Pure and Applied Mathematics Quarterly
Volume 18, Number 2, 773–791, 2022

The sharp Gevrey Kotake-Narasimhan theorem with an elementary proof

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Abstract: We study the regularity of Gevrey vectors for Hörmander operators

$$P = \sum_{j=1}^{m} X_j^2 + X_0 + c$$

where the X_j are real, smooth vector fields and $c(x)$ is a smooth function, all in Gevrey class G^s. P is assumed to satisfy a subelliptic estimate in an open set Ω_0: for some $\varepsilon > 0$ there exists a constant C such that

$$\|v\|_\varepsilon^2 \leq C\left(|(Pv, v)| + \|v\|_0^2\right) \qquad \forall v \in C_0^\infty(\Omega_0).$$

We prove directly that for $s \geq 1, G^s(P, \Omega_0) \subset G^{s/\varepsilon}(\Omega_0)$, i.e.,

$$\begin{aligned} &\forall K \Subset \Omega_0,\ \exists C_K : \|P^j u\|_{L^2(K)} \leq C_K^{j+1}(2j)!^s,\ \forall j \\ \implies &\forall K' \Subset \Omega_0,\ \exists \tilde{C}_{K'} : \|D^\ell u\|_{L^2(K')} \leq \tilde{C}_{K'}^{\ell+1} \ell!^{s/\varepsilon},\ \forall \ell. \end{aligned}$$

In other words, Gevrey growth of derivatives of u as measured by *iterates* of P yields Gevrey regularity for u in a larger Gevrey class dictated by the size of ε in the *a priori* estimate.

When $\varepsilon = 1$, P is elliptic and so we recover the original Kotake-Narasimhan theorem ([9]), which has been studied in many other classes, including the class of ultradifferentiable functions ([1]).

Our result has appeared previously ([17]) but with a proof that one colleague referred to as 'incomplete', perhaps recalling their initial reaction that our approach would be 'very long' if written out in all detail. We have chosen to come up with a less 'detailed' but more intuitive proof, in the last section, that should leave no doubt of the complete adequacy of this approach.

We are indebted to the referee for insightful observations.

Received September 30, 2020.

1. Background

In 1972, Derridj and Zuily [5] proved G^s hypoellipticity ($Pu \in G^s \implies u \in G^s$) for

$$P = \sum_{j=1}^{m} X_j^2 + X_0 + c$$

satisfying

$$\|v\|_\varepsilon^2 \le C\left(|(Pv,v)| + \|v\|_0^2\right) \qquad \forall v \in C_0^\infty, \tag{1.1}$$

whenever $s > 1/\varepsilon \in \mathbb{Q}$ and very recently, for P with G^k coeffients, $k \in \mathbb{N}^+$, by studying Gevrey vectors for such operators (see below), Derridj was able to sharpen this result to include $s = 1/\varepsilon \in \mathbb{Q}$, but still with *rational* ε and G^k coefficients, $k \in \mathbb{N}^+$ (announced in [4] and proven in [3]).

Consider a linear partial differential operator P of order 2 with real analytic coefficients. An *analytic* vector for P is a distribution u such that u behaves analytically when differentiated by powers of P alone: locally, $\|P^j u\| \le C^j (2j)!$ that is, not all derivatives of u are assumed to behave as though u were analytic, only those sums occurring together precisely as in P.

Similarly a Gevrey-s vector u for P (with P only assumed to have Gevrey-s coefficients now) satisfies (locally) $\|P^j u\| \le C^j (2j)!^s$, or more precisely,

$$\forall K \Subset \Omega_0,\ \exists C_K : \|P^j u\|_{L^2(K)} \le C_K^{j+1} (2j)!^s,\ \forall j.$$

Derridj proved that Gevrey-s vectors for P under (1.1) belong to $G^{s/\varepsilon}$ (for $s > 1/\varepsilon$ with ε rational) and, to accomplish this, followed the classical method of adding a variable and showing (local) Gevrey hypoellipticity in $G^{1,s/\varepsilon}_{t,x}$ for the operator

$$Q = -D_t^2 - P. \tag{1.2}$$

This yields the result since the (convergent) homogeneous solution

$$U(t,x) = \sum_{\ell \ge 0} (-1)^\ell \frac{t^{2\ell}}{(2\ell)!} P^\ell u(x)$$

for Q just above has trace $U(0,x) = u(x)$.

Slightly earlier, N. Braun Rodrigues, G. Chinni, P. D. Cordaro and M. R. Jahnke [11] had obtained a global result on a torus for a restricted subclass of such operators P.

The methods we use also apply to prove the anisotropic hypoellipticity for (1.2) even for non-rational ε.

G^s *functions* are always Gevrey-s *vectors* for any P.

2. General considerations

There are two main results of this paper. First, the subellipticity index ε will no longer need to be rational, and, secondly, we are able to let s equal $1/\varepsilon$. From a technical point of view, the proof is no harder for Gevrey-k coefficients than for analytic coefficients, so we shall take the vector fields to have analytic coefficients.

And from a more personal point of view, in reading Derridj's preprint ([4]) we could not find a reason why the result should not follow from the direct lines we have established over many decades and which in fact avoid the need to add a variable and deal with (1.2), despite the historical significance of that approach which in some sense deals with iterates of P in a less obvious way.

In the elliptic case ($\varepsilon = 1$ in (2.3) just below), we recover the original Kotake-Narasimhan theorem ([9]).

The only hypothesis, aside from Gevrey smoothness of the coefficients of P in Ω_0, is the subelliptic estimate: for some $\varepsilon \in (0, 1]$,

$$\|v\|_\varepsilon^2 \; (+\sum_1^n \|X_j v\|_{L^2}^2) \leq C\{|(Pv, v)_{L^2}| + \|v\|_{L^2}^2\}, \quad \forall v \in C_0^\infty(\Omega_0) \tag{2.3}$$

3. Smoothness

From the basic *a priori* estimate (2.3) and those that will follow from it, we have $u \in C^\infty$: given $u \in G^s(P, \Omega_0)$ with $Pu \in L^2_{loc}$ it will follow from (2.3) that $u \in H^\varepsilon_{loc}$. From our estimate (4.5) below (for $\|u\|^2_{2\varepsilon}$), it will follow that $Pu \in H^\varepsilon_{loc}$, (since $P^2 u \in L^2_{loc}$) and hence that $u \in H^{2\varepsilon}_{loc}$, and similarly from $P^n u \in L^2_{loc}$, that $P^{n-1}u \in H^\varepsilon_{loc}$, $\ldots$, and finally that $u \in H^{(n+1)\varepsilon}_{loc}$ (for all n, and hence $u \in C^\infty$). Thus we will assume that u is smooth.

