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Twisted Milnor Hypersurfaces

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Abstract: In this paper, we study **twisted Milnor hypersurfaces** and compute their $\widehat{A}$-genus and Atiyah-Singer-Milnor α-invariant. Our tool to compute the α-invariant is Zhang's analytic Rokhlin congruence formula. We also give some applications about the geometry, more precisely, group actions and existence of Riemannian metrics of positive scalar curvature on twisted Milnor hypersurfaces.

Keywords: Twisted Milnor hypersurface, $\widehat{A}$-genus, α invariant, positive scalar curvature.

1. Introduction

1.1. Twisted Milnor hypersurfaces

Denote by H_{n_1,n_2} the Milnor hypersurface, which is the smooth hypersurface in $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$, the product of two complex projective spaces, defined by the equation

$$x_0y_0 + \cdots + x_ky_k = 0,\ k = \min\{n_1, n_2\}, \tag{1.1}$$

where $[x_0 : x_1 : \cdots : x_{n_1}]$ and $[y_0 : y_1 : \cdots : y_{n_2}]$ are the homogeneous coordinates on $\mathbb{C}P^{n_1}$ and $\mathbb{C}P^{n_2}$ respectively. Then H_{n_1,n_2} is Poincaré dual to the cohomology class $u + v \in H^2(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2})$, where u and v are the generators of $H^*(\mathbb{C}P^{n_1}, \mathbb{Z})$ and $H^*(\mathbb{C}P^{n_2}, \mathbb{Z})$ respectively. It is well-known that Milnor hypersurfaces can be used as generators in the unitary bordism ring Ω^U (cf [16, Section 4.1]).

In this paper, we consider a generalization of the Milnor hypersurfaces, namely hypersurfaces in certain classes of quasitoric manifolds with polytope being the product of two simplicies $\Delta^{n_1} \times \Delta^{n_2}$ and characteristic matrices

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being block lower triangular rather than block diagonal. For details and the background of quasitoric manifolds, see Section 2.

The quasitoric manifold discussed here can also be regarded as the projective bundle over $\mathbb{C}P^{n_1}$ with fiber $\mathbb{C}P^{n_2}$, i.e.

$$V = \mathbb{C}P(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}) \to \mathbb{C}P^{n_1},$$

where η is the tautological line bundle over $\mathbb{C}P^{n_1}$ and $\underline{\mathbb{C}}$ is the trivial line bundle. Let γ be the vertical tautological line bundle over V. Let $u = c_1(\overline{\eta}), v = c_1(\overline{\gamma}) \in H^2(V, \mathbb{Z})$ be the first Chern classes of $\overline{\eta}$ and $\overline{\gamma}$, the complex conjugations of η and γ respectively. Denote $\mathbf{I} = (i_1, \cdots, i_{n_2})$.

Definition 1.1. *Denote by $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ a smooth hypersurface in V Poincaré dual to $d_1u + d_2v$, which we call a* **twisted Milnor hypersurface**.

Remark 1.

1. *When $\mathbf{I} = \mathbf{0} = (0, \cdots, 0)$, $H^{\mathbf{0}}_{n_1,n_2}(1, 1)$ is the classical Milnor hypersurface H_{n_1,n_2}.*
2. *When $\mathbf{I} = \mathbf{0}$, $n_1 \leq n_2$, the smooth hypersurface $H^{\mathbf{0}}_{n_1,n_2}(d_1, 1)$ has a model as the zero locus of the equation*

$$x_0^{d_1} y_0 + x_1^{d_1} y_1 + \cdots + x_{n_1}^{d_1} y_{n_1} = 0,$$

which is the generalization of equation (1.1). However, for general twisted Milnor hypersurface $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$, such algebraic geometric models do not exist. Furthermore, whether the twisted Milnor hypersurface can be realized as an algebraic subvariety of V is still unknown since it is related to Hodge conjecture.

3. *$V = \mathbb{C}P(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}})$ is a quasitoric manifold over the product $\Delta^{n_1} \times \Delta^{n_2}$ of two simplices. In particular, by the classification of two stage generalized Bott manifolds in [9], V can represent all two stage Bott towers up to diffeomorphism. Furthermore, each V is also a projective toric variety [15, p. 306] and when $\mathbf{I}$ is negative, the complex structures on these projective toric varieties coincide with the natural complex structures on the projective bundles.*
4. *When $\mathbf{I} = (1, 0, \cdots, 0)$, V becomes $L(n_1, n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$. It was shown in [25] that any generator of the unitary bordism ring Ω^U can be found in $\mathbb{Z}\langle L(n_1, n_2)\rangle$.*

The main purpose of this paper is to study some index theoretical invariants of the twisted Milnor hypersurfaces $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$. More precisely, we will first pay much attention on the calculations of the $\widehat{A}$-genus and the Atiyah-Milnor-Singer α-invariant for $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ and then give some applications by applying classical results in geometry.

1.2. $\widehat{A}$-genus and Atiyah-Milnor-Singer α-invariant

Let M be a $4k$-dimensional closed oriented smooth manifold. Let the formal Chern roots of $TM \otimes \mathbb{C}$ be $\{\pm x_j, 1 \leq j \leq 2k\}$. The Hirzebruch $\widehat{A}$-genus is the characteristic number of M defined by

$$\widehat{A}(M) = \left\langle \prod_{j=1}^{2k} \frac{x_j/2}{\sinh(x_j/2)}, [M] \right\rangle.$$

If the dimension of M is not divisible by 4, define the $\widehat{A}$-genus of M to be 0. The $\widehat{A}$-genus gives a ring homomorphism $\widehat{A} : \Omega^{SO}_* \to \mathbb{Q}$.

Let M be an n-dimensional closed smooth spin manifold. The projection map $\pi : M \to pt$ induces an Umkehr homomorphism

$$\pi_! : KO(M) \to KO^{-n}(pt) = KO_n(pt),$$

which is constructed by using the Thom isomorphism for spin bundles in KO-theory. The Atiyah-Milnor-Singer α-invariant is defined to be $\alpha(M) = \pi_!(1)$. The α-invariant gives a ring homomorphism

$$\alpha : \Omega^{spin}_* \to KO_*(pt)$$

([5, §6, Chapter V], c.f. [23, §16, Chapter III]).

For spin manifolds, the $\widehat{A}$-genus and α-invariant are the topological indices of the Atiyah-Singer Dirac operators. By the Bott periodicity, KO-theory is 8 periodic. One has

$$KO_n(pt) = \begin{cases} 0, & \text{for } n \equiv 3,5,6,7 \bmod 8, \\ \mathbb{Z}, & \text{for } n \equiv 0,4 \bmod 8, \\ \mathbb{Z}_2, & \text{for } n \equiv 1,2 \bmod 8. \end{cases}$$

and

$$\alpha(M) = \begin{cases} \widehat{A}(M), & \text{for } n \equiv 0 \bmod 8, \\ \frac{1}{2}\widehat{A}(M) & \text{for } n \equiv 4 \bmod 8. \end{cases}$$

See [23, §7, Chapter II] for details.

