leq \| \enspace \|$. By Lemma 7.5

$$\|\xi\| = \sup_{f\in\mathcal{F}} \int |f\xi| \leq 4 \sup_{f\in\mathcal{F}} \Big| \int f\xi \Big| + 3 \sup_{f\in\mathcal{F}} ⨍ |f| \int |\xi|, \qquad \xi \in C^\infty(X).$$

Equivalence follows, because the first supremum on the right is $p(\xi)$ and the last term is $\leq Cp(\xi)$ by Lemma 4.3 and Theorem 1.3.

8. Proof of Theorem 1.5

The construction of a smooth, $\mathrm{Ham}(\omega)$ invariant function $p : C^\infty(X) \to \mathbb{R}$ that is not invariant under volume preserving diffeomorphisms is based on symplectic rigidity; but linear rigidity, the easy kind, suffices. Let V be a $2n \geq 4$ dimensional symplectic vector space over $\mathbb{R}$, and $\mathfrak{Q}$ the vector space of quadratic forms $Q : V \to \mathbb{R}$. Linear maps of V act on $\mathfrak{Q}$ by composition. It is easy to construct a smooth function $t : \mathfrak{Q} \to \mathbb{R}$ that is invariant under the symplectic group $\mathrm{Sp}(V)$, but not under $\mathrm{SL}(V)$. For Poisson bracket $\{\ ,\ \}$ turns $\mathfrak{Q}$ into a Lie algebra, and induces the adjoint action $\mathrm{ad}_Q : \mathfrak{Q} \to \mathfrak{Q}$,

$$\mathrm{ad}_Q(R) = \{Q, R\} = (\mathrm{sgrad}\, Q)R, \qquad Q, R \in \mathfrak{Q}.$$

We let $t(Q) = \mathrm{tr}\, \mathrm{ad}_Q^2$. Thus t is a polynomial on $\mathfrak{Q}$. If $V \to V'$ is an isomorphism of symplectic vector spaces under which quadratic forms Q, Q' correspond, then $t(Q) = t(Q')$.

For example, suppose that V is $\mathbb{R}^{2n}$ with coordinates x_ν, y_ν and symplectic form $\sum_1^n dx_\nu \wedge dy_\nu$. Consider

$$Q(x,y) = \sum q_\nu x_\nu y_\nu, \qquad q_\nu \in \mathbb{R}.$$

As $\operatorname{sgrad} Q = \sum_\nu q_\nu(x_\nu \partial_{x_\nu} - y_\nu \partial_{y_\nu})$, monomials $x_\lambda x_\mu$, $x_\lambda y_\mu$, and $y_\lambda y_\mu$ form an eigenbasis of ad_Q, with eigenvalues $q_\lambda + q_\mu$, resp. $q_\lambda - q_\mu$, resp. $-q_\lambda - q_\mu$. Hence

$$\begin{aligned} t(Q) &= \sum_{\lambda\le\mu}(q_\lambda+q_\mu)^2 + \sum_{\lambda,\mu}(q_\lambda-q_\mu)^2 + \sum_{\lambda\ge\mu}(q_\lambda+q_\mu)^2 \\ &= \sum_{\lambda=\mu}(2q_\lambda)^2 + \sum_{\lambda,\mu}\big((q_\lambda+q_\mu)^2+(q_\lambda-q_\mu)^2\big) \\ &= 4\sum_\lambda q_\lambda^2 + 2\sum_{\lambda,\mu}(q_\lambda^2+q_\mu^2) = (4n+4)\sum_\lambda q_\lambda^2. \end{aligned} \tag{8.1}$$

Note that Q and $R = \sum r_\nu x_\nu y_\nu$ are on the same $\mathrm{SL}(V)$ orbit whenever $\prod q_\nu = \prod r_\nu$. We conclude t is not $\mathrm{SL}(V)$ invariant.

We need to introduce one more player. If a general quadratic form $Q : V \to \mathbb{R}$ is written in linear symplectic coordinates z_ν, $\nu = 1, \dots, 2n$, as $Q(z) = \sum a_{\lambda\nu} z_\lambda z_\nu$, with $a_{\lambda\nu} = a_{\nu\lambda}$, we let

$$\operatorname{Det} Q = \det(a_{\lambda\nu}).$$

Thus $\operatorname{Det} Q$ is independent of the choice of the coordinates, and is even $\mathrm{SL}(V)$ invariant.

Fix a smooth function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(s) = 0$ for $|s| \le 1/2$ and $\varphi(s) = s$ for $|s| \ge 1$. If $\xi \in C^\infty(X)$ and x is a critical point of ξ, let $Q_x = Q_{\xi,x}$ stand for the quadratic Taylor polynomial of $\xi - \xi(x)$ at x, a quadratic form on the symplectic vector space T_xX (the Hessian). Given $\varepsilon > 0$, critical points x of ξ for which $|\operatorname{Det} Q_x| \ge \varepsilon$ form a discrete and compact, hence finite set. In particular ξ has countably many nondegenerate critical points, that we denote x_i. Define $p\colon C^\infty(X) \to \mathbb{R}$ by letting

$$p(\xi) = \sum_i \varphi(\operatorname{Det} Q_{x_i}) t(Q_{x_i}); \tag{8.2}$$

we are summing over all nondegenerate critical points x_i of ξ, or only over those for which $|\operatorname{Det} Q_{x_i}| > 1/2$. We claim that p is smooth.