And since the proof is unchanged if one assumes that the coefficients of P are real analytic functions instead of merely belonging to a Gevrey class, we will not mention their precise smoothness again.

4. Estimates

Unless otherwise specified, norms and inner products will be in L^2 and we will frequently employ a fractional power of the Laplacian Λ

$$\widehat{\Lambda^\mu w}(\xi) = (1+|\xi|^2)^{\mu/2}\hat{w}(\xi),$$

so that $\lambda^\mu(\xi) = (1+|\xi|^2)^{\mu/2}$ is the symbol of Λ^μ.

In order to obtain estimates at higher levels, we want to replace v by $\varphi(x)\Lambda^\varepsilon v$ above, with $\varphi \in C_0^\infty(\Omega_0)$, $\varphi \equiv 1$ on $\Omega' \Subset \Omega_0$ so that we are inserting suitably supported functions into the norm, and we denote by the special notation $\langle A|B\rangle$ *both* AB and BA (i.e., the order of A and B is unspecified). In all cases we will add the two terms and then explicitly include their bracket on the right hand side. Using the common expression $\lesssim$ for $\le C_0$ when C_0 will only take on a finite, fixed number of values (involving the spatial dimension and the number of vector fields in P), (2.3) now may be written, omitting the 'junk' (L^2) term $\|\varphi\Lambda^\varepsilon v\|^2_{L^2}$ on the right:

$$\|\varphi\Lambda^\varepsilon v\|^2_\varepsilon + \|\langle X|\varphi\Lambda^\varepsilon\rangle v\|^2_{L^2} \lesssim |(P\varphi\Lambda^\varepsilon v, \varphi\Lambda^\varepsilon v)_{L^2}| + \|[X,\varphi\Lambda^\varepsilon]v\|^2_{L^2}. \tag{4.4}$$

We will never need to distinguish between the various $X_j, j = 1,\dots m$ or explicitly sum over them so we have dropped that index. Finally, a right hand side as above will be taken to include a uniformly 'junk' term, in this case $\|\varphi\Lambda^\varepsilon v\|^2_{L^2}$, arising from the last term in (2.3).

For starters, we keep both norms *and inner products* in the estimate (this is crucial) as we try to estimate 2ε derivatives instead of just ε:

$$\begin{aligned} &\|\varphi\Lambda^\varepsilon v\|^2_\varepsilon + \|\langle X|\varphi\Lambda^\varepsilon\rangle v\|^2_{L^2} \le \\ &\lesssim |(\varphi\Lambda^\varepsilon Pv, \varphi\Lambda^\varepsilon v)| + |([P,\varphi\Lambda^\varepsilon]v, \varphi\Lambda^\varepsilon v)| + \|[X,\varphi\Lambda^\varepsilon]v\|^2_{L^2}. \end{aligned} \tag{4.5}$$

We shall write φ' for *any* first derivative of φ (such as $X\varphi$):

$$\begin{aligned}[P,\varphi\Lambda^\varepsilon] = [X^2,\varphi\Lambda^\varepsilon] = X[X,\varphi\Lambda^\varepsilon] + [X,\varphi\Lambda^\varepsilon]X = \\ = X[X,\varphi\Lambda^\varepsilon] + \varphi'\Lambda^\varepsilon X + \varphi[X,\Lambda^\varepsilon]X\end{aligned}$$

and (this is the essential but tricky step)

$$\varphi[X,\Lambda^\varepsilon]X = X\varphi[X,\Lambda^\varepsilon] + \varphi[[X,\Lambda^\varepsilon],X] - \varphi'[X,\Lambda^\varepsilon],$$

so that expanding the second term on the right in (4.5), integrating by parts, using $X^* \sim -X$ (since the X_j are real) and swapping the order of φ and φ'

since they are functions and hence commute,

$$([P,\varphi\Lambda^\varepsilon]v,\varphi\Lambda^\varepsilon v)\sim\pm([X,\varphi\Lambda^\varepsilon]v,X\varphi\Lambda^\varepsilon v)\pm(\varphi\Lambda^\varepsilon Xv,\varphi'\Lambda^\varepsilon v)$$
$$\pm(\varphi[X,\Lambda^\varepsilon]v,X\varphi\Lambda^\varepsilon v)\pm(\varphi[X,\Lambda^\varepsilon]v,\varphi'\Lambda^\varepsilon v)\pm(\varphi[[X,\Lambda^\varepsilon],X]v,\varphi\Lambda^\varepsilon v)$$

and thus, after a usual weighted Schwarz inequality to absorb (a small multiple of) $\|\langle X|\varphi\Lambda^\varepsilon\rangle v\|^2$ on the left, estimate (4.5) becomes

$$\|\varphi\Lambda^\varepsilon v\|_\varepsilon^2+\|\langle X|\varphi\Lambda^\varepsilon\rangle v\|_{L^2}^2\lesssim|(\varphi\Lambda^\varepsilon Pv,\varphi\Lambda^\varepsilon v)|$$

$$(+\|[X,\varphi\Lambda^\varepsilon]v\|_{L^2}^2)+\|\varphi'\Lambda^\varepsilon v\|_{L^2}^2+\|\varphi\Lambda_1^\varepsilon v\|_{L^2}^2+\|\varphi\Lambda_2^\varepsilon v\|_{-\varepsilon}^2 \tag{4.6}$$

where

$$\Lambda_1^\varepsilon=[X,\Lambda^\varepsilon]$$

and

$$\Lambda_2^\varepsilon=[[X,\Lambda^\varepsilon],X],$$

both fairly elementary pseudodifferential operators of order ε. We have suppressed the term $\|\varphi[X,\Lambda^\varepsilon]\|_{L^2}^2$, since $\varphi[X,\Lambda^\varepsilon]=[X,\varphi\Lambda^\varepsilon]-X(\varphi)\Lambda^\varepsilon$ both of which already appeared above. And we could have omitted the term $\|[X,\varphi\Lambda^\varepsilon]v\|_{L^2}^2$ since the two next terms contain it, but we will preserve it for now because it is suggestive and helps make sense of the second term on the left.

