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Kohn-Rossi cohomology and complex Plateau problem

Xiankui Meng* and Stephen Shing-Toung Yau†

Dedicate to Professor Joseph J. Kohn on the occasion of his 90th birthday

Abstract: In the present work, we investigate the relationship between compact strongly pseudoconvex CR manifolds and the singularities of their Stein fillings. We compute the dimensions of Kohn-Rossi cohomology groups with values in holomorphic vector bundles in terms of local cohomology groups. As an application, we solve the classical complex Plateau problem for compact strongly pseudoconvex CR manifold X when its Stein filling V has only isolated complete intersection singularities. This generalizes earlier results of Yau.
Keywords: Kohn-Rossi cohomology, local cohomology, isolated complete intersection singularity, projective resolution.

1. Introduction

The classical complex Plateau problem is one of the fundamental questions of complex geometry. It asks which odd dimensional real submanifolds of $\mathbb{C}^N$ are boundaries of Stein manifolds. CR manifolds are abstract models of boundaries of complex manifolds. In fact, Boutet de Monvel ([3]) proved that any compact strongly pseudoconvex CR manifold of dimension at least five can be CR embedded in some complex Euclidean space. A beautiful theorem of Harvey and Lawson ([11], [12]) says that these CR manifolds are the boundaries of Stein spaces with only isolated normal singularities.

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Throughout this paper we shall assume that X is a compact connected CR manifold of real dimension $2n-1 \geqslant 5$ and V is the Stein filling of X. What remains to be determined is the necessary and sufficient conditions on X for nonexistence of singularities inside V.

One of the most important invariants in CR geometry is the so-called Kohn-Rossi cohomology groups introduced by Kohn and Rossi in [16]. Of course it would be of interest to compute the dimensions of these groups. In [29], the second named author related the Kohn-Rossi cohomology group $H^{p,q}(X)$ to the local cohomology groups at the singularities of V and answered affirmatively a conjecture of Kohn and Rossi from [16]. In case the singularities of V are hypersurface singularities, the Kohn-Rossi cohomology groups were computed explicitly. This allows him to solve the complex Plateau problem in the hypersurface case.

It has been an interesting question to compute the $\bar{\partial}_b$-cohomology groups of forms with values in a holomorphic vector bundle (cf. [16], [24]). In the first part of this paper, following the ideas of Yau ([29]), we shall consider the dimensions of these groups in terms of local cohomology.

Theorem 1.1. *Let V be an n-dimensional reduced irreducible Stein space with smooth boundary X. We assume that V is imbedded in a slightly larger reduced irreducible complex space V' with V' smooth near $X = \partial V$. Suppose $\mathscr{F}$ is a coherent analytic sheaf on V' such that $\mathscr{F}$ is locally free near X. If V is strongly pseudoconvex, then the dimensions of the Kohn-Rossi cohomology groups*

$$\dim H^{p,q}\left(X, \mathscr{F}\right) = \sum_{x \in Z \cup S} \dim H^{q+1}_{\{x\}}\left(V, \Omega^p \otimes \mathscr{F}\right) \quad \textit{for} \quad 1 \leqslant q \leqslant n-2,$$

where S is the singular locus of V and Z is the set of all points of V where $\mathscr{F}$ is not locally free.

Corollary 1.2. *Suppose that V is strongly pseudoconvex. If V is perfect (i.e., the stalks $\mathscr{O}_x$ of the structure sheaf are Cohen-Macaulay rings) and $\mathscr{F}$ is locally free, then*

$$\dim H^{0,q}\left(X, \mathscr{F}\right) = 0 \quad \textit{for} \quad 1 \leqslant q \leqslant n-2.$$

If V is smooth and $\mathscr{F}$ is locally free, then

$$\dim H^{p,q}\left(X, \mathscr{F}\right) = 0 \quad \textit{for} \quad 1 \leqslant q \leqslant n-2.$$

We next study the relationship between the vanishing of Kohn-Rossi cohomology groups of X and the corresponding properties of V.

Proposition 1.3. *Suppose, with the above notations, that V is strongly pseudoconvex and $n \geqslant 3$. Then the following statements are equivalent:*

1. $H^{0,q}(X) = 0$ *for* $1 \leqslant q \leqslant n-2$;
2. *The Stein filling V of X is perfect;*
3. $\operatorname{depth} \omega_{V,x} = n$, $x \in V$, *where ω_V is the canonical sheaf of V.*

Corollary 1.4. *Let (V, x) be a normal isolated singularity of dimension $n \geqslant 3$. If (V, x) is Cohen-Macaulay, then it is Gorenstein if and only if the projective dimension $\mathrm{pd}_{\mathscr{O}_x}(\omega_{V,x})$ is finite.*

Proposition 1.5. *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1 \geqslant 5$. Suppose X is the boundary of a strongly pseudoconvex manifold which is a modification of a Stein space at normal isolated singularities. If one of the following conditions hold,*

- $H^{1,q}(X) = 0$ *for* $1 \leqslant q \leqslant n-2$ *and the projective dimension of* Ω^{**} *(the double dual of Ω) is finite;*
- $H^{0,q}(X, \Theta) = 0$ *for* $1 \leqslant q \leqslant n-2$ *and the projective dimension of the tangent sheaf Θ is finite,*

then V is smooth.

The theory of Buchsbaum-Eisenbud ([5]) gives free resolutions of the exterior products of certain modules. These resolutions can be used to calculate local cohomology groups.

Proposition 1.6. *Let (V, x) be an isolated Gorenstein singularity of dimension n and $\mathscr{F}$ a coherent analytic sheaf on V. Suppose $\mathscr{F}_x$ is given by the following exact sequence*

$$0 \longrightarrow \mathscr{O}_x^m \xrightarrow{\phi} \mathscr{O}_x^N \to \mathscr{F}_x \longrightarrow 0,$$

and $\mathscr{F}$ is locally free on $V \setminus \{x\}$. Then

$$\dim H^q_{\{x\}}(V, \Lambda^p \mathscr{F}) = \begin{cases} 0 & \text{if} \quad p+q \leqslant n-1, \\ \dim \mathcal{S}_p(\operatorname{coker} \phi^*) & \text{if} \quad p+q = n. \end{cases}$$

Here $\phi^* : \left(\mathscr{O}_x^N\right)^* \to \left(\mathscr{O}_x^m\right)^*$ is the dual map of ϕ, $\operatorname{coker} \phi^* = \left(\mathscr{O}_x^m\right)^* / \operatorname{Im} \phi^*$, and $\mathcal{S}_p(\operatorname{coker} \phi^*)$ is the p-th symmetric power of the $\mathscr{O}_x$-module $\operatorname{coker} \phi^*$.

As a result of Theorem 1.1 and Proposition 1.6, we can obtain the following:

Theorem 1.7. *Suppose X is the boundary of a strongly pseudoconvex manifold of dimension $n \geqslant 3$ which is a modification of a Stein space V at isolated singularities. Let S be the singular set of V. If the singularities (V, x), $x \in S$ are complete intersections, then*

$$\dim H^{p,q}(X) = \begin{cases} 0 & \textit{if} \quad p+q \leqslant n-2,\ 1 \leqslant q \leqslant n-2, \\ \sum\limits_{x \in S} \tau_x^p & \textit{if} \quad p+q = n-1,\ 1 \leqslant q \leqslant n-2, \\ \sum\limits_{x \in S} \tau_x^{n-p} & \textit{if} \quad p+q = n,\ 1 \leqslant q \leqslant n-2, \\ 0 & \textit{if} \quad p+q \geqslant n+1,\ 1 \leqslant q \leqslant n-2,\ 0 \leqslant p \leqslant n, \end{cases}$$

where

$$\begin{aligned} \tau_x^p &= \dim_{\mathbb{C}} \mathcal{S}_p \left(\operatorname{Ext}^1_{\mathscr{O}_{V,x}} \left(\Omega_{V,x}, \mathscr{O}_{V,x} \right) \right) \\ &= \dim_{\mathbb{C}} \frac{\mathcal{S}_p \left(\mathscr{O}^m_{U,x} \right)}{\sum_{i=1}^m f_i \cdot \mathcal{S}_p \left(\mathscr{O}^m_{U,x} \right) + J_p(f_1, \cdots, f_m) \left(\mathcal{S}_{p-1} \left(\mathscr{O}^m_{U,x} \right) \otimes \mathscr{O}^{m+n}_{U,x} \right)}. \end{aligned}$$

Moreover, the complex space V is smooth if and only if

$$H^{n-q-1,q}(X) = 0$$

for some $1 \leqslant q \leqslant n-2$.

When the Stein filling V has only isolated complete intersection singularities, Theorem 1.7 answers the complex Plateau problem in the affirmative sense.

2. Preliminaries

In this section, we shall recall some basic notations and definitions.

2.1. Depth

Let R be a commutative ring with unit, M an R-module and $a_1, \cdots, a_r$ a sequence of elements of R. We say $a_1, \cdots, a_r$ is an M-regular sequence if the following conditions are satisfied:

- For each $1 \leqslant i \leqslant r$, a_i is not a zero-divisor on the module

$$M/(a_1, \cdots, a_{i-1})M,$$

- $M \neq (a_1, \cdots, a_r)M$.

When all a_i belong to an ideal I, we say $a_1, \cdots, a_r$ is an M-regular sequence in I. If, moreover, there is no $b \in I$ such that $a_1, \cdots, a_r, b$ is M-regular, then $a_1, \cdots, a_r$ is said to be a maximal M-regular sequence in I.

If R is a noetherian ring, M is a finite R-module and I is an ideal of R with $IM \neq M$, we call the length of the maximal M-regular sequence in I the I-depth of M and denote it by $\operatorname{depth}_I(M)$. When $(R, \mathfrak{m})$ is a local ring we write $\operatorname{depth} M$ for $\operatorname{depth}_{\mathfrak{m}} M$ and call it simply the depth of M. Moreover, if $(R, \mathfrak{m})$ is a noetherian local ring, then R is said to be Cohen-Macaulay if $\operatorname{depth} R = \dim R$.

2.2. Projective dimensions

Given a module M, a projective resolution of M is an infinite exact sequence of modules

$$\cdots \to P_n \to \cdots P_2 \to P_1 \to P_0 \to M \to 0$$

with all the P_i projective. Every module possesses a projective resolution. The length of a finite resolution is the subscript n such that P_n is nonzero and $P_i = 0$ for i greater than n. If M admits a finite projective resolution, the minimal length among all finite projective resolutions of M is called its projective dimension and denoted $\operatorname{pd}_R(M)$. If M does not admit a finite projective resolution, then by convention the projective dimension is said to be infinite.

Theorem 2.1 (Auslander-Buchsbaum)**.** *If R is a commutative Noetherian local ring and M is a non-zero finitely generated R-module of finite projective dimension, then*

$$\operatorname{pd}_R(M) + \operatorname{depth} M = \operatorname{depth} R.$$

2.3. Symmetric and exterior algebras

We follow the exposition of [5]. Let R be a commutative ring with unit. The tensor algebra of the R-module M is the graded, noncommutative algebra

$$T_R(M) = R \oplus M \oplus (M \otimes_R M) \oplus \cdots,$$

where the product of $x_1 \otimes \cdots \otimes x_m$ and $y_1 \otimes \cdots \otimes y_n$ is $x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$.

The symmetric algebra of M is the algebra $\mathcal{S}_R(M)$ obtained from $T_R(M)$ by imposing the commutative law, that is, by factoring out the two-sided ideal generated by the relations $x \otimes y - y \otimes x$ for all $x, y \in M$.