The $\widehat{A}$-genus and α-invariants have profound applications in geometry. Atiyah and Hirzebruch [1] proved that if a compact group acts non-trivially on a compact spin manifold, then the equivariant index of the spin Dirac operator vanishes, and in particular, the $\widehat{A}$-genus of the compact manifold vanishes. Gromov-Lawson [14] proved that a simply connected closed non-spin manifold of dimension ≥ 5 always carries a Riemannian metric of positive scalar curvature. In the spin case, it is well known that the α-invariant vanishes when the manifolds carry a Riemannian metric of positive scalar curvature (due to Lichnerowicz [22] in dimension $4k$ and Hitchin [18] in dimension $8k+1, 8k+2$). Stolz [27] proved that a simply connected, closed spin manifold of dimension≥ 5 carries a Riemannian metric of positive scalar curvature if and only if the α-invariant vanishes.

The $\widehat{A}$-genus and α-invariant have been computed on some classes of manifolds. Brooks [6] computed the $\widehat{A}$-genus of complex hypersurfaces and complete intersections in complex projective spaces. Applying his analytic Rokhlin congruence formula established in [29, 31], Zhang [32] computed the α-invariant of hypersurfaces in complex projective spaces and characterized all the $8k+2$ dimensional hypersurfaces carrying a Riemannian metric of positive scalar curvature. In [13], H. Feng, B. Zhang generalized the result in [32] to $8k+2$ dimensional complete intersections of two hypersurfaces in complex projective spaces. Applying the analytic Rokhlin congruence formula to general complete intersections and using the Seiberg-Witten invariant in dimension 4, Fang and Shao [12] gave the the necessary and sufficient condition for a complete intersection complex projective spaces carrying a Riemannian metric of positive scalar curvature. Recently, Baraglia [2] recovered the formula for the α-invariant of general complete intersections obtained in [12] with a different approach.

In this paper, we will compute the $\widehat{A}$-genus as well as the α-invariant of twisted Milnor hypersurfaces and give some applications. The real dimension of twisted Milnor hypersurfaces is always even. We will compute the $\widehat{A}$-genus without mentioning dimensions (when the dimension is not divisible by 4, the $\widehat{A}$-genus is automatically 0), and compute the α-invariant when the dimension is congruent to 2 mod 8, i.e. when $n_1 + n_2 \equiv 2 \bmod 4$.

In the famous book [16, §3.2], the $\widehat{A}$-genus and the elliptic genus of Milnor hypersurfaces have been computed by using the **universal genus**. The universal genus method in [16] does not work for the computation of the $\widehat{A}$-genus of twisted Milnor hypersurfaces due to the twistings. We will directly compute it here by combining the characteristic functions and the twisting information. To compute the α-invariant, our main tool is Zhang's analytic

Rokhlin congruence formula. In the application to the existence of Riemannian metric of positive scalar curvature on twisted Minor hypersurfaces, we express the α-invariants by using the dyadic expansion following Zhang [32]. During the calculation, we discover a very interesting number $A(n, l)$, which is closely related to several classical numbers in number theory. These are summarized in Appendix A.

In a forthcoming paper, we will study the elliptic genus and Witten genus of the twisted Milnor hypersurfaces.

1.3. Main results

Given $\mathbf{I} = (i_1, \cdots, i_{n_2})$. Denote $\sigma_1 = \sum_{j=1}^{n_2} i_j$. Set

(1.2)

$$F_{n_1,n_2,\mathbf{I}}(d_1, d_2) = \sum_{\substack{0\le r\le n_2 \\ \forall 1\le j\le r \\ l_j\ge 1, \sum_{j=1}^r l_j\le n_1, 0\le m_j\le l_j \\ 1\le s_1<s_2<\cdots\, s_r\le n_2}} (-1)^{\sum_{j=1}^r m_j}\binom{\vec{l}}{\vec{m}}\binom{\frac{d_1+n_1-1+\sigma_1}{2}-\vec{s}\cdot\vec{m}}{n_1}\binom{\frac{d_2+n_2-1}{2}+\sum_{j=1}^r l_j - r}{n_2+\sum_{j=1}^r l_j},$$

where

$$\vec{l} = (l_1, \cdots, l_r), \vec{m} = (m_1, \cdots, m_r), \vec{s} = (i_{s_1}, \cdots, i_{s_r})$$

and

$$\binom{\vec{l}}{\vec{m}} := \binom{l_1}{m_1}\cdots\binom{l_r}{m_r},\ \ \vec{s}\cdot\vec{m} := \sum_{j=1}^r i_{s_j} m_j.$$

Theorem 1.2 (Theorem 3.6)**.** *For $n_1 + n_2 \equiv 1 \mod 2$, one has*

$$\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)) = F_{n_1,n_2,\mathbf{I}}(d_1, d_2) - F_{n_1,n_2,\mathbf{I}}(-d_1, -d_2). \tag{1.3}$$

Since V is a complex manifold, it carries a canonical spin^c structure.

Proposition 1.3 (Proposition 2.5)**.** *If there exist $k_1, k_2 \in \mathbb{Z}$ such that*

$$d_1 = 2k_1 + n_1 + 1 - \sigma_1, d_2 = 2k_2 + n_2 + 1,$$

then $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ is a spin manifold carrying the induced spin structure from the spin^c structure of V.

Convention Throughout this paper, when we say $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ is spin, it always means $H^{\mathbf{I}}_{n_1,n_2}-$ (d_1,d_2) carries the induced spin structure from the spin^c structure of V.

Remark 2. *If* $\mathbf{I}$ *is negative and* d_1, d_2 *are positive, then* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *is a hyperplane section of* V*. By the Lefschetz hyperplane theorem (c.f. [21, Chapter V]),* $i^*: H^2(V,\mathbb{Z}) \to H^2(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2),\mathbb{Z})$ *is an isomorphism for* $n_1+n_2>3$*. Then Proposition 1.3 actually gives the necessary and suffcient condition for* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *to be spin. Furthermore, the embedding* i *induces an isomorphism on their fundamental groups. Since* $\pi_1(V)=0$, $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *is simply connected, and hence the spin structure is unique.*

Theorem 1.4 (Theorem 3.8)**.** *Assume that* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *is spin and* $n_1+n_2=4k+2$ *(i.e.,* $\dim H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)\equiv 2 \bmod 8$*). Then one has*

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv F_{n_1,n_2,\mathbf{I}}(d_1,d_2) \bmod 2. \tag{1.4}$$

Corollary 1.5 (Example 3.1, 3.2)**.** *Take* $\mathbf{I}=\mathbf{0}$*. For* $n_1+n_2\equiv 1 \bmod 2$*, we have*

$$\widehat{A}(H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)) = 2\binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}; \tag{1.5}$$

and when $H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)$ *is spin and* $n_1+n_2\equiv 2 \bmod 4$*, we have*

$$\alpha(H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)) \equiv \binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2} \bmod 2. \tag{1.6}$$

In particular, we have $\widehat{A}(H^{\mathbf{0}}_{n_1,n_2}(1,1))=0$*, which coincides with the result in [16, p. 40] that the* $\widehat{A}$*-genus of a Milnor hypersurface always vanishes.*