The essential feature of (4.6) is that a gain of ε results in at most *one* derivative on φ, and it is for this reason that we have retained the inner product with P in the estimate, since when an extra derivative threatens, we are able to exchange the φ's on the two sides of the inner product and avoid an extra derivative on φ when we have gained only *one* ε power of Λ.

And a small note: while v is a test function of compact support, our 'solution' u will not have compact support. Thus we will introduce a 'largest' localizing function, denoted Ψ, equal to 1 near the supports of all the other localizing functions, which will sit beside u everywhere but in the end be removable modulo infinitely smoothing brackets with precise bounds since there will be other functions of smaller support, such as φ, to render this Ψ superfluous.

5. Personal heuristics

This paper has an unusual formulation, but one that I hope will make it unusually readable, and I have written it this way because the proof is somewhat intricate and could surely benefit from any available help.

It has become my conviction over the years that a mathematical paper that contains every symbol, and every derivative of a localizing function explicitly notated becomes unreadable. I personally require more sense of reader-friendly 'flow' in reading a technical paper to render the formulas accessible. Perhaps, to paraphrase Frege in [8], anyone who understands the flow of the argument and the justification of the flow well enough probably does not actually need all the detailed calculations. [1]

I would not go that far. But the challenge of following every bracket and every derivative and writing it down would test the strongest stomach and I prefer to omit that much detail and ask the reader to honor the author's honesty and track record and precision and to let the flow suffice in a few places.

I took this approach in a recent paper [16] and in fact the referee wrote that "I guess the author is trying to explain the ideas in his technical calculations by describing them in words with a minimum of symbols, but the words pile on to the point where one needs to be almost as familiar with the calculations as the author himself for them to make sense. A reader might wonder if the author is trying to pull a fast one by substituting a lot of hand-waving for honest computation - if it weren't for some of the subsequent pages where the symbols swamp the words. Can't one strike a better balance?" I have tried for many years to find a better balance and concluded that in this material, and for this author, the answer is "Sadly, no."

6. Derivatives in terms of powers of P

The algorithm we will use to achieve estimates in terms of pure powers of P on u is as follows: essentially as above, although now of higher order β, modulo uniform, lower order errors, with

$$\|\langle X|\varphi\Lambda^\beta\rangle v\|^2_{L^2} \underset{\text{def}}{\equiv} \|X\varphi\Lambda^\beta v\|^2_{L^2} + \|\varphi\Lambda^\beta X v\|^2_{L^2},$$

[1]In Frege's work, when trying to define the concept of number, he went to great lengths to define the number "n" as the equivalence class of all sets whose elements could be put in one to one correspondence with the first n natural numbers (which he defined carefully). To a mathematician this is not a difficult concept (a set with n cows is in the same equivalence class as a collection of n objects of any sort, and it is this 'equivalence' *class of sets* that is the 'number' n.) It is roughly at this point where Frege declares that anyone who has understood what he has written to that point (though few would have in his day) wouldn't have needed to read his somewhat lengthy but beautiful philosophical works.

where now the notation $\lesssim_\beta$ will be used if there is any dependence on β and $\lesssim$ alone will be independent of β,

(1) First we estimate, as above, for arbitrary β and all $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}&\|\varphi\Lambda^\beta v\|_\varepsilon^2 + \|\langle X|\varphi\Lambda^\beta\rangle v\|_{L^2}^2\\ &\lesssim |(P\varphi\Lambda^\beta v, \varphi\Lambda^\beta v)_{L^2}| \quad (+\,\|[X,\varphi\Lambda^\beta]v\|_{L^2}^2)\end{aligned}$$

(2) Then we commute P past $\varphi\Lambda^\beta$ until it lands beside v, to obtain $(\varphi\Lambda^\beta Pv, \varphi\Lambda^\beta v)_{L^2}$, thus requiring treatment of the bracket

$$([P,\varphi\Lambda^\beta]v, \varphi\Lambda^\beta v)_{L^2}.$$

(3) To expand this bracket, we write $P = X^2$ generically, so that with $\varphi' = \pm[X,\varphi]$,

$$\begin{aligned}&[P,\varphi\Lambda^\beta] = [P,\varphi]\,\Lambda^\beta + \varphi[P,\Lambda^\beta] =\\ &= \varphi' X\Lambda^\beta + X\varphi'\Lambda^\beta + 2\varphi X[X,\Lambda^\beta] + \varphi[[X,\Lambda^\beta],X]\end{aligned}$$

and thus, integrating X by parts and/or switching φ and φ', and using a weighted Schwarz inequality, uniformly in β, and modulo a small constant times the LHS in (1),

$$|([P,\varphi\Lambda^\beta]v,\varphi\Lambda^\beta v)| \sim \|\varphi'\Lambda^\beta v\|_{L^2}^2 + \|\varphi\Lambda_1^\beta v\|_{L^2}^2 + \|\varphi\Lambda_2^\beta v\|_{-\varepsilon}^2$$

where we recall the notation

$$\Lambda_1^\beta = [X,\Lambda^\beta] \text{ and } \Lambda_2^\beta = [[X,\Lambda^\beta],X],$$

both of which are of order β.

(4) We gather these steps and freely move φ past powers of Λ, since any bracket (whether applied to v or Pv) will introduce $\mathbb{N} \ni k \geq 1$ derivatives on φ but also *decrease* the power of Λ by at least k (and not merely by $\varepsilon \times k$), i.e., create junk terms, (together with the corresponding remainders - see the next section):

$$\begin{aligned}&\|\varphi\Lambda^{\beta+\varepsilon}v\|_{L^2}^2 + \|\langle X|\varphi\Lambda^\beta\rangle v\|_{L^2}^2 \sim \|\varphi\Lambda^\beta v\|_\varepsilon^2 + \|\langle X|\varphi\Lambda^\beta\rangle v\|_{L^2}^2\\ &\leq C\|\varphi\Lambda^{\beta-\varepsilon}Pv\|_{L^2}^2 + \|\varphi'\Lambda^\beta v\|_{L^2}^2 + \|\varphi\Lambda_1^\beta v\|_{L^2}^2 + \|\varphi\Lambda_2^\beta v\|_{-\varepsilon}^2.\end{aligned}$$

