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The ∂-operator and real holomorphic vector fields

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Abstract: Let (M, h) be a Hermitian manifold and ψ a smooth weight function on M. The ∂-complex on weighted Bergman spaces $A^2_{(p,0)}(M, h, e^{-\psi})$ of holomorphic $(p, 0)$-forms was recently studied in [10] and [9]. It was shown that if h is Kähler and a suitable density condition holds, the ∂-complex exhibits an interesting holomorphicity/duality property when $(\bar{\partial}\psi)^\sharp$ is holomorphic (i.e., when the real gradient field $\mathrm{grad}_h \psi$ is a real holomorphic vector field.) For general Hermitian metrics, this property does not hold without the holomorphicity of the torsion tensor $T_p{}^{rs}$.

In this paper, we investigate the existence of real-valued weight functions with real holomorphic gradient fields on Kähler and conformally Kähler manifolds and their relationship to the ∂-complex on weighted Bergman spaces. For Kähler metrics with multi-radial potential functions on $\mathbb{C}^n$, we determine all multi-radial weight functions with real holomorphic gradient fields. For conformally Kähler metrics on complex space forms, we first identify the metrics having holomorphic torsion leading to several interesting examples such as the Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ and the "half" hyperbolic metric on the unit ball. For some of these metrics, we further determine weight functions ψ with real holomorphic gradient fields. They provide a wealth of triples $(M, h, e^{-\psi})$ of Hermitian non-Kähler manifolds with weights for which the ∂-complex exhibits the aforementioned holomorphicity/duality property. Among these examples, we study in detail the ∂-complex on the unit ball with the half hyperbolic metric and derive a new estimate for the ∂-equation.

Keywords: ∂-complex, weighted Bergman spaces, conformally Kähler metrics, holomorphic vector fields.

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1. Introduction

Let (M,h) denote a manifold of complex dimension n with a Hermitian metric h, and let ψ be a smooth real-valued function on M. Consider the Segal–Bargmann spaces of $(p,0)$-forms

$$A^2_{(p,0)}(M,h,e^{-\psi}) \\ = \left\{ u = \sum_{|J|=p}{}' u_J\, dz^J : \int_M |u|_h^2 e^{-\psi} d\mathrm{vol}_h < \infty,\ u_J \text{ holomorphic} \right\}.$$

Here $J=(j_1,\dots,j_p)$ are multiindices of length p and the summation is taken over increasing indices; in holomorphic coordinates, the metric h has the form $h_{j\bar{k}} dz^j \otimes dz^{\bar{k}}$, where $[h_{j\bar{k}}]$ is a positive definite Hermitian matrix with smooth coefficients; the volume element induced by the metric is denoted by $d\mathrm{vol}_h := \det(h_{j\bar{l}})\, d\lambda$; the metric h induces a metric on tensors of each degree, so for $(1,0)$-forms $u = u_j dz^j$ and $v = v_j dz^j$ one has $\langle u, v\rangle_h = h^{j\bar{k}} u_j v_{\bar{k}}$ and $|u|_h^2 = \langle u, u\rangle_h$, where $[h^{j\bar{k}}]$ is the transpose of the inverse matrix of $[h_{j\bar{k}}]$.

Under suitable conditions (see [9], [10]) the complex derivative

$$\partial u := \sum_{|J|=p}{}' \sum_{j=1}^n \frac{\partial u_J}{\partial z_j}\, dz^j \wedge dz^J$$

is a densely defined, in general unbounded operator

$$\partial : A^2_{(p,0)}(M,h,e^{-\psi}) \longrightarrow A^2_{(p+1,0)}(M,h,e^{-\psi}), \quad 0 \le p \le n-1.$$

In order to determine the adjoint operator

$$\partial^* : A^2_{(p+1,0)}(M,h,e^{-\psi}) \longrightarrow A^2_{(p,0)}(M,h,e^{-\psi})$$

it is necessary to consider the nonvanishing Christoffel symbols for the Chern connection in local coordinates $z^1,\dots,z^n$:

$$\Gamma^i_{jk} = h^{i\bar{l}} \partial_j h_{k\bar{l}}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}. \tag{1.1}$$

For a general Hermitian metric, the torsion tensor T^i_{jk} may be nontrivial; it is defined by

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad T^{\bar{i}}_{\bar{j}\bar{k}} = \overline{T^i_{jk}}, \tag{1.2}$$

the torsion $(1,0)$-form is then obtained by taking the trace:

$$\tau = T^i_{ji} dz^j. \tag{1.3}$$

We use $h_{j\bar{k}}$ and its inverse $h^{\bar{k}l}$ to lower and raise indices. For example, raising and lowering indices of the torsion, we have

$$T_q{}^{pr} := T^{\bar{i}}_{\bar{j}\bar{k}} h_{q\bar{i}} h^{p\bar{j}} h^{r\bar{k}}. \tag{1.4}$$

In particular, for a $(0,1)$ form $w = w_{\bar{k}}\, d\bar{z}^k$, raising indices gives the "musical" operator $\sharp$ acting on w and produces a $(1,0)$ vector field $w^\sharp := h^{k\bar{j}} w_{\bar{j}}\, \partial_k$. Now, if $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is a holomorphic vector field, the adjoint operator ∂^* on $\mathrm{dom}(\partial^*) \subset A^2_{(1,0)}(M, h, e^{-\psi})$ can be expressed in the form

$$\partial^* u = \langle u, \partial\psi - \tau\rangle_h, \tag{1.5}$$

see [10] for more details. If, in addition, the metric h is Kählerian one has $\tau = 0$ and thus

$$\partial^* u = h^{j\bar{k}} u_j \frac{\partial\psi}{\partial\bar{z}^k}, \tag{1.6}$$

which means the complex vector field

$$X := h^{j\bar{k}} \frac{\partial\psi}{\partial\bar{z}^k} \frac{\partial}{\partial z^j} \tag{1.7}$$

is holomorphic. In this case, the gradient field $\mathrm{grad}_h\psi$ is a *real holomorphic* vector field in the terminology of [13]. There are important classes of Kähler manifolds admitting a function with real holomorphic gradient vector field, for instance the gradient Kähler–Ricci solitons, see [2] and [13]. The existence of real holomorphic gradient vector fields is also related to Calabi's extremal Kähler metric [1] and to strong hypercontractivity of the weighted Laplacian [6]. In [13] it is shown that the real holomorphicity of the gradient vector field of a weight function implies Liouville theorems for weighted holomorphic, or more generally, weighted harmonic functions and mappings. We shall see quickly that the holomorphicity of the gradient field of a conformal factor is also related to the holomorphicity of the torsion of the conformally Kähler metric.

Here we continue our investigation of the ∂-complex

$$A^2(M, h, e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(1,0)}(M, h, e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(2,0)}(M, h, e^{-\psi}), \tag{1.8}$$

and the corresponding complex Laplacian

$$\widetilde{\Box}_p = \partial\partial^* + \partial^*\partial : A^2_{(1,0)}(M,h,e^{-\psi}) \longrightarrow A^2_{(1,0)}(M,h,e^{-\psi}), \tag{1.9}$$

which, under suitable assumptions, will be a densely defined self-adjoint operator, see [10] and [9], where the classical case of the Segal–Bargmann space with the Euclidean metric is treated.

For $(p,0)$-forms with $p \geqslant 2$, the holomorphicity of $(\bar\partial\psi - \bar\tau)^\sharp$ is not enough for the adjoint ∂^* to have a simple formula analogous to (1.5). In order to describe the formula for ∂^* on $(2,0)$-forms, we write

$$v = \frac{1}{2}\sum_{j,k} v_{jk} dz^j \wedge dz^k = \sum_{j<k} v_{jk} dz^j \wedge dz^k, \tag{1.10}$$

where $v_{jk} = -v_{kj}$. Define an operator $T^\sharp \colon \Lambda^{2,0}(M) \to \Lambda^{1,0}(M)$ by

$$T^\sharp(v) = \frac{1}{2} T_p{}^{rs} v_{rs} dz^p, \tag{1.11}$$

where $T_p{}^{rs}$ is given by (1.4). If $u = u_j dz^j$, then

$$\partial u = \frac{1}{2}\sum_{j,k}\left(\frac{\partial u_k}{\partial z^j} - \frac{\partial u_j}{\partial z^k}\right) dz^j \wedge dz^k. \tag{1.12}$$

Moreover, since $v_{pq} = -v_{qp}$, we find that

$$\langle \partial u, v\rangle_h = \sum_{j,k,p,q} \overline{v_{pq}} h^{k\bar p} h^{j\bar q} \left(\frac{\partial u_k}{\partial z^j}\right). \tag{1.13}$$

The formula for ∂^* is then given by

$$\partial^* v = P_{h,\psi}\left(-(\psi_{\bar j} - \tau_{\bar j}) v_{pq} h^{q\bar j} dz^p + T^\sharp(v)\right). \tag{1.14}$$

Here, $P_{h,\psi}$ is the orthogonal projection from $L^2_{(1,0)}(M,h,e^{-\psi})$ onto$A^2_{(1,0)}(M,h,e^{-\psi})$, see [10]. If h is Kähler and $(\bar\partial\psi)^\sharp$ is holomorphic then, as in the case of 1-forms,

$$\partial^* v = -\psi_{\bar j} v_{pq} h^{q\bar j} dz^p. \tag{1.15}$$

In this case, the non-local orthogonal projection $P_{h,\psi}$ plays no role and ∂^* reduces essentially to a “multiplication” operator. In the non-Kähler case, by inspecting (1.14), we find that the relevant condition is the holomorphicity of the torsion tensor; the precise definition is as follows.

Definition 1.1. Let h be a Hermitian metric on a complex manifold. We say that h has *holomorphic torsion* if

$$\nabla_{\bar{l}} T_p{}^{rs} = 0, \tag{1.16}$$

where ∇ is the Chern connection.

Clearly, h has holomorphic torsion if and only if the components of the torsion $T_p{}^{rs}$ (in any holomorphic coordinate system) are holomorphic. Moreover, it implies that $\bar{\tau}^\sharp$ is a holomorphic $(1,0)$-vector field.

Let D_p^* and ∂_p^* be the Hilbert space adjoints of ∂ in the Lebesgue space $L^2_{(p+1,0)}(M, h, e^{-\psi})$ and the weighted Begman space $A^2_{(p+1,0)}(M, h, e^{-\psi})$, respectively. In summary, we have the following theorem which generalizes [10, Theorem 1.1].

Theorem 1.2. *Let (M, h) be a complete Hermitian manifold with weight $e^{-\psi}$. Assume that the torsion $T_p{}^{rs}$ of the Chern connection is holomorphic. If $(\bar{\partial}\psi)^\sharp$ is holomorphic, then for $\eta \in \operatorname{dom}(D_p^*)$, $p \geqslant 0$, that is holomorphic in an open set $U \subset M$, $D_p^*\eta$ is also holomorphic in U. In particular, if ∂_p is densely defined in the Bergman space $A^2_{(p,0)}(M, h, e^{-\psi})$, then*

$$D_p^*\eta = \partial_p^*\eta \tag{1.17}$$

for $\eta \in \operatorname{dom}(\partial_p^*)$.

In the following, we give two examples when the theorem applies. The first example shows that in some situations it is necessary to consider non-Kähler Hermitian metrics.

Example 1.3 (Hopf manifolds)**.** The simplest examples of Hermitian non-Kähler metrics with holomorphic torsion are conformal flat metrics. On $\mathbb{C}^n$, these metrics are described explicitly in Proposition 3.3. They are of the form $g_{j\bar{k}} = \phi^{-1}\delta_{jk}$ in the standard coordinates of $\mathbb{C}^n$, where ϕ is given in (3.7). For example, in (3.7), if we take $c_{j\bar{k}}$ to be $\frac{1}{4}\times$ the identity matrix and $\gamma = 0$, then we obtain the following metric on $\mathbb{C}^n \setminus \{0\}$ with holomorphic torsion:

$$g_{j\bar{k}} = \frac{4\delta_{jk}}{|z|^2}. \tag{1.18}$$

Let $M := \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional Hopf manifold. It is diffeomorphic to $(\mathbb{C}^n \setminus \{0\})/G$, where G is the infinite cyclic group generated by $z \mapsto \frac{1}{2}z$ acting freely and properly discontinuously on $\mathbb{C}^n \setminus \{0\}$, and has the induced complex structure; see, e.g., [11] for more details. The Hermitian metric $g_{j\bar{k}}$ in (1.18) is invariant under the action of G and descents to a natural locally conformal Kähler metric with holomorphic torsion on the standard compact Hopf manifold. It is well-known that for $n \geqslant 2$ the first Betti number $b_1(M) = 1$ and hence M admits *no* Kähler metric; see [11].