The exterior algebra of M is the algebra $\Lambda_R M$ obtained from $T_R(M)$ by imposing skew-commutativity, that is, by factoring out the two-sided ideal generated by the elements $x^2 = x \otimes x$ for all $x \in M$. (From the formula $(x+y)\otimes(x+y) = x\otimes x + x\otimes y + y\otimes x + y\otimes y$ we see that $x\otimes y + y\otimes x$ goes to 0 in $\Lambda_R M$ for all $x, y \in M$, so that $\Lambda_R M$ really is skew-commutative.)

We define the d-th symmetric power of M, written $\mathcal{S}_{R,d}(M)$ or $\mathcal{S}_d(M)$, to be the image in $\mathcal{S}_R(M)$ of $M \otimes \cdots \otimes M$ (d factors) in $T_R(M)$ and the d-th exterior power $\Lambda^d M$ to be the image in $\Lambda_R M$ of $M \otimes \cdots \otimes M$ (d factors) in $T_R(M)$.

2.4. Local cohomology

Let A be a closed subset in a topological space Y and $\mathscr{A}$ a sheaf of abelian groups on Y. We define $\Gamma_A(Y, \mathscr{A})$ as the subgroup of all elements of $\Gamma(Y, \mathscr{A})$ whose supports are contained in A. If

$$0 \to \mathscr{A} \to \mathscr{C}^0 \to \mathscr{C}^1 \to \mathscr{C}^2 \to \cdots$$

is the canonical (or any other) flabby resolution of $\mathscr{A}$, we define the groups $H^i_A(Y, \mathscr{A})$ as the cohomology groups of the complex

$$0 \to \Gamma_A\left(Y, \mathscr{C}^0\right) \to \Gamma_A\left(Y, \mathscr{C}^1\right) \to \Gamma_A\left(Y, \mathscr{C}^2\right) \to \cdots$$

and call them the groups of local cohomology with supports in A and coefficients in $\mathscr{A}$.

2.5. CR manifolds and pseudoconvexity

Let X be a connected real manifold of dimension $2n-1$ and S an $(n-1)$-dimensional subbundle of $\mathbb{C}T_X$ such that

- $S \cap \overline{S} = \{0\}$.
- If L, L' are local sections of S, then so is $[L, L]$.

The manifold X, together with the structure S, is called a CR manifold.

Let $L_1, \cdots, L_{n-1}$ be a local frame of S. Choose a purely imaginary local section N of $\mathbb{C}T_X$ such that $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}, N$ span $\mathbb{C}T_X$. Then the matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \sum a^k_{ij} L_k + \sum b^k_{ij} \overline{L}_k + c_{ij} N.$$

is Hermitian, and is called Levi form. The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \cdots, L_{n-1}, N$. X is said to be strongly pseudoconvex if the Levi form is definite at each point of X.

Throughout this paper, we always assume that X is a real hypersurface of a complex manifold M. Suppose that X is locally defined by $r = 0$, where r is a real smooth function on M with $|dr| = 1$ on X. For each point $x \in X$, the Levi form at x is the Hermitian form on the $(n-1)$-dimensional space $T^{1,0}_{M,x} \cap \mathbb{C}T_{X,x}$ given by

$$(L_1, L_2) \mapsto 2\left\langle \partial\bar{\partial} r, L_1 \wedge \overline{L}_2 \right\rangle,$$

where $T^{1,0}_{M,x}$ is the space of holomorphic vectors at x.

Let M be a complex manifold with smooth boundary $X = \partial M$ such that $\overline{M} = M \cup X$ is compact. M is said to be strongly pseudoconvex if the Levi form is positive definite at each point of X. If M is strongly pseudoconvex, then it is a modification of a Stein space V with isolated singularities. In this case, we also say V is strongly pseudoconvex.

2.6. Cotangent sheaf and tangent sheaf

We shall define the sheaf of germs of holomorphic 1-forms for arbitrary complex space V. Let us first consider a model space W in a domain $D \subset \mathbb{C}^n$ with ideal $\mathscr{I} \subset \mathscr{O}_D$. Let Ω_D be the sheaf of germs of holomorphic 1-forms on D. Then the map

$$\mathscr{I} \to \Omega_D, \quad f \to df$$

sends $\mathscr{I}^2$ into $\mathscr{I}\Omega_D$ and hence, by passing to residue classes, a morphism

$$\alpha : \mathscr{I}/\mathscr{I}^2 \to \Omega_D/\mathscr{I}\Omega_D.$$

We put $\Omega_W = \operatorname{coker} \alpha$, this is a coherent sheaf on V. The case of an arbitrary complex space is handled by gluing. Let $\{U_i\}$ be an open covering of V such that there exists a biholomorphic map $\tau_i : U_i \to V_i$ onto a model space V_i. Then we can define the sheaf $\Omega_i = \Omega_{U_i} \cong \Omega_{V_i}$ via τ_i. The isomorphisms $\tau_i^{-1} \circ \tau_j : U_i \cap U_j \to U_i \cap U_j$ give rise to isomorphisms $\theta_{ij} : \Omega_i|_{U_i \cap U_j} \to \Omega_j|_{U_i \cap U_j}$ such that $\theta_{ij}\theta_{jk} = \theta_{ik}$. Hence we have an $\mathscr{O}_V$-sheaf Ω_V on X such that $\Omega_V|_{U_i} = \Omega_i$. This sheaf is coherent on V and called the sheaf of germs of holomorphic 1-forms on V or the cotangent sheaf of V. Note that Ω^1_V is locally free if and only if V is regular.

We write $\Omega_V^p = \Lambda^p \Omega_V$ (with the usual convention that $\Omega^0 = \mathscr{O}_V$) and refer to it as the sheaf of holomorphic p-forms. The tangent sheaf Θ_V of V is defined to be the dual sheaf of Ω_V, i.e. $\Theta_V = \Omega_V^*$.

If V is normal, then we can define the canonical sheaf of V as $\omega_V = \theta_* \Omega^n_{V_{sm}}$, where V_{sm} is the regular part of V and $\theta : V_{sm} \to V$ is the inclusion map.

2.7. Isolated singularities

We shall often denote by (V, x) the pair of an analytic space V with a point $x \in V$ such that $V \backslash \{x\}$ is smooth and pure dimensional. We call such a pair an isolated singularity (even in case V is smooth). Two pairs (V, x) and (W, y) are equivalent if there exist a neighborhood $V' \subset V$ of x, a neighborhood $W' \subset W$ of y and an isomorphism $f : V' \to W'$ such that $f(x) = y$. An equivalent class of such pairs is called a germ of isolated singularities and denoted also by (V, x).

The singularity (V, x) is said to be Cohen-Macaulay if the local ring $\mathscr{O}_{V,x}$ is Cohen-Macaulay. Moreover, if (V, x) is Cohen-Macaulay and the canonical sheaf ω_V is free, then we say (V, x) is a Gorenstein singularity.

2.8. Isolated complete intersection singularities

The conventions followed are those of Looijenga ([18]). Let (U, x) be a complex manifold germ of dimension N, and $(V, x) \subset (U, x)$ an analytic subgerm of dimension n which is given by an ideal $\mathcal{I} \subset \mathscr{O}_{U,x}$. We say that $\mathcal{I}$ defines a complete intersection at x if $\mathcal{I}$ admits $m = N - n$ generators $f_1, \cdots, f_{N-n}$.

Moreover, if the common zero set of $f_1, \cdots, f_{N-n}$ and the $N - n$ form $df_1 \wedge \cdots \wedge df_{N-n}$ is contained in $\{x\}$, then we say that (V, x) is an isolated complete intersection singularity (this includes the case that (V, x) is regular). Given a coordinate $z_1, \cdots, z_N$ for (U, x), let $\mathcal{J}$ be the ideal in $\mathscr{O}_{U,x}$ generated by the determinants of the $(N - n) \times (N - n)$ submatrices of the Jacobian matrix $\left(\frac{\partial f_j}{\partial z_i}\right)$. The definition of $\mathcal{J}$ is independent of the choices of generators and the singularity (V, x) is isolated if and only if $\mathcal{J} \supset \mathfrak{m}_x^k$ for some $k \geqslant 1$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{U,x}$. The number $\dim \mathfrak{m}_x/\mathfrak{m}_x^2$ is called the embedding dimension of (V, x) and $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 - n$ is called the embedding codimension of (V, x).

If (V, x) is a complete intersection at x, then the ring $\mathscr{O}_x = \mathscr{O}_{U,x}/\mathcal{I}$ is a Gorenstein ring of dimension n and the sequence $f_1, \cdots, f_{N-n}$ is $\mathscr{O}_x$-regular. Moreover, if (V, x) is an isolated complete intersection singularity, then $\operatorname{depth}_{\mathcal{J}/\mathcal{I}} \mathscr{O}_x = n$. For a proof one may refer to [19].

3. Kohn-Rossi's $\bar{\partial}_b$-complex

In this section, we shall recall the theory of Kohn and Rossi. Let M' be a complex manifold and $X \subset M'$ a real hypersurface. Assume that X is locally defined by $r = 0$, where r is a real smooth function on M' with $|dr| = 1$ on X. Let $\mathscr{A}^{p,q}$ be the sheaf of germs of smooth differential forms of type (p, q) on M'. Let E be a holomorphic vector bundle over M' and $\mathscr{O}(E)$ the sheaf of germs of holomorphic sections of E. Let

$$\mathcal{A}^{p,q}(M', E) = \{\text{sections of } \mathscr{A}^{p,q} \otimes \mathscr{O}(E) \text{ over } M'\}$$

and

$$\mathcal{C}^{p,q}(M', E) = \{\phi \in \mathcal{A}^{p,q}(M', E);\ \bar{\partial} r \wedge \phi = 0 \text{ on } X\}.$$

It is easy to shown that

$$\bar{\partial}\mathcal{C}^{p,q}(M', E) \subset \mathcal{C}^{p,q+1}(M', E).$$

Let $\mathscr{C}^{p,q}(E)$ denote the sheaves of germs of $\mathcal{C}^{p,q}(M', E)$ on M'. Then there is a natural injection

$$0 \longrightarrow \mathscr{C}^{p,q}(E) \longrightarrow \mathscr{A}^{p,q} \otimes \mathscr{O}(E).$$

The quotient sheaf

$$\mathscr{B}^{p,q}(E) = \left(\mathscr{A}^{p,q} \otimes \mathscr{O}(E)\right) / \mathscr{C}^{p,q}(E)$$

is a locally free sheaf supported on X. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}^{p,q}(E) \arrow[r] \arrow[d, "\bar{\partial}"] & \mathscr{A}^{p,q} \otimes \mathscr{O}(E) \arrow[r] \arrow[d, "\bar{\partial}"] & \mathscr{B}^{p,q}(E) \arrow[r] \arrow[d, "\bar{\partial}_b"] & 0 \\
0 \arrow[r] & \mathscr{C}^{p,q+1}(E) \arrow[r] & \mathscr{A}^{p,q+1} \otimes \mathscr{O}(E) \arrow[r] & \mathscr{B}^{p,q+1}(E) \arrow[r] & 0
\end{tikzcd}$$

where $\bar{\partial}_b$ is the quotient map which is induced by $\bar{\partial}$. Let $\mathcal{B}^{p,q}(X, E)$ denote the space of sections of $\mathscr{B}^{p,q}(E)$. Since $\mathscr{C}^{p,q}(E)$ is fine, the induced sequence of global sections

$$0 \longrightarrow \mathcal{C}^{p,*}(M', E) \longrightarrow \mathcal{A}^{p,*}(M', E) \longrightarrow \mathcal{B}^{p,*}(X, E) \longrightarrow 0$$

is exact. Since $\bar{\partial}^2 = 0$, it follows that $\bar{\partial}_b^2 = 0$, so we have the boundary complex

$$0 \to \mathcal{B}^{p,0}(X,E) \xrightarrow{\bar{\partial}_b} \mathcal{B}^{p,1}(X,E) \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} \mathcal{B}^{p,n-1}(X,E) \to 0.$$

In fact, following Tanaka [24], the boundary complex can be reformulated in a way independent of the imbedding $X \subset M'$.