(5) Both Λ_1^β and Λ_2^β are of order β and will be expanded in the next section (*Expanding the Brackets*). Looking ahead to (7.8) below, however, for

the moment with $\mu = \beta$ and any r, and (abusively) writing $(\Lambda^\beta)^{(\ell)}$ for the operator with symbol $(\lambda^\beta)^{(\ell)}$,

$$\varphi\Lambda_1^\beta v = \varphi[X, \Lambda^\beta]v = \varphi \sum_{\ell=1}^{r-1} \frac{1}{\ell!} a^{(\ell)} (\Lambda^\beta)^{(\ell)} Dv + {}_1R_r v$$

since $X = aD$,

$$\varphi\Lambda_1^\beta v = \varphi[X, \Lambda^\beta]v = \sum_{\ell=1}^{r-1} \varphi \frac{a^{(\ell)}}{\ell!} (\Lambda^\beta)^{(\ell)} Dv + {}_1R_r v$$

and so

$$\|\varphi\Lambda_1^\beta v\|_{L^2} \lesssim \sum_{\ell=1}^{\beta-1} C_a^\ell \beta^\ell \|\varphi(\Lambda^{\beta-\ell})Dv\|_{L^2} + \|{}_1R_r v\|_{L^2}$$

and the similar but slightly more complicated expression for

$$\begin{gathered}
\varphi\Lambda_2^\beta v = \varphi[[aD, \Lambda^\beta], aD]v = \varphi[[a, \Lambda^\beta]D, aD]v \\
= \varphi([a, \Lambda^\beta]a'D + [[a, \Lambda^\beta], aD]D)v \\
= \varphi([a, \Lambda^\beta]a'D + [[a, \Lambda^\beta], a]D^2 + a[a', \Lambda^\beta]D)v \\
\sim \varphi([[a, \Lambda^\beta], a]D^2 + 2a[a', \Lambda^\beta]D)v \\
\sim \varphi \sum_{\ell=1}^{r-1} \sum_{\ell'=1}^{r'-1} \frac{1}{\ell!\,\ell'!} a^{(\ell)} a^{(\ell')} (\Lambda^\beta)^{(\ell+\ell')} D^2 v \\
+ \varphi \sum_{\ell=1}^{r-1} \frac{1}{\ell!} a^{(\ell+1)} a (\Lambda^\beta)^{(\ell)} Dv
\end{gathered}$$

so that, and bringing the coefficients out of the norm,

$$\varphi\Lambda_2^\beta v \sim \sum_{\tilde\ell=2}^{r-1} C_a^{\tilde\ell} \beta^{\tilde\ell} \varphi\,\Lambda^{\beta-\tilde\ell} D^2 v + \sum_{\ell=1}^{r-1} C_a^\ell \beta^\ell \varphi\,\Lambda^{\beta-\ell} Dv$$

or

$$\|\Lambda^{-\varepsilon}\varphi\Lambda_2^\beta v\|_{L^2}^2 \le \sum_{\ell=0}^{r-1} C_a^\ell \beta^\ell \|\varphi\,\Lambda^{\beta-\ell} v\|_{-\varepsilon}^2.$$

As always with pseudodifferential operators, there will be a sum of terms of lower and lower order as dictated by Leibniz' formula for brackets, and remainders.

(6) We repeat the above steps by applying the estimate in (4) to the terms on the right in (4) producing $\varphi\Lambda^{\beta-3\varepsilon}P^2v$, $\varphi'\Lambda^{\beta-2\varepsilon}Pv$ and $\varphi''\Lambda^{\beta-\varepsilon}v$, etc. On the right hand side each of the four terms will lead to a 'spray' of additional more terms, about four times as many at each next step. The resulting paradigm may be simplified to read

$$\|\varphi\Lambda^{\beta+\varepsilon}v\|_{L^2}^2 \rightsquigarrow \|\varphi\Lambda^{\beta-\varepsilon}Pv\|_{L^2}^2 + \|\varphi'\Lambda^{\beta}v\|_{L^2}^2$$
$$\rightsquigarrow \|\varphi\Lambda^{\beta-3\varepsilon}P^2v\|_{L^2}^2 + \|\varphi'\Lambda^{\beta-2\varepsilon}Pv\|_{L^2}^2 + \|\varphi''\Lambda^{\beta-\varepsilon}v\|_{L^2}^2,$$

and in general, after k iterations, there will be C^k terms of the form

$$\|\varphi^{(k_1)}\Lambda^{\beta+\varepsilon-(k_1+2k_2)\varepsilon}P^{k_2}v\|_{L^2}^2$$

with $k = k_1 + k_2$.

(7) We continue each iteration until we just get to $\beta+\varepsilon-(k_1+2k_2)\varepsilon \le 0$, (but at the previous step, $\beta+\varepsilon-(k_1+2k_2)\varepsilon \ge 0$, i.e., $k_1+2k_2 = \lceil\frac{\beta+\varepsilon}{\varepsilon}\rceil$), so that

$$\|\varphi\Lambda^{\beta+\varepsilon}v\|_{L^2}^2 \le C^k\|\varphi^{(k_1)}\Lambda^{\beta+\varepsilon-(k_1+2k_2)\varepsilon}P^{k_2}v\|_{L^2}^2$$

where the power of Λ in each term on the right is non-positive.

(8) It remains to apply all of this to our 'solution' u, which is subject to the growth of $P^k u$, not functions like v which are 'test' functions and have compact support:

$$\|P^j u\|_{L^2(K)} \le C_K^{2j+1}(2j)!^s, \ \forall j \text{ for suitable } C_K.$$

But in the estimate in item (7) we are free to replace v by Ψu where $\Psi \equiv 1$ near the support of φ, since any error committed in then bringing Ψ out of the norm will be of order $-\infty$. Modulo this error, then,

$$\|\varphi\Lambda^{\beta+\varepsilon}\Psi u\|_{L^2}^2 \le C^k\|\varphi^{(k_1)}\Lambda^{\beta+\varepsilon-(k_1+2k_2)\varepsilon}P^{k_2}u\|_{L^2(K)}^2$$