Example 1.4. We revisit the following example in [10]. Let $M = \mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and let $h_{j\bar{k}} = (1-|z|^2)^{-1}\delta_{jk}$ be a conformally flat metric. By direct computations, we find that the torsion

$$T_q{}^{pr} = z^p \delta_q^r - z^r \delta_q^p \tag{1.19}$$

is nontrivial (unless $n = 1$) and holomorphic. Let $\psi = \alpha \log(1-|z|^2)$. Then

$$(\bar{\partial}\psi)^\sharp = -\alpha \sum_{j=1}^n z_j \frac{\partial}{\partial z^j} \tag{1.20}$$

is a holomorphic vector field. The triple $(M, h, e^{-\psi})$ satisfies the hypothesis of Theorem 1.2, except that h is not complete. The ∂-complex on the Bergman spaces $A^2_{(p,0)}(M, h, e^{-\psi})$ of holomorphic $(p,0)$-forms exhibits an interesting holomorphicity/duality property similar to that on the Segal-Bargmann space; see [10].

In this paper, we investigate conformally Kähler manifolds with holomorphic torsion and weight functions whose gradients are real holomorphic vector fields. The first part is devoted to Kähler metrics with multi-radial potential functions. It is also shown that in many cases the real holomorphic vector field is of the form

$$Z = \sum_{j=1}^n C_j z_j \frac{\partial}{\partial z^j}, \tag{1.21}$$

where C_j are real constants. In addition, we exploit an example where some constants C_j are zero, which means that the adjoint of ∂ "forgets" some of the variables.

In the second part we consider conformally Kähler metrics. Let (M, h) be a Kähler manifold and let $g = \phi^{-1}h$ be a conformal metric. We study the condition on ϕ such that g has holomorphic torsion. This is the case precisely when $\mathrm{grad}_h \phi$ is a real holomorphic vector field. We determine all conformally Kähler metrics having holomorphic torsion on Kähler spaces of constant holomorphic sectional curvature. We thus obtain a wealth of examples of Hermitian manifolds with holomorphic torsion. On some of these examples, we also determine all real-valued functions ψ whose real gradient fields $\operatorname{grad}_g\psi$ are real holomorphic. On such a triple $(M, g, e^{-\psi})$, the ∂-complex on the weighted Bergman spaces exhibits an interesting holomorphicity/duality property. We analyze the ∂-complex on the unit ball $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z|^2 < 1\}$ endowed with the "half" hyperbolic metric,

$$h_{j\bar{k}} = \delta_{jk} + \frac{\bar{z}_j z_k}{1 - |z|^2}, \tag{1.22}$$

and obtain the following result.

Theorem 1.5 (= Theorem 4.2)**.** *Let h be the half hyperbolic metric on the unit ball $\mathbb{B}^n$, $\alpha < 0$, and $\psi(z) = \alpha \log(1-|z|^2)$. Then the complex Laplacian $\widetilde{\square}_1$ has a bounded inverse $\widetilde{N}_1$, which is a compact operator on $A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$ with discrete spectrum. If*

$$\nu = \begin{cases} -\alpha, & \text{if } n = 1, \\ \min\{1 - \alpha, -2\alpha\}, & \text{if } n = 2, \\ n - \alpha - 1, & \text{if } n \geqslant 3, \end{cases} \tag{1.23}$$

then

$$\left\| \widetilde{N}_1 u \right\| \leqslant \frac{1}{\nu} \|u\|, \tag{1.24}$$

for each $u \in A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$. In fact, the first positive eigenvalue of $\widetilde{\square}_1$ is $\lambda_1 = \nu$.

Consequently, if $\eta = \eta_j dz_j \in A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$ with $\partial\eta = 0$, then $f := \partial^* \widetilde{N}_1 \eta$ is the canonical solution of $\partial f = \eta$, this means $\partial f = \eta$ and $f \in (\ker \partial)^\perp$. Moreover,

$$\begin{aligned} &\int_{\mathbb{B}^n} |f|^2 (1 - |z|^2)^{-\alpha-1} d\lambda \\ &\qquad \leqslant \frac{1}{\nu} \int_{\mathbb{B}^n} \left(\sum_{j=1}^n |\eta_j|^2 - \left| \sum_j^n \eta_j z_j \right|^2 \right) (1 - |z|^2)^{-\alpha-1} d\lambda. \end{aligned} \tag{1.25}$$

We also consider $U(n)$-invariant metrics on $\mathbb{C}^n$ in a conformal class of a given $U(n)$-invariant Kähler metric. It is shown that there exists essentially a 2-parameter family of $U(n)$-invariant conformal metrics with holomorphic and nontrivial torsion. Moreover, with respect to such a metric, there exists essentially a 2-parameter family of weight functions with real holomorphic gradient fields.

2. Kähler metrics with multi-radial potential functions

We consider Kähler metrics on $\mathbb{C}^n$ with *multi-radial* potential functions

$$\chi(z_1, z_2, \ldots, z_n) = \tilde{\chi}(r_1, r_2, \ldots, r_n) \tag{2.1}$$

where $r_j = |z_j|^2$, $j = 1, \ldots, n$. For these metrics, we can determine explicitly the multi-radial weight functions ψ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic.

Theorem 2.1. *Let $\chi(z) = \tilde{\chi}(|z_1|^2, \ldots, |z_n|^2)$ be a multi-radial potential function for a Kähler metric in $\mathbb{C}^n$. If $\psi(z) = \widetilde{\psi}(|z_1|^2, \ldots, |z_n|^2)$ is a multi-radial weight function such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then*

$$(\bar{\partial}\psi)^\sharp = \sum_{j=1}^{n} C_j z_j \frac{\partial}{\partial z_j}, \tag{2.2}$$

where C_k's are real constant and

$$\tilde{\psi} = C_0 + \sum_{j=1}^{n} C_j r_j \frac{\partial \tilde{\chi}}{\partial r_j}. \tag{2.3}$$

Proof. By direct computation, we find that

$$h_{j\bar{k}} = \frac{\partial \tilde{\chi}}{\partial r_j} \delta_{jk} + \bar{z}_j z_k \left(\frac{\partial^2 \tilde{\chi}}{\partial r_j \partial r_k} \right). \tag{2.4}$$

Observe that $\partial\tilde{\chi}/\partial r_j$ and $\partial^2\tilde{\chi}/\partial r_j \partial r_k$ are real-valued. Observe that $\partial\tilde{\chi}/\partial r_j > 0$ for all j near the origin.

We claim that the inverse transpose matrix has the form

$$h^{j\bar{k}} = \left(\frac{\partial \tilde{\chi}}{\partial r_j} \right)^{-1} \delta_{jk} + V_{jk} z_j \bar{z}_k \tag{2.5}$$

for some matrix V_{jk} with real-valued entries. Indeed, consider the system of equations with unknowns V_{jk},

$$\left(\left(\frac{\partial\tilde{\chi}}{\partial r_j}\right)^{-1}\delta_{jk}+V_{jk}z_j\bar{z}_k\right)h_{l\bar{k}}=\delta_l^j, \tag{2.6}$$

which is equivalent to a system with real coefficients

$$\left(\frac{\partial\tilde{\chi}}{\partial r_j}\right)^{-1}\left(\frac{\partial^2\tilde{\chi}}{\partial r_j\partial r_l}\right)+\left(\frac{\partial\tilde{\chi}}{\partial r_j}\right)V_{jl}+\sum_{k=1}^{n}V_{jk}r_k\left(\frac{\partial^2\tilde{\chi}}{\partial r_k\partial r_l}\right)=0. \tag{2.7}$$

For fixed j, the system of equation for $V_{jk}, k=1,2,\dots,n$, can be written as

$$\begin{pmatrix} a_j+r_1b_{11} & r_2b_{21} & \cdots & r_nb_{n1} \\ r_1b_{12} & a_j+r_2b_{22} & \cdots & r_nb_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ r_1b_{1n} & r_2b_{2n} & \cdots & a_j+r_nb_{nn} \end{pmatrix}\cdot\begin{pmatrix} V_{j1} \\ V_{j2} \\ \vdots \\ V_{jn}\end{pmatrix}=\begin{pmatrix} -a_j^{-1}b_{j1} \\ -a_j^{-1}b_{j2} \\ \vdots \\ -a_j^{-1}b_{jn}\end{pmatrix}, \tag{2.8}$$

where

$$a_j=\frac{\partial\tilde{\chi}}{\partial r_j}>0, \qquad b_{kl}=\frac{\partial^2\tilde{\chi}}{\partial r_k\partial r_l}, \tag{2.9}$$

all are real-valued. Clearly, at the origin $r_1=r_2=\cdots=r_n=0$, the determinant of the coefficient matrix is $a_j^n>0$. Thus, this system of linear equations is uniquely solvable near the origin and the solution is real. The claim follows.

On the other hand, since ψ is multi-radial, we have

$$\frac{\partial\psi}{\partial\bar{z}_k}=\frac{\partial\tilde{\psi}}{\partial r_k}z_k. \tag{2.10}$$

This and (2.5) imply that

$$(\bar{\partial}\psi)^\sharp=\sum_{j=1}^{n}\left(\left(\frac{\partial\tilde{\chi}}{\partial r_j}\right)^{-1}\left(\frac{\partial\widetilde{\psi}}{\partial r_j}\right)+\sum_{k=1}^{n}r_kV_{jk}\left(\frac{\partial\widetilde{\psi}}{\partial r_k}\right)\right)z_j\frac{\partial}{\partial z_j}. \tag{2.11}$$

Since for each j the expression in the parenthesis is real-valued, it is holomorphic if and only if it is a constant. Thus

$$(\bar{\partial}\psi)^\sharp=\sum_{j=1}^{n}C_jz_j\frac{\partial}{\partial z_j}, \tag{2.12}$$

where $C_1, C_2, \dots, C_n$ are real constants. Thus, (2.2) holds. Applying the flat "musical" operator $\flat$ to both sides, we find that $\tilde{\psi}$ must satisfy the PDE

$$z_l \frac{\partial \tilde{\psi}}{\partial r_l} = \frac{\partial \psi}{\partial \bar{z}_l} = \sum_{j=1}^{n} C_j z_j h_{j\bar{l}} = z_l C_l \frac{\partial \tilde{\chi}}{\partial r_l} + z_l \sum_{j=1}^{n} C_j r_j \frac{\partial^2 \tilde{\chi}}{\partial r_j \partial r_l} = z_l \frac{\partial}{\partial r_l} \left(\sum_{j=1}^{n} C_j r_j \frac{\partial \tilde{\chi}}{\partial r_j} \right) \tag{2.13}$$

whose general solution is

$$\tilde{\psi} = C_0 + \sum_{j=1}^{n} C_j r_j \frac{\partial \tilde{\chi}}{\partial r_j}. \tag{2.14}$$

The proof is complete. □

For example, let $\tilde{\chi}$ have the following form

$$\tilde{\chi}(r_1, r_2, \dots, r_n) = F_1(r_1) + F_2(r_2) + \dots + F_n(r_n),$$

where $r_j = |z_j|^2$, $j = 1, \dots, n$, with smooth real valued functions F_j, $j = 1, \dots, n$. Then we have a diagonal matrix

$$h_{j\bar{k}} = \delta_{jk}(F_j' + r_j F_j'').$$

We have to suppose that all entries satisfy $F_j' + r_j F_j'' > 0$. For the determinant we get

$$\delta = \prod_{j=1}^{n} (F_j' + r_j F_j'').$$

For $h^{j\bar{k}}$ we get

$$(h^{j\bar{k}}) = 1/\delta \begin{pmatrix} \prod_{j\neq 1}(F_j' + r_j F_j'') & 0 & \dots & 0 \\ 0 & \prod_{j\neq 2}(F_j' + r_j F_j'') & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \prod_{j\neq n}(F_j' + r_j F_j'') \end{pmatrix}. \tag{2.15}$$

For this metric, we can always find a weight function ψ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic. In fact, we can determine all such multi-radial weight functions ψ.