Definition 3.1. The cohomology of the above boundary complex is called Kohn-Rossi cohomology and is denoted by $H^{p,q}(X, \mathscr{O}(E))$. In the special case when E is a trivial line bundle, we may write $H^{p,q}(X)$ for $H^{p,q}(X, \mathscr{O})$.

Let M be a Hermitian complex manifold of complex dimension n with smooth boundary $X = \partial M$ such that $\overline{M} = M \cup X$ is compact. We shall assume, without loss of generality, that M is imbedded in a slightly larger open manifold M' and that X is locally defined by the equation $r = 0$, where r is a real smooth function with $r < 0$ inside M, $r > 0$ outside $\overline{M}$, and $|dr| = 1$ on X. Suppose E is a holomorphic vector bundle over M'. Let

$$\mathcal{A}^{p,q}(M,E) = \{\text{sections of } \mathscr{A}^{p,q} \otimes \mathscr{O}(E) \text{ over } M\},$$
$$\mathcal{A}^{p,q}\left(\overline{M},E\right) = \{\text{sections of } \mathscr{A}^{p,q} \otimes \mathscr{O}(E) \text{ over } \overline{M}\}.$$

Let g be a Hermitian metric on M' and let dV stand for the Riemannian volume form on M'. Then one can define a natural inner product on $\Lambda^{p,q}\mathbb{C}T^*_{M'}$. Let h be a Hermitian metric on E and we denote by $\langle \bullet, \bullet \rangle$ the corresponding inner product on $\Lambda^{p,q}\mathbb{C}T^*_{M'} \otimes E$. We define global scalar product for E-valued forms by

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle dV, \quad \text{for} \quad \phi, \psi \in \mathcal{A}^{p,q}\left(\overline{M}, E\right).$$

Let $L^{(p,q)}$ denote the Hilbert space obtained by completing $\mathcal{A}^{p,q}\left(\overline{M},E\right)$ under the above inner product. We shall henceforth use the symbol $\bar{\partial}$ to mean the closure of $\bar{\partial}|_{\mathcal{A}^{p,q}\left(\overline{M},E\right)}$ with respect to $L^{(p,q)}$. Let $\bar{\partial}^*$ be the Hilbert space adjoint of $\bar{\partial}$. Further we define the unbounded operator $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Finally we define the space $\mathcal{H}^{p,q}(M,E)$ by

$$\mathcal{H}^{p,q}(M,E) = \{\phi \in \mathrm{Dom}(\Delta_{\bar{\partial}});\ \Delta_{\bar{\partial}}\phi = 0\}.$$

There are several natural cohomology groups associated to the $\bar{\partial}$-complex on the holomorphic vector bundle E over Hermitian manifold M'. Consider the following two vector spaces:

$$H^{p,q}(M,E) = \frac{\left\{\phi \in \mathcal{A}^{p,q}(M,E);\ \bar{\partial}\phi = 0\right\}}{\bar{\partial}\mathcal{A}^{p,q-1}(M,E)},$$
$$H^{p,q}\left(\overline{M},E\right) = \frac{\left\{\phi \in \mathcal{A}^{p,q}\left(\overline{M},E\right);\ \bar{\partial}\phi = 0\right\}}{\bar{\partial}\mathcal{A}^{p,q-1}\left(\overline{M},E\right)}.$$

As a consequence of his beautiful solution of the $\bar{\partial}$-Neumann problem, Kohn proved the following:

Theorem 3.2 ([15, 16])**.** *If M is strongly pseudoconvex and $q > 0$, then*

$$H^{p,q}\left(\overline{M},E\right) \cong \mathcal{H}^{p,q}(M,E),$$

and they are finite dimensional.

On the other hand, the Dolbeault theorem asserts that

$$H^{p,q}(M,E) \cong H^{q}(M,\Omega^{p}\otimes\mathscr{O}(E)),$$

where Ω^p denotes the sheaf of germs of holomorphic p-forms on M. The relationship between these important groups and the preceding one is the following theorem.

Theorem 3.3 ([15, 14])**.** *If M is strongly pseudoconvex and $q > 0$, then*

$$H^{p,q}\left(\overline{M},E\right) \cong H^{p,q}(M,E).$$

Proof. For the convenience of the reader, we will show that this theorem follows from Theorem 3.2 and the Andreotti-Grauert theory. Since M is strongly pseudoconvex, one can find a single strictly plurisubharmonic defining function r for all of ∂M by Grauert [7]. So there is a neighborhood U of ∂M in M' and a smooth strictly plurisubharmonic function r on U such that $U \cap M = \{x \in U;\ r(x) < 0\}$. Since ∂M is compact, there exists $\delta_0 > 0$ such that $\{-3\delta_0 \leqslant r \leqslant \delta_0\} \Subset U$. Let χ be a smooth function on M' such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $\{r \geqslant -\delta_0\}$ and $\chi \equiv 0$ on $M \setminus \{r > -2\delta_0\}$. Then

$$\widetilde{r} = (r + 3\delta_0)\,\chi - 3\delta_0$$

is a smooth function on $M \cup U$ and $\widetilde{r} = r$ on $\{r > -\delta_0\}$. Let

$$M_\delta = \{\widetilde{r} < \delta\} \subset M', \quad 0 \leqslant \delta \leqslant \delta_0.$$

Then $M_0 = M$ and M_δ is 1-convex since $\widetilde{r} = r$ is strictly plurisubharmonic on $\{r > -\delta_0\}$.

Let us consider the restriction maps

$$H^{p,q}\left(M_\delta, E\right) \longrightarrow H^{p,q}\left(\overline{M}, E\right) \longrightarrow H^{p,q}\left(M, E\right).$$

By Theorem 3.2, $H^{p,q}\left(\overline{M}, E\right)$ is finite dimensional. So we can choose $\bar{\partial}$-closed $\phi_i \in \mathcal{A}^{p,q}\left(\overline{M}, E\right)$, $i = 1, \cdots, d$, such that their cohomology classes generate a basis of $H^{p,q}\left(\overline{M}, E\right)$. If $\delta > 0$ is small enough, then we may assume that $\phi_i \in \mathcal{A}^{p,q}\left(M_\delta, E\right)$ for $1 \leqslant i \leqslant d$. So we can conclude that

$$H^{p,q}\left(M_\delta, E\right) \longrightarrow H^{p,q}\left(\overline{M}, E\right)$$

is surjective. On the other hand, the restriction map

$$H^{q}\left(M_\delta, \Omega^p \otimes \mathscr{O}(E)\right) \longrightarrow H^{q}\left(M, \Omega^p \otimes \mathscr{O}(E)\right)$$

is isomorphic by the Andreotti-Grauert theory ([1]) and hence

$$H^{p,q}\left(M_\delta, E\right) \longrightarrow H^{p,q}\left(M, E\right)$$

is isomorphic by Dolbeault theorem. Then we can conclude that the restriction map

$$H^{p,q}\left(\overline{M}, E\right) \longrightarrow H^{p,q}\left(M, E\right)$$

is an isomorphism. $\square$

We next consider the duality theorem. Let

$$\mathcal{C}^{p,q}\left(\overline{M}, E\right) = \{\phi \in \mathcal{A}^{p,q}(\overline{M}, E);\ \bar{\partial} r \wedge \phi = 0 \text{ on } X\}.$$

Since

$$\bar{\partial}\mathcal{C}^{p,q}\left(\overline{M}, E\right) \subset \mathcal{C}^{p,q+1}\left(\overline{M}, E\right),$$

one can therefore form the cohomology

$$H_0^{p,q}\left(\overline{M}, E\right) = \frac{\left\{\phi \in \mathcal{C}^{p,q}\left(\overline{M}, E\right);\ \bar{\partial}\phi = 0\right\}}{\bar{\partial}\mathcal{C}^{p,q-1}\left(\overline{M}, E\right)}.$$

Theorem 3.4 ([16]). *If M is strongly pseudoconvex, then*

$$H_0^{p,q}\left(\overline{M}, E\right) \cong \left(H^{n-p,n-q}\left(\overline{M}, E^*\right)\right)^* \quad \textit{for} \quad 0 \leqslant q < n,$$

where E^* is the dual bundle of E.

4. Computation of Kohn-Rossi's $\bar{\partial}_b$-cohomology

In this section, we will compute Kohn-Rossi's $\bar{\partial}_b$-cohomology in terms of local cohomology. Let us fix the notations. Let V be an n-dimensional reduced irreducible Stein space with smooth boundary $X = \partial V$. We assume that V is imbedded in a slightly larger reduced irreducible complex space V' with V' smooth near X and that X is defined by the equation $r = 0$, where r is a real smooth function with $r < 0$ inside V, $r > 0$ outside $\overline{V}$, and $|dr| = 1$ on X.

Suppose $\mathscr{F}$ is a coherent analytic sheaf on V' and $\mathscr{F}$ is locally free near X. Let Ω^p denote the sheaf of germs of holomorphic p-forms on V'. Let S be the singular locus of V and Z the set of all points of V where the coherent sheaf $\mathscr{F}$ is not locally free. Then S and Z are compact analytic sets in V. Since V is Stein, we can conclude

$$W = S \cup Z = \{x_1, \cdots, x_m\}$$

is a finite set. For every $x_i \in W$ $(1 \leqslant i \leqslant m)$, the local cohomology group

$$H^q_{\{x_i\}}\left(V, \Omega^p \otimes \mathscr{F}\right)$$

is finite dimensional for $0 \leqslant q \leqslant n-1$. Note that $\dim H^q_{\{x_i\}}\left(V, \Omega^p\right)$ is the so-called Brieskorn number of type (p, q) at the point x_i.

Theorem 4.1. *Suppose, with the above notations, that V is strongly pseudoconvex, then the dimensions of the Kohn-Rossi cohomology groups*

$$\dim H^{p,q}\left(X, \mathscr{F}\right) = \sum_{i=1}^{m} \dim H^{q+1}_{\{x_i\}}\left(V, \Omega^p \otimes \mathscr{F}\right) \quad \textit{for} \quad 1 \leqslant q \leqslant n-2. \tag{1}$$

Proof. By the desingularization theorem of Hironaka, we can find a proper modification $\mu : N' \to V'$ such that N' is smooth, $N \Subset N'$ is strongly pseudoconvex and $\mu : \overline{N} \setminus \mu^{-1}(S) \to \overline{V} \setminus S$ is biholomorphic. Then the coherent sheaf $\mu^*\mathscr{F}$ is locally free on $N \setminus \mu^{-1}(Z)$. By a theorem of Rossi (cf. [21], Theorem 3.5), we can find another modification $\tau : M' \to N'$ such that

1. M' is smooth, $M \Subset M'$ is strongly pseudoconvex and $\partial M \cong X$;
2. $\tau : M \setminus (\mu \circ \tau)^{-1}(Z) \longrightarrow N \setminus \mu^{-1}(Z)$ is biholomorphic;
3. The coherent sheaf

$$\mathscr{E} = (\mu \circ \tau)^* \mathscr{F} \Big/ \mathscr{T}\left((\mu \circ \tau)^* \mathscr{F}\right)$$

is locally free on M', where $\mathscr{T}\left((\mu \circ \tau)^* \mathscr{F}\right) \subset (\mu \circ \tau)^* \mathscr{F}$ is the torsion subsheaf. Hence there exists a holomorphic vector bundle E over M' such that $\mathscr{E} = \mathscr{O}(E)$.