Corollary 1.6 (Example 3.3)**.** $H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1)$ *is spin for* $j\equiv 0 \bmod 2$*, and*

$$\widehat{A}(H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1)) \neq 0 \iff j\neq 0,-2.$$

Remark 3. *From this corollary, we see that for the non-twisted case we have* $\widehat{A}(H^{\mathbf{0}}_{2,n_2}(1,n_2+1))=0$*, for the twisted case we have* $\widehat{A}(H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1))\neq 0 (j\neq -2)$*. This provides a good example and illustrate the difference between twisted Milnor hypersurfaces and the untwisted Milnor hypersurfaces. Actually, we give a family of twisted Milnor hypersurfaces with non vanishing* $\widehat{A}$*-genus, see Subsection 3.4.*

Corollary 1.7 (Example 3.5)**.** *For* $n_1 = 1, n_2 \equiv 1 \bmod 4$,

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv \binom{k_1+1}{1}\binom{n_2+k_2}{n_2} + \sigma_1\binom{n_2+k_2}{n_2+1} \bmod 2.$$

Corollary 1.8 (Example 3.6)**.** *For* $n_1 = 2, n_2 \equiv 0 \bmod 4$, *denote* $\sigma_2 = \sum_{1\le j<k\le n_2} i_j \cdot i_k$,

$$\begin{aligned}&\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))\\ &\quad\equiv \binom{k_1+2}{2}\binom{n_2+k_2}{n_2} + \left(-\frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\right)\binom{n_2+k_2}{n_2+1}\\ &\qquad + (\sigma_1^2-2\sigma_2)\binom{n_2+k_2+1}{n_2+2} + \sigma_2\binom{n_2+k_2}{n_2+2} \bmod 2.\end{aligned}$$

The Atiyah-Hirzebruch vanishing theorem [1] asserts that if the circle S^1 acts nontrivially on a connected spin manifold M, then $\widehat{A}(M) = 0$. We therefore have

Corollary 1.9. *If* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *is spin and* $F_{n_1,n_2,\mathbf{I}}(d_1,d_2) - F_{n_1,n_2,\mathbf{I}}(-d_1, -d_2)$ *does not vanish, then* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *does not admit a nontrivial circle action.*

Based on formula (1.2), we have

Corollary 1.10 (Example 3.4)**.** *If* n_2 *is even, then* $\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(d_1,1)) = 0, \ \forall\, \mathbf{I}$.

In fact, when $\mathbf{I} = 0$, Corollary 1.10 has a geometric interpretation. Observe that if $n_1 \le n_2$ and $d_2 = 1$, the smooth hypersurface $H^{\mathbf{0}}_{n_1,n_2}(d_1,1)$ can be described as the zero locus of equation

$$x_0^{d_1}y_0 + x_1^{d_1}y_1 + \cdots + x_{n_1}^{d_1}y_{n_1} = 0,$$

where $[x_0 : x_1 : \cdots : x_{n_1}]$ and $[y_0 : y_1 : \cdots : y_{n_2}]$ are the homogeneous coordinates on $\mathbb{C}P^{n_1}$, reps. $\mathbb{C}P^{n_2}$. There exists a natural circle action on $H^{\mathbf{0}}_{n_1,n_2}(d_1,1)$ defined by

$$\lambda \cdot [x_0 : x_1 : \cdots : x_{n_1}] = [x_0 : \lambda x_1 : \cdots : \lambda x_{n_1}]$$

$$\lambda \cdot [y_0 : y_1 : \cdots : y_{n_2}] = [y_0 : \lambda^{-d_1} y_1 : \cdots : \lambda^{-d_1} y_{n_2}]$$

where $\lambda \in S^1$.

Naturally we would like to ask

Problem 1. *Does there exist a non-trivial circle action on* $H^{\mathbf{I}}_{n_1,n_2}(d_1,1)$ *for* $\mathbf{I}\neq\mathbf{0}$*?*

Assume $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ is spin and $\dim H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)\geq 5$. By [19, Theorem A], for any oriented manifold M^{2n}, any codimension 2 homology class is represented by a submanifold $K\subset M$, and (M,K) is n-connected. In the following, let us assume $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ to be simply connected.

Applying Stolz theorem [27], a spin $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ admits a Riemannian metric of positive scalar curvature (PSC) if and only if $\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))=0$.

Corollary 1.11. *The spin twisted Milnor hypersurface* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *admits a Riemannian metric of PSC iff* $F_{n_1,n_2,\mathbf{I}}(d_1,d_2)\equiv 0 \mod 2$*.*

Motivated by Zhang's results in [32], we use the dyadic expansion coefficients to characterize the existence of Riemannian metric of PSC for twisted Milnor hypersurfaces. Let $a_k(n)$ be the coefficient in the dyadic expansion of $n\in\mathbb{Z}$:

$$n=a_0(n)+a_1(n)2^1+a_2(n)2^2+\cdots+a_k(n)2^k, \exists\ k\in\mathbb{Z}.$$

We are able to give the characterisation for the existence of PSC on two types of twisted Milnor hypersurfaces.

Corollary 1.12 (Corollary 4.9)**.** *Assume* $n_1=1$*,* $n_1+n_2\equiv 2\mod 4$*, and* $k_1=-\frac{n_1+1-d_1-\sigma_1}{2}$*,* $k_2=-\frac{n_2+1-d_2}{2}$ *are integers. Then the spin* $H^{\mathbf{I}}_{1,n_2}(d_1,d_2)$ *does not admit a Riemannian metric of PSC if and only if one of the following conditions holds,*

- $k_2\geq 0$, $k_2\equiv 0\mod 4$, $k_1\equiv 0\mod 2$, *and* $\forall\ i, a_i([\frac{k_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$;
- $k_2\geq 0$, $k_2\equiv 1\mod 4$, $\sigma_1\equiv 1\mod 2$, *and* $\forall\ i, a_i([\frac{k_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$;
- $k_2\geq 0$, $k_2\equiv 2\mod 4$, $k_1+\sigma_1\equiv 0\mod 2$, *and* $\forall\ i, a_i([\frac{k_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$;
- $k_2\leq -n_2-1$, $-k_2\equiv 0\mod 4$, $k_1\equiv 0\mod 2$, *and* $\forall\ i, a_i([\frac{-k_2-1-n_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$;
- $k_2\leq -n_2-1$, $-k_2\equiv 2\mod 4$, $k_1+\sigma_1\equiv 0\mod 2$, $\forall\ i, a_i([\frac{-k_2-1-n_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$;
- $k_2\leq -n_2-1$, $-k_2\equiv 3\mod 4$, $\sigma_1\equiv 1\mod 2$, *and* $\forall\ i, a_i([\frac{-k_2-1-n_2}{4}])+a_i([\frac{n_2}{4}])\leq 1$.

