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On the Levi problem on Kähler manifolds under the negativity of canonical bundles on the boundary

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Dedicated to Professor Joseph J. Kohn on his 90th birthday

Abstract: It is proved that a bounded C^2-smooth pseudoconvex domain Ω in a Kähler manifold M can be mapped onto a locally closed analytic set in $\mathbb{C}^N$ holomorphically and properly with connected fibers if the canonical bundle of M is negative on a neighborhood of $\partial\Omega$. A similar result is obtained for Zariski open domains in compact manifolds.

Introduction

Let M be a complex manifold and let $\Omega \Subset M$ be a pseudoconvex domain with a C^∞ smooth boundary. In [K-2] Kohn proved the following: Assume that there exists a strictly plurisubharmonic function on a neighborhood of $\partial\Omega$. Then, given a $\bar\partial$-closed (p,q)-form α, which is C^∞ in $\overline{\Omega}$ and which is $\bar\partial$-exact on Ω then for every m there exists a $(p,q-1)$-form $u_{(m)}$ which is C^m on $\overline{\Omega}$ such that $\bar\partial u_{(m)} = \alpha$. This extends the analysis in the solution [K-1] (see also [K-N]) of the complex boundary value problem, the $\bar\partial$-Neumann problem, on strongly pseudoconvex domains. It was pointed out by Barret [B] that the assumption on the existence of strictly plurisubharmonic functions cannot be removed. Even with this condition, the question on the smoothness of L^2 canonical solutions has been answered negatively in general (cf. [Ch]). On the other hand, Kohn [K-3] looked for sufficient conditions for the estimates implying the smoothness of the canonical solutions. For that he introduced the notion of subelliptic multipliers which turned out to be effective for the domains of finite type (cf. [S-2], [K-Z]) and also useful in complex geometry (cf. [Dm-2], [S-1]). Positive answers to the regularity problem have been also given for other classes of weakly pseudoconvex domains (cf. [B-S] and [F-S]).

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Analysis on pseudoconvex domains on complex manifolds is related to questions in algebraic geometry (cf. [G-2], [G-R]) and differential geometry (cf. [T], [Nd], [L-Sz]). The Levi problem on certain weakly pseudoconvex domains has been solved in such contexts (e.g. [K-S], [Fk], [Ty]).

Recall that the Levi problem on a complex manifold M asks basically whether or not a pseudoconvex domain in M is holomorphically convex. In the situation of [K-2] as above, it was first proved by Grauert [G-1] that Ω is holomorphically convex, by generalizing Oka's theorem on the holomorphic convexity of pseudoconvex domains over $\mathbb{C}^n$. Narashimhan [N] raised a question whether or not a bounded smooth pseudoconvex domain with at least one strongly pseudoconvex boundary point admits a nonconstant holomorphic function, which is still open. The original question asked the holomorphic convexity of such domains, but Grauert [G-3] found a counterexample. For a partial answer see [D-Oh].

The purpose of the present note is to prove the following.

Theorem 0.1. *Let M be a complex manifold and let $\Omega \Subset M$ be a pseudoconvex domain with C^2-smooth boundary. Assume that M admits a Kähler metric and the canonical bundle K_M of M admits a fiber metric whose curvature form is negative on a neighborhood of $\partial\Omega$. Then there exists a holomorphic map with connected fibers from Ω to $\mathbb{C}^N$ for some $N \in \mathbb{N}$ which is proper onto the image.*

This is an extension of Grauert's theorem in the spirit of Fujita [F] and Takeuchi [Tk], where the Levi problem on (and over) $\mathbb{CP}^n$ was solved in full generality (see also [Oh-S]). As is well known, the condition on K_M cannot be removed (cf. [G-3]) but the Kähler condition may be superfluous.

The proof of Theorem 0.1 is based on a bundle convexity theorem in [Oh-5], a continuation of [Oh-1,3,4], by which singular fiber metrics on K_M^{-1} with positive curvature currents are constructed. The Kähler condition is needed to produce nonconstant holomorphic functions by solving the $\bar\partial$-equations.

1. Preliminary

Let us recall a bundle convexity theorem from [Oh-5]. It was stated in a general form, but the content is summarized as follows for bounded smooth pseudoconvex domains.