Corollary 2.2. *Let h be a Kähler metric on $\mathbb{C}^n$ with a potential function*

$$\chi(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} F_j(|z_j|^2). \tag{2.16}$$

If $\psi(z_1, \ldots, z_n) = \tilde{\psi}(|z_1|^2, \ldots, |z_n|^2)$ is a multi-radial weight, then $(\bar{\partial}\psi)^\sharp$ is holomorphic if and only if

$$\psi(z_1, \ldots, z_n) = C_0 + \sum_{j=1}^{n} C_j |z_j|^2 F_j'(|z_j|^2). \tag{2.17}$$

If this is the case, then we obtain the real holomorphic vector field

$$h^{j\bar{k}} \frac{\partial \psi}{\partial \bar{z}^k} \frac{\partial}{\partial z^j} = \sum_{j=1}^{n} C_j z_j \frac{\partial}{\partial z^j}. \tag{2.18}$$

Proof. Using (2.15), we find that

$$h^{j\bar{k}} \psi_{\bar{k}} = \sum_{j=1}^{n} z_j \frac{\tilde{\psi}_{r_j}(r_1, r_2, \ldots, r_n)}{F_j' + r_j F_j''}. \tag{2.19}$$

Then $(\bar{\partial}\psi)^\sharp$ is holomorphic if and only if

$$\frac{\tilde{\psi}_{r_j}(r_1, r_2, \ldots, r_n)}{F_j' + r_j F_j''} = C_j \tag{2.20}$$

for some real constant C_j. This PDE can be solved easily and the solutions are given as in (2.17). The proof is complete. $\square$

Example 2.3. We consider the polydisk

$$\mathbb{D}^n := \left\{ z \in \mathbb{C}^n \colon |z_j|^2 < 1, j = 1, 2, \ldots, n \right\}.$$

The Bergman metric on $\mathbb{D}^n$ is the Kähler metric with potential function

$$\chi(z) = \log K(z, z) = -2 \sum_{j=1}^{n} \log(1 - |z_j|^2), \tag{2.21}$$

which is decoupled and multi-radial. Applying Corollary 2.2 with $F_j(r) = -\log(1 - r)$, we see that all multi-radial weight functions ψ with $(\bar{\partial}\psi)^\sharp$ holomorphic are of the form

$$\psi = \gamma_0 + \sum_{j=1}^{n} \frac{\gamma_j}{1 - |z_j|^2}. \tag{2.22}$$

Under a suitable condition on γ_j, the $\bar\partial$-complex on the Bergman spaces $A^2(\mathbb{D}^n,$
$h, e^{-\psi})$ is similar to that on the Bergman spaces on the unit ball with complex hyperbolic metric, studied earlier in [10].

Another interesting decoupled multi-radial potential function is given in the form

$$\tilde\chi(r_1, r_2, \dots, r_n) = \prod_{j=1}^{n} G_j(r_j), \tag{2.23}$$

where $G_j(r)$'s are real-valued function of a real variable. We have

$$\begin{aligned} h_{j\bar{j}} &= \partial_j \partial_{\bar{j}} \chi(z_1, z_2, \dots, z_n) \\ &= G_1 \cdots G_{j-1}(G_j' + r_j G_j'') G_{j+1} \cdots G_n \\ &= \chi \frac{G_j' + r_j G_j''}{G_j} \end{aligned}$$

and for $k \neq j$, we have

$$h_{j\bar{k}} = \partial_j \partial_{\bar{k}} \chi(z_1, z_2, \dots, z_n) = \chi \frac{G_j' \bar{z}_j G_k' z_k}{G_j G_k}. \tag{2.24}$$

Thus, the Kähler metric is given by a rank-1 perturbation of a diagonal metric. Precisely,

$$h_{j\bar{k}} = \chi \left(\frac{G_j G_j' + r_j G_j G_j'' - (G_j')^2 r_j}{G_j^2} \delta_{jk} + \frac{G_j' \bar{z}_j G_k' z_k}{G_j G_k} \right). \tag{2.25}$$

Theorem 2.1 gives the following:

Corollary 2.4. *Let h be a Kähler metric on $\mathbb{C}^n$ with a potential function*

$$\chi(z_1, \dots, z_n) = \prod_{j=1}^{n} G_j(|z_j|^2). \tag{2.26}$$

Then a multi-radial weight function $\psi(z_1,\ldots,z_n)=\tilde{\psi}(|z_1|^2,\ldots,|z_n|^2)$ *has the property that* $(\bar{\partial}\psi)^\sharp$ *is holomorphic if and only if*

$$\psi(z)=C_0+\sum_{j=1}^{n}C_j|z_j|^2G_j'(|z_j|^2)\prod_{k\neq j}G_k(|z_k|^2), \tag{2.27}$$

where $C_0,C_1,\ldots,C_n$ *are real constants, and* $G_j'=\partial G_j/\partial r_j$. *In this case, the holomorphic vector field is*

$$h^{j\bar{k}}\frac{\partial\psi}{\partial\bar{z}^k}\frac{\partial}{\partial z^j}=\sum_{j=1}^{n}C_jz_j\frac{\partial}{\partial z^j}. \tag{2.28}$$

In the rest of this section, we study in detail an example of Kähler metric given by a multi-radial non-decoupled function, yet the weight function can be chosen so that the adjoint ∂^*-operator "forgets" one variable.

Example 2.5. In the following we consider a non-decoupled example on $\mathbb{C}^2$ with potential function

$$\chi(z_1,z_2)=\frac{1}{4}|z_1|^4+|z_1|^2|z_2|^2+|z_1|^2+|z_2|^2. \tag{2.29}$$

In the standard coordinates of $\mathbb{C}^2$, the metric is given by the matrix

$$\left[h_{j\bar{k}}\right]=\begin{pmatrix}|z_1|^2+|z_2|^2+1 & \bar{z}_1z_2\\ z_1\bar{z}_2 & |z_1|^2+1\end{pmatrix}, \tag{2.30}$$

with the determinant

$$\delta=\det\left[h_{j\bar{k}}\right]=|z_1|^4+2|z_1|^2+|z_2|^2+1. \tag{2.31}$$

Therefore,

$$\left[h^{j\bar{k}}\right]=\frac{1}{\delta}\begin{pmatrix}|z_1|^2+1 & -z_1\bar{z}_2\\ -\bar{z}_1z_2 & |z_1|^2+|z_2|^2+1\end{pmatrix}. \tag{2.32}$$

If $\psi(z_1,z_2)=\tilde{\psi}(r_1,r_2)$ is a multi-radial weight with real holomorphic gradient field, then Theorem 2.1 shows that

$$(\bar{\partial}\psi)^\sharp=C_1z_1\frac{\partial}{\partial z_1}+C_2z_2\frac{\partial}{\partial z_2}, \tag{2.33}$$

where C_1 and C_2 are two constants, and

$$\psi(z_1,z_2) = C_1|z_1|^2 + C_2|z_2|^2 + (C_1+C_2)|z_1z_2|^2 + \frac{1}{2}C_1|z_1|^4. \tag{2.34}$$

We consider the case $C_1 = 1$ and $C_2 = 0$ so that

$$\psi(z_1,z_2) = \frac{|z_1|^4}{2} + |z_1|^2|z_2|^2 + |z_1|^2 \tag{2.35}$$

and the corresponding Bergman spaces

$$A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi}) = \left\{ f : \mathbb{C}^2 \longrightarrow \mathbb{C} \text{ entire} : \int_{\mathbb{C}^2} |f|^2 e^{-\psi}\delta\, d\lambda < \infty \right\} \tag{2.36}$$

and

$$\begin{aligned} & A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \\ & \quad = \left\{ u = u_1\, dz_1 + u_2\, dz_2, u_1, u_2 \text{ entire} : \int_{\mathbb{C}^2} |u|^2_h e^{-\psi}\delta\, d\lambda < \infty \right\}, \end{aligned} \tag{2.37}$$

where $|u|^2_h = h^{j\overline{k}} u_j u_{\overline{k}}$. It is easily seen that these spaces are non-trivial.

We claim that $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$ does not contain monomials in z_2: consider the function $f(z_1,z_2) = z_2^m$, for $m \in \mathbb{N}$. Using polar coordinates we get

$$\begin{aligned} \|f\|^2 &= 4\pi^2 \int_0^\infty \int_0^\infty r_2^{2m}(r_1^4 + 2r_1^2 + r_2^2 + 1)\, e^{-r_1^4/2 - r_1^2 r_2^2 - r_1^2} r_1 r_2\, dr_1 dr_2 \\ &= 4\pi^2 \int_0^\infty \left(\int_0^\infty r_2^{2m+3} e^{-r_1^2 r_2^2}\, dr_2 \right) (r_1^5 + 2r_1^3 + r_1) e^{-r_1^4/2 - r_1^2}\, dr_1, \end{aligned}$$

for the inner integral we substitute $s = r_1^2 r_2^2$ and get

$$\frac{1}{2r_1^{2m+4}} \int_0^\infty s^{m+1}\, e^{-s}\, ds, \tag{2.38}$$

which shows that integration with respect to r_1 is divergent and hence the claim follows.

In a similar way, we show that all functions $z_1^k z_2^\ell$, for $k \in \mathbb{N}, k \geq 2$ and $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$, belong to $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$. They even belong to $\mathrm{dom}(\partial)$. Here, we have to take care for the slightly different norm in $A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi})$:

We have to consider the integral

$$\int_0^\infty \int_0^\infty r_1^{2k} r_2^{2\ell} (r_1^4 + 2r_1^2 + r_2^2 + 1)\, e^{-r_1^4/2 - r_1^2 r_2^2 - r_1^2} r_1 r_2 \; dr_1 dr_2;$$

the critical summand is

$$\int_0^\infty \int_0^\infty r_1^{2k} r_2^{2\ell+2}\, e^{-r_1^4/2 - r_1^2 r_2^2 - r_1^2} r_1 r_2 \; dr_1 dr_2;$$

integration with respect to r_2 gives

$$\begin{aligned} \int_0^\infty r_1^{2k+1} r_2^{2\ell+2}\, e^{-r_1^2 r_2^2} r_2 dr_2 &= \frac{1}{2} \int_0^\infty r_1^{2k-1} r_1^{-2\ell-2} s^{\ell+1} e^{-s}\, ds \\ &= \frac{1}{2} \int_0^\infty r_1^{2k-2\ell-3} s^{\ell+1} e^{-s}\, ds, \end{aligned}$$

and we observe that $2k - 2\ell - 3 \geq 0$, whenever $\ell \leq k-2$.

In order to show that the functions $z_1^k z_2^\ell$, for $k \in \mathbb{N}, k \geq 2$ and $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$ belong to $\mathrm{dom}(\partial)$, we first have to consider

$$\partial(z_1^k z_2^\ell) = k z_1^{k-1} z_2^\ell \, dz_1 + \ell z_1^k z_2^{\ell-1}\, dz_2, \tag{2.39}$$

now we compute

$$\begin{aligned} |\partial(z_1^k z_2^\ell)|_h^2 &= \frac{|z_1|^2+1}{\delta} |k z_1^{k-1} z_2^\ell|^2 - \frac{z_1 \overline{z}_2}{\delta} k z_1^{k-1} z_2^\ell\, \overline{(\ell z_1^k z_2^{\ell-1})} \\ &- \frac{\overline{z}_1 z_2}{\delta} \overline{(k z_1^{k-1} z_2^\ell)}\, (\ell z_1^k z_2^{\ell-1}) + \frac{|z_1|^2 + |z_2|^2 + 1}{\delta} |\ell z_1^k z_2^{\ell-1}|^2 \end{aligned}$$

and observe that in the first term the exponent for r_1 after integration with respect to r_2 is again $2k - 2\ell - 3$ and in the last term we have the right exponents for z_1 and z_2, namely $|z_1|^{2k}|z_2|^{2\ell}$. Hence the functions $z_1^k z_2^\ell$, for $k \in \mathbb{N}, k \geq 2$ and $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$ belong to $\mathrm{dom}(\partial)$.

It is clear that $\left\{ z_1^k z_2^\ell \ : k \in \mathbb{N}, k \geq 2,\ \ell \in \mathbb{Z}, 0 \leq \ell \leq k-2 \right\}$ is an orthogonal system in $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$.