Our conclusion is that for any $K' \Subset \Omega_0$, $\exists\, C_{K'} : \|D^m u\|_{L^2(K')} \le \tilde{C}^{m+1}m!^{s/\varepsilon}$, $\forall m$. Indeed, taking $\beta+\varepsilon = m$, we have, (since $\beta+\varepsilon-(k_1+2k_2)\varepsilon \le 0$)

$$\|D^m u\|_{L^2(K')} \le \tilde{C}^{m+1} \sup_{k_1+2k_2=\lceil\frac{m}{\varepsilon}\rceil} \|\varphi^{(k_1)}\|_\infty \|P^{k_2}u\|_{L^2(K)};$$

in particular, with $\varphi \in G^s$, and different constants in each instance, but independent of everything but u,

$$\|D^m u\|_{L^2(K')} \le \tilde{C}^{m+1} \sup_{k_1+2k_2=\lceil\frac{m}{\varepsilon}\rceil} k_1!^s \|P^{k_2}u\|_{L^2(K)}$$

$$\leq \tilde{C}^{m+1}\lceil\frac{m}{\varepsilon}\rceil!^s \leq C^m(\frac{m}{\varepsilon}+1)!^s \leq C_1^{m/\varepsilon}(\frac{m}{\varepsilon})!^s$$

7. Expanding the brackets

In order to write out the above brackets of the previous section concretely, we use a Taylor expansion of the symbol $\lambda^\mu(\xi)$ of Λ^μ: $\forall \mu, r$, and write, with $f = a$ (a coefficient of one of the X's, which will always be accompanied by φ) or by $f = \varphi(x)$ itself,

$$\begin{aligned}([f,\Lambda^\mu]v)^\wedge(\xi) &= \int \hat{f}(\xi-\eta)\sum_{\ell=1}^{r-1}\frac{(\xi-\eta)^\ell\lambda^{\mu(\ell)}(\eta)}{\ell!}\hat{v}(\eta)d\eta + \widehat{{}_fR_r v}(\xi)\\ &= \sum_{\ell=1}^{r-1}\int\frac{\widehat{f^{(\ell)}}(\xi-\eta)}{\ell!}\lambda^{\mu(\ell)}(\eta)\hat{v}(\eta)d\eta + \widehat{{}_fR_r v}(\xi)\end{aligned}$$

where

$$\widehat{{}_fR_r v}(\xi) = \int\frac{\widehat{f^{(r)}}(\xi-\eta)}{r!}\underbrace{\int_0^1 dp\cdots\int_0^1 dt}_{r}\ \lambda^{\mu(r)}(\eta+t\cdots p(\xi-\eta))\hat{v}(\eta)d\eta$$

so that, writing $(\Lambda^\mu)^{(\ell)}$ for the operator with symbol $(\lambda^\mu)^{(\ell)}(\cdot)$ and taking $f = \varphi$ we have

$$\|[\varphi(x),\Lambda^\mu]v\|_{L^2} \leq \sum_{\ell=1}^{r-1}\frac{1}{\ell!}\|\varphi^{(\ell)}(\Lambda^\mu)^{(\ell)}v\|_{L^2} + \|{}_\varphi R_r v\|_{L^2}. \tag{7.7}$$

Recalling that we write $X = aD$, with $f = a$ (localized):

$$\|\varphi[a,\Lambda^\mu]Dv\|_{L^2} \leq \sum_{\ell=1}^{r-1}\frac{1}{\ell!}\|\varphi a^{(\ell)}(\Lambda^\mu)^{(\ell)}Dv\|_{L^2} + \|\varphi_a R_r v\|_{L^2}. \tag{7.8}$$

For the last term in (4) above, $\|\varphi\Lambda_2^\beta v\|_{-\varepsilon}^2$, we write

$$\begin{aligned}\Lambda^{-\varepsilon}\varphi\Lambda_2^\mu v &= \Lambda^{-\varepsilon}\varphi[[a,\Lambda^\mu]D, aD]v\\ &= \Lambda^{-\varepsilon}\varphi([a,\Lambda^\mu]a'D + [[a,\Lambda^\mu],aD]D)v\\ &= \Lambda^{-\varepsilon}\varphi([a,\Lambda^\mu]a'D + [[a,\Lambda^\mu],a]D^2 + a[a',\Lambda^\mu]D)v\\ &\sim \Lambda^{-\varepsilon}\varphi([[a,\Lambda^\mu],a]D^2 + 2a[a',\Lambda^\mu]D)v\end{aligned}$$

$$\sim \Lambda^{-\varepsilon}\varphi\sum_{\ell=1}^{r-1}\frac{1}{\ell!}\sum_{\ell'=1}^{r'-1}\frac{1}{\ell'!}a^{(\ell)}a^{(\ell')}(\Lambda^{\mu})^{(\ell+\ell')}D^2 v$$
$$+\Lambda^{-\varepsilon}\varphi\sum_{\ell=1}^{r-1}\frac{1}{\ell!}a^{(\ell+1)}a(\Lambda^{\mu})^{(\ell)}Dv$$

and then take the L^2 norms.

To treat the remainders, we divide up the region of integration as we did in ([13]) into two parts, the first where $|\xi-\eta|\leq\frac{1}{10}|\eta|$, and hence the action of R_r is bounded by the L^1 norm of derivatives of the coefficients of total order r times $\|\Lambda^{\mu-r}v\|_{L^2}$ and the region where $|\xi|$ (and hence $|\eta|$) is bounded by a multiple of $|\xi-\eta|$ and so that $|\lambda^{\mu}(\xi)-\lambda^{\mu}(\eta)|\leq C^{\mu}|\xi-\eta|^{\mu}$, whence for any M,

$$|([\Lambda^{\mu},a(x)]v)^{\wedge}(\xi)| = |((\lambda^{\mu}\hat{a})*\hat{v})(\xi)-(\hat{a}*(\lambda^{\mu}\hat{v}))(\xi)|$$
$$=\left|\lambda^{\mu}(\xi)\int\hat{a}(\xi-\eta)\hat{v}(\eta)d\eta-\int\hat{a}(\xi-\eta)\lambda^{\mu}(\eta)\hat{v}(\eta)\,d\eta\right|$$
$$=\left|\int\hat{a}(\xi-\eta)[\lambda^{\mu}(\xi)-\lambda^{\mu}(\eta)]\hat{v}(\eta)d\eta\right|$$
$$\leq C^{M}\left|\int\widehat{a^{(M+\mu)}}(\xi-\eta)(1+|\eta|^2)^{-M/2}\hat{v}(\eta)d\eta\right|.$$

8. Calculations of $(\boldsymbol{\lambda}^{\mu})^{(\ell)}$

We devote the rest of the paper to obtaining transparent expressions for derivatives of powers of the symbol of λ, which will render the expressions in the previous section utterly standard, since in the usual treatment of the algebra of pseudo-differential operators, their brackets, adjoints, etc and composition of properly supported operators depends on accurate expressions for derivatives of their symbols which are well known in the familiar classes but we find that easy bounds may not suffice in our case or be that 'easy' to compute.