Theorem 1.1. (cf. [Oh-5, Theorem 0.3 and Theorem 4.1]) *Let M be a complex manifold, let $\Omega \Subset M$ be a pseudoconvex domain with a C^2-smooth boundary and let B be a holomorphic line bundle over M with a fiber metric h whose* *curvature form is positive on a neighborhood of $\partial\Omega$. Then there exists a positive integer m_0 such that for all $m \geq m_0$ $\dim H^{0,0}(\Omega, B^m) = \infty$ and that, for any compact set $K \subset \Omega$ and for any positive number R, one can find a compact set $\tilde{K} \subset \Omega$ such that for any point $x \in \Omega \setminus \tilde{K}$ there exists an element s of $H^{0,0}(\Omega, B^m)$ satisfying*

$$\sup_K |s|_{h^m} < 1 \ \ \text{and} \ \ |s(x)|_{h^m} > R. \tag{1.1}$$

Corollary 1.1. *Let M and Ω be as in Theorem 0.1 and let κ be any fiber metric of K_M. Then, for any compact set $K \subset \Omega$, $K_M^{-1}|_\Omega$ admits a singular fiber metric whose curvature current is C^∞ and positive on $\Omega \setminus A$ for some proper analytic set $A \subset \Omega$.*

For the proof of Theorem 0.1, somewhat more than Corollary 1.1 is used because we need a complete Kähler metric on $M \setminus A$ in order to produce holomorphic functions by solving $\bar{\partial}$-equations. Construction of a complete Kähler metric on $M \setminus A$ will be described after that of the above singular fiber metric in the proof of Theorem 0.1 later.

2. Proof of Theorem 0.1

Let the situation be as in Theorem 0.1, so that Ω is a relatively compact pseudoconvex domain with a C^2-smooth boundary. Let κ_M be a fiber metric of K_M whose curvature form is negative on a neighborhood U of $\partial\Omega$. Let $\rho : M \to (-1, 1)$ be a C^2 function satisfying $U \cap \Omega = \{x \in U; \rho(x) < 0\}$ and $(d\rho)^{-1}(0) \cap \partial\Omega = \varnothing$. We may assume that ρ is C^∞ on Ω.

For the convenience of the reader, we recall below an outline of the proof of Theorem 1.1.

Let (M, g) be a Hermitian manifold of dimension n. For any point $x \in M$ we denote by Π_x the set of C^∞ functions $\Psi : M \setminus \{x\} \to (-\infty, 0)$ such that $e^{-\Psi}$ is not integrable on any neighborhood of x and the length of $d\Psi$ with respect to g is bounded on the set $\{y \in M; \Psi(y) > -1\}$.

Definition 1.

$$\varrho(x) := \sup\{r^{-1}; rg + \partial\bar{\partial}\Psi > 0 \ \text{ on } M \setminus \{x\} \text{ for some } \Psi \in \Pi_x\} \in (0, \infty].$$

*ϱ is called the **$\partial\bar{\partial}$-radius function**.*

Definition 2. *A complete Hermitian manifold (M, g) is said to be **$\partial\bar{\partial}$-uniformly complete** if its $\partial\bar{\partial}$-radius function is bounded from below by a positive constant.*

We put

$$\Pi_x^r = \{\Psi \in \Pi_x; rg + \partial\bar{\partial}\Psi > 0 \text{ on } M \setminus \{x\}\}.$$

An open set U in M is said to be an ***r-admissible neighborhood*** of x if one can find a $\Psi \in \Pi_x^r$ such that $U \supset \{y \in M \setminus \{x\}; \Psi(y) < -1\}$.

Definition 3. *A holomorphic Hermitian line bundle (B, h) over a complete Hermitian manifold (M, g) is said to be **tame** if there exist $r > 0$, $C > 0$ and a system of open sets*

$$\mathscr{U} = \{U_x; x \in M \text{ and } U_x \text{ is an r-admissible neighborhood of } x\}$$

*such that (B, h) is equivalent over every $U \in \mathscr{U}$ to the trivial bundle $U \times \mathbb{C}$ in such a way that there exists an equivalence $\tau : B|_U \to U \times \mathbb{C}$ satisfying $C^{-1}|p_{\mathbb{C}}(\tau(\zeta))| \leq |\zeta|_h \leq C|p_{\mathbb{C}}(\tau(\zeta))|$ for all $\zeta \in B|_U$, where $p_{\mathbb{C}} : U \times \mathbb{C} \to \mathbb{C}$ denotes the projection. The covering $\mathscr{U}$ is then said to be **r-adapted to** (B, h). A tame line bundle (B, h) is called **strictly tame** if Θ_h is bounded with respect to g.*