Then $\pi = \mu \circ \tau : M \to V$ is a modification of V at the points $x_1, \cdots, x_m$. Let $A = \pi^{-1}(W)$ be the exceptional set of π. Then $\pi : M \setminus A \to V \setminus W$ is biholomorphic and $\mathscr{E} = \pi^* \mathscr{F}$ on $M \setminus A$. For the simplicity of notation, we also denote by Ω^p the sheaf of germs of holomorphic p-forms on M.

We claim that there exists a smooth nonnegative strictly plurisubharmonic exhaustion function ϕ on V such that $W = \{\phi = 0\}$. In fact, suppose the maximal ideal $\mathfrak{m}_{x_i} \subset \mathscr{O}_{x_i}$ is generated by $f_{i1}, \cdots, f_{in_i}$. We may assume that $f_{ik} \in \mathscr{O}(U_i)$ and $\{x \in U_i;\ f_{i1}(x) = \cdots = f_{in_i}(x) = 0\} = \{x_i\}$, where U_i is an open neighborhood of x_i such that $U_i \cap U_j = \emptyset$. Let $\lambda_i,\ 1 \leqslant i \leqslant m$ be cut-off functions such that $\operatorname{Supp} \lambda_i \Subset U_i$, $0 \leqslant \lambda_i \leqslant 1$ and $\lambda_i = 1$ near x_i. Then the function

$$\sum_{i=1}^{m} \lambda_i \log\left(\sum_k |f_{ik}|^2\right)$$

is quasi-plurisubharmonic on V. If ψ is a smooth strictly plurisubharmonic exhaustion function on V, we may select a convex increasing function χ such that

$$\sum_{i=1}^{m} \lambda_i \log\left(\sum_k |f_{ik}|^2\right) + \chi \circ \psi$$

is strictly plurisubharmonic and exhaustion. Then we can take

$$\phi = \exp\left\{\sum_{i=1}^{m} \lambda_i \log\left(\sum_k |f_{ik}|^2\right) + \chi \circ \psi\right\}.$$

It is obvious $\phi(x) = 0$ if and only if $x \in W$. So there exists a smooth nonnegative plurisubharmonic exhaustion function $\varphi = \phi \circ \pi$ on M such that $A = \{\varphi = 0\}$ and φ is strictly plurisubharmonic on $M \setminus A$. Put

$$M_r = \{x \in M;\ \varphi(x) \leqslant r\}.$$

Let $\mathcal{A}_c^{p,q}(M,E)$ be the space of E-valued (p,q)-forms with compact supports in M and $H_c^{p,q}(M,E)$ the cohomology group with compact support. We claim that the natural inclusion map

$$i:\mathcal{A}_c^{p,*}(M,E)\longrightarrow \mathcal{C}^{p,q}\left(\overline{M},E\right)$$

induce isomorphisms

$$H_c^{p,q}(M,E)\xrightarrow{\cong} H_0^{p,q}\left(\overline{M},E\right)\quad \text{for}\quad 1\leqslant q\leqslant n-1. \tag{2}$$

In fact, our claim follows from the following commutative diagram:

$$\begin{tikzcd}
H_c^{p,q}(M,E) \arrow[r] \arrow[d, "\cong"', "\text{Serre duality}"] & H_0^{p,q}\left(\overline{M},E\right) \arrow[d, "\cong"', "\text{Theorem 3.4}"] \\
\left(H^{n-p,n-q}(M,E^*)\right)^* \arrow[r, "\cong", "\text{Theorem 3.3}"'] & \left(H^{n-p,n-q}\left(\overline{M},E^*\right)\right)^*.
\end{tikzcd}$$

Following Laufer [17], we consider the sheaf cohomology with support at infinity. There is a natural exact sequence

$$0\longrightarrow \mathcal{A}_c^{p,*}(M,E)\longrightarrow \mathcal{A}^{p,*}(M,E)\longrightarrow \mathcal{A}_\infty^{p,*}(M,E)\longrightarrow 0.$$

Then the sheaf cohomology with support at infinity $H_\infty^q(M,\Omega^p\otimes\mathscr{E})$ is the cohomology of the quotient complex $\left(\mathcal{A}_\infty^{p,*}(M,E),\bar\partial\right)$.

Another natural exact sequence is

$$0\longrightarrow \mathcal{A}_c^{p,*}(M,E)\longrightarrow \mathcal{A}^{p,*}\left(\overline{M},E\right)\longrightarrow \mathcal{A}_\infty^{p,*}\left(\overline{M},E\right)\longrightarrow 0.$$

The cohomology of $\left(\mathcal{A}_\infty^{p,*}\left(\overline{M},E\right),\bar\partial\right)$ is denoted by $H_\infty^q\left(\overline{M},\Omega^p\otimes\mathscr{E}\right)$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_c^{p,*}(M,E) \arrow[r] \arrow[d] & \mathcal{A}^{p,*}\left(\overline{M},E\right) \arrow[r] \arrow[d] & \mathcal{A}_\infty^{p,*}\left(\overline{M},E\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{A}_c^{p,*}(M,E) \arrow[r] & \mathcal{A}^{p,*}(M,E) \arrow[r] & \mathcal{A}_\infty^{p,*}(M,E) \arrow[r] & 0
\end{tikzcd}$$

It follows from Theorem 3.3 and the five lemma that

$$H_\infty^q\left(\overline{M},\Omega^p\otimes\mathscr{E}\right)\cong H_\infty^q(M,\Omega^p\otimes\mathscr{E})\quad \text{for}\quad q\geqslant 1. \tag{3}$$

Now the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}_c^{p,*}(M,E) & \longrightarrow & \mathcal{A}^{p,*}\left(\overline{M},E\right) & \longrightarrow & \mathcal{A}_\infty^{p,*}\left(\overline{M},E\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^{p,q}\left(\overline{M},E\right) & \longrightarrow & \mathcal{A}^{p,*}\left(\overline{M},E\right) & \longrightarrow & \mathcal{B}^{p,*}(X,E) & \longrightarrow & 0
\end{array}$$

gives

$$H_\infty^q\left(\overline{M},\Omega^p\otimes\mathscr{E}\right)\cong H^{p,q}(X,\mathscr{E})\quad\text{for}\quad 1\leqslant q\leqslant n-2 \tag{4}$$

by (2) and the five lemma.

We need to compute the sheaf cohomology with support at infinity. By Laufer [17],

$$H_\infty^q\left(M,\Omega^p\otimes\mathscr{E}\right)\cong\varinjlim_r H^q\left(M\setminus M_r,\Omega^p\otimes\mathscr{E}\right).$$

On the other hand, by Andreotti and Grauert [1],

$$H^q\left(M\setminus A,\Omega^p\otimes\mathscr{E}\right)\longrightarrow H^q\left(M\setminus M_r,\Omega^p\otimes\mathscr{E}\right)$$

is isomorphic for $q\leqslant n-2$ and $r>0$. So we have

$$H_\infty^q\left(M,\Omega^p\otimes\mathscr{E}\right)\cong H^q\left(M\setminus A,\Omega^p\otimes\mathscr{E}\right)\quad\text{for}\quad q\leqslant n-2. \tag{5}$$

Since $\pi:\overline{M}\setminus A\longrightarrow\overline{V}\setminus W$ is biholomorphic and

$$\Omega^p\otimes\mathscr{E}|_{M'\setminus A}=\pi^*\left(\left(\Omega^p\otimes\mathscr{F}\right)|_{V'\setminus W}\right),$$

we have

$$H^q\left(M\setminus A,\Omega^p\otimes\mathscr{E}\right)\cong H^q\left(V\setminus W,\Omega^p\otimes\mathscr{F}\right), \tag{6}$$

and

$$H^{p,q}(X,\mathscr{F})=H^{p,q}(X,\mathscr{E}). \tag{7}$$

Let us consider the following local cohomology exact sequence:

$$\begin{aligned}\to H^q\left(V,\Omega^p\otimes\mathscr{F}\right)\to H^q\left(V\setminus W,\Omega^p\otimes\mathscr{F}\right)\to H_W^{q+1}\left(V,\Omega^p\otimes\mathscr{F}\right)\\ \to H^{q+1}\left(V,\Omega^p\otimes\mathscr{F}\right)\to H^{q+1}\left(V\setminus W,\Omega^p\otimes\mathscr{F}\right)\to\cdots\end{aligned}$$

By Cartan's Theorem B, we have

$$H^q(V \setminus W, \Omega^p \otimes \mathscr{F}) \cong H_W^{q+1}(V, \Omega^p \otimes \mathscr{F}) \quad \text{for} \quad q \geqslant 1. \tag{8}$$

Finally, we can conclude

$$H^{p,q}(X, \mathscr{F}) \cong H_W^{q+1}(V, \Omega^p \otimes \mathscr{F}) \quad \text{for} \quad 1 \leqslant q \leqslant n-2. \tag{9}$$

by the equations from (3) to (8), and hence

$$\dim H^{p,q}(X, \mathscr{F}) = \sum_{i=1}^{m} \dim H_{\{x_i\}}^{q+1}(V, \Omega^p \otimes \mathscr{F}) \quad \text{for} \quad 1 \leqslant q \leqslant n-2. \tag{10}$$

Finally, we note that the sheaf $\Omega^p \otimes \mathscr{F}$ in (1) can be replaced by its double dual. □

To compute local cohomology groups, let us recall the following vanishing theorem.

Proposition 4.2 (cf. [2]). *Let W be a complex space and $\mathscr{G}$ a coherent analytic sheaf on W. Suppose $x \in W$ and m is an integer. Then* $\operatorname{depth} \mathscr{G}_x \geqslant m$ *if and only if*

$$H_{\{x\}}^q(V, \mathscr{G}) = 0 \quad \textit{for} \quad q < m.$$

If V is perfect, then $\operatorname{depth} \mathscr{O}_x = \dim \mathscr{O}_x$ for $x \in V$ by definition. If $\mathscr{F}$ is locally free, then

$$\operatorname{depth} \mathscr{F}_x = \operatorname{depth} \mathscr{O}_x = \dim \mathscr{O}_x.$$

Moreover, if V is smooth, then V is Cohen-Macaulay and Ω^p is locally free. So we have the following corollary.

Corollary 4.3. *Suppose, with the above notations, that V is strongly pseudoconvex. If V is perfect and $\mathscr{F}$ is locally free, then*

$$\dim H^{0,q}(X, \mathscr{F}) = 0 \quad \textit{for} \quad 1 \leqslant q \leqslant n-2.$$

If V is smooth and $\mathscr{F}$ is locally free, then

$$\dim H^{p,q}(X, \mathscr{F}) = 0 \quad \textit{for} \quad 1 \leqslant q \leqslant n-2.$$

Let us recall the Serre duality theorem for strongly pseudoconvex CR manifolds. In fact, by Theorem 4.1, this theorem is equivalent to the duality theorem of Naruki [20] for local cohomology groups.

Theorem 4.4 ([24])**.** *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$. Then for any (p,q) we have*

$$H^{p,q}(X) \cong H^{n-p,n-q-1}(X).$$

Now we can prove the following proposition.