We have, since $\lambda(\rho)=(1+|\rho|^2)^{1/2}$,

$$\lambda'=\rho\lambda^{-1} \tag{8.9}$$

$$(\lambda^{a})'=a\rho\lambda^{a-2} \tag{8.10}$$

and, since $\rho^2=\lambda^2-1$, for any integral value of b,

$$(\rho\lambda^{b})'=(1+b)\lambda^{b}-b\lambda^{b-2}. \tag{8.11}$$

Thus (using (8.10) and (8.11),

$$(\lambda^a)^{('')} = ((\lambda^a)')' = a\left(\rho\lambda^{a-2}\right)' = a\{(1+(a-2))\lambda^{a-2} - (a-2)\lambda^{a-4}\}$$

or

$$(8.12) \qquad (\lambda^a)^{('')} = a(a-1)\lambda^{a-2} - a(a-2)\lambda^{a-4}$$

which will form the pattern for higher derivatives of powers of λ.

Evidently, products of monomials involving $\mu - a_j$ with $a_j \in \mathbb{N}^+ \cup 0$ and will pile up and will use the notation $A = (a_1, a_2, \ldots, a_{|A|})$ and

$$(8.13) \qquad [A]_\mu = (\mu - a_1)(\mu - a_2)\ldots(\mu - a_{|A|}).$$

Thus simplified, we may write

$$(8.14) \qquad (\lambda^a)^{('')} = [0,1]_\mu \lambda^{a-2} - [0,2]_\mu \lambda^{a-4}.$$

For the third derivative we have at once, from (8.10),

$$(8.15) \qquad (\lambda^\mu)^{(''')}(\rho) = \rho\{[0,1,2]_\mu \lambda^{\mu-4} - [0,2,4]_\mu \lambda^{\mu-6}\}$$

But the fourth derivative is a bit more complicated, since new $\lambda^{\mu-6}$ terms come from *both* terms in the third derivative. (And in computing higher derivatives *many* terms come into play, which became very hard to estimate together and led to the following, explicit, expressions.)

Using (8.11) and our notation $(\mu - d)[a,b,c]_\mu = [a,b,c,d]_\mu$,

$$(8.16) \qquad \begin{aligned} (\lambda^\mu)^{(iv)}(\rho) = \{&(1+(\mu-4))[0,1,2]_\mu\, \lambda^{\mu-4} - (\mu-4)[0,1,2]_\mu\, \lambda^{\mu-6}\} \\ &- \{(1+(\mu-6))[0,2,4]_\mu\, \lambda^{\mu-6} - (\mu-6)[0,2,4]_\mu\, \lambda^{\mu-8}\} \\ &= [0,1,2,3]_\mu\, \lambda^{\mu-4} - [0,1,2,4]_\mu\, \lambda^{\mu-6} \\ &- [0,2,4,5]_\mu\, \lambda^{\mu-6} + [0,2,4,6]_\mu\, \lambda^{\mu-8} \\ = [0,1,2,3]_\mu\, \lambda^{\mu-4} &- 2[0,2,3,4]_\mu\, \lambda^{\mu-6} + [0,2,4,6]_\mu\, \lambda^{\mu-8}. \end{aligned}$$

Here the coefficients of the first and last terms are transparent, but the middle one is less so, resulting from the somewhat mysterious fact that the two coefficients of $\lambda^{\mu-6}$ may be combined:

$$\begin{aligned} -[0,1,2,4]_\mu - [0,2,4,5]_\mu &= -(\mu-1)[0,2,4]_\mu - (\mu-5)[0,2,4]_\mu \\ &= -2(\mu-3)[0,2,4]_\mu = -2[0,2,3,4]_\mu. \end{aligned}$$

Thus $[0,1,2,4]_\mu$ and $[0,2,4,5]_\mu$ have a 'common factor' of $[0,2,4]_\mu$. We will exploit similar relationships below. Continuing, with (8.10),

$$\begin{gathered}(\lambda^\mu)^{(v)}(\rho) = \\ = \rho\{[0,1,2,3,4]_\mu\, \lambda^{\mu-6} - 2[0,2,3,4,6]_\mu\, \lambda^{\mu-8} + [0,2,4,6,8]_\mu\, \lambda^{\mu-10}\}\end{gathered}$$

and so, using (8.11) as we did in (8.16) above,

$$\begin{gathered}(\lambda^\mu)^{(vi)}(\rho) = \\ = (1+(\mu-6))[0,1,2,3,4]_\mu\, \lambda^{\mu-6} - (\mu-6)[0,1,2,3,4]_\mu\, \lambda^{\mu-8} \\ -2\left\{(1+(\mu-8))[0,2,3,4,6]_\mu\, \lambda^{\mu-8} - (\mu-8)[0,2,3,4,6]_\mu\, \lambda^{\mu-10}\right\} \\ +(1+(\mu-10))[0,2,4,6,8]_\mu\, \lambda^{\mu-10} - (\mu-10)[0,2,4,6,8]_\mu\, \lambda^{\mu-12} \\ = [0,1,2,3,4,5]_\mu\, \lambda^{\mu-6} \\ -\left([0,1,2,3,4,6]_\mu\, + 2[0,2,3,4,6,7]_\mu\,\right)\lambda^{\mu-8} \\ +\left(2[0,2,3,4,6,8]_\mu\, + [0,2,4,6,8,9]_\mu\,\right)\lambda^{\mu-10} \\ -[0,2,4,6,8,10]_\mu\, \lambda^{\mu-12}\end{gathered}$$

so that, after gathering $(\mu-1)+2(\mu-7)=3(\mu-5)$, we have

$$-([0,1,2,3,4,6]_\mu + 2[0,2,3,4,6,7]_\mu) = -3[0,2,3,4,5,6]_\mu$$

and similarly, gathering $2(\mu-3)+(\mu-9)=3(\mu-5)$, we have

$$2[0,2,3,4,6,8]_\mu\, + [0,2,4,6,8,9]_\mu = 3[0,2,4,5,6,8]_\mu$$

and so

$$\begin{gathered}(\lambda^\mu)^{(vi)}(\rho) = \\ = [0,1,2,3,4,5]_\mu\, \lambda^{\mu-6} \\ -3[0,2,3,4,5,6]_\mu\, \lambda^{\mu-8} \\ +3[0,2,4,5,6,8]_\mu\, \lambda^{\mu-10} \\ -[0,2,4,6,8,10]_\mu\, \lambda^{\mu-12}.\end{gathered}$$