Corollary 1.13 (Corollary 4.10)**.** *Assume* $n_1=2$*,* $n_1+n_2\equiv 2\mod 4$*, and* $k_1=-\frac{n_1+1-d_1-\sigma_1}{2}$*,* $k_2=-\frac{n_2+1-d_2}{2}$ *are integers. Then the spin* $H^{\mathbf{I}}_{2,n_2}(d_1,d_2)$ *does not admit a Riemannian metric of PSC if and only if one of the following condition holds*

- $k_2\geq 0$, $k_2\equiv 0\mod 4$, $k_1\equiv 0$ *or* $1\mod 4$, $\forall\ i, a_i([\frac{k_2}{4}])+a_i(\frac{n_2}{4})\leq 1$;

- $k_2 \geq 0$, $k_2 \equiv 1 \bmod 4$, $\binom{k_1+2}{2}+\frac{\sigma_1^2-2\sigma_2}{2}+\frac{(2k_1+3)\sigma_1}{2} \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}])+a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \geq 0$, $k_2 \equiv 2 \bmod 4$, $\binom{k_1+2}{2}+\sigma_1^2-\sigma_2 \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}])+a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \geq 0$, $k_2 \equiv 3 \bmod 4$, $\binom{k_1+2}{2} + \frac{\sigma_1(2k_1+3-\sigma_1)}{2} \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 = -n_2 - 1$, $\frac{(k_1+1)(k_1+2)}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \bmod 2$;
- $k_2 \leq -n_2-2$, $-k_2 \equiv 0 \bmod 4$, $k_1 \equiv 0$ or $1 \bmod 4$, $\forall\, i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \leq -n_2 - 2$, $-k_2 \equiv 1 \bmod 4$, $\binom{k_1+2}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \bmod 2$, $\forall\, i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \leq -n_2 - 2$, $-k_2 \equiv 2 \bmod 4$, $\binom{k_1+2}{2} + \sigma_1^2 - \sigma_2 \equiv 1 \bmod 2, \forall\, i$, $a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \leq -n_2-2$, $-k_2 \equiv 3 \bmod 4$, $\binom{k_1+2}{2} + \frac{(2k_1+3+3\sigma_1)\sigma_1}{2} - \sigma_2 \equiv 1 \bmod 2$, $\forall\, i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;

This paper is organized as follows. In Section 2, we give some topological preliminaries on twisted Milnor hypersurfaces. In Section 3, we give explicit formulas for $\widehat{A}$-genus and α invariant of $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ by $F_{n_1,n_2,\mathbf{I}}(d_1,d_2)$. During the computation, we use the binomial number $A(n,l)$ and its properties. In Section 4, we give some applications of these two index invariants in understanding the geometry of twisted Milnor hypersurfaces. We collect some interesting relations between the number $A(n,l)$ with four classical numbers as well as some properties about the parity of combinatorial numbers in Appendix A and Appendix B.

2. Topological preliminaries on twisted Milnor hypersurfaces

Let P^n be a simple convex polytope of dimension n. A **quasitoric manifold** [11] M^{2n} over P^n is a smooth T^n-manifold M^{2n} satisfying the following two conditions:

1. the action is locally standard;
2. there is continuous projection $\pi : M^{2n} \to P^n$ whose fibers are T^n-orbits.

Note that 2 says that the orbit space of T^n on M^{2n} is homeomorphic to P^n.

Denote by $\mathcal{F}$ the set of codimension one faces of P^n. For every $F \in \mathcal{F}$, let $x \in \mathrm{Int}(F)$. The isotropy group of x is a codimension one subgroup of T^n; this isotropy subgroup is determined by a primitive vector $v \in \mathbb{Z}^n$; thus defines a **characteristic function** $\lambda : \mathcal{F} \to \mathbb{Z}^n$, and we call its corresponding matrix the **characteristic matrix** (more details can be found in [11].)

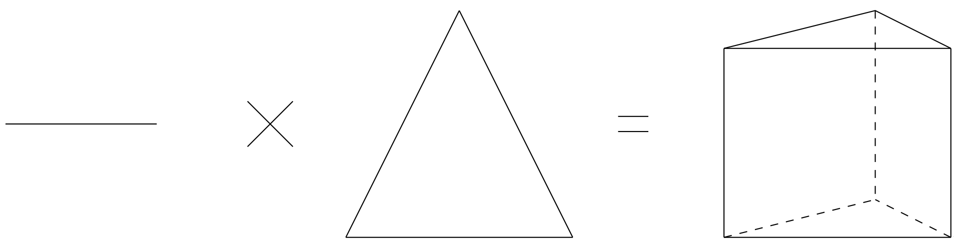
For example, identify T^n with $T^{n+1}/\langle(g,\cdots,g), g\in T^n\rangle$ and T^n acts on $\mathbb{C}P^n$ in the usual manner, the quotient space is then the simplex Δ^n, where

$$\Delta^n=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|x_i\geq 0,\ i=1,2,\cdots,n,\ x_1+\cdots+x_n\leq 1\}.$$

Then $\mathbb{C}P^n$ can be viewed as a quasitoric manifold over the simplex Δ^n with characteristic matrix:

$$\begin{pmatrix}1 & 0 & \cdots & 0 & -1\\ 0 & 1 & \cdots & & -1\\ \vdots & & \ddots & & \vdots\\ 0 & & & 1 & -1\end{pmatrix}.$$

Now let's consider more complicated combinatorics: the product of two simplices $\Delta^{n_1}\times\Delta^{n_2}$. For example, $\Delta^1\times\Delta^2$ looks like



The product of two projective spacess $\mathbb{C}P^{n_1}\times\mathbb{C}P^{n_2}$ is a quasitoric manifold over the polytope $\Delta^{n_1}\times\Delta^{n_2}$ with the block diagonal characteristic matrix:

$$\begin{pmatrix}\overbrace{1 \quad 0 \quad \cdots \quad 0}^{n_1} & -1 & & & \\ 0 \quad 1 \quad \cdots \quad\ \ & -1 & & & \\ \vdots \qquad \ddots \qquad & \vdots & & & \\ 0 \qquad\qquad 1 & -1 & & & \\ & & 1 & & -1\\ & & & \ddots & \cdots\\ & & & 1 & -1\end{pmatrix}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} n_2.$$

Twisting the block diagonal characteristic matrix to be a block lower triangular characteristic matrix can give interesting new quasitoric manifolds. Let V be a quasitoric manifold, whose corresponding polytope is $\Delta^{n_1}\times\Delta^{n_2}$ while the characteristic matrix is

$$\begin{pmatrix} \overbrace{1 \quad 0 \quad \cdots \quad 0}^{n_1} & -1 & & & \\ 0 \quad 1 \quad \cdots \quad & -1 & & & \\ \vdots \quad \quad \ddots \quad & \vdots & & & \\ 0 \quad \quad \quad \quad 1 & -1 & & & \\ & i_1 & 1 & & -1 \\ & \vdots & & \ddots & \cdots \\ & i_{n_2} & & 1 & -1 \end{pmatrix} \Bigg\} n_2.$$

V can also be considered as the total space of the projectivisation

$$\mathbb{C}P(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}) \to \mathbb{C}P^{n_1}$$

(c.f. [7, Section 7.8]), where η is the tautological line bundle over $\mathbb{C}P^{n_1}$. Denote by $\overline{\eta}$ the conjugate bundle of η. Let γ be the tautological vertical line bundle over V. The manifold V is the total space of a bundle over $\mathbb{C}P^{n_1}$ with fiber $\mathbb{C}P^{n_2}$.