Proposition 4.5. *The following statements are equivalent:*

1. $H^{0,q}(X) = 0$ *for* $1 \leqslant q \leqslant n-2$;
2. *The Stein filling V of X is perfect;*
3. $\operatorname{depth} \omega_{V,x} = n$, $x \in V$, *where ω_V is the canonical sheaf of V.*

Proof. By Theorem 4.1, the condition

$$H^{0,q}(X) = 0 \quad \text{for} \quad 1 \leqslant q \leqslant n-2$$

is equivalent to

$$H^q_{\{x\}}(V, \mathscr{O}) = 0 \quad \text{for} \quad 2 \leqslant q \leqslant n-1, \quad x \in S.$$

Since V is normal by assumption, we have

$$H^0_{\{x\}}(V, \mathscr{O}) = H^1_{\{x\}}(V, \mathscr{O}) = 0.$$

By Proposition 4.2, we can conclude that

$$H^{0,q}(X) = 0 \quad \text{for} \quad 1 \leqslant q \leqslant n-2$$

if and only if

$$\operatorname{depth}(\mathscr{O}_x) = n, \quad \forall x \in V.$$

By duality theorem for strongly pseudoconvex CR manifold,

$$H^{0,q}(X) \cong H^{n,n-1-q}(X) \quad \text{for} \quad 1 \leqslant q \leqslant n-2.$$

So the condition

$$H^{0,q}(X) = 0 \quad \text{for} \quad 1 \leqslant q \leqslant n-2$$

is equivalent to

$$H^q_{\{x\}}(V, \omega_V) = 0 \quad \text{for} \quad 2 \leqslant q \leqslant n-1, \quad x \in S.$$

However, the canonical sheaf ω_V is reflexive and hence

$$H^0_{\{x\}}(V,\omega_V) = H^1_{\{x\}}(V,\omega) = 0.$$

So we can get the desired conclusion by Proposition 4.2. $\square$

Corollary 4.6. *Let (V,x) be a normal isolated singularity of dimension $n \geqslant 3$. If (V,x) is Cohen-Macaulay, then it is Gorenstein if and only if the projective dimension $\operatorname{pd}_{\mathscr{O}_x}(\omega_{V,x})$ is finite.*

Proof. We have shown that (V,x) is Cohen-Macaulay iff $\operatorname{depth}\omega_{V,x} = n$. If $\operatorname{pd}_{\mathscr{O}_x}(\omega_{V,x})$ is finite, then $\operatorname{pd}_{\mathscr{O}_x}(\omega_{V,x}) = 0$ by the Auslander–Buchsbaum formula

$$\operatorname{pd}_{\mathscr{O}_x}(\omega_{V,x}) + \operatorname{depth}\omega_{V,x} = \operatorname{depth}\mathscr{O}_x.$$

Thus $\omega_{V,x}$ is a free $\mathscr{O}_x$-module and hence (V,x) is Gorenstein. $\square$

Duco van Straten and Joseph Steenbrink solved Zariski–Lipman conjecture in case of isolated singularities of dimension at least three.

Theorem 4.7 ([23])**.** *If (V,x) is an isolated singularity of dimension $n > 2$ and $\Theta_x = \Omega^*_x$ is free, then (V,x) is in fact smooth.*

Proposition 4.8. *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1 \geqslant 5$. Suppose X is the boundary of a strongly pseudoconvex manifold which is a modification of a Stein space at normal isolated singularities. If one of the following conditions hold,*

- *$H^{1,q}(X) = 0$ for $1 \leqslant q \leqslant n-2$ and the projective dimension of Ω^{**} (the double dual of Ω) is finite;*
- *$H^{0,q}(X,\Theta) = 0$ for $1 \leqslant q \leqslant n-2$ and the projective dimension of the tangent sheaf Θ is finite,*

then V is smooth.

Proof. Let $\Omega^{[1]} = \Omega^{**}$. The condition $H^{1,q}(X) = 0$, $1 \leqslant q \leqslant n-2$ implies

$$H^q_{\{x\}}\left(V,\Omega^{[1]}\right) = 0 \quad \text{for} \quad 2 \leqslant q \leqslant n-1, \quad x \in S.$$

Here S is the singular locus of V. Since $\Omega^{[1]}$ is reflexive,

$$H^0_{\{x\}}\left(V,\Omega^{[1]}\right) = H^1_{\{x\}}\left(V,\Omega^{[1]}\right) = 0.$$

By Proposition 4.2, we have $\operatorname{depth}\Omega^{[1]} = n$. If the projective dimension of $\Omega^{[1]}$ is finite, then

$$\mathrm{pd}_{\mathscr{O}_x}(\Omega_x^{[1]}) + \operatorname{depth}\Omega_x^{[1]} = \operatorname{depth}\mathscr{O}_x, \quad x \in S$$

by the formula of Auslander–Buchsbaum. So we can conclude that the projective dimension of $\Omega^{[1]}$ is zero and hence $\Omega^{[1]}$ is locally free. In this case, the tangent sheaf $\Theta = \Omega^* = \left(\Omega^{[1]}\right)^*$ is locally free too. By Theorem 4.7, we can conclude V is smooth. Similarly, one can prove the second statement of this Proposition. □

5. Free resolutions of the exterior powers of a module

Let R be a noetherian commutative ring with unit and M a free R-module of finite rank N. Let $\mathcal{S}_k\left(R^m\right)$ be the k-th symmetric power of R^m. It is a free module of rank $\binom{m+k-1}{m-1}$. Let $\{e_1, \cdots, e_m\}$ be a basis of R^m and let $e_{i_1} \odot \cdots \odot e_{i_k}$ be the symmetric product of $e_{i_1}, \cdots, e_{i_k}$. We denote by $\Lambda^p M$ the p-th exterior product of M. Then any element of $\mathcal{S}_k\left(R^m\right) \otimes \Lambda^{p-k}M$ can be written as

$$\sum_{1 \leqslant i_1, \cdots, i_k \leqslant m} \Gamma_{i_1, \cdots, i_k}\ e_{i_1} \odot \cdots \odot e_{i_k},$$

where $\Gamma_{i_1, \cdots, i_k} \in \Lambda^{p-k}M$ and $\Gamma_{i_{\tau(1)}, \cdots, i_{\tau(k)}} = \Gamma_{i_1, \cdots, i_k}$ for every permutation τ of the symbols $\{1, \cdots, k\}$. Let $\omega_1, \cdots, \omega_m$ be given elements of M. Then we can define a sequence $\mathcal{C}^p(\omega_1, \cdots, \omega_m)$:

$$\begin{aligned} 0 \longrightarrow \mathcal{S}_p\left(R^m\right) \xrightarrow{\partial_p} \mathcal{S}_{p-1}\left(R^m\right) \otimes \Lambda^1 M \longrightarrow \cdots \\ \cdots \longrightarrow \mathcal{S}_1\left(R^m\right) \otimes \Lambda^{p-1}M \xrightarrow{\partial_1} \Lambda^p M \longrightarrow 0, \end{aligned}$$

where each operation ∂_k is a R-linear operator defined by

$$\begin{aligned} &\partial_k \left(\sum_{1 \leqslant i_1, \cdots, i_k \leqslant m} \Gamma_{i_1, \cdots, i_k} e_{i_1} \odot \cdots \odot e_{i_k} \right) \\ &= \sum_{1 \leqslant j_1, \cdots, j_{k-1} \leqslant m} \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i, j_1, \cdots, j_{k-1}} e_{j_1} \odot \cdots \odot e_{j_{k-1}}. \end{aligned} \tag{11}$$

The sequence $\mathcal{C}^p(\omega_1, \cdots, \omega_k)$ is a complex since

$$\begin{aligned}
&\partial_k \circ \partial_{k+1} \Big(\sum_{1 \leqslant i_1, \cdots, i_{k+1} \leqslant m} \Gamma_{i_1, \cdots, i_{k+1}} e_{i_1} \odot \cdots \odot e_{i_{k+1}} \Big) \\
&= \partial_k \Big(\sum_{1 \leqslant i'_1, \cdots, i'_k \leqslant m} \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i, i'_1, \cdots, i'_k} e_{i'_1} \odot \cdots \odot e_{i'_k} \Big) \\
&= \sum_{1 \leqslant j_1, \cdots, j_{k-1} \leqslant m} \sum_{1 \leqslant i, j \leqslant m} \omega_j \wedge \omega_i \wedge \Gamma_{i, j, j_1, \cdots, j_{k-1}} e_{j_1} \odot \cdots \odot e_{j_{k-1}} \\
&= 0.
\end{aligned} \tag{12}$$

Here, we use the fact that $\Gamma_{i,j,j_1,\cdots,j_{k-1}} = \Gamma_{j,i,j_1,\cdots,j_{k-1}}$.

We need to compute the homology of $\mathcal{C}^p(\omega_1, \cdots, \omega_m)$. For this, let us recall a beautiful theorem of Saito. Let $\omega_1, \cdots, \omega_m$ be given elements of M and $(\theta_1, \cdots, \theta_N)$ a basis of M, then we can write

$$\omega_1 \wedge \cdots \wedge \omega_m = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant N} a_{i_1, \cdots, i_m} \, \theta_{i_1} \wedge \cdots \wedge \theta_{i_m}.$$

Let I be the ideal of R generated by the coefficients $a_{i_1,\cdots,i_m}, 1 \leqslant i_1 < \cdots < i_m \leqslant N$. (We put $I = R$ when $m = 0$.) Then we define

$$\begin{aligned}
Z^k &:= \{\omega \in \Lambda^k M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_m = 0\}, \quad k = 0, 1, 2, \cdots, \\
H^k &:= Z^k \Big/ \Big(\sum_{i=1}^{m} \omega_i \wedge \Lambda^{k-1} M \Big), \quad k = 0, 1, 2, \cdots.
\end{aligned}$$

In the case when $m = 0$, we understand $Z^k = 0$, $H^k = 0$ for $k = 0, 1, 2, \cdots$. Now Saito's theorem can be stated:

Theorem 5.1 ([22])**.**

1. *There exists an integer $s \geqslant 0$ such that*

$$I^s H^k = 0 \quad \textit{for} \quad k = 0, 1, 2, \cdots.$$

2. *If $0 \leqslant k < \mathrm{depth}_I(R)$, then $H^k = 0$.*

Now we can prove the following theorem which is due to Buchsbaum and Eisenbud. For a generalization of this result, one can refer to [27].

Theorem 5.2 ([4])**.** *Let R be a noetherian commutative ring with unit, and M a free R-module of finite rank. Let $\omega_1, \cdots, \omega_m$ be given elements of M*

and I the ideal of R generated by the coefficients of $\omega_1 \wedge \cdots \wedge \omega_m$. Then $\mathcal{C}^p(\omega_1, \cdots, \omega_m)$ is a free resolution of

$$\Lambda^p M \Big/ \left(\sum_{i=1}^{m} \omega_i \wedge \Lambda^{p-1} M \right) = \Lambda^p \left(M \Big/ (\omega_1, \cdots, \omega_m) \right)$$

in case of $p \leqslant \mathrm{depth}_I(R)$.