Notice that the digit 5 appears in each 'gathering' (as a factor) and was the smallest digit initially missing from each coefficient in the expansion of $(\lambda^\mu)^{(vi)}(\rho) =$ above and which, in the end, seems to 'migrate' from the rightmost position (in the coefficient of $(\lambda^\mu)^{(vi)}$ above) one position to the left in each line until there is no room in the last line, which will always be the product of monomials with even values of a.

Thus the seventh derivative, using (8.9):

$$\begin{gathered}(\lambda^\mu)^{(vii)}(\rho)\\ = \rho\,[0,1,2,3,4,5,6]_\mu\,\lambda^{\mu-8}\\ -3\,\rho\,[0,2,3,4,5,6,8]_\mu\,\lambda^{\mu-10}\\ +3\,\rho\,[0,2,4,5,6,8,10]_\mu\,\lambda^{\mu-12}\\ -\rho\,[0,2,4,6,8,10,12]_\mu\,\lambda^{\mu-14}.\end{gathered}$$

And for the eighth derivative, using (8.9) and recalling (8.13), we get

$$\begin{gathered}(\lambda^\mu)^{(viii)} = (1+(\mu-8))\lambda^{\mu-8}[0,1,2,3,4,5,6]_\mu\\ -(\mu-8)\lambda^{\mu-10}[0,1,2,3,4,5,6]_\mu\\ -3\left((1+(\mu-10))\lambda^{\mu-10}-(\mu-10)\lambda^{\mu-12}\right)[0,2,3,4,5,6,8]_\mu\\ +3\left((1+(\mu-12))\lambda^{\mu-12}-(\mu-12)\lambda^{\mu-14}\right)[0,2,4,5,6,8,10]_\mu\\ -\left((1+(\mu-14))\lambda^{\mu-14}-(\mu-14)\lambda^{\mu-16}\right)[0,2,4,6,8,10,12]_\mu\\ = (\mu-7)[0,1,2,3,4,5,6]_\mu\lambda^{\mu-8}\\ -\{(\mu-8)[0,1,2,3,4,5,6]_\mu+3(\mu-9)[0,2,3,4,5,6,8]_\mu\}\lambda^{\mu-10}\\ +3\{(\mu-10)[0,2,3,4,5,6,8]_\mu+(\mu-11)[0,2,4,5,6,8,10]_\mu\}\lambda^{\mu-12}\\ -\{3(\mu-12)[0,2,4,5,6,8,10]_\mu+(\mu-13)[0,2,4,6,8,10,12]_\mu\}\lambda^{\mu-14}\\ +(\mu-14)[0,2,4,6,8,10,12]_\mu\lambda^{\mu-16}\end{gathered}$$

or

$$\begin{gathered}(\lambda^\mu)^{(viii)}(\rho) =\\ = [0,1,2,3,4,5,6,7]_\mu\lambda^{\mu-8}\\ -4[0,2,3,4,5,6,7,8]_\mu\lambda^{\mu-10}\\ +6[0,2,4,5,6,7,8,10]_\mu\lambda^{\mu-12}\\ -4[0,2,4,6,7,8,10,12]_\mu\lambda^{\mu-14}\\ +[0,2,4,6,8,10,12,14]_\mu\lambda^{\mu-16}\end{gathered}$$

after bringing $(\mu-1)$ out of the first $\lambda^{\mu-10}$ term and inserting $(\mu-8)$ to get $-4(\mu-7)[0,2,3,4,5,6,8]_\mu\lambda^{\mu-10} = -4[0,2,3,4,5,6,7,8]_\mu\lambda^{\mu-10}$ as coefficient of $\lambda^{\mu-10}$, then bringing $(\mu-3)$ out of the first $\lambda^{\mu-12}$ term and inserting $(\mu-10)$ to get $6[0,2,4,6,7,8,10]_\mu$ as coefficient of $\lambda^{\mu-12}$, and finally inserting the $(\mu-12)$ and 'factoring out' $(\mu-5)$, together with $(\mu-13)$, yields $(-3(\mu-5)-(\mu-13)) = -4(\mu-7))$ times $[0,2,4,6,8,10,12]_\mu$ or $-4[0,2,4,6,7,8,10,12]$ as the coefficient of $\lambda^{\mu-14}$.

Once again the pattern seems to be that after $\mu-7$ is introduced, it migrates to the left as lower odds factors vanish and are replaced by successive evens on the right. Thus we might expect that for the tenth derivative, with 9 taking the place of 7, we would have the following series of coefficients of powers of $\{\lambda\}$

$$\begin{gathered}
[0,1,2,3,4,5,6,7,8,9]_\mu,\\
-5[0,2,3,4,5,6,7,8,9,10]_\mu,\\
10[0,2,4,5,6,7,8,9,10,12]_\mu,\\
-10[0,2,4,6,7,8,9,10,12,14]_\mu,\\
5[0,2,4,6,8,9,10,12,14,16]_\mu,\\
-[0,2,4,6,8,10,12,14,16,18]_\mu.
\end{gathered}$$

This turns out to be the case and also for all higher (even) derivatives, the odd ones following at once from the evens. Note that the 'coefficients' (here 0,-5,10,-10,5,-1) result from sums of analogous coefficients at the previous level.