The tangent bundle and cohomology ring structure of V are clear from the following theorem.

Theorem 2.1 (Borel and Hirzebruch [4])**.** *Let $p : \mathbb{C}P(\xi) \to X$ be the projectivization of a complex n-plane bundle ξ over a complex manifold X, and γ the tautological vertical line bundle over $\mathbb{C}P(\xi)$. Then there is an isomorphism of vector bundles*

$$T\mathbb{C}P(\xi) \oplus \underline{\mathbb{C}} \cong p^*TX \oplus (\overline{\gamma} \otimes p^*\xi),$$

where $\underline{\mathbb{C}}$ denotes a trivial line bundle over $\mathbb{C}P(\xi)$. Futhermore,

$$H^*(\mathbb{C}P(\xi); \mathbb{Z}) \cong H^*(X)[c_1(\overline{\gamma})]/\langle c_n(\overline{\gamma} \otimes p^*\xi)\rangle.$$

In this paper, $X = \mathbb{C}P^{n_1}, \xi = \eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}$. Let $u = p^*c_1(\overline{\eta}), v = c_1(\overline{\gamma})$. By the above theorem, we have

$$TV \oplus \underline{\mathbb{C}} \cong p^*T\mathbb{C}P^{n_1} \oplus \overline{\gamma} \otimes p^*(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}),$$

$$H^*(V) \cong H^*(\mathbb{C}P^{n_1}; \mathbb{Z})[v]/c_{n_2+1}(\overline{\gamma} \otimes p^*(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}})).$$

Furthermore

$$\overline{\gamma} \otimes p^*(\eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}) = (\overline{\gamma} \otimes p^*\eta^{\otimes i_1}) \oplus \cdots \oplus (\overline{\gamma} \otimes p^*\eta^{\otimes i_{n_2}}) \oplus (\overline{\gamma} \otimes \underline{\mathbb{C}}).$$

and

$$c(\overline{\gamma}\otimes\eta^{\otimes i_1}) = 1 + c_1(\overline{\gamma}) + c_1(\eta^{\otimes i_1}) = 1 + v - i_1u.$$

Thus we have

$$c(V) = (1+u)^{n_1+1}(1+v)\prod_{j=1}^{n_2}(1+v-i_ju).$$

Therefore

$$c_1(V) = (n_1+1-\sigma_1)u + (n_2+1)v$$

and the total Pontryagin class

$$p(V) = (1+u^2)^{n_1+1}(1+(v-i_1u)^2)\cdots(1+(v-i_{n_2}u)^2)(1+v^2).$$

We also have $H^*(\mathbb{C}P^{n_1};\mathbb{Z}) \cong \mathbb{Z}[u]/\langle u^{n_1+1}\rangle$ and therefore

$$H^*(V) \cong \mathbb{Z}[u,v]/\langle u^{n_1+1}, v(v-i_1u)\cdots(v-i_{n_2}u)\rangle.$$

The following result should be known to experts, although we didn't find it in the literature. We state it here and give a proof.

Lemma 2.2.

$$\langle u^{n_1}v^{n_2}, [V]\rangle = 1.$$

Proof. There is a fibration $\mathbb{C}P^{n_2} \to V \to \mathbb{C}P^{n_1}$. By the multiplicative property of Euler characteristics for fibration [26, p. 481], we have $\chi(V) = \chi(\mathbb{C}P^{n_1})\chi(\mathbb{C}P^{n_2})$. Since $H^*(\mathbb{C}P^n,\mathbb{Z}) = \mathbb{Z}[u]/\langle u^{n+1}\rangle$ with deg $u = 2$, the odd dimension of $H^*(\mathbb{C}P^n,\mathbb{Z})$ vanishes,

$$\chi(\mathbb{C}P^n) = \sum_{i=0}^{n}(-1)^{2i}\mathrm{Rank}(H^{2i}(\mathbb{C}P^n,\mathbb{Z})) = n+1.$$

Based on $\langle c_{n_1+n_2}(V), [V]\rangle = \chi(V)$,

$$\chi(V) = (n_1+1)(n_2+1).$$

And $u^{n_1+1} = 0$, so

$$c_{n_1+n_2}(V) = (n_1+1)(n_2+1)u^{n_1}v^{n_2}.$$

Therefore

$$\langle (n_1+1)u^{n_1}(n_2+1)v^{n_2}, [V]\rangle = (n_1+1)(n_2+1)$$

and we have $\langle u^{n_1}v^{n_2}, [V]\rangle = 1$. $\square$

Since $H^{2(n_1+n_2)}(V) \cong \mathbb{Z}[u^{n_1}v^{n_2}]$, assume $u^{n_1-k}v^{n_2+k} = \beta_k u^{n_1}v^{n_2}$ for $0 \leq k \leq n_1$.

Lemma 2.3.

$$\beta_k = \sum_{\substack{\forall 1\leq i\leq n_2,\ p_i\geq 0 \\ p_1+p_2+\cdots+p_{n_2}=k}} i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}.$$

Proof. If $k = 0$, obviously $\beta_0 = 1$.

If $0 < k \leq n_1$, since $v(v - i_1u)\cdots(v - i_{n_2}u) = 0$, we have

$$\begin{aligned} 0 &= v^k(v - i_1u)\cdots(v - i_{n_2}u) \\ &= v^{n_2+k} - \sigma_1 uv^{n_2+k-1} + \cdots + (-1)^k\sigma_k u^k v^{n_2} \\ &= (\beta_k - \sigma_1\beta_{k-1} + \cdots + (-1)^k\sigma_k\beta_0)u^k v^{n_2} \end{aligned}$$

where σ_i is the i-th elementary symmetric polynomial in $\{i_1, \cdots, i_{n_2}\}$. So

$$\beta_k - \sigma_1\beta_{k-1} + \cdots + (-1)^k\sigma_k\beta_0 = 0.$$

Thus β_k is the unique solution of the equation.

On the other hand, since $u^{n_1+1} = 0$, we have

$$\begin{aligned} 1 &= (1 - i_1u)\cdots(1 - i_{n_2}u)\frac{1}{(1 - i_1u)}\cdots\frac{1}{(1 - i_{n_2})} \\ &= (\sum_{i=0}^{n_2}(-1)^i\sigma_i u^i)\prod_{j=1}^{n_2}(1 + i_ju + \cdots + (i_j)^{n_1}u^{n_1}) \\ &= (\sum_{i=0}^{n_2}(-1)^i\sigma_i u^i)\sum_{m=0}^{n_1}(\sum_{p_1+\cdots+p_{n_2}=m} i_1^{p_1}\cdots i_{n_2}^{p_{n_2}})u^m \\ &= \sum_{k=0}^{n_1}\sum_{m+i=k}((-1)^i\sigma_i\beta_m)u^{m+i} \\ &= \sum_{k=0}^{n_1}(\beta_k + (-1)\sigma_1\beta_{k-1} + \cdots + (-1)^k\sigma_k\beta_0)u^k, \end{aligned}$$

from which we see that $\beta_k = \sum_{\substack{\forall 1\leq i\leq n_2,\ p_i\geq 0 \\ p_1+p_2+\cdots+p_{n_2}=k}} i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}$ as desired. $\square$

Remark 4. *In this paper, we will mainly use the expression in Lemma 2.3. One can also give an expression of* β_k *by the elementary symmetry polynomials of* $\{i_1, \cdots, i_{n_2}\}$,

$$\beta_k = \sum_{p_1+2p_2+\cdots+n_2p_{n_2}=k} (-1)^{k-p} \sigma_1^{p_1}\sigma_2^{p_2}\cdots\sigma_{n_2}^{p_{n_2}} \frac{p!}{p_1!\cdots p_{n_2}!},$$

where $p = p_1 + \cdots + p_{n_2}$.