Proof. It is obvious that $H_0(\mathcal{C}^p(\omega_1, \cdots, \omega_m)) = \Lambda^p M \Big/ \left(\sum_{i=1}^{m} \omega_i \wedge \Lambda^{p-1} M \right)$. For higher homology, we prove it by double induction on (p, m). When $p = 0$, the theorem is trivially valid. Inductively, suppose the theorem holds for $p - 1 \geqslant 0$ and all m. We need to prove the theorem for p and all m. For fixed p, we again use induction on m. If $m = 1$, then the complex $\mathcal{C}^p(\omega_1)$ is given by

$$0 \longrightarrow \Lambda^0 M \xrightarrow{\omega_1 \wedge \bullet} \Lambda^1 M \longrightarrow \cdots \longrightarrow \Lambda^{p-1} M \xrightarrow{\omega_1 \wedge \bullet} \Lambda^p M \longrightarrow 0. \tag{13}$$

So by Theorem 5.1, the homology

$$H_k(\mathcal{C}^p(\omega_1)) = \frac{\{\omega \in \Lambda^{p-k} M : \omega \wedge \omega_1 = 0\}}{\omega_1 \wedge \Lambda^{p-k-1} M} = 0 \quad \text{for} \quad k \geqslant 1. \tag{14}$$

Suppose that the theorem is also valid for p and $m - 1$. We need to show that the theorem holds for p and m. Let

$$\Gamma = \sum_{1 \leqslant i_1, \cdots, i_k \leqslant m} \Gamma_{i_1, \cdots, i_k} \; e_{i_1} \odot \cdots \odot e_{i_k} \in \ker \partial_k, \;\; k \geqslant 1.$$

Then

$$\sum_{i=1}^{m} \omega_i \wedge \Gamma_{i, j_1, \cdots, j_{k-1}} = 0 \quad \text{for} \quad 1 \leqslant j_1, \cdots, j_{k-1} \leqslant m. \tag{15}$$

If $k = 1$, then

$$\sum_{i=1}^{m} \omega_i \wedge \Gamma_i = 0. \tag{16}$$

After acting by the operation $\omega_2 \wedge \cdots \omega_m \wedge \bullet$, we obtain

$$\omega_1 \wedge \cdots \omega_m \wedge \Gamma_1 = 0. \tag{17}$$

By assumption, $p \leqslant \operatorname{depth}_I(R)$, Theorem 5.1 implies that there exist $\Gamma_{i,1} \in \Lambda^{p-2}M,\ 1 \leqslant i \leqslant m$ such that

$$\Gamma_1 = \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i,1}. \tag{18}$$

Substituting this into (16) gives

$$\sum_{i=2}^{m} \omega_i \wedge (\Gamma_i - \omega_1 \wedge \Gamma_{i,1}) = 0. \tag{19}$$

We may think

$$\sum_{i=2}^{m} (\Gamma_i - \omega_1 \wedge \Gamma_{i,1})\ e_i \in \mathcal{S}_1 \left(R^{m-1}\right) \otimes \Lambda^{p-1}M$$

Let $\tilde{I}$ be the ideal generated by the coefficients of $\omega_2 \wedge \cdots \omega_m$. Then $\tilde{I} \supset I$ and hence

$$\operatorname{depth}_{\tilde{I}}(R) \geqslant \operatorname{depth}_I(R) \geqslant p.$$

By inductive assumption, the first homology of $\mathcal{C}^p(\omega_2, \cdots, \omega_m)$ is zero. So we can conclude that there exist $\Gamma_{i,j} \in \Lambda^{p-2}M,\ 2 \leqslant i, j \leqslant m$ such that $\Gamma_{i,j} = \Gamma_{j,i}$ and

$$\Gamma_j - \omega_1 \wedge \Gamma_{j,1} = \sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j} \quad \text{for} \quad 2 \leqslant j \leqslant m. \tag{20}$$

Setting $\Gamma_{1,j} = \Gamma_{j,1}$, then we have

$$\Gamma_j = \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i,j} \quad \text{for} \quad 1 \leqslant j \leqslant m. \tag{21}$$

The element $\Theta := \sum_{1 \leqslant i,j \leqslant m} \Gamma_{i,j}\ e_i \odot e_j \in \mathcal{S}_2\left(R^m\right) \otimes \Lambda^{p-2}M$ satisfies $\partial_2(\Theta) = \Gamma$.

If $k \geqslant 2$, then we can define

$$\widetilde{\Gamma} := \sum_{1 \leqslant i_1, \cdots, i_{k-1} \leqslant m} \Gamma_{i_1, \cdots, i_{k-1}, 1}\ e_{i_1} \odot \cdots \odot e_{i_{k-1}} \in \mathcal{S}_{k-1}\left(R^m\right) \otimes \Lambda^{p-k}M.$$

The equations

$$\sum_{i=1}^{m} \omega_i \wedge \Gamma_{i, j_1, \cdots, j_{k-2}, 1} = 0, \quad 1 \leqslant j_1, \cdots, j_{k-2} \leqslant m \tag{22}$$

imply $\widetilde{\Gamma} \in \ker \tilde{\partial}_{k-1}$, where $\tilde{\partial}_{k-1}$ is the differential of $\mathcal{C}^{p-1}(\omega_1, \cdots, \omega_m)$. By inductive assumption, the homology $H_{k-1}(\mathcal{C}^{p-1}(\omega_1, \cdots, \omega_m)) = 0$. Therefore, one can find $\Gamma_{i_1,\cdots,i_k,1} \in \Lambda^{p-k-1}M,\ 1 \leqslant i_1, \cdots, i_k \leqslant m$ so that

$$\Gamma_{j_1,\cdots,j_{k-1},1} = \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1},1}, \quad 1 \leqslant j_1, \cdots, j_{k-1} \leqslant m, \tag{23}$$

and $\Gamma_{i_{\tau(1)},\cdots,i_{\tau(k-1)},1} = \Gamma_{i_1,\cdots,i_{k-1},1}$ for every permutation τ of the symbols $\{1, \cdots, k-1\}$. Plugging (23) into (15), we compute that

$$\begin{aligned}
&\sum_{i=1}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1}} \\
&= \omega_1 \wedge \Gamma_{1,j_1,\cdots,j_{k-1}} + \sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1}} \\
&= \omega_1 \wedge \Gamma_{j_1,\cdots,j_{k-1},1} + \sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1}} \\
&= \omega_1 \wedge \Big(\sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1},1} \Big) + \sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_{k-1}} \\
&= \sum_{i=2}^{m} \omega_i \wedge \Big(\Gamma_{i,j_1,\cdots,j_{k-1}} - \omega_1 \wedge \Gamma_{i,j_1,\cdots,j_{k-1},1} \Big)
\end{aligned} \tag{24}$$

for $1 \leqslant j_1, \cdots, j_{k-1} \leqslant m$. If we set

$$\widehat{\Gamma} := \sum_{2 \leqslant i_1,\cdots,i_k \leqslant m} \Big(\Gamma_{i_1,\cdots,i_k} - \omega_1 \wedge \Gamma_{i_1,\cdots,i_k,1} \Big)\, e_{i_1} \odot \cdots \odot e_{i_k} \in \mathcal{S}_k \Big(R^{m-1} \Big) \otimes \Lambda^{p-k} M,$$

then we have $\widehat{\Gamma} \in \ker \hat{\partial}_k$, where $\hat{\partial}_k$ is the differential of $\mathcal{C}^p(\omega_2, \cdots, \omega_m)$. However, by inductive assumption, the homology $H_k(\mathcal{C}^p(\omega_2, \cdots, \omega_m)) = 0$. So we can conclude that there exist $\Gamma_{i_1,\cdots,i_{k+1}} \in \Lambda^{p-k-1}M,\ 2 \leqslant i_1, \cdots, i_{k+1} \leqslant m$ such that

$$\Gamma_{j_1,\cdots,j_k} - \omega_1 \wedge \Gamma_{j_1,\cdots,j_k,1} = \sum_{i=2}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_k} \quad \text{for} \quad 2 \leqslant j_1, \cdots, j_k \leqslant m \tag{25}$$

and $\Gamma_{i_{\tau(1)},\cdots,i_{\tau(k+1)}} = \Gamma_{i_1,\cdots,i_{k+1}}$ for every permutation τ of $\{2, \cdots, k+1\}$.

Finally, we can set

$$\Gamma_{i_1,\cdots,i_{k+1}} = \Gamma_{i_1,\cdots,\hat{i}_s,\cdots,i_{k+1},1} \tag{26}$$

if there exists some $1 \leqslant s \leqslant k$ such that $i_s = 1$. Then we have

$$\Gamma_{j_1,\cdots,j_k} = \sum_{i=1}^{m} \omega_i \wedge \Gamma_{i,j_1,\cdots,j_k} \quad \text{for} \quad 1 \leqslant j_1, \cdots, j_k \leqslant m. \tag{27}$$

If we define

$$\Theta := \sum_{1\leqslant i_1,\cdots,i_k\leqslant m} \Gamma_{i_1,\cdots,i_{k+1}}\ e_{i_1} \odot \cdots \odot e_{i_{k+1}} \in \mathcal{S}_{k+1}(R^m) \otimes \Lambda^{p-k-1}M,$$

then $\partial_{k+1}(\Theta) = \Gamma$. This finish the inductive step. □

For the following applications, let us consider the map

$$\partial_p : \mathcal{S}_p(R^m) \longrightarrow \mathcal{S}_{p-1}(R^m) \otimes M$$

and its dual map

$$\partial_p^* : (\mathcal{S}_{p-1}(R^m) \otimes M)^* \longrightarrow (\mathcal{S}_p(R^m))^*.$$

Let $\{e_1, \cdots, e_m\}$ be a basis of R^m and $\{\theta_1, \cdots, \theta_N\}$ a basis of M. Then

$$\{e_{i_1} \odot \cdots \odot e_{i_p}; \quad m \geqslant i_1 \geqslant i_2 \geqslant \cdots \geqslant i_p \geqslant 1\}$$

is a basis of $\mathcal{S}_p(R^m)$. The multi-indices $(i_1, \cdots, i_p) \prec (i'_1, \cdots, i'_p)$ if and only if

$$i_s < i'_s, \quad \text{and} \quad i_k = i'_k \quad \text{when} \quad k > s$$

for some $1 \leqslant s \leqslant p$. This gives an order of the set $\{(i_1, \cdots, i_p)\}$. Similarly, let

$$\{e_{j_1} \odot \cdots \odot e_{j_{p-1}} \otimes \theta_j; \quad m \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_p \geqslant 1,\ N \geqslant j \geqslant 1\}$$

be a basis of $\mathcal{S}_{p-1}(R^m) \otimes M$. The element $\omega_k \in M,\ 1 \leqslant k \leqslant m$ can be written as

$$\omega_k = \sum_{j=1}^{N} \omega_{kj}\theta_j,$$

where $\omega_{kj} \in R$ for $1 \leqslant j \leqslant N$. Under the isomorphism $M \cong R^N$, ω_k can be represented by the $1 \times N$ matrix

$$\omega_k = (\omega_{k1}, \cdots, \omega_{kN}).$$

Then the map

$$\partial_p : \ \sum \Gamma_{i_1,\cdots,i_p}\ e_{i_1} \odot \cdots \odot e_{i_p} \longmapsto \sum \omega_k \wedge \Gamma_{k,j_1,\cdots,j_{p-1}} e_{j_1} \odot \cdots \odot e_{j_{p-1}}$$

can be represented by a matrix.