The general formulas for the derivatives of λ^μ, are

$$\begin{aligned}
(\lambda^\mu)^{(2h)}(\rho) = \sum_{k=0}^{h-1}(-1)^k\binom{h}{k}\prod_{s=0}^{h+k-1}(\mu-2s)\prod_{\sigma=k}^{h-1}(\mu-2\sigma-1)\lambda^{\mu-2h-2k}\\
+\prod_{s=0}^{2h-1}(\mu-2s)\lambda^{\mu-4h}
\end{aligned}$$

and

$$\begin{aligned}
(\lambda^\mu)^{(2h+1)}(\rho) = \rho\sum_{k=0}^{h-1}(-1)^k\binom{h}{k}\prod_{s=0}^{h+k}(\mu-2s)\prod_{\sigma=k}^{h-1}(\mu-2\sigma-1)\lambda^{\mu-2h-2k-2}\\
+\rho\prod_{s=0}^{2h}(\mu-2s)\lambda^{\mu-4h-2}.
\end{aligned}$$

These formulas may be proved by induction on h by using (8.10), (8.11), the well-known relation

$$\binom{h-1}{k-1}+\binom{h-1}{k}=\binom{h}{k}$$

and the equality

$$\binom{h-1}{k-1}(2k-1)+\binom{h-1}{k}(2h+2k-1)=\binom{h}{k}(2h-1)$$

which can be easily proved by direct calculation.

Indeed, the second formula above follows immediately from the first by (8.10). Computing then $(\lambda^\mu)^{(2h+2)}$ from the second by (8.11) we get

$$\begin{aligned}(\lambda^\mu)^{(2h+2)} &= \sum_{k=0}^{h-1}(-1)^k\binom{h}{k}\prod_{s=0}^{h+k}(\mu-2s)\prod_{\sigma=k}^{h-1}(\mu-2\sigma-1)\\ &\cdot\{[1+(\mu-2h-2k-2)]\lambda^{\mu-2h-2k-2}-(\mu-2h-2k-2)\lambda^{\mu-2h-2k-4}\}\\ &+\prod_{s=0}^{2h}(\mu-2s)\{[1+(\mu-4h-2)]\lambda^{\mu-4h-2}-(\mu-4h-2)\lambda^{\mu-4h-4}\}.\end{aligned}$$

At each level $\lambda^{\mu-\alpha}$ we have two terms coming from two different steps: the term $-(\mu-2h-2k-2)\lambda^{\mu-2h-2k-4}$ of step k goes together with the term $[1+(\mu-2h-2k-4)]\lambda^{\mu-2h-2k-4}$ of step $k+1$, and hence, collecting common factors and using the binomial identities just above we have:

$$\begin{aligned}&-(-1)^k\binom{h}{k}\prod_{s=0}^{h+k}(\mu-2s)\prod_{\sigma=k}^{h-1}(\mu-2\sigma-1)(\mu-2h-2(k+1))\lambda^{\mu-2(h+1)-2(k+1)}\\ &\qquad+(-1)^{k+1}\binom{h}{k+1}\prod_{s=0}^{h+k+1}(\mu-2s)\prod_{\sigma=k+1}^{h-1}(\mu-2\sigma-1)\\ &\qquad\cdot[1+(\mu-2h-2(k+1)-2)]\lambda^{\mu-2(h+1)-2(k+1)}\\ &\qquad=(-1)^{k+1}\prod_{s=0}^{h+k+1}(\mu-2s)\prod_{\sigma=k+1}^{h-1}(\mu-2\sigma-1)\\ &\cdot\left\{\binom{h}{k}(\mu-2k-1)+\binom{h}{k+1}[\mu-(2(h+1)+2(k+1)-1)]\right\}\lambda^{\mu-2(h+1)-2(k+1)}\\ &=(-1)^{k+1}\prod_{s=0}^{h+k+1}(\mu-2s)\prod_{\sigma=k+1}^{h-1}(\mu-2\sigma-1)\cdot\left\{\left[\binom{h}{k}+\binom{h}{k+1}\right]\mu\right.\end{aligned}$$

$$- \left[\binom{h}{k}(2(k+1)-1) + \binom{h}{k+1}(2(h+1)+2(k+1)-1) \right] \Bigg\} \lambda^{\mu-2(h+1)-2(k+1)}$$

$$= (-1)^{k+1} \prod_{s=0}^{h+k+1} (\mu - 2s) \prod_{\sigma=k+1}^{h-1} (\mu - 2\sigma - 1)$$

$$\cdot \left\{ \binom{h+1}{k+1}\mu - \binom{h+1}{k+1}(2h+1)] \right\} \lambda^{\mu-2(h+1)-2(k+1)}$$

$$= (-1)^{k+1} \binom{h+1}{k+1} \prod_{s=0}^{h+k+1} (\mu - 2s) \prod_{\sigma=k+1}^{h} (\mu - 2\sigma - 1) \lambda^{\mu-2(h+1)-2(k+1)},$$

obtaining the term that must appear in the formula for $(\lambda^\mu)^{2(h+1)}(\rho)$. Lastly, with such transparent expressions for these derivatives, the needed bounds to handle the brackets with remainders in the previous section are easy to establish.

9. Adding a variable

Previous proofs concerning Gevrey vectors have often, as in Derridj's paper, proven and then used the Gevrey hypoellipticity of the operator

$$Q = -\frac{\partial^2}{\partial t^2} - P.$$

The proof that a homogeneous solution for Q satisfies $U \in G^{1,s}_{t,x}$ locally for $s \geq 1/\varepsilon$ follows using the above techniques and the evident *a priori* inequality

$$\begin{aligned} \|W(t,x)\|^2_{L^2(t),\,\varepsilon(x)} + \sum \|X_j W(t,x)\|^2_{L^2(t,x)} + \|W(t,x)\|^2_{1(t),\,L^2(x)} \\ \leq C\{|(QW,W))|_{L^2} + \|W(t,x)\|^2_{L^2(t,x)}\} \end{aligned}$$

for W of small support and smooth since the variables are completely separated.

Then observing that under our hypothesis on the iterates of P on u, the homogeneous convergent series

$$U(t,x) = \sum_{\ell \geq 0} (-1)^\ell \frac{t^{2\ell}}{(2\ell)!} P^\ell u(x)$$

satisfies the above equation in some interval about $t = 0$, and hence, restricted to $t = 0$ where it is equal to u, will have the desired regularity in Gevrey class.

Finally, since the variables t and x are totally separated in the problem, localizing functions may be taken as products $\varphi_1(t)\varphi_2(x)$ with φ_1 of Ehrenpreis type or using nested open sets in t while in x Gevrey localization is familiar (and the fact that coefficients now depend on t as well as x presents no new obstacles, even in brackets with D_t or Λ_2^β).

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