Corollary 2.4.

$$\langle u^{n_1-k}v^{n_2+k}, [V]\rangle = \beta_k = \sum_{\substack{\forall 1\le i\le n_2,\ p_i\ge 0\\ p_1+p_2+\cdots+p_{n_2}=k}} i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}.$$

We set

$$k_1 = \frac{d_1-(n_1+1-\sigma_1)}{2},\ \ k_2 = \frac{d_2-(n_2+1)}{2},$$

where $\sigma_1 = \sum_{j=1}^{n_2} i_j$.

Proposition 2.5. *If* $k_1, k_2 \in \mathbb{Z}$, *then* $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ *is a spin manifold carrying the induced spin structure (c.f. [31]).*

Proof. Since V is a complex manifold, it carries a canonical spin^c structure [20].

As $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ is Poincaré dual to $d_1u + d_2v$, by adjunction formula, there exists a complex line bundle ξ over V with $c_1(\xi) = d_1u+d_2v$, such that the normal bundle ν of the inclusion $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2) \overset{i}{\hookrightarrow} V$ is the pullback of ξ.

Therefore we have

$$TV|_{H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)} \cong TH^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)\oplus\nu \cong TH^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)\oplus i^*(\xi)$$

and so

$$i^*c(V) = c(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))i^*c(\nu)$$

Futhermore, we have

$$c(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) = i^*\{c(V)(1+d_1u+d_2v)^{-1}\}$$

and

$$c_1(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) = (n_1+1-\sigma_1-d_1)i^*u + (n_2+1-d_2)i^*v.$$

Thus $\omega_2 \equiv c_1 \equiv 0 \bmod 2$, $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ is spin. ∎

3. $\hat{A}$-genus and α-invariant of twisted Milnor hypersurfaces

In this section, we compute the $\widehat{A}$-genus and α-invariant of $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$. The computation involves two combinatoric numbers, including $A(n,l)$ and $F_{n_1,n_2,\mathbf{I}}(d_1,d_2)$, which we will deal with in Subsections 3.1 and 3.2 first.

3.1. The number $A(n,l)$

Define

$$A(n,l)=\begin{cases}\frac{1}{n!}\sum\limits_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^n, & 0\le l\le n;\\ 0, & \text{otherwise.}\end{cases}$$

Denote by

$$T(x)=\frac{x}{1-e^{-x}}=\sum_{m=0}^{\infty}\frac{B_m(1)}{m!}x^m,$$

the Todd series (c.f. [10]). Let $T^{(n)}(x)$ be the n-th derivative of $T(x)$.

Lemma 3.1. *The following identity holds,*

$$\sum_{m=0}^{n}\frac{(-v)^m}{m!}T^{(m)}(v)=(T(v)-vT'(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}.$$

Proof. When $n=1$,

$$T(v)-vT'(v)=(T(v)-vT'(v))\{A(1,1)(-v)^0\}$$

holds since $A(1,1)=1$.

When $n=2$, since $T(v)=\frac{ve^v}{e^v-1}$, we have

$$T(v)-vT'(v)+\frac{v^2}{2}T''(v)=(T(v)-vT'(v))\{\frac{-v}{2}+T(v)\}$$

Since $A(2,1)=\frac{1}{2}$ and $A(2,2)=1$, the lemma holds.

Assume the lemma holds for any integer $\le n$. Multiply v to both sides of

$$\sum_{m=0}^{n}\frac{(-v)^m}{m!}T^{(m)}(v)=(T(v)-vT'(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\},$$

we have

$$-\sum_{m=0}^{n}\frac{(-v)^{m+1}}{m!}T^{(m)}(v)=v(T(v)-vT'(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}.$$

Differentiating the left hand side gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}v}\{vT(v)-v^2T'(v)+\cdots+(-1)^{n-1}\frac{v^n}{(n-1)!}T^{(n-1)}(v)+(-1)^n\frac{v^{n+1}}{n!}T^{(n)}(v)\}\\
=&(T(v)+vT'(v))-(2vT'(v)+v^2T''(v))+\cdots+(-1)^{n-1}\{\frac{nv^{n-1}}{(n-1)!}T^{(n-1)}(v)\\
&+\frac{v^n}{(n-1)!}T^{(n)}(v)\}+(-1)^n\left\{\frac{(n+1)v^n}{n!}T^{(n)}(v)+\frac{v^{n+1}}{n!}T^{(n+1)}(v)\right\}\\
=&T(v)-(-1+2)vT'(v)+\cdots+(-1)^n(-1+\frac{n+1}{n})\frac{v^n}{(n-1)!}T^{(n)}(v)\\
&+(-1)^n\frac{v^{n+1}}{n!}T^{(n+1)}(v)\\
=&T(v)-vT'(v)+\cdots+(-1)^n\frac{v^n}{n!}T^{(n)}(v)+(-1)^n\frac{v^{n+1}}{n!}T^{(n+1)}(v)\\
=&(T(v)-vT'(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}-\frac{(-v)^{n+1}}{n!}T^{(n+1)}(v),
\end{aligned}$$

which equals to the differentiation of the right hand side:

$$\begin{aligned}
&(T(v)-vT'(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
&-v^2T''(v)\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
&+v(T(v)-vT'(v))\frac{\mathrm{d}}{\mathrm{d}v}\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}.
\end{aligned}$$

So we get

$$\begin{aligned}
&\frac{(-v)^{n+1}}{n!}T^{(n+1)}(v)=v(vT''(v))\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
&-v(T(v)-vT'(v))\frac{\mathrm{d}}{\mathrm{d}v}\left\{\sum_{l=1}^{n}A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}.
\end{aligned}$$