Indeed, given $\eta \in C^\infty(X)$, let C consist of its critical points y for which $|\operatorname{Det} Q_y| \le 1/4$, a compact subset of X, and let y_i, $1 \le i \le k$ denote the rest of its critical points. It is possible that $k = 0$, and even that η has no nondegenerate critical point at all. About each y_i construct a neighborhood U_i so that the only critical point within $\overline{U}_i$ is y_i. About each $y \in C$ construct a neighborhood $V \subset X$ with local coordinates $z_1, \dots, z_{2n}$ so that $\omega|V = \sum_\nu dz_\nu \wedge dz_{n+\nu}$. Let $U \subset\subset V$ be a neighborhood of y consisting of x such that the quadratic form $Q(z) = \sum \partial_\lambda \partial_\nu \eta(x) z_\lambda z_\nu$ has determinant $|\mathrm{Det}\, Q| < 1/3$. Choose a finite cover $\{U_{k+1}, \dots, U_l\}$ of C by such neighborhoods U. If $\xi \in C^\infty(X)$ is in a sufficiently small neighborhood of η,

in each $\overline{U}_j$, $j \le k$, ξ has a single critical point, which depends smoothly on ξ;

all critical points x of ξ in $\bigcup_{j>k} \overline{U}_j$ satisfy $|\mathrm{Det}\, Q_{\xi,x}| < 1/2$; and

ξ has no critical points outside $\bigcup_1^l \overline{U}_j$.

Therefore p in (8.2) is a smooth function in this neighborhood of η, hence everywhere.

Invariance of Det and t implies that p is $\mathrm{Ham}(\omega)$ invariant. It is, however, not invariant under general volume preserving diffeomorphisms for the following reason. Fix a coordinate system x_ν, y_ν on an open $W \subset X$, centered at some $o \in W$, such that $\omega|W = \sum dx_\nu \wedge dy_\nu$. Let $\xi \in C^\infty(X)$ be given by $\xi = 2\sum x_\nu y_\nu$ on W.

The local flow of a vector field $v = \sum a_\nu(x,y)\partial_{x_\nu} + b_\nu(x,y)\partial_{y_\nu}$ preserves ω^n if and only if $\mathrm{div}\, v = 0$; that is, if the $(2n-1)$–form

$$\alpha = \sum_\nu (a_\nu dx_\nu - b_\nu dy_\nu) \wedge \bigwedge_{\lambda \ne \nu} dx_\lambda \wedge dy_\lambda$$

is closed, or if locally $\alpha = d\beta$. This shows that the germ of any volume preserving flow at o can be continued to a volume preserving flow on all of X, that will be supported in our coordinate neighborhood. With $c_\nu \in \mathbb{R}$ consider the germ of a diffeomorphism at o

$$(x,y) \mapsto (e^{c_\nu} x_\nu, e^{c_\nu} y_\nu)_{1 \le \nu \le n}. \tag{8.3}$$

This is the time 1 map of a volume preserving flow if $\sum c_\nu = 0$. If so, there is a volume preserving diffeomorphism $g : X \to X$, supported in W, whose germ at o is (8.3). Now ξ and $\eta = \xi \circ g$ have the same critical points, and even their germs agree at all critical points except possibly at o. Hence the contributions to $p(\xi)$ and $p(\eta)$ of critical points different from o are the same. At o

$$Q_{\xi,o} = \xi = 2\sum x_\nu y_\nu, \qquad Q_{\eta,o} = \eta = 2\sum e^{2c_\nu} x_\nu y_\nu.$$

This means that $\operatorname{Det} Q_{\xi,o} = \operatorname{Det} Q_{\eta,o} = \pm 1$, while in general, in view of (8.1)

$$t(Q_{\xi,o}) = 4(4n+4)n \neq 4(4n+4)\sum e^{4c_\nu} = t(Q_{\eta,o}).$$

Therefore $p(\xi) \neq p(\eta)$, as claimed.

Note also that p is discontinuous in the sup norm topology, since arbitrarily $\|\ \|_\infty$–close to $0 \in C^\infty(X)$ there are ξ with a unique nondegenerate critical point x, where the Hessian $Q_{\xi,x}$ can be arbitrarily prescribed.

9. Appendix

Here we reproduce the definition of rearrangement invariant Banach spaces from [BS, pp. 2–3, 59], in the case of finite measure spaces.

Let (X, μ) be a finite measure space and $\mathcal{M}^+$ the space of measurable functions $X \to [0, \infty]$. A map $q : \mathcal{M}^+ \to [0, \infty]$ is a rearrangement invariant function norm if with some $c, C \in (0, \infty)$ the following hold for all $a \in [0, \infty)$ and $f, g, f_j \in \mathcal{M}^+$:

$$q(f) = 0 \text{ if and only if } f = 0 \text{ a.e}; \quad q(af) = aq(f); \quad q(f+g) \leq q(f) + q(g);$$
$$\text{if } f \leq g \text{ then } q(f) \leq q(g); \quad c\int_X f\, d\mu \leq q(f) \leq C \operatorname{ess\,sup} f;$$
$$\text{if } f_1 \leq f_2 \leq \ldots \leq f_j \leq \ldots \to f \text{ a.e, then } q(f_j) \to q(f);$$
$$\text{if } f, g \text{ are equidistributed, then } q(f) = q(g).$$

Given such q, the collection of measurable functions f that satisfy $q(|f|) < \infty$, modulo a.e. equivalence, form a Banach space with norm $\|f\| = q(|f|)$. These are the rearrangement invariant Banach spaces.

For example, L^p spaces are rearrangement invariant, but Sobolev spaces $W^{k,p}$ on a compact manifold are not if the order $k > 0$..

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