Let $f \in A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$. Then f can be written as its Taylor series

$$f(z_1, z_2) = \sum_{\alpha,\beta} c_{\alpha,\beta} z_1^\alpha z_2^\beta, \tag{2.40}$$

which is uniformly convergent on compact subsets of $\mathbb{C}^2$. Hence, using polar coordinates we get

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(r_1e^{i\phi_1}, r_2e^{i\phi_2})\, e^{-i\alpha\phi_1}e^{-i\beta\phi_2}\, d\phi_1 d\phi_2 = c_{\alpha,\beta}\, r_1^{\alpha} r_2^{\beta}, \tag{2.41}$$

and by Parseval's formula

$$\int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{i\phi_1}, r_2e^{i\phi_2})|^2\, d\phi_1 d\phi_2 = 4\pi^2 \sum_{\alpha,\beta} |c_{\alpha,\beta}|^2 r_1^{2\alpha} r_2^{2\beta}. \tag{2.42}$$

Computing the norm of f in $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$, we see that

$$\|f\|^2 = 4\pi^2 \int_0^{\infty}\int_0^{\infty} \sum_{\alpha,\beta} |c_{\alpha,\beta}|^2 r_1^{2\alpha} r_2^{2\beta} (r_1^4 + 2r_1^2 + r_2^2 + 1)\, e^{-r_1^4/2 - r_1^2 r_2^2 - r_1^2} r_1 r_2\, dr_1 dr_2,$$

and Lebesgue's dominated convergence theorem implies that we can interchange integration and summation, so we have

$$\|f\|^2 = 4\pi^2 \sum_{\alpha,\beta} \int_0^{\infty}\int_0^{\infty} |c_{\alpha,\beta}|^2 r_1^{2\alpha} r_2^{2\beta} (r_1^4 + 2r_1^2 + r_2^2 + 1)\, e^{-r_1^4/2 - r_1^2 r_2^2 - r_1^2} r_1 r_2\, dr_1 dr_2.$$

This implies that the system $\{z_1^k z_2^\ell \ : k \in \mathbb{N}, k \geq 2,\ \ell \in \mathbb{Z}, 0 \leq \ell \leq k-2\}$ is an orthogonal basis of $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$, as all other functions $z_1^k z_2^\ell$ do not belong to $A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi})$. In addition we have that the operator ∂ is densely defined.

Since $(\bar\partial\psi)^\sharp = z_1 \partial/\partial z_1$, we have for $u = u_1\, dz_1 + u_2\, dz_2 \in \operatorname{dom}(\partial^*)$

$$\partial^* u = z_1 u_1. \tag{2.43}$$

Thus, the adjoint ∂^* "forgets" the z_2-variable, although the weight and the metric both depend on z_2.

Now let $u = u_1 dz_1 + u_2 dz_2 \in A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi})$. Then

$$|\partial u|_h^2 = \left|\frac{\partial u_2}{\partial z_1} - \frac{\partial u_1}{\partial z_2}\right|^2 \frac{1}{\delta}, \tag{2.44}$$

therefore

$$\partial : A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \longrightarrow A^2_{(2,0)}(\mathbb{C}^2, h, e^{-\psi}) \tag{2.45}$$

is also densely defined.

Let

$$v = v_{12}\, dz_1 \wedge dz_2 \in A^2_{(2,0)}(\mathbb{C}^2, h, e^{-\psi}). \tag{2.46}$$

Then, by the same computation as above, we get

$$\partial^* v = P_{h,\psi}\left(-\psi_{\bar{j}} v_{12} h^{2\bar{j}}\right) dz_1 + P_{h,\psi}\left(-\psi_{\bar{j}} v_{21} h^{1\bar{j}}\right) dz_2 = z_1 v_{12}\, dz_2. \tag{2.47}$$

So we obtain for $\tilde{\Box} = \partial^*\partial + \partial\partial^*$ and $u \in A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \cap \operatorname{dom}(\tilde{\Box})$ that

$$\tilde{\Box} u = \left(u_1 + z_1 \frac{\partial u_1}{\partial z_1}\right) dz_1 + z_1 \frac{\partial u_2}{\partial z_1}\, dz_2. \tag{2.48}$$

Proposition 2.6. *The operator*

$$\tilde{\Box} : A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \longrightarrow A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \tag{2.49}$$

is densely defined and its spectrum consists of point eigenvalues with finite multiplicities. Precisely, for $k = 1, 2, \dots,$ the eigenvalues are $\lambda_k = k+1$, with multiplicity $2k-1$.

Proof. In order to determine the eigenvalues of $\tilde{\Box}$, we consider the basis elements $z_1^k z_2^\ell$, for $k \in \mathbb{N}, k \geq 2$ and $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$ and define

$$v^1_{k,\ell} = z_1^k z_2^\ell\, dz_1 \text{ and } v^2_{k,\ell} = z_1^k z_2^\ell\, dz_2. \tag{2.50}$$

Then we have

$$\tilde{\Box} v^1_{k,\ell} = (k+1) v^1_{k,\ell} \text{ and } \tilde{\Box} v^2_{k,\ell} = k v^2_{k,\ell}. \tag{2.51}$$

Since $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$, the eigenvalues k and $k+1$ are of finite multiplicity and as the functions $z_1^k z_2^\ell$, for $k \in \mathbb{N}, k \geq 2$ and $\ell \in \mathbb{Z}, 0 \leq \ell \leq k-2$ constitute an orthogonal basis in the components of $A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi})$ the operator $\tilde{\Box}$ has a compact resolvent. $\square$

3. Conformally Kähler metrics

Let (M, h) be a Kähler manifold and let $g = \phi^{-1} h$ be a conformal metric. In this section, we study the question when g has holomorphic torsion. Our motivation comes from Theorem 1.2 which says essentially that if g has holomorphic torsion and if ψ is a weight function such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then the ∂-complex on the Bergman spaces $A^2_{(p,0)}(M, h, e^{-\psi})$ exhibits an interesting holomorphicity/duality property, provided that some additional density conditions hold; see also [10]. We first consider the case when (M, h) is a complex space form of constant (negative, zero, or positive) curvature. Using a result in [7], we determine all conformal metrics with holomorphic torsion. We further determine the real-valued function whose gradient with respect to the conformal metrics are real holomorphic. These results provide several interesting examples in which the ∂-complex has the aforementioned holomorphicity property.

Proposition 3.1. *Let (M, h) be a Kähler manifold of dimension $n \geqslant 2$ and let $g = \phi^{-1}h$ be a conformally Kähler metric. Let τ^g be the torsion form of g and $\sharp_g$ the sharp "musical" operator associated to g. Then the following are equivalent.*

(i) g has holomorphic torsion,
(ii) $(\overline{\tau^g})^{\sharp_g}$ is holomorphic,
(iii) $(\bar{\partial}\phi)^{\sharp}$ is holomorphic.

Proof. "(i) $\Longrightarrow$ (ii)" is simple and explained in the introduction. Now let $\hat{\Gamma}^j_{kl}$ and $\hat{T}^j_{kl}$ be the Christoffel symbols and the components of the torsion of g and let $\sigma = -\log\phi$. Then by direct calculation, we have $\hat{\Gamma}^j_{kl} = \Gamma^j_{kl} + \sigma_k\delta^j_l$. Thus,

$$\hat{T}^j_{kl} = \sigma_k\delta^j_l - \sigma_l\delta^j_k, \tag{3.1}$$

$$\tau^g = (n-1)\sum_{k=1}^{n}\sigma_k dz_k. \tag{3.2}$$

Lowering and raising the indices using $g_{j\bar{k}} = e^{\sigma}h_{j\bar{k}}$ and its inverse

$$\hat{T}_p{}^{rs} = \hat{T}^{\bar{j}}{}_{\bar{k}\bar{l}}g^{r\bar{k}}g^{s\bar{l}}g_{p\bar{j}} = e^{-\sigma}\left(\sigma_{\bar{k}}h^{r\bar{k}}\delta^s_p - \sigma_{\bar{l}}h^{s\bar{l}}\delta^r_p\right) \tag{3.3}$$

and

$$(\overline{\tau^g})^{\sharp_g} = (n-1)e^{-\sigma}h^{j\bar{k}}\sigma_{\bar{k}}\frac{\partial}{\partial z_j} = (n-1)(\bar{\partial}\phi)^{\sharp}. \tag{3.4}$$

This shows that "(ii) $\iff$ (iii)". Finally, from (3.3), $\hat{T}_p{}^{rs}$ is holomorphic if and only if for each r, $e^{-\sigma}\sigma_{\bar{k}}h^{r\bar{k}} = \phi_{\bar{k}}h^{r\bar{k}}$ is holomorphic. This shows that (iii) implies (i). The proof is complete. $\square$

Thus, the existence of a conformal metric with holomorphic torsion is equivalent to that of a nonvanishing real-valued solution ϕ to the equation $\nabla_j\nabla_k\phi = 0$. In many cases considered in this paper, non-constant solutions exist locally or globally on open manifolds. However, we point out that for compact manifolds, the existence of global conformally Kähler metrics with nontrivial holomorphic torsion is related to the geometry of the manifolds. In fact, as an application of the "Bochner technique" in differential geometry, we have the following

Corollary 3.2. *Let (M, h) be a compact Kähler manifold, and let $R_{j\bar{k}}$ be the Ricci curvature:*

$$R_{j\bar{k}} = -\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log\det(h_{\ell\bar{m}}). \tag{3.5}$$

Suppose that $(R_{j\bar{k}})$ is non-positive. If $g = \phi^{-1}h$ is a conformally Kähler metric having holomorphic torsion, then g is homothetic to h.

For example, there is *no* conformally flat metric with holomorphic torsion on complex flat tori $\mathbb{C}^n/\Lambda$, Λ being a lattice in $\mathbb{C}^n$, other than the flat metrics.

Proof. If g has holomorphic torsion, then by Proposition 3.1, $(\bar{\partial}\phi)^\sharp$ is holomorphic. By a result of Bochner (see [5, Theorem 2.4.1]), $(\bar{\partial}\phi)^\sharp$ is parallel. In particular, $\bar{\partial}\partial\phi = 0$ and hence ϕ is pluriharmonic. But M is compact and the maximum principle implies that ϕ is a constant. □

Remark 1. The proof of Proposition 3.1 above is purely local. Thus, we can state a version of "(i) $\Longleftrightarrow$ (ii)" for locally conformally Kähler manifolds as follows. Recall that if (M, g) is locally conformally Kähler, then there exists a closed 1-form θ, the *Lee form*, that satisfies

$$d\omega = \theta \wedge \omega, \tag{3.6}$$

where $\omega = ig_{j\bar{k}} dz_j \wedge d\bar{z}_k$ is the fundamental $(1,1)$-form in local coordinates (see [12]). Condition (ii) is equivalent to the real holomorphicity of the Lee field $\theta^\sharp$. Thus, g has holomorphic torsion if and only if the *Lee vector field* $\theta^\sharp$ is holomorphic. This property was studied in, e.g., [12], which also gives an abundance of conformally Kähler metrics on a Hopf manifold (as in Example 1.3) with holomorphic Lee field and hence they all have holomorphic torsion.

3.1. Conformal flat metrics on $\mathbb{C}^n$

Proposition 3.3. *Let ϕ be a smooth function such that the set $\{\phi > 0\}$ is a nonempty open set in $\mathbb{C}^n$. Then, a conformally flat Hermitian metric*

$g_{j\bar{k}} = \phi^{-1}\delta_{jk}$ on $\{\phi > 0\}$ has holomorphic torsion if and only if

$$\phi = \sum_{j,k=1}^{n} c_{j\bar{k}} z_j \bar{z}_k + \operatorname{Re}\sum_{k=1}^{n} \alpha_k z_k + \gamma, \tag{3.7}$$

where $c_{j\bar{k}}$ is a Hermitian matrix, $\alpha_k \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

Proof. From Proposition 3.1, the metric g has holomorphic torsion if and only if $\partial\phi/\partial\bar{z}_k$ is holomorphic for each k, or equivalently,

$$\frac{\partial^2 \phi}{\partial z_j \partial z_j} = 0. \tag{3.8}$$

This PDE has been solved explicitly by Gross and Qian in [7]. Real-valued solutions to this equation are known to have the form (3.7). The proof is complete. $\square$

Example 3.4. In (3.7), if we take $c_{j\bar{k}}$ to be the identity matrix, $\alpha_k = 0$, and $\gamma = 1$, then we obtain on $\mathbb{C}^n$ a conformally flat Hermitian metric

$$g_{j\bar{k}} = \frac{\delta_{jk}}{1+|z|^2}. \tag{3.9}$$

On the other hand, if we take $c_{j\bar{k}}$ to be minus the identity matrix, $\alpha_k = 0$, and $\gamma = 1$, then we obtain the metric

$$g_{j\bar{k}} = \frac{\delta_{jk}}{1-|z|^2}, \tag{3.10}$$

which is a conformally flat Hermitian metric on the unit ball $\mathbb{B}^n := \{|z| < 1\}$, cf. [10]. Both metrics have holomorphic torsion.