Let us choose the dual basis $\{e_1^*, \cdots, e_m^*\}$ for $(R^m)^*$ such that $e_i^*(e_j) = \delta_{ij}$. Then the set $\left\{e_{i_1}^* \odot \cdots \odot e_{i_p}^*\right\}$ forms a basis of $(\mathcal{S}_p(R^m))^*$. Similarly, let $\{\theta_1^*, \cdots, \theta_N^*\}$ be the basis of M^* such that $\theta_k^*(\theta_l) = \delta_{kl}$. Then

$$\left\{e_{j_1}^* \odot \cdots \odot e_{j_{p-1}}^* \otimes \theta_j^*\right\}$$

is a basis of $(\mathcal{S}_{p-1}(R^m) \otimes M)^*$. Under these bases, let $J_p(\omega_1, \cdots, \omega_m)$ be the matrix determined by

$$\partial_p^* : (\mathcal{S}_{p-1}(R^m) \otimes M)^* \longrightarrow (\mathcal{S}_p(R^m))^* .$$

Then $J_p(\omega_1, \cdots, \omega_m)$ is the transpose of the matrix corresponding to ∂_p.

If $p = 1$, then

$$J_1(\omega_1, \cdots, \omega_m) = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix}. \tag{28}$$

If $m = 1$, then

$$J_p(\omega_1) = \omega_1 \quad \text{for} \quad p \geqslant 1. \tag{29}$$

When $m = 2$, we have

$$J_p(\omega_1, \omega_2) = \begin{pmatrix} \omega_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \omega_2 & \omega_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2 & \omega_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega_2 & \omega_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \omega_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \omega_2 & \omega_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \omega_2 \end{pmatrix}. \tag{30}$$

In general, the matrix $J_p(\omega_1, \cdots, \omega_m)$ can be given inductively by

$$
(31) \qquad J_p(\omega_1, \cdots, \omega_m) = \left(\begin{array}{c:cccc} & & & 0 & \\ \hdashline & \omega_1 & & & \\ J_{p-1}(\omega_1, \cdots, \omega_m) & & \omega_1 & & \\ & & & \ddots & \\ & & & & \omega_1 \\ \hdashline 0 & & J_p(\omega_2, \cdots, \omega_m) & & \end{array}\right).
$$

Proposition 5.3. *Under the isomorphisms*

$$
(\mathcal{S}_p(R^m))^* \cong \mathcal{S}_p(R^m), \quad (\mathcal{S}_{p-1}(R^m) \otimes M)^* \cong \mathcal{S}_{p-1}(R^m) \otimes R^N,
$$

we have

$$
\operatorname{coker} \partial_p^* = (\mathcal{S}_p(R^m))^* / \operatorname{Im} \partial_p^* \cong \frac{\mathcal{S}_p(R^m)}{J_p(\omega_1, \cdots, \omega_m)(\mathcal{S}_{p-1}(R^m) \otimes R^N)}.
$$

Given a module whose projective dimension is equal or less than one, the local cohomology groups with values in the exterior products of this module can be computed by the above resolutions given by Buchsbaum-Eisenbud.

Proposition 5.4. *Let (V, x) be an isolated Gorenstein singularity of dimension n and $\mathscr{F}$ a coherent analytic sheaf on V. Suppose $\mathscr{F}_x$ is given by the following exact sequence*

$$
0 \longrightarrow \mathscr{O}_x^m \xrightarrow{\phi} \mathscr{O}_x^N \to \mathscr{F}_x \longrightarrow 0,
$$

and $\mathscr{F}$ is locally free on $V \setminus \{x\}$. Then

$$
(32) \qquad \dim H^q_{\{x\}}(V, \Lambda^p \mathscr{F}) = \begin{cases} 0 & \text{if} \quad p + q \leqslant n - 1, \\ \dim_{\mathbb{C}} \mathcal{S}_p(\operatorname{coker} \phi^*) & \text{if} \quad p + q = n, \end{cases}
$$

and the module

$$
\mathcal{S}_p(\operatorname{coker} \phi^*) = [\mathcal{S}_p(\mathscr{O}_x^m)]^* / \operatorname{Im} \partial_p^*,
$$

where ∂_p^* is the dual map of $\partial_p : \mathcal{S}_p(\mathscr{O}_x^m) \to \mathcal{S}_{p-1}(\mathscr{O}_x^m) \otimes \mathscr{O}_x^N$.

Proof. We may assume that $M = \mathscr{O}_x^N$ and ϕ is given by $(\omega_1, \cdots, \omega_m)$, where $\omega_1, \cdots, \omega_m$ are elements of M. Then

$$
\mathscr{F}_x \cong M / (\omega_1, \cdots, \omega_m).
$$

Let I be the ideal of $\mathscr{O}_x$ generated by the coefficients of $\omega_1 \wedge \cdots \wedge \omega_m$. Then

$$\operatorname{depth}_I(\mathscr{O}_x) = \dim V = n,$$

since (V, x) is Cohen-Macaulay and $\mathscr{F}$ is locally free on $V \setminus \{x\}$. For a proof, see for instance Theorem 30 and Theroem 31 of [19]. By Theorem 5.2, the complex $\mathcal{C}^p(\omega_1, \cdots, \omega_m)$

$$0 \longrightarrow \mathcal{S}_p\left(\mathscr{O}_x^m\right) \xrightarrow{\partial_p} \mathcal{S}_{p-1}\left(\mathscr{O}_x^m\right) \otimes \Lambda^1 M \longrightarrow \cdots \xrightarrow{\partial_1} \Lambda^p M \longrightarrow \Lambda^p \mathscr{F}_x$$

is a free resolution of the $\mathscr{O}_x$-module $\Lambda^p \mathscr{F}_x$ for $p \leqslant n$. Then we have

$$\operatorname{Ext}^k_{\mathscr{O}_x}(\Lambda^p \mathscr{F}_x, \mathscr{O}_x) = 0 \quad \text{for} \quad k > p \tag{33}$$

since the length of $\mathcal{C}^p(\omega_1, \cdots, \omega_m)$ is p and

$$\operatorname{Ext}^p_{\mathscr{O}_x}(\Lambda^p \mathscr{F}_x, \mathscr{O}_x) = [\mathcal{S}_p\left(\mathscr{O}_x^m\right)]^* / \operatorname{Im} \partial_p^* = \operatorname{coker} \partial_p^*. \tag{34}$$

By the right exactness of the symmetric algebra (cf. [5]), we have

$$\operatorname{Ext}^p_{\mathscr{O}_x}(\Lambda^p \mathscr{F}_x, \mathscr{O}_x) = \mathcal{S}_p(\operatorname{coker} \phi^*). \tag{35}$$

Since (V, x) is Gorenstein, by local duality (cf. [9]),

$$\dim H^q_{\{x\}}\left(V, \Lambda^p \mathscr{F}\right) = \dim \operatorname{Ext}^{n-q}_{\mathscr{O}_x}(\Lambda^p \mathscr{F}_x, \mathscr{O}_x). \tag{36}$$

Then

$$\dim H^q_{\{x\}}\left(V, \Lambda^p \mathscr{F}\right) = \begin{cases} 0 & \text{if} \quad p + q \leqslant n - 1, \\ \dim \mathcal{S}_p(\operatorname{coker} \phi^*) & \text{if} \quad p + q = n. \end{cases} \tag{37}$$

Note that $\mathcal{S}_p(\operatorname{coker} \phi^*) = 0$ iff $\operatorname{coker} \phi^* = 0$ iff $\mathscr{F}_x$ is free as an $\mathscr{O}_x$-module. $\square$

In view of Theorem 4.1 and Proposition 5.4, we can conclude the following result:

Theorem 5.5. *Let V be an n-dimensional strongly pseudoconvex reduced irreducible Stein space with smooth boundary X. We assume that V is imbedded in a slightly larger reduced irreducible complex space V' with V' smooth near $X = \partial V$. Suppose $\mathscr{F}$ is a reflexive sheaf ($\mathscr{F} = \mathscr{F}^{**}$) on V' such that $\mathscr{F}$ is locally free near X and let Z be the set of all points of V where $\mathscr{F}$ is not* *locally free. Let* $r = \operatorname{rank}\left(\mathscr{F}|_{V\setminus Z}\right)$. *Suppose that, for* $x \in Z$, *the singularity* (V, x) *is Gorenstein and* $\mathscr{F}_x$ *is given by*

$$0 \longrightarrow \mathscr{O}_x^m \xrightarrow{\phi_x} \mathscr{O}_x^N \to \mathscr{F}_x \longrightarrow 0.$$

Let $p \leqslant r$ *and* $1 \leqslant q \leqslant n-2$. *Then the dimensions of the Kohn-Rossi cohomology groups*

$$\dim H^{0,q}\left(X, \Lambda^p \mathscr{F}\right) = \begin{cases} 0 & \text{if} \quad p+q \leqslant n-2, \\ \sum\limits_{x\in Z} \dim_{\mathbb{C}} \mathcal{S}_p(\operatorname{coker} \phi_x^*) & \text{if} \quad p+q = n-1. \end{cases}$$

Moreover, $\mathscr{F}$ *is locally free on* V *if and only if* $H^{0,n-p-1}\left(X, \Lambda^p \mathscr{F}\right) = 0$ *for some integer* $1 \leqslant p \leqslant \min\{n-2, r\}$.

6. The complex Plateau problem

Let U be a complex manifold of dimension N and let V be a complex analytic subvariety of U. Let $\mathscr{I}$ be the ideal sheaf of V in U and let Ω_U^p be the sheaf of germs of holomorphic p-forms on U. In case of $p < 0$, we may understand $\Omega_U^p = 0$. Then the sheaf of germs of holomophic p-forms on V is given by

$$\Omega_V^p = \Omega_U^p \Big/ \left\{ f\alpha + dg \wedge \beta : \ f, g \in \mathscr{I}, \ \alpha \in \Omega_U^p, \ \beta \in \Omega_U^{p-1} \right\}.$$

It is a sheaf of $\mathscr{O}_V$-modules. By the construction of cotangent sheaf, there is an exact sequence of $\mathscr{O}_V$-modules

$$\mathcal{N}_V^* \xrightarrow{\alpha} \Omega_U|_V \to \Omega_V \to 0,$$

where $\mathcal{N}_V^* = \left(\mathscr{I}/\mathscr{I}^2\right)|_V$ is the conormal sheaf of V in U, and $\Omega_U|_V = \left(\Omega_U/\mathscr{I}\Omega_U\right)|_V$ denotes the analytic restriction of Ω_U to V. Call a closed complex subspace V of a complex manifold U locally a complete intersection if the ideal $\mathscr{I}$ can be generated, locally, by $\operatorname{codim}(V, U)$ holomorphic functions. In this case $\mathscr{I}/\mathscr{I}^2$ is locally free of rank $\operatorname{codim}(V, U)$. If now, in addition, the space V is reduced, then the sequence

$$0 \to \mathcal{N}_V^* \xrightarrow{\alpha} \Omega_U|_V \to \Omega_V \to 0$$

is exact, cf. [8]. Note that $\mathcal{N}_V^*$ and $\Omega_U|_V$ are locally free sheaves of $\mathscr{O}_V$-modules.