Since $v^2T''(v) = (T(v) - vT'(v))(2T(v) - v - 2)$, we have

$$\begin{aligned}
&\frac{(-v)^{n+1}}{(n+1)!}T^{(n+1)}(v)\\
=&\frac{v^2T''(v)}{n+1}\left\{\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
&-\frac{v(T(v)-vT'(v))}{n+1}\cdot\frac{\mathrm{d}}{\mathrm{d}v}\left\{\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
=&\frac{(T(v)-vT'(v))(2T(v)-v-2)}{n+1}\left\{\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
&-\frac{v(T(v)-vT'(v))}{n+1}\cdot\frac{\mathrm{d}}{\mathrm{d}v}\left\{\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\right\}\\
=&\frac{T(v)-vT'(v)}{n+1}\\
&\cdot\left\{(2T(v)-v-2)\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}-v\frac{\mathrm{d}}{\mathrm{d}v}[\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}]\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&T(v)-vT'(v)+\cdots+\frac{(-v)^n}{n!}T^{(n)}(v)+\frac{(-v)^{n+1}}{(n+1)!}T^{(n+1)}(v)\\
=&(T(v)-vT'(v))\{\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\}\\
&+\frac{T(v)-vT'(v)}{n+1}\cdot\{(2T(v)-v-2)\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\\
&-v\frac{\mathrm{d}}{\mathrm{d}v}[\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}]\}\\
=&\frac{T(v)-vT'(v)}{n+1}\cdot\{(2T(v)-v+n-1)\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\\
&-v\cdot\frac{\mathrm{d}}{\mathrm{d}v}[\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}]\}\\
=&\frac{T(v)-vT'(v)}{n+1}\cdot\{(2T(v)-v+(n-1))\sum_{l=1}^{n} A(l,n)(-v)^{n-l}T(v)^{l-1}
\end{aligned}$$

$$
\begin{aligned}
&+ v\sum_{l=1}^{n} A(n,l)\cdot(n-l)(-v)^{n-l-1}T(v)^{l-1}\\
&- v\sum_{l=1}^{n} A(n,l)(-v)^{n-l}(l-1)T(v)^{l-2}T'(v)\}\\
=&\frac{T(v)-vT'(v)}{n+1}\{(2T(v)-v+(n-1))\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l-1}\\
&+\sum_{l=1}^{n}(l-n)A(n,l)\cdot(-v)^{n-l}T(v)^{l-1}\\
&- v\sum_{l=1}^{n} A(n,l)(-v)^{n-l}(l-1)T(v)^{l-2}T'(v)\}\\
=&\frac{T(v)-vT'(v)}{n+1}\cdot\{\sum_{l=1}^{n}(2T(v)-v+l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
&- vT'(v)\sum_{l=1}^{n} A(n,l)\cdot(-v)^{n-l}(l-1)T(v)^{l-2}\}.
\end{aligned}
$$

For the first part in the bracket, we have

$$
\begin{aligned}
&\sum_{l=1}^{n}(2T(v)-v+l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&2\sum_{l=1}^{n} A(n,l)(-v)^{n-l}T(v)^{l}+\sum_{l=1}^{n} A(n,l)(-v)^{n-l+1}T(v)^{l-1}\\
&+\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&\sum_{l=2}^{n+1} 2A(n,l-1)(-v)^{n-l+1}T(v)^{l-1}+\sum_{l=1}^{n} A(n,l)(-v)^{n-l+1}T(v)^{l-1}\\
&+\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&\sum_{l=1}^{n+1}(A(n,l)+2A(n,l-1))(-v)^{n-l+1}T(v)^{l-1}\\
&+\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}.
\end{aligned}
$$

For the second part in the bracket, Since $vT'(v)=T(v)(v-T(v)+1)$, we have

$$
\begin{aligned}
&-vT'(v)\sum_{l=1}^{n}A(n,l)(-v)^{n-l}(l-1)T(v)^{l-2}\\
=&(-v+T(v)-1)\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l+1}T(v)^{l-1}+\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l}\\
&+\sum_{l=1}^{n}(-1)(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l+1}T(v)^{l-1}+\sum_{l=2}^{n+1}(l-2)A(n,l-1)(-v)^{n-l+1}T(v)^{l-1}\\
&+\sum_{l=1}^{n}(-1)(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}\\
=&\sum_{l=1}^{n+1}\{(l-1)A(n,l)+(l-2)A(n,l-1)\}(-v)^{n-l+1}T(v)^{l-1}\\
&-\sum_{l=1}^{n}(l-1)A(n,l)(-v)^{n-l}T(v)^{l-1}
\end{aligned}
$$

Combining them together, we have

$$
\begin{aligned}
&T(v)-vT'(v)+\cdots+\frac{(-v)^n}{n!}T^{(n)}(v)+\frac{(-v)^{n+1}}{(n+1)!}T^{(n+1)}(v)\\
=&(T(v)-vT'(v))\sum_{l=1}^{n+1}\frac{l}{n+1}(A(n,l)+A(n,l-1))(-v)^{n-l+1}T(v)^{l-1}\\
=&\sum_{l=1}^{n+1}A(n+1,l)(-v)^{n+1-l}T(v)^{l-1}. \qquad \square
\end{aligned}
$$

3.2. The number $F_{n_1,n_2,\mathbf{I}}(d_1,d_2)$

Let

$$
j(x)=\frac{\sinh(x/2)}{x/2} \tag{3.1}
$$

be the j-function (c.f. [3, p. 167]). Denote $Q(x) := j^{-1}(x)$. It is not hard to see that

$$Q(x) = e^{-\frac{x}{2}} \frac{x}{1-e^{-x}} = e^{-\frac{x}{2}} T(x). \tag{3.2}$$

Denote

$$F_{n_1,n_2,\mathbf{I}}(d_1,d_2) = \sum_{\substack{0\le r\le n_2 \\ \forall 1\le j\le r \\ l_j\ge 1, \sum_{j=1}^r l_j \le n_1, 0\le m_j\le l_j \\ 1\le s_1<s_2<\cdots\ s_r\le n_2}} (-1)^{\sum_{j=1}^r m_j} \binom{\vec{l}}{\vec{m}} \binom{\frac{d_1+n_1-1+\sigma_1}{2} - \vec{s}\cdot\vec{m}}{n_1} \binom{\frac{d_2+n_2-1}{2} + \sum_{j=1}^r l_j - r}{n_2 + \sum_{j=1}^r l_j}, \tag{3.3}$$

where we denote $\vec{l} = (l_1, \cdots, l_r), \vec{m} = (m_1, \cdots, m_r), \vec{s} = (i_{s_1}, \cdots, i_{s_r})$ and

$$\binom{\vec{l}}{\vec{m}} := \binom{l_1}{m_1} \cdots \binom{l_r}{m_r}, \ \vec{s}\cdot\vec{m} := \sum_{j=1}^r m_j i_{s_j}. \tag{3.4}$$

Let $k_1 = \frac{d_1-n_1-1+\sigma_1}{2}, k_2 = \frac{d_2-n_2-1}{2}$, where k_1, k_2 are not necessary integer.

Proposition 3.2. *Let V, u, v, d_1, d_2 be as in Section 2. One has*

$$F_{n_1,n_2,\mathbf{I}}(d_1,d_2) = \langle Q(u)^{n_1+1} Q(v) \prod_{j=1}^{n_2} Q(v - i_j u) e^{\frac{d_1 u + d_2 v}{2}}, [V] \rangle.$$

The rest of this subsection is devoted to the proof of this proposition. We will proceed by two steps.