Theorem 3.5. *Let $M = \mathbb{B}^n$ and let*

$$g_{j\bar{k}} = \frac{\delta_{jk}}{1-|z|^2} \tag{3.11}$$

be a conformally flat metric on $\mathbb{B}^n$. If ψ is a real-valued function on $\mathbb{B}^n$ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then

$$\psi(z) = A + B\log(1-|z|^2) \tag{3.12}$$

for some real constants A and B.

Proof. Let ψ be a weight function on the Hermitian manifold $(\mathbb{B}^n, g)$ such that $(\bar\partial\psi)^\sharp$ is holomorphic. Since $g^{\bar k l} = (1-|z|^2)\delta_{kl}$, we have

$$(\bar\partial\psi)^\sharp = (1-|z|^2)\sum_{k=1}^{n}\psi_{\bar k}\partial_k. \tag{3.13}$$

Thus, the holomorphicity of $(\bar\partial\psi)^\sharp$ is equivalent to

$$f^{(k)} := (1-|z|^2)\psi_{\bar k} \tag{3.14}$$

is holomorphic for each k. Now, we compute

$$\frac{\partial(z_k\psi)}{\partial\bar z_k} = \frac{z_k f^{(k)}}{1-|z|^2} = \frac{\partial}{\partial\bar z_k}\left(-f^{(k)}\log(1-|z|^2)\right). \tag{3.15}$$

Therefore,

$$z_k\psi = -f^{(k)}\log(1-|z|^2) + v^{(k)}, \tag{3.16}$$

where $v^{(k)}$ is holomorphic in z_k. Thus, both sides of (3.16) are real-analytic in z_k. Expanding in power series at $z_k = 0$ (keeping other variables fixed), we obtain

$$f^{(k)}(z) = \sum_{l=0}^{\infty} A_l z_k^l, \quad v^{(k)}(z) = \sum_{s=0}^{\infty} C_s z_k^s, \quad \psi(z) = \sum_{p,q=0}^{\infty} c_{pq} z_k^p \bar z_k^q. \tag{3.17}$$

Plugging these into equation (3.16) above, we get

$$\sum_{p,q} c_{pq} z_k^{p+1}\bar z_k^q = -\left(\sum_{l=0}^{\infty} A_l z_k^l\right)\left(\sum_{m=0}^{\infty} B_m z_k^m \bar z_k^m\right) + \sum_{s=0}^{\infty} C_s z_k^s, \tag{3.18}$$

where

$$\log(1-|z|^2) = \sum_{m=0}^{\infty} B_m z_k^m \bar z_k^m, \quad B_0 = \log\left(1 - \sum_{j\neq k}|z_j|^2\right). \tag{3.19}$$

For each set $(z_j : j \neq k)$ fixed, the series involved in equation (3.18) above are uniformly and absolutely convergent in a small disc $\{|z_k| < r\}$. In particular, we can expand the product of infinite sums on the right-hand side and equate the coefficients of monomials $z_k^p \bar{z}_k^q$. Thus, comparing the terms with bi-degree $(p+1,0)$, we have

$$c_{p,0} = -A_{p+1}B_0 + C_{p+1}, \quad p = 0,1,2,\ldots. \tag{3.20}$$

Comparing terms of bi-degree $(1,q)$ we get

$$c_{0,0} = -A_1 B_0 + C_1, \quad c_{0,1} = -A_0 B_1, \quad c_{0,q} = 0 \text{ for } q \geqslant 2. \tag{3.21}$$

Thus, by the reality of ψ, we have

$$c_{p,0} = \overline{c_{0,p}} = 0, \quad \forall\, p \geqslant 2. \tag{3.22}$$

Then we find that

$$C_p = B_0 A_p \text{ for } p = 0 \text{ and } p \geqslant 3. \tag{3.23}$$

Hence,

$$v^{(k)} = B_0 f^{(k)} + c_{0,0} z_k + c_{1,0} z_k^2. \tag{3.24}$$

Plugging this into the original equation (3.16), we find that

$$-A_0 B_1 |z_k|^2 + \sum_{p+q\geqslant 2} c_{p,q} z_k^{p+1} \bar{z}_k^q = -\left(\sum_{l=0}^{\infty} A_l z_k^l\right)\left(\sum_{m=1}^{\infty} B_m z_k^m \bar{z}_k^m\right). \tag{3.25}$$

Equating the terms of bi-degree $(p+1,p)$ we have

$$c_{p,p} = -A_1 B_p, \quad p \geqslant 1. \tag{3.26}$$

Equating the terms of bi-degree (p,p) we have

$$c_{p-1,p} = -A_0 B_p, \quad p \geqslant 1. \tag{3.27}$$

Taking the conjugate, we have

$$c_{p+1,p} = \overline{c_{p,p+1}} = -\overline{A_0} B_{p+1}. \tag{3.28}$$

There are no terms of bi-degree (p,q) if $p < q$ in (3.25). Thus, $A_3 = A_4 = \cdots = 0$. On the other hand, equating the terms of bi-degree $(p+2,p)$, we find that

$$-A_2 B_p = c_{p+1,p} = -\overline{A_0} B_{p+1}. \tag{3.29}$$

This holds for all p if and only if $A_0 = A_2 = 0$ and hence $c_{0,1} = c_{1,0} = 0$. Consequently,

$$z_k(\psi - c_{0,0}) = -f^{(k)}(z)\left[\log(1-|z|^2) - B_0\right]. \tag{3.30}$$

By the reality of ψ, $c_{0,0}$, and $\log(1-|z|^2)$, and holomorphicity of $f^{(k)}(z)$ in all variables, we must have

$$f^{(k)}(z) = A_1 z_k, \tag{3.31}$$

where A_1 does not depend on $z_1, z_2, \ldots, z_n$. Thus,

$$\psi(z) = c_{0,0} + A_1B_0 + A_1\log(1-|z|^2) = C_1 + A_1\log(1-|z|^2), \tag{3.32}$$

where C_1 does not depend on z_k. To show that C_1 is a constant, we assume that $l \neq k$. By the same argument with k is replaced by l, we have

$$\psi(z) = \tilde{C}_1 + \tilde{A}_1\log(1-|z|^2) \tag{3.33}$$

for $\tilde{A}_1$ a constant and $\tilde{C}_1$ does not depend on z_l. We have

$$\tilde{C}_1 - C_1 = (A_1 - \tilde{A}_1)\log(1-|z|^2). \tag{3.34}$$

Applying $\partial^2/\partial z_l \partial z_k$ to both sides, we have

$$\begin{aligned} 0 = \frac{\partial^2}{\partial z_l \partial z_k}\left(\tilde{C}_1 - C_1\right) &= \frac{\partial^2}{\partial z_l \partial z_k}\left((A_1 - \tilde{A}_1)\log(1-|z|^2)\right) \\ &= (A_1 - \tilde{A}_1)\bar{z}_k\bar{z}_l(1-|z|^2)^{-2}. \end{aligned}$$

This shows that $A_1 = \tilde{A}_1$ and $C_1 = \tilde{C}_1$. In particular, C_1 does not depend on z_l, for any l. This completes the proof. $\square$

In Section 5.2 of [10], the authors studied the ∂-complex on the weighted Bergman spaces $A^2_{(p,0)}(\mathbb{B}^n, g_{j\bar{k}}, e^{-\psi})$ where g is given in (3.11) above and $\psi(z) = \alpha\log(1-|z|^2)$. Theorem 3.5 shows that this choice of the weight function is essentially the only one that makes the ∂-complex having the holomorphicity/duality property.

3.2. Conformal metrics on the complex projective space

The complex projective space $\mathbb{CP}^n$ is the quotient space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim \tag{3.35}$$

where $\sim$ is the equivalent relation

$$(Z_0, Z_1, \ldots, Z_n) \sim (Z_0', Z_1', \ldots, Z_n') \tag{3.36}$$

if and only if $Z_j = \lambda Z_j'$ for some $\lambda \in \mathbb{C}$. We denote by $[Z_0\colon Z_1\colon \cdots \colon Z_n]$ the equivalence class of $(Z_0, Z_1, \ldots, Z_n)$ and by $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ the canonical projection. Then π induces a natural complex manifold structure on $\mathbb{CP}^n$. Moreover, $\mathbb{CP}^n$ is covered by $n+1$ coordinate charts $U_j := \{[Z_0\colon Z_1\colon \cdots\colon Z_n] \in \mathbb{CP}^n\colon Z_j \neq 0\}$, $j = 0, 1, \ldots, n$, each of which is biholomorphic to $\mathbb{C}^n$ via the map

$$\phi_j([Z_0\colon Z_1\colon \cdots \colon Z_n]) \to \left(Z_0/Z_j, Z_1/Z_j, \ldots, \widehat{Z_j/Z_j}, \ldots, Z_n/Z_j\right), \tag{3.37}$$

where the j^{th} coordinate in the right-hand side is removed. The Fubini–Study metric on $\mathbb{CP}^n$ can be described in each coordinate chart $U_j \cong \mathbb{C}^n$. For example, on U_0 the Fubini–Study metric h_{FS} reduces to the Kähler metric on $\mathbb{C}^n$ given by

$$h_{j\bar{k}} = \partial_j \partial_{\bar{k}} \log(1 + |z|^2), \quad z_k = Z_k/Z_0,\ k = 1, 2, \ldots, n. \tag{3.38}$$

Then h_{FS} is a Kähler metric of constant holomorphic sectional curvature $K = 2$; see [11].

Proposition 3.1 and a result of Gross–Qian [7, §3.3] give the following

Proposition 3.6. *Let ϕ be a smooth function such that the set $\{\phi > 0\}$ is a nonempty open set in $\mathbb{C}^n$. Then, a conformally Fubini–Study Hermitian metric $g_{j\bar{k}} = \phi^{-1} h_{j\bar{k}}$ on $\{\phi > 0\}$ has holomorphic torsion if and only if*

$$(1 + |z|^2)\phi = \sum_{j,k=1}^{n} c_{j\bar{k}} z_j \bar{z}_k + \mathrm{Re} \sum_{k=1}^{n} \alpha_k z_k + \gamma, \tag{3.39}$$

where $c_{j\bar{k}}$ is a Hermitian matrix, $\alpha_k \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

Remark 2. Each function ϕ in (3.39) gives rise to a Hermitian metric conformal to the Fubini–Study metric on a subset $\Omega = \{\phi > 0\}$ of $\mathbb{C}^n \subset \mathbb{CP}^n$.

Depending on the choice of coefficients, Ω may be bounded, unbounded, or the whole $\mathbb{C}^n$. All conformal metrics on the whole $\mathbb{CP}^n$ with holomorphic torsion can be found using a result of Futaki [5]. They arise as $\phi^{-1}h_{FS}$ (h_{FS} is the Fubini–Study metric) where $\phi = \phi_0 + C$, where ϕ_0 is in the first eigenspace of the Laplacian, and C is a real constant, $C > -\min \phi_0$.

A particularly interesting case is when $c_{j\bar{k}} = 0$, $\alpha_k = 0$ and $\gamma = 1$. In this case we have

$$g_{j\bar{k}} = \delta_{j\bar{k}} - \frac{\bar{z}_j z_k}{1 + |z|^2} \tag{3.40}$$

is a Hermitian non-Kähler metric on $\mathbb{C}^n$ with holomorphic torsion. This metric is analogous to the "half" hyperbolic metric on the unit ball discussed in the next section. In the special case $n = 1$, this is the same as (3.9) and the metric is the well-known Hamilton's "cigar" soliton (a.k.a. the Witten's blackhole.)

Theorem 3.7. *Let $g_{j\bar{k}}$ be as in (3.40). If ψ is real-valued function on $(\mathbb{C}^n, g_{j\bar{k}})$ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then*

$$\psi(z) = A + B \log(1 + |z|^2) \tag{3.41}$$

for A and B are two real constants.

The proof of this theorem is similar to that of Theorem 3.9 below. We omit the details.