Theorem 2.1. (cf. [Oh-5, Theorem 3.1]) *Let (M, g) be a $\partial\bar{\partial}$-uniformly complete Hermitian manifold whose canonical line bundle K_M has a fiber metric κ such that (K_M, κ) is tame. Let (B, h) be a holomorphic strictly tame Hermitian line bundle over M such that $g = \Theta_h$ holds outside a compact subset of M. Then, for any C^2 function $\varphi : M \to [1, \infty)$ such that $\partial\bar{\partial}\varphi + \Theta_h > \ell(\varphi)^2 \partial\varphi\bar{\partial}\varphi$ holds for some C^∞ function $\ell : [1, \infty) \to (0, \infty)$ satisfying*

$$\int_1^\infty \ell(t)dt = \infty$$

and such that the least eigenvalue of $\partial\bar{\partial}\varphi$ at $x \in M$ with respect to g does not accumulate to any negative number as $\varphi(x) \to \infty$, one can find $m_0 \in \mathbb{N}$ such that

$$\dim H^{0,0}(M, K_M \otimes B^m) = \infty \text{ for all } m \geq m_0$$

and, for any sequence $\{x_\mu \in M; \mu \in \mathbb{N}\}$ satisfying

$$\lim_{\mu\to\infty} \inf \{\varphi(x); x \in U_{x_\mu} \in \mathscr{U}\} = \infty \tag{2.1}$$

for some open covering $\mathscr{U}$ of M m-adapted to (B, h) and (K_M, κ), there exists an element s of $H^{0,0}(M, K_M \otimes B^m)$ satisfying

$$\limsup_{\mu\to\infty} |s(x_\mu)|_{\kappa h^m} = \infty.$$

Since the section s is found by solving a $\bar\partial$-equation with L^2 norm estimates in such a way that the L^2 norm of s on a fixed compact subset K of M is bounded by a constant depending only on K and $\sup_K \varphi$, it can be also concluded that one can find $\hat{K}$ for any $R > 0$ that for any $x \in M \setminus \hat{K}$ there exists $s \in H^{0,0}(M, K_M \otimes B^m)$ satisfying $\sup_K |s|_{\kappa h^m} < 1$ and $|s(x)|_{\kappa h^m} > R$.

Furthermore, since $\dim H^{0,0}(M, K_M \otimes B^m) = \infty$ holds for sufficiently large m, for any point $y \in M$ and for any $N \in \mathbb{N}$ one may assume that the above section s satisfies a side condition that s has the zero of order N at y, expanding $\hat{K}$ if necessary.

The proof of Theorem 1.1 is based on all these preparations. The rest is to find g, κ, h and φ so that, among all, (2.1) is satisfied for some $\mathscr{U}$. For the detail, see [Oh-5].

Now we proceed to the proof of Theorem 0.1.

Proof of Theorem 0.1. By Theorem 1.1 and by the assumption on K_M, there exist holomorphic sections $s_1, \dots, s_N$ of K_M^{-m} over Ω for sufficiently large m, such that the function $\psi := \log \sum_{j=1}^N |s_j|^2_{\kappa_M^{-m}}$ satisfies

$$\partial\bar\partial\psi - m\Theta_{\kappa_M} > 0 \tag{2.2}$$

on $\Omega \setminus A$, where $A := \{x \in \Omega; \psi(x) = -\infty\}$ and the connected components of A are compact by the K_M^{-m} convexity property (1.1) of Ω. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ convex increasing function satisfying $\lambda(t) = -\log(-t)$ for $t < -1$ and $\lambda''(t) = 0$ for $t > 0$. Then, for any Kähler metric g on M,

$$\epsilon\partial\bar\partial(\lambda(\psi) - \log(-\rho)) + g$$

is a complete Kähler metric on $\Omega \setminus A$ for sufficiently small $\epsilon > 0$. Moreover, by exploiting that $\dim H^{0,0}(\Omega, K_M^{-m})$ is infinite-dimensional, we may assume that, for any finitely many points $y_1, \dots, y_\ell$ given in advance, $y_j \in A$ for all j and $\exp(-\frac{\psi}{m})$ is not integrable on any neighborhood of y_j. Under this condition, let A_j be the connected component of A containing y_j and set $\mathcal{A} = \{A_j; 1 \le j \le \ell\}$.

Then, by applying the L^2 vanishing theorem in [Dm-1] or [Oh-2] to the bundle K_M^{-1} on $\Omega \setminus A$ with respect to the fiber metric $\kappa_M^{-1}\exp(\frac{\psi}{m})$, and taking the negligibility of analytic sets with respect to L^2 holomorphic functions into account, one has the surjectivity of the restriction map

$$H^{0,0}(\Omega) \to \mathbb{C}^{\mathcal{A}}.$$

From this, the conclusion of Theorem 0.1 is straightforward. □

Remark 2.3. By a similar method as above one can show the following.

Theorem 2.2. *Let M be a compact Kähler manifold and let D be an effective divisor on M such that the restriction of the associated line bundle $[D]$ to the support $|D|$ of D is semipositive. If $K_M|_{|D|}$ is negative, then $M \setminus |D|$ is mapped properly and holomorphically with connected fibers onto a locally closed analytic set in some $\mathbb{C}^N$.*

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