Let (U,x) be a complex manifold germ of dimension N, and $(V,x)\subset(U,x)$ an isolated complete intersection singularity of dimension n. We may suppose V is given by $f_1=\cdots=f_m=0$ and $df_1\wedge\cdots\wedge df_m\neq 0$ on $V\setminus\{x\}$. Then we have the following exact sequence of $\mathscr{O}_{V,x}$-modules

$$0\to\mathscr{O}_{V,x}^m\xrightarrow{\phi}\left(\Omega_U|_V\right)_x\to\Omega_{V,x}\to 0$$

and $\left(\Omega_U|_V\right)_x$ is a free $\mathscr{O}_{V,x}$-module of rank N. Let $\widetilde{df_k}$ be the restriction of df_k to V. Then the map ϕ is given by $\widetilde{df_1},\cdots,\widetilde{df_m}$ and

$$\Omega_{V,x}\cong\left(\Omega_U|_V\right)_x\Big/\left(\widetilde{df_1},\cdots,\widetilde{df_m}\right).$$

Let us write

$$df_k=\left(\frac{\partial f_k}{\partial z_1},\cdots,\frac{\partial f_k}{\partial z_N}\right),\quad 1\leqslant k\leqslant m,$$

and

$$J_1(f_1,\cdots,f_m)=\begin{pmatrix}df_1\\ \vdots\\ df_m\end{pmatrix}=\begin{pmatrix}\frac{\partial f_1}{\partial z_1}&\cdots&\frac{\partial f_1}{\partial z_N}\\ \vdots&\ddots&\vdots\\ \frac{\partial f_m}{\partial z_1}&\cdots&\frac{\partial f_m}{\partial z_N}\end{pmatrix}. \tag{38}$$

As in Section 5, we can define the matrix $J_p\left(f_1,\cdots,f_m\right)$ inductively by

$$J_p\left(f_1,\cdots,f_m\right)=\left(\begin{array}{c:cccc} & & & 0 & \\ \hdashline & df_1 & & & \\ J_{p-1}\left(f_1,\cdots,f_m\right) & & df_1 & & \\ & & & \ddots & \\ & & & & df_1\\ \hdashline 0 & & & J_p\left(f_2,\cdots,f_m\right) & \end{array}\right). \tag{39}$$

As a consequence of Theorem 5.5 and Serre duality theorem for compact strongly pseudoconvex CR manifolds, we have

Theorem 6.1. *Suppose X is the boundary of a strongly pseudoconvex manifold of dimension $n\geqslant 3$ which is a modification of a Stein space V at isolated singularities. Let S be the singular set of V. If the singularities (V,x), $x\in S$*

are complete intersections, then

$$\dim H^{p,q}(X)=\begin{cases}0 & \textit{if}\quad p+q\leqslant n-2,\ 1\leqslant q\leqslant n-2,\\ \sum\limits_{x\in S}\tau_x^p & \textit{if}\quad p+q=n-1,\ 1\leqslant q\leqslant n-2,\\ \sum\limits_{x\in S}\tau_x^{n-p} & \textit{if}\quad p+q=n,\ 1\leqslant q\leqslant n-2,\\ 0 & \textit{if}\quad p+q\geqslant n+1,\ 1\leqslant q\leqslant n-2,\ 0\leqslant p\leqslant n,\end{cases}$$

where

$$\begin{aligned}\tau_x^p&=\dim_{\mathbb{C}}\mathcal{S}_p\left(\operatorname{Ext}^1_{\mathscr{O}_{V,x}}\left(\Omega_{V,x},\mathscr{O}_{V,x}\right)\right)\\ &=\dim_{\mathbb{C}}\frac{\mathcal{S}_p\left(\mathscr{O}^m_{U,x}\right)}{\sum_{i=1}^m f_i\cdot\mathcal{S}_p\left(\mathscr{O}^m_{U,x}\right)+J_p(f_1,\cdots,f_m)\left(\mathcal{S}_{p-1}\left(\mathscr{O}^m_{U,x}\right)\otimes\mathscr{O}^{m+n}_{U,x}\right)}.\end{aligned}$$

Moreover, the complex space V is smooth if and only if $H^{n-q-1,q}(X)=0$ for some $1\leqslant q\leqslant n-2$.

Proof. It suffices to compute $\dim H^{p,q}(X)$ in case $p+q\leqslant n-1$ because

$$\dim H^{p,q}(X)=\dim H^{n-p,n-q-1}(X)$$

by Theorem 4.4. In view of Theorem 4.1,

$$\dim H^{p,q}(X)=\sum_{x\in S}\dim H^{q+1}_{\{x\}}(V,\Omega^p),$$

for $1\leqslant q\leqslant n-2$, so we only need to compute the dimension of $H^q_{\{x\}}(V,\Omega^p)$.

Now we can apply Proposition 5.4 to get the desired conclusion. For the convenience of the reader, we give here the proof again. Let I be the ideal of $\mathscr{O}_{V,x}$ generated by $\widetilde{df_1}\wedge\cdots\wedge\widetilde{df_m}$. Then $\operatorname{depth}_I(\mathscr{O}_{V,x})=n$, since (V,x) is an isolated complete intersection singularity. By Theorem 5.2, the complex $\mathcal{C}^p\left(\widetilde{df_1},\cdots,\widetilde{df_m}\right)$:

$$0\longrightarrow\mathcal{S}_p\left(\mathscr{O}^m_{V,x}\right)\xrightarrow{\partial_p}\mathcal{S}_{p-1}\left(\mathscr{O}^m_{V,x}\right)\otimes\Lambda^1\left(\Omega_U|_V\right)_x\longrightarrow\cdots\xrightarrow{\partial_1}\Lambda^p\left(\Omega_U|_V\right)_x$$

is a free resolution of the $\mathscr{O}_{V,x}$-module $\Omega^p_{V,x}$. Then we have

$$\operatorname{Ext}^k_{\mathscr{O}_{V,x}}\left(\Omega^p_{V,x},\mathscr{O}_{V,x}\right)=\begin{cases}0, & \text{if}\quad k>p;\\ \operatorname{coker}\partial_p^*, & \text{if}\quad k=p,\end{cases}$$

where

$$\partial_p^* : \left(\mathcal{S}_{p-1}\left(\mathscr{O}_{V,x}^m\right) \otimes \Lambda^1\left(\Omega_U|_V\right)_x\right)^* \longrightarrow \left(\mathcal{S}_p\left(\mathscr{O}_{V,x}^m\right)\right)^*$$

is the dual map of ∂_p. By local duality,

$$\dim H^q_{\{x\}}\left(V, \Omega^p\right) = \dim \operatorname{Ext}^{n-q}_{\mathscr{O}_{V,x}}(\Omega^p_{V,x}, \mathscr{O}_{V,x}) \tag{40}$$

since isolated complete intersection singularities are Gorenstein. Thus

$$\dim H^q_{\{x\}}(V, \Omega^p) = \begin{cases} 0, & \text{if} \quad p+q<n; \\ \operatorname{coker} \partial_p^*, & \text{if} \quad p+q=n. \end{cases}$$

Under the isomorphisms

$$\begin{aligned} &\left(\mathcal{S}_p\left(\mathscr{O}_{V,x}^m\right)\right)^* \cong \mathcal{S}_p\left(\mathscr{O}_{V,x}^m\right), \\ &\left(\mathcal{S}_{p-1}\left(\mathscr{O}_{V,x}^m\right) \otimes \Lambda^1\left(\Omega_U|_V\right)_x\right)^* \cong \mathcal{S}_{p-1}\left(\mathscr{O}_{V,x}^m\right) \otimes \mathscr{O}_{V,x}^N, \end{aligned}$$

the map ∂_p^* can be represented by

$$J_p\left(\widetilde{df_1}, \cdots, \widetilde{df_m}\right) : \ \mathcal{S}_{p-1}\left(\mathscr{O}_{V,x}^m\right) \otimes \mathscr{O}_{V,x}^N \longrightarrow \mathcal{S}_p\left(\mathscr{O}_{V,x}^m\right).$$

So we have

$$\begin{aligned} \operatorname{coker} \partial_p^* &\cong \frac{\mathcal{S}_p\left(\mathscr{O}_{V,x}^m\right)}{\operatorname{Im} J_p\left(\widetilde{df_1}, \cdots, \widetilde{df_m}\right)} \\ &\cong \frac{\mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right)}{\sum_{i=1}^m f_i \cdot \mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right) + \operatorname{Im} J_p(df_1, \cdots, df_m)} \end{aligned}$$

and hence

$$\tau_x^p = \dim_{\mathbb{C}} \frac{\mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right)}{\sum_{i=1}^m f_i \cdot \mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right) + J_p(f_1, \cdots, f_m)\left(\mathcal{S}_{p-1}\left(\mathscr{O}_{U,x}^m\right) \otimes \mathscr{O}_{U,x}^N\right)}.$$

If V is smooth, then $H^{p,q}(X) = 0$ for $1 \leqslant q \leqslant n-2$ by Theorem 4.1. For the other direction, suppose $H^{n-q-1,q}(X) = 0$ for some $1 \leqslant q \leqslant n-2$. Then for every $x \in S$, we have

$$\frac{\mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right)}{\sum_{i=1}^m f_i \cdot \mathcal{S}_p\left(\mathscr{O}_{U,x}^m\right) + J_p(f_1, \cdots, f_m)\left(\mathcal{S}_{p-1}\left(\mathscr{O}_{U,x}^m\right) \otimes \mathscr{O}_{U,x}^N\right)} = 0. \tag{41}$$

So the rank of the matrix

$$J_p(f_1,\cdots,f_m)=\left(\begin{array}{c:c} J_{p-1}(f_1,\cdots,f_m) & \begin{array}{c} 0 \\ \hdashline \begin{matrix} df_1 & & & \\ & df_1 & & \\ & & \ddots & \\ & & & df_1 \end{matrix} \end{array} \\ \hdashline 0 & J_p(f_2,\cdots,f_m) \end{array}\right) \tag{42}$$

is equal to the rank of $\mathcal{S}_p\left(\mathscr{O}^m_{U,x}\right)$. In other words, the columns of the matrix $J_p(f_1,\cdots,f_m)$ are linearly independent at x. In particular, the columns of

$$\left(\begin{array}{c} J_{p-1}(f_1,\cdots,f_m) \\ \hdashline 0 \end{array}\right) \tag{43}$$

are linearly independent at x. But this implies the rank of $J_{p-1}(f_1,\cdots,f_m)$ is equal to the rank of $\mathcal{S}_{p-1}\left(\mathscr{O}^m_{U,x}\right)$. By induction, we can conclude that the Jacobi matrix

$$J_1(f_1,\cdots,f_m)=\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_N} \end{pmatrix} \tag{44}$$

has rank m at x. So x is a smooth point of V and hence V is a smooth manifold. □

We should note that the computations of the local cohomology group $H^q_{\{x\}}(V,\Omega^p)$ is due to Naruki [20] and Vosegaard [25]. In fact, τ^p_x is the p-th generalized Tjurina number at x defined by Vosegaard.

By the above formula, it is obvious that the numbers τ^p_x $(0\leqslant p\leqslant n-1)$ coincide for hypersurface singularities. However, one can find concrete examples to show these numbers are different in general. The results of Greuel ([9]) and Naruki ([20]) show that all of the these numbers coincide for weighted homogeneous isolated complete intersection singularities. Thus we have the following corollary:

Corollary 6.2. *Suppose X is the boundary of a strongly pseudoconvex manifold of dimension $n\geqslant 3$ which is a modification of a Stein space V at isolated* *singularities* $x_1, \cdots, x_s$. *If there exist integers* $1 \leqslant q, q' \leqslant n-2$ *and* $q \neq q'$ *such that*

$$\dim H^{n-q-1,q}(X) \neq \dim H^{n-q'-1,q'}(X),$$

then (V, x_i) *is not a hypersurface singularity for some* $1 \leqslant i \leqslant s$.

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Xiankui Meng
School of Science
Beijing University of Posts and Telecommunications
Beijing
100876
P.R. China
E-mail: mengxiankui@amss.ac.cn

Stephen Shing-Toung Yau
Department of Mathematical Sciences
Tsinghua University
Beijing
100084
P.R. China
Yanqi Lake Beijing Institute of Mathematical Sciences and Applications
Beijing
101400
P.R. China
E-mail: yau@uic.edu