3.3. Conformally complex hyperbolic metrics

Combining Proposition 3.1 and Gross and Qian [7, Theorem 3.4], we have the following

Proposition 3.8. *Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and let*

$$h_{j\bar{k}} = (1 - |z|^2)^{-1} \left(\delta_{jk} + \frac{\bar{z}_j z_k}{1 - |z|^2} \right) \tag{3.42}$$

be the complex hyperbolic metric on $\mathbb{B}^n$. Let $g = \phi^{-1}h$ be a conformal metric on $\mathbb{B}^n$. Then g has holomorphic torsion if and only if

$$(1 - |z|^2)\phi = \sum_{j,k} c_{j\bar{k}} z^j \bar{z}^k + \operatorname{Re}\left(\sum_k \alpha_k z^k \right) + \gamma, \tag{3.43}$$

where $c_{j\bar{k}}$ is a Hermitian matrix, $\alpha_k \in \mathbb{C}$, and $\gamma \in \mathbb{R}$.

Remark 3. In [7], the following example was briefly discussed. For each $\beta \in \mathbb{R}$, put

$$h_{j\bar{k}} = (1 - |z|^2)^{\beta - 1} \left(\delta_{jk} + \frac{\bar{z}_j z_k}{1 - |z|^2} \right). \tag{3.44}$$

By the Sherman–Morrison formula, we find that the inverse transpose is

$$h^{k\bar{l}} = (1 - |z|^2)^{1-\beta} \left(\delta^{kl} - \bar{z}_l z_k \right). \tag{3.45}$$

Thus, the torsion tensor takes the following form

$$T^l_{jk} = \Gamma^l_{jk} - \Gamma^l_{kj} = \frac{\beta(\bar{z}_k \delta^l_j - \bar{z}_j \delta^l_k)}{1 - |z|^2} \tag{3.46}$$

and h is not Kähler, unless $\beta = 0$ or $n = 1$. Tracing over the indices l and k, we find that

$$\tau_j = -\frac{\beta(n-1)\bar{z}_j}{1 - |z|^2}. \tag{3.47}$$

Thus, h has holomorphic torsion if and only if $\beta = 0$ (Kähler case) or $\beta = 1$. In the latter case, h is the "half" hyperbolic metric, which is the only one in this family having holomorphic torsion.

Theorem 3.9. *Let ψ be a function on $\mathbb{B}^n$ with the half hyperbolic metric, then $(\bar{\partial}\psi)^\sharp$ is holomorphic if and only if*

$$\psi(z) = A + B \log(1 - |z|^2), \tag{3.48}$$

where A and B are real constants.

Proof. Let $Z = Z^k \partial_k = (\bar{\partial}\psi)^\sharp$. Since

$$h^{k\bar{l}} = \delta^{kl} - \bar{z}_l z_k, \tag{3.49}$$

we have

$$Z^k = h^{k\bar{l}} \psi_{\bar{l}} = \psi_{\bar{k}} - z_k \sum_{l=1}^n \bar{z}_l \psi_{\bar{l}}. \tag{3.50}$$

If Z is holomorphic, then

$$0 = \partial_{\bar{j}} Z^k = \psi_{\bar{k}\bar{j}} - z_k \psi_{\bar{j}} - z_k \sum_{l=1}^n \bar{z}_l \psi_{\bar{l}\bar{j}}. \tag{3.51}$$

Thus,

$$\psi_{\bar{k}\bar{j}} = z_k \psi_{\bar{j}} + z_k \sum_{l=1}^{n} \bar{z}_l \psi_{\bar{l}\bar{j}}. \tag{3.52}$$

Multiplying both sides with $\bar{z}_k$ and summing over k, we obtain

$$\sum_{l=1}^{n} \bar{z}_k \psi_{\bar{k}\bar{j}} = |z|^2 \psi_{\bar{j}} + |z|^2 \sum_{l=1}^{n} \bar{z}_l \psi_{\bar{l}\bar{j}}. \tag{3.53}$$

Therefore,

$$(1 - |z|^2) \sum_{l=1}^{n} \bar{z}_k \psi_{\bar{k}\bar{j}} = |z|^2 \psi_{\bar{j}}. \tag{3.54}$$

Combining this with (3.52), we obtain

$$\psi_{\bar{k}\bar{j}} = z_k \left(\psi_{\bar{j}} + \frac{|z|^2}{1 - |z|^2} \psi_{\bar{j}} \right) = \frac{z_k \psi_{\bar{j}}}{1 - |z|^2}. \tag{3.55}$$

Equivalently,

$$\frac{\partial}{\partial \bar{z}_k} \left[(1 - |z|^2) \psi_{\bar{j}} \right] = 0. \tag{3.56}$$

Thus, ψ satisfies the conditions in Theorem 3.5. Consequently,

$$\psi(z) = A + B \log(1 - |z|^2), \tag{3.57}$$

where A and B are real constants. The proof is complete. □

3.4. Conformally $U(n)$-invariant Kähler metrics

In the sequel, we consider $U(n)$-invariant Kähler metrics and radial weights. Suppose that $h_{j\bar{k}}$ is a Kählerian metric induced by a radial potential $h(z) = \tilde{h}(|z|^2)$, where $\tilde{h}(r)$ is a real-valued function of a real variable. Precisely, we have

$$h_{j\overline{k}} = \partial_j \partial_{\overline{k}} \tilde{h}(|z|^2) = \tilde{h}'(|z|^2)\, \delta_{jk} + \tilde{h}''(|z|^2)\, \overline{z}_j z_k. \tag{3.58}$$

Thus, $h_{j\bar{k}}$ is a rank-one perturbation of a multiple of the identity matrix. For $h_{j\bar{k}}$ to be positive definite, we assume that $\tilde{h}'(r) > 0$ and $r\tilde{h}''(r) + \tilde{h}'(r) > 0$. The Sherman–Morrison formula give the formula for the (transposed) inverse

$$h^{k\bar{j}} = \frac{1}{h'}\left(\delta_{jk} - \frac{\tilde{h}'' z_k \overline{z}_j}{\tilde{h}' + r\tilde{h}''}\right), \quad r = |z|^2, \tag{3.59}$$

so that $h_{\bar{l}k}h^{k\bar{j}} = \delta_{\bar{l}}^{\bar{j}}$, the Kronecker symbol.

Proposition 3.10. *Let g be the conformally $U(n)$-invariant Kähler metric*

$$g_{j\bar{k}} = e^{\tilde{\sigma}(|z|^2)}\partial_j\partial_{\bar{k}}\tilde{h}(|z|^2) \tag{3.60}$$

and $\psi(z) = \tilde{\psi}(|z|^2)$ is a real-valued radial weight function. Then $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is holomorphic if and only if

$$\tilde{\psi}(r) = (n-1)\tilde{\sigma}(r) + C_1\int_0^r e^{\tilde{\sigma}(s)}(\tilde{h}'(s) + s\tilde{h}''(s))\, ds + \tilde{C}. \tag{3.61}$$

where C and C_1 are real constants.

Proof. We have

$$T^i_{jk} = \sigma_j\delta^i_k - \sigma_k\delta^i_j = \tilde{\sigma}'\left(\bar{z}_j\delta^i_k - \bar{z}_k\delta^i_j\right). \tag{3.62}$$

Then it follows that the torsion $(1,0)$-form of $g_{j\bar{k}}$ is

$$\tau = \tau_k dz^k = (n-1)\tilde{\sigma}'\overline{z}_k dz^k. \tag{3.63}$$

If $\psi(z) = \tilde{\psi}(r)$, $r = |z|^2$, is a radial weight, then $\partial_{\bar{j}}\psi = \tilde{\psi}'(r)z_j$. For

$$(\bar{\partial}\psi - \bar{\tau})^\sharp = g^{j\bar{k}}\left(\psi_{\bar{k}} - \tau_{\bar{k}}\right)\frac{\partial}{\partial z^j} \tag{3.64}$$

we get

$$g^{j\bar{k}}\left(\psi_{\bar{k}} - \tau_{\bar{k}}\right) = \frac{\tilde{\psi}' - (n-1)\tilde{\sigma}'}{e^{\tilde{\sigma}}(\tilde{h}' + r\tilde{h}'')}z_j. \tag{3.65}$$

Therefore $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is holomorphic if and only if

$$\tilde{\psi}' = (n-1)\tilde{\sigma}' + C_1 e^{\tilde{\sigma}}(\tilde{h}' + r\tilde{h}''), \tag{3.66}$$

for some constant C_1. So for another constant $\tilde{C}$ we have

$$\tilde{\psi}(r) = (n-1)\tilde{\sigma}(r) + C_1 \int_0^r e^{\tilde{\sigma}(s)}(\tilde{h}'(s) + s\tilde{h}''(s))\, ds + \tilde{C}.$$

The proof is complete. □

Example 3.11. Considering the unit ball in $\mathbb{C}^n$ and the hyperbolic metric induced by the potential function $\tilde{h}(r) = -\log(1-r)$, we get $\tilde{h}'(r)+r\tilde{h}''(r) = (1-r)^{-2}$ and

$$\tilde{\psi}(r) = (n-1)\tilde{\sigma}(r) + C\int_0^r \frac{e^{\tilde{\sigma}(s)}}{(1-s)^2}\, ds + C_1. \tag{3.67}$$

Take, for example, $\tilde{\sigma}(r) = \alpha\log(1-r)$, with $\alpha > 1$ and

$$\tilde{\psi}(r) = \alpha(n-1)\log(1-r) - A(1-r)^{\alpha-1} + B. \tag{3.68}$$

If D^* denote the $L^2(M, h, e^{-\psi})$-space adjoint of ∂, then D^*u is holomorphic if u is a holomorphic $(1,0)$-form. However, if $\alpha \neq 0$ and $n \geqslant 3$, then for a holomorphic $(2,0)$-form v, D^*v need not be holomorphic.

Proposition 3.12. *Let ϕ be a radial positive function on $\mathbb{C}^n$ $(n \geqslant 2)$, $\phi(z) = \tilde{\phi}(|z|^2)$. The Hermitian metric $g_{j\bar{k}} := \phi^{-1}(|z|^2)\partial_j\partial_{\bar{k}}\tilde{h}(|z|^2)$ has holomorphic torsion if and only if*

$$\tilde{\phi}(r) = A + Br\tilde{h}'(r), \tag{3.69}$$

where A and B are two real constants.

Proof. From Proposition 3.1, g has holomorphic torsion if and only if $(\bar{\partial}\phi)^\sharp$ is holomorphic. By direct computation,

$$h^{j\bar{k}}\phi_{\bar{k}} = \frac{\tilde{\phi}'(r)z_j}{\tilde{h}'(r) + r\tilde{h}''(r)}, \quad r = |z|^2. \tag{3.70}$$

This is holomorphic for all l if and only if $\tilde{\phi}'(r)/(\tilde{h}'(r) + r\tilde{h}''(r))$ is constant:

$$\tilde{\phi}' = B(\tilde{h}' + r\tilde{h}'') = B(r\tilde{h}')'. \tag{3.71}$$

Integrating this we complete the proof. □

Hence, the conformally $U(n)$-invariant Kähler metric

$$g_{j\bar{k}} = e^{\tilde{\sigma}(|z|^2)}\partial_j\partial_{\bar{k}}\tilde{h}(|z|^2) \tag{3.72}$$

has holomorphic torsion if and only if

$$\tilde{\sigma}(r) = -\log(C_2 r\tilde{h}'(r) + C_3), \tag{3.73}$$

where the constant C_3 has to be chosen such that $C_2 r\tilde{h}'(r) + C_3 > 0$.

This also determines the weight function ψ: we use (3.61) and get

$$\tilde{\psi}(r) = -C_4\log(C_2 r\tilde{h}'(r) + C_3) + C_5, \tag{3.74}$$

where $C_4 = n - 1 - (C_1/C_2)$.

With this choice of $\tilde{\sigma}$ and $\tilde{\psi}$ we get for a $(1,0)$-form $u = u_j dz^j \in \operatorname{dom}(\partial^*)$ that

$$\partial^* u = C_1 \sum_{j=1}^{n} z_j dz^j, \tag{3.75}$$

and for a $(2,0)$-form $v = v_{pq} dz^p \wedge dz^q \in \operatorname{dom}(\partial^*)$ that

$$\partial^* v = -(C_1 - C_2)\sum_{q=1}^{n} z^q v_{pq} dz^p. \tag{3.76}$$

Finally we have shown the following

Theorem 3.13. *Let g be the conformally $U(n)$-invariant Kähler metric given as in (3.72) together with a radial real-valued weight function $\psi(z) = \tilde{\psi}(|z|^2)$. The vector field $(\bar{\partial}\psi - \bar{\tau})^\sharp$ and the torsion operator $T^\sharp$ are holmorphic if and only if*

$$\tilde{\sigma}(r) = -\log(C_2 r\tilde{h}'(r) + C_3) \tag{3.77}$$

and

$$\tilde{\psi}(r) = -C_4\log(C_2 r\tilde{h}'(r) + C_3) + C_5, \tag{3.78}$$

where $C_4 = n - 1 - (C_1/C_2)$ and the constant C_3 has to be chosen such that $C_2 r\tilde{h}'(r) + C_3 > 0$.

In this case we have for the vector field $(\bar{\partial}\psi - \bar{\tau})^\sharp = C_1\sum_{j=1}^{n} z^j\partial_j$ and for the torsion operator

$$T^\sharp(v) = -C_2\sum_{q=1}^{n} z^q v_{pq} dz^p.$$

4. The ∂-complex on the unit ball with the half hyperbolic metric

4.1. The half hyperbolic metric on the unit ball

Consider the half hyperbolic metric on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ given in the "standard" coordinate by

$$h_{j\bar{k}} = \delta_{jk} + \frac{\bar{z}_j z_k}{1-|z|^2}. \tag{4.1}$$

If $g_{j\bar{k}} = -\partial_j \partial_{\bar{k}} \log(1-|z|^2)$ is the complex hyperbolic metric, then $h_{j\bar{k}} = (1-|z|^2) g_{j\bar{k}}$, i.e., h is conformally Kähler.

For some motivations, we list several basic curvature properties of this metric as follows. We have,

$$\partial_i h_{j\bar{l}} = \frac{\bar{z}_j}{1-|z|^2}\left(\delta_{il} + \frac{\bar{z}_i z_l}{1-|z|^2}\right), \tag{4.2}$$

and therefore,

$$\Gamma^k_{ij} = h^{k\bar{l}} \partial_i h_{j\bar{l}} = \frac{\bar{z}_j \delta_{ik}}{1-|z|^2}. \tag{4.3}$$

Thus, the curvature of h is

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -h_{p\bar{l}} \partial_{\bar{j}} \Gamma^p_{ik} \\ &= -\frac{1}{1-|z|^2}\left(\delta_{il}\delta_{jk} + \frac{\delta_{jk}\bar{z}_i z_l}{1-|z|^2} + \frac{\delta_{il}\bar{z}_k z_j}{1-|z|^2} + \frac{\bar{z}_i z_j \bar{z}_k z_l}{(1-|z|^2)^2}\right) \\ &= -\frac{h_{i\bar{l}} h_{k\bar{j}}}{1-|z|^2}. \end{aligned} \tag{4.4}$$

Thus, the half hyperbolic metric has negative pointwise constant holomorphic sectional curvature

$$K(\xi)\big|_z = \left.\frac{R_{i\bar{j}k\bar{l}}\xi^i \xi^{\bar{j}} \xi^k \xi^{\bar{l}}}{|\xi|^4}\right|_z = -\frac{1}{1-|z|^2}, \quad \text{for } \xi = \xi^j \partial_j \in T_z^{(1,0)}(M), \tag{4.5}$$

which is unbounded on $\mathbb{B}^n$. The curvature satisfies additional symmetry

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}, \tag{4.6}$$

and thus the first two Chern–Ricci curvatures are equal:

$$R^{(1)}_{i\bar{j}} := h^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{1}{1-|z|^2} h_{i\bar{j}}, \tag{4.7}$$

$$R^{(2)}_{k\bar{l}} := h^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{1}{1-|z|^2} h_{k\bar{l}}, \tag{4.8}$$

and the third Chern-Ricci curvature is

$$R^{(3)}_{k\bar{j}} := h^{i\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{n}{1-|z|^2} h_{k\bar{j}}. \tag{4.9}$$

The half hyperbolic metric is (weak) Chern-Einstein with two different unbounded and negative Chern scalar curvatures

$$s := h^{i\bar{j}} R^{(1)}_{i\bar{j}} = -\frac{n}{1-|z|^2}, \quad \hat{s} := h^{k\bar{j}} R^{(3)}_{k\bar{j}} = -\frac{n^2}{1-|z|^2}. \tag{4.10}$$

Using (3.3), we find that

$$T_p{}^{rs} = z_s \delta^r_p - z_r \delta^s_p \tag{4.11}$$

is holomorphic. Furthermore,

$$\bar{\tau}^\sharp = -(n-1) \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} \tag{4.12}$$

is also holomorphic.

4.2. The ∂-complex

Theorem 3.9 suggests that we should choose the weight function

$$\psi(z) = \alpha \log(1-|z|^2), \tag{4.13}$$

whose gradient is real holomorphic. Since

$$\det[h_{j\bar{k}}] = \frac{1}{1-|z|^2} \tag{4.14}$$

the weighted measure is

$$e^{-\psi} d\mathrm{vol}_h = (1-|z|^2)^{-\alpha-1} d\lambda. \tag{4.15}$$

Then the corresponding Bergman space

(4.16)
$$A^2_{(0,0)}(\mathbb{B}^n, h, e^{-\psi}) := \left\{ f \in \mathcal{O}(\mathbb{B}^n) \colon \|f\|^2 := \int_{\mathbb{B}^n} |f|^2 (1-|z|^2)^{-\alpha-1} d\lambda < \infty \right\}$$

is the "usual" Bergman space $A^2_{-\alpha-1}(\mathbb{B}^n)$ in the ball with parameter $-\alpha - 1$, which is of infinite dimension if $\alpha < 0$. We thus assume that $\alpha < 0$ from now on.

For $u = \sum_{k=1}^n u_k dz_k$, we have

$$|u|_h^2 := u_j u_{\bar{k}} h^{j\bar{k}} = \sum_{k=1}^n |u_k|^2 - \left| \sum_{k=1}^n z_k u_k \right|^2 . \tag{4.17}$$

The Bergman space $A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$ consists of $(1,0)$-forms with holomorphic coefficients $u = \sum_{j=1}^n u_j dz_j$ such that

$$\|u\|^2 := \int_{\mathbb{B}^n} \left(\sum_{k=1}^n |u_k|^2 - \left| \sum_{k=1}^n z_k u_k \right|^2 \right) (1-|z|^2)^{-\alpha-1} d\lambda < \infty. \tag{4.18}$$

Since the restrictions of polynomials onto $\mathbb{B}^n$ are dense in each Bergman spaces $A^2(\mathbb{B}^n, (1-|z|^2)^\gamma)$ for $\gamma > -1$, the polynomials as well as $(p,0)$-forms with polynomial coefficients are dense in the respective Bergman spaces. Thus ∂-operator is densely defined in $A^2_{(p,0)}(\mathbb{B}^n, h, e^{-\psi})$ for each $0 \leqslant p \leqslant n$.

Observe that

$$(\bar{\partial}\psi)^\sharp = -\alpha \sum_{k=1}^n z_k \frac{\partial}{\partial z_k}, \tag{4.19}$$

is holomorphic, and by (4.12) we have that

$$(\bar{\partial}\psi - \bar{\tau})^\sharp = (n-1-\alpha) \sum_{j=1}^n z_j \partial_j . \tag{4.20}$$

This, together with an integration by parts argument, gives the formula for ∂^*:

Proposition 4.1. *Let $u = u_j dz_j \in A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$. If $\sum_{k=1}^n u_k z_k$ belongs to $A^2_{(0,0)}(\mathbb{B}^n, h, e^{-\psi})$, then $u \in \operatorname{dom}(\partial^*)$ and*

$$\partial^* u = (n-1-\alpha) \sum_{j=1}^n z_j u_j . \tag{4.21}$$

Proof. The proof is essentially an integration by parts argument. But the metric h is not complete and thus we need to verify the vanishing of the "boundary" term directly. Let χ_R $(0 < R < 1)$ be a family of smooth functions of a real variable such that $\chi_R \equiv 1$ on $(-\infty, R]$, the support of χ_R is contained in $(-\infty, 1)$, and $|\chi_R'| < 2/(1-R)$. By abuse of notation we write $\chi_R(z) = \chi_R(|z|^2)$, so that $\partial\chi_R/\partial\bar{z}_k = \chi_R'(|z|^2)z_k$.

Let $v \in A^2_{(0,0)}(\mathbb{B}^n, h, e^{-\psi})$, then by integration by parts,

(4.22)

$$\begin{aligned}(\chi_R u, \partial v)_{L^2(\mathbb{B}^n, h, \psi)} &= \int_{\mathbb{B}^n} h^{j\bar{k}} \chi_R u_j \overline{v_k} e^{-\psi} d\mathrm{vol}_h \\ &= \int_{\mathbb{B}^n} \sum_{k=1}^n \overline{v_k} \left(u_k - \bar{z}_k \sum_{j=1}^n u_j z_j \right) \chi_R(|z|^2)(1-|z|^2)^{-1-\alpha} d\lambda \\ &= \int_{\mathbb{B}^n} \overline{v} \sum_{k=1}^n \frac{\partial}{\partial \bar{z}_k} \left(\left(u_k - \bar{z}_k \sum_{j=1}^n u_j z_j \right) \chi_R(|z|^2)(1-|z|^2)^{-1-\alpha} \right) d\lambda \\ &= (n-\alpha-1) \int_{\mathbb{B}^n} \left(\sum_k u_k z_k \right) \chi_R(|z|^2)\overline{v}(1-|z|^2)^{-\alpha-1} d\lambda \\ &\quad - \int_{\mathbb{B}^n} \overline{v} \left(\sum_k u_k z_k \right) \chi_R'(|z|^2)(1-|z|^2)^{-\alpha} d\lambda.\end{aligned}$$

Since $\chi_R'(|z|^2) = 0$ for $|z|^2 < R$ and $\chi_R'(|z|^2) < 2(1-|z|^2)^{-1}$ for $0 \leqslant |z| < 1$, we can estimate the last integral as follows:

$$\begin{aligned}\text{(4.23)} \quad &\left| \int_{\mathbb{B}^n} \overline{v} \left(\sum_k u_k z_k \right) \chi_R'(|z|^2)(1-|z|^2)^{-\alpha} d\lambda \right| \\ &\qquad \leqslant 2 \int_{R<|z|<1} \left| \sum_k u_k z_k \right| |v|(1-|z|^2)^{-\alpha-1} d\lambda.\end{aligned}$$

On the other hand, since both $\sum_k u_k z_k$ and v belong to $A^2_{(0,0)}(\mathbb{B}^n, h, e^{-\psi}) = A^2_{-\alpha-1}(\mathbb{B}^n)$, the "standard" weighted Bergmann space in the ball with weight $(1-|z|^2)^{-\alpha-1}$, the Hölder inequality implies that

$$\text{(4.24)} \quad \int_{\mathbb{B}^n} \left| \sum_k u_k z_k \right| |v|(1-|z|^2)^{-\alpha-1} d\lambda$$

$$\leqslant \left\| \sum_k u_k z_k \right\|_{A^2_{-\alpha-1}(\mathbb{B}^n)} \cdot \|v\|_{A^2_{-\alpha-1}(\mathbb{B}^n)} < \infty.$$

This implies that the right-hand side (and hence both sides) of (4.23) tends to 0 as $R \to 1^-$. Letting $R \to 1^-$ in (4.22), using the denominated Lebesgue convergence theorem, we obtain

$$(4.25)\qquad \begin{aligned} (u, \partial v)_{h,\psi} &= (n-\alpha-1) \int_{\mathbb{B}^n} \overline{v} \left(\sum_k u_k z_k \right) (1-|z|^2)^{-\alpha-1} d\lambda \\ &= \left((n-\alpha-1) \sum_k u_k z_k, v \right)_{h,\psi}. \end{aligned}$$

Consequently, the map $v \mapsto (u, \partial v)_{h,\psi}$ is continuous and thus $u \in \mathrm{dom}(\partial^*)$. Moreover,

$$(4.26)\qquad \partial^* u = (n-\alpha-1) \sum_k u_k z_k.$$

The proof is complete. □

For two-form $v_{rs} dz_r \wedge dz_s$, with $v_{rs} = -v_{sr}$, we have by (4.11),

$$(4.27)\qquad T^\sharp(v) := \frac{1}{2} T_p{}^{rs} v_{rs} dz_p = \sum_{s=1}^n z_s v_{ps} dz_p.$$

Therefore, by (1.14), we can verify as in Proposition 4.1 that

$$(4.28)\qquad \partial^* v = -(n-\alpha-2) \sum_{s=1}^n z_s v_{rs} dz_r.$$

For $u = u_j dz_j$, we have

$$\partial u = \frac{1}{2} \sum_{j,k} \left(\frac{\partial u_k}{\partial z_j} - \frac{\partial u_j}{\partial z_k} \right) dz_j \wedge dz_k,$$

and thus

$$(4.29)\qquad \partial^* \partial u = (n-\alpha-2) \sum_{k=1}^n \sum_{j=1}^n \left(\frac{\partial u_k}{\partial z_j} - \frac{\partial u_j}{\partial z_k} \right) z_j dz_k.$$

On the other hand,

$$\partial\partial^* u = (n-\alpha-1)\sum_{k=1}^{n}\left(u_k + \sum_{j=1}^{n} z_j \frac{\partial u_j}{\partial z_k}\right) dz_k. \tag{4.30}$$

Consequently,

$$\widetilde{\Box}_1 u = (n-\alpha-1)u + \sum_{k=1}^{n}\sum_{j=1}^{n}\left((n-\alpha-2)\frac{\partial u_k}{\partial z_j} + \frac{\partial u_j}{\partial z_k}\right) z_j dz_k. \tag{4.31}$$

Unlike the cases of Segal-Bargmann space [9] and weighted Bergman space with hyperbolic metric [10], this is not a diagonal operator. Nevertheless we can apply the methods from Theorem 5.4 of [10] to get the following

Theorem 4.2. *Let h be the half hyperbolic metric on the unit ball $\mathbb{B}^n$, $\alpha < 0$, and $\psi(z) = \alpha \log(1-|z|^2)$. Then the complex Laplacian $\widetilde{\Box}_1$ has a bounded inverse $\widetilde{N}_1$, which is a compact operator on $A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$ with discrete spectrum. In addition, if*

$$\nu = \begin{cases} -\alpha, & \text{if } n = 1,\\ \min\{1-\alpha, -2\alpha\}, & \text{if } n = 2,\\ n-\alpha-1, & \text{if } n \geqslant 3.\end{cases} \tag{4.32}$$

then

$$\left\|\widetilde{N}_1 u\right\| \leqslant \frac{1}{\nu}\|u\|, \tag{4.33}$$

for each $u \in A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$.

Consequently, if $\eta = \eta_j dz_j \in A^2_{(1,0)}(\mathbb{B}^n, h, e^{-\psi})$ with $\partial\eta = 0$, thenf $:= \partial^*\widetilde{N}_1\eta$ is the canonical solution of $\partial f = \eta$, this means $\partial f = \eta$ and $f \in (\ker\partial)^{\perp}$. Moreover,

(4.34)

$$\int_{\mathbb{B}^n} |f|^2 (1-|z|^2)^{-\alpha-1} d\lambda \leqslant \frac{1}{\nu}\int_{\mathbb{B}^n}\left(\sum_{j=1}^{n}|\eta_j|^2 - \left|\sum_{j}^{n}\eta_j z_j\right|^2\right)(1-|z|^2)^{-\alpha-1}d\lambda.$$

Remark 4. If $n = 1$ or $n \geqslant 3$, then the first positive eigenvalue of $\widetilde{\Box}_1$ is $\lambda_1 = n-1-\alpha$ with the multiplicity n. If $n = 2$, there are three subcases:

If $-1 < \alpha < 0$, then $\lambda_1 = -2\alpha$ is a simple eigenvalue and the corresponding eigenspace E_1 is spanned by $z_1 dz_2 - z_2 dz_1$; if $\alpha = -1$, then $\lambda_1 = 2$ with multiplicity 3 and E_1 is spanned by dz_1, dz_2, and $z_1 dz_2 - z_2 dz_1$; if $\alpha < -1$, then $\lambda_1 = 1 - \alpha$ with multiplicity 2 and E_1 is spanned by dz_1 and dz_2.

Proof. The subspaces

$$A^2_{(1,0)}(m) := \text{span}\left\{c_J z^J dz_l \colon, |J| = m, l = 1, 2, \ldots, n\right\}, \tag{4.35}$$

$m = 0, 1, 2, \ldots$, are invariant under the action of $\widetilde{\Box}_1$. Using a standard result in spectral theory (see Lemma 5.1 of [10] or [3]), we can study the spectrum of $\widetilde{\Box}_1$ by studying the spectra of its restrictions onto finite dimensional subspaces $A^2_{(1,0)}(m)$. If $n = 1$, then each subspace is one-dimensional. Moreover, write $z_1 = z$, we have

$$\widetilde{\Box}_1(z^k dz) = -(k+1)\alpha z^k dz. \tag{4.36}$$

We find that, when $n = 1$, $\widetilde{\Box}_1$ has simple eigenvalues $-\alpha, -2\alpha, \cdots \to +\infty$ since $\alpha < 0$.

Consider the case $n \geqslant 2$. When $m = 0$, $A^2_{(1,0)}(0)$ is spanned by$dz_1, dz_2, \ldots, dz_n$ and $\widetilde{\Box}_1(dz_k) = (n - \alpha - 1)\, dz_k$ and hence $n - \alpha - 1$ is an eigenvalue for $\widetilde{\Box}_1$. When $m = 1$, $A^2_{(1,0)}(1)$ has dimension n^2 and is spanned by $z_j dz_k$, $j, k = 1, \ldots n$. For example, if $n = 2$ then the matrix representation of $\widetilde{\Box}_1$ in the basis $e_1 := z_1 dz_1, e_2 := z_1 dz_2, e_3 := z_2 dz_1$, and $e_4 := z_2 dz_2$ is the following constant column-sum matrix

$$\begin{pmatrix} 2-2\alpha & 0 & 0 & 0 \\ 0 & 1-2\alpha & 1 & 0 \\ 0 & 1 & 1-2\alpha & 0 \\ 0 & 0 & 0 & 2-2\alpha \end{pmatrix} \tag{4.37}$$

whose eigenvalues are -2α and $2(1-\alpha)$, the latter has multiplicity 3, and the matrix is diagonalizable. Observe that -2α is an eigenvalue for all $n \geqslant 2$.

Consider the case $m = 2$ and $n = 2$, $A^2_{(1,0)}(2)$ has a basis of 6 vectors: $e_1 = z_1^2 dz_1$, $e_2 = z_1^2 dz_2$, $e_3 = z_1 z_2 dz_1$, $e_4 = z_1 z_2 dz_2$, $e_5 = z_2^2 dz_1$, and$e_6 =$

$z_2^2 dz_2$. The matrix representation of $\widetilde{\Box}_1$ in this basis is

$$\begin{pmatrix} 3-3\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-3\alpha & 1 & 0 & 0 & 0 \\ 0 & 2 & 2-3\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-3\alpha & 2 & 0 \\ 0 & 0 & 0 & 1 & 1-3\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 3-3\alpha \end{pmatrix}. \tag{4.38}$$

The eigenvalues of this matrix are $3(1-\alpha)$ (multiplicity 4) and -3α (multiplicity 2).

Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a multi-index and let $|\Lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. If $k \neq l$, we define the multi-index

$$\Lambda_{j,l} = (\lambda_1, \ldots, \lambda_{l-1}, \lambda_l + 1, \lambda_{l+1}, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_n), \tag{4.39}$$

when $j > l$ and similarly for $l < j$. That is, the operation $\Lambda \mapsto \Lambda_{j,l}$ adds 1 to l^{th}-index and subtracts 1 from j^{th}-index. Clearly, $|\Lambda_{j,l}| = |\Lambda|$.

If $u = z^\Lambda dz_l$ where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a multi-index, then

$$\widetilde{\Box}_1 u = \left((|\Lambda| + 1)(n - \alpha - 1) - |\Lambda| + \lambda_l\right) z^\Lambda dz_l + \sum_{j \neq l} \lambda_j z^{\Lambda_{j,l}} dz_j. \tag{4.40}$$

Suppose that $e_\gamma = z^{\Lambda^\gamma} dz_{l_\gamma}$, $\Lambda^\gamma = (\lambda_1^\gamma, \ldots, \lambda_n^\gamma),\ |\Lambda_\gamma| = m,\ \gamma = 1, 2, \ldots, N$, be a basis for the space $A^2_{(1,0)}(m)$. Write

$$\widetilde{\Box}_1(e_\beta) = \sum_\gamma a_{\gamma\beta} e_\gamma. \tag{4.41}$$

The matrix representation for $\widetilde{\Box}_1$ on $A^2_{(1,0)}(m))$ is a constant sum column matrix; the sum of the entries of each column is

$$\sum_\beta^N a_{\gamma\beta} = (m+1)(n-\alpha-1), \quad N = n\binom{n+m-1}{n-1}, \tag{4.42}$$

while the diagonal entries are of the form

$$(m+1)(n-\alpha-1) + \lambda_l - m. \tag{4.43}$$

Take $\gamma \neq \beta$. Clearly, if $l_\gamma = l_\beta$ then $a_{\gamma\beta} = 0$. If $l_\gamma \neq l_\beta$ and if $\Lambda^{\beta}_{l_\gamma, l_\beta} \neq \Lambda^\gamma$, then $a_{\gamma\beta} = 0$. Finally, if $l_\gamma \neq l_\beta$ and $\Lambda^{\beta}_{l_\gamma, l_\beta} = \Lambda^\gamma$, then

$$a_{\gamma\beta} = \lambda^{\beta}_{l_\gamma} = \lambda^{\gamma}_{l_\gamma} + 1. \tag{4.44}$$

Thus, we have for each fixed γ,

$$\sum_{\beta} a_{\gamma\beta} = \sum_{\beta, l_\gamma \neq l_\beta, \Lambda^{\beta}_{l_\gamma, l_\beta} = \Lambda^\gamma} (\lambda^{\gamma}_{l_\gamma} + 1) = q_\gamma(\lambda^{\gamma}_{l_\gamma} + 1). \tag{4.45}$$

where q_γ equals the number of nonzero indices in the multi-index Λ^γ other than λ_{l_γ}; in particular, $q_\gamma \leqslant n-1$. We first consider the case $\lambda^{\gamma}_{l_\gamma} \leqslant m-2$. Then (4.43) shows that

$$\begin{aligned} \delta_\gamma &:= a_{\gamma\gamma} - \sum_{\beta \neq \gamma} a_{\gamma\beta} \\ &\geqslant ((m+1)(n-\alpha-1) + \lambda^{\gamma}_{l_\gamma} - m) - (n-1)(\lambda^{\gamma}_{l_\gamma} + 1) \\ &= -\alpha(m+1) + (n-2)(m - \lambda^{\gamma}_{l_\gamma}) \\ &\geqslant -\alpha(m+1) + 2(n-2) \\ &\geqslant 2(n-\alpha-2). \end{aligned} \tag{4.46}$$

If $\lambda^{\gamma}_{l_\gamma} = m-1$, then $q_\gamma = 1$ and in this case $\delta_\gamma = (m+1)(n-\alpha-2)$. If $\lambda^{\gamma}_{l_\gamma} = m$, then $q_\gamma = 0$ and $\delta_\gamma = (m+1)(n-\alpha-1)$. Thus, in any case

$$\delta_\gamma \geqslant 2(n-\alpha-2). \tag{4.47}$$

By theorem of Geršgorin [4], the eigenvalues of $[a_{\alpha\beta}]$ must be in the union of the circles centered at $a_{\gamma\gamma}$ with radius $R_\gamma = a_{\gamma\gamma} - \delta_\gamma$, $\gamma = 1, 2, \ldots, N$. Consequently, the eigenvalues must be larger than $2(n-\alpha-2)$. Moreover, for $m \geqslant 2$, these eigenvalues of $\widetilde{\Box}_1$ on $A^2_{1,0}(m)$ are larger than $-\alpha(m+1) \to \infty$. This shows that the inverse operator $\widetilde{N}_1$ is bounded and compact.

When $n = 2$, $2(n-\alpha-2) = -2\alpha$ is an eigenvalue and the corresponding eigenspace in $A^2_{(1,0)}(1)$ is spanned by $z_1 dz_2 - z_2 dz_1$. Thus the first positive eigenvalue in this case is

$$\lambda_1 = \min\{1-\alpha, -2\alpha\}. \tag{4.48}$$

When $n \geqslant 3$, we always have $2(n-\alpha-2) > n-\alpha-1$ since $\alpha < 0$ and thus $\lambda_1 = n-\alpha-1$. The proof is complete. ∎

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