In the expansion of $\widehat{A}(M) \cdot e^{\frac{d_1 u + d_2 v}{2}}$ with respect to u, observe that only $n_1 + 1$ terms

$$\{u^{n_1} v^{n_2}, u^{n_1-1} v^{n_2+1}, \cdots, v^{n_2+2}\}$$

survive, as $u^m = 0, \forall\ m > n_1$.

Let γ_1, γ_2 be small circles around $u = 0$ (resp. $v = 0$), we have

$$\begin{aligned} &\langle Q(u)^{n_1+1} Q(v) \prod_{j=1}^{n_2} Q(v - i_j u) e^{\frac{d_1 u + d_2 v}{2}}, [V] \rangle \\ &= \sum_{k=0}^{n_1} \beta_k \cdot (\frac{1}{2\pi i})^2 \oint_{\gamma_1} \oint_{\gamma_2} \frac{Q(u)^{n_1+1} Q(v) \prod_{j=1}^{n_2} Q(v - i_j u) e^{\frac{d_1 u + d_2 v}{2}}}{u^{n_1-k+1} v^{n_2+k+1}} dudv. \end{aligned} \tag{3.5}$$

Recall that $d_1u + d_2v = (2k_1 + n_1 + 1 - \sigma_1)u + (2k_2 + n_2 + 1)v$ and $Q(x) = e^{-\frac{x}{2}}T(x)$. So

$$\begin{aligned}
&\langle Q(u)^{n_1+1} \prod_{j=1}^{n_2} Q(v - i_j u)Q(v)e^{\frac{d_1u+d_2v}{2}}, [V]\rangle \\
=&\langle Q(u)^{n_1+1}Q(v) \prod_{j=1}^{n_2} Q(v - i_j u)e^{\frac{d_1u+d_2v}{2}}, [V]\rangle \\
=&\langle T(u)^{n_1+1}T(v) \prod_{j=1}^{n_2} T(v - i_j u)e^{\frac{d_1u+d_2v}{2} - \frac{v}{2} - \frac{(n_1+1)u}{2} - \frac{\sum_{j=1}^{n_2}(v-i_ju)}{2}}, [V]\rangle \\
=&\langle T(u)^{n_1+1}T(v) \prod_{j=1}^{n_2} T(v - i_j u)e^{k_1u+k_2v}, [V]\rangle \\
=&\langle T(u)^{n_1+1}e^{k_1u}T(v) \prod_{j=1}^{n_2} \{T(v) - i_j uT'(v) + \cdots + \frac{(-i_j u)^{n_1}}{n_1!}T^{(n_1)}(v)\}e^{k_2v}, [V]\rangle,
\end{aligned}$$

where the last equation uses the Taylor expansion:

$$T(v - iu) \equiv T(v) + \frac{-i}{1!}T'(v)u + \cdots + \frac{(-i)^{n_1}}{n_1!}T^{(n_1)}(v)u^{n_1} \bmod u^{n_1+1}.$$

The rest will be calculated by two steps.

3.2.1. The first step In this step, to make the notations simpler, denote

$$b_m := \frac{1}{2\pi i} \oint_\gamma \frac{T(u)^{n_1+1}e^{k_1u}}{u^{m+1}}du$$

and thus

$$T(u)^{n_1+1}e^{k_1u} = 1 + b_1u + \cdots + b_m u^m + \cdots,$$

where γ is a small circle around $u = 0$, $0 \le m \le n_1$.

Lemma 3.3.

$$b_{n_1} = \frac{1}{2\pi i} \oint_\gamma \frac{T(u)^{n_1+1}e^{ku}}{u^{n_1+1}}du = \binom{n_1 + k}{n_1},$$

where $\binom{n_1+k}{n_1}$ *is the generalised binomial coefficient defined as*

$$\binom{n_1 + k}{n_1} = \frac{(n_1 + k) \cdots (k + 1)}{n_1!}, \quad \forall\ n_1 \in \mathbb{N},\ k \in \mathbb{Z}.$$

Proof.

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_\gamma \frac{T(u)^{n_1+1}e^{ku}}{u^{n_1+1}}du =&\frac{1}{2\pi i}\oint_\gamma (\frac{u}{1-e^{-u}})^{n_1+1}e^{ku}\frac{1}{u^{n_1+1}}du\\
=&\frac{1}{2\pi i}\oint_\gamma \frac{e^{(n_1+k+1)u}}{(e^u-1)^{n_1+1}}du\\
=&\frac{1}{2\pi i}\oint_\gamma \frac{(1+t)^{n_1+k}}{t^{n_1+1}}dt\ (let\ e^u=t+1)\\
=&\binom{n_1+k}{n_1}.
\end{aligned}
$$

□

Furthermore,

$$
\begin{aligned}
&\langle Q(u)^{n_1+1}\prod_{j=1}^{n_2}Q(v-i_ju)Q(v)e^{\frac{d_1u+d_2v}{2}},[V]\rangle\\
=&\langle(\sum_{m=0}^{n_1}b_mu^m)T(v)\prod_{k=1}^{n_2}\{\sum_{p_k=0}^{n_1}\frac{(-i_ku)^{p_k}}{p_k!}T^{(p_k)}(v)\}e^{k_2v},[V]\rangle\\
=&(\sum_{m=0}^{n_1}b_mu^m)T(v)\\
&\cdot\langle\{\sum_{p=0}^{n_1}u^p\sum_{\substack{p_1+\cdots+p_{n_2}=p\\ \forall 1\le k\le n_2,p_k\ge 0}}(-1)^p\cdot i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}\}e^{k_2v},[V]\rangle\\
=&\langle\sum_{m=0}^{n_1}u^m\{\sum_{p=0}^{m}b_{m-p}\sum_{\substack{p_1+\cdots+p_{n_2}=p\\ \forall 1\le k\le n_2,p_k\ge 0}}(-1)^p\cdot i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}\}\\
&\cdot T(v)e^{k_2v},[V]\rangle\\
=&\sum_{m=0}^{n_1}\beta_{n_1-m}\sum_{p=0}^{m}b_{m-p}\sum_{\substack{p_1+\cdots+p_{n_2}=p\\ \forall 1\le k\le n_2,p_k\ge 0}}(-1)^p\cdot i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}\frac{1}{2\pi i}\\
&\cdot\oint_\gamma\frac{\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}}{v^{n_2+n_1-m+1}}T(v)e^{k_2v}dv\\
=&\sum_{m=0}^{n_1}\sum_{p=0}^{m}b_{m-p}\sum_{\substack{p_1+\cdots+p_{n_2}=p\\ q_1+\cdots+q_{n_2}=n_1-m\\ \forall 1\le k\le n_2,p_k,q_k\ge 0}}i_1^{q_1}\cdots i_{n_2}^{q_{n_2}}\cdot i_1^{p_1}\cdots i_{n_2}^{p_{n_2}}\frac{(-1)^p}{2\pi i}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \oint_\gamma \frac{\{\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}\}T(v)e^{k_2v}}{v^{n_2+n_1+1-m}}dv \\
=&\sum_{m=0}^{n_1}\sum_{p=0}^{m} b_{m-p} \sum_{\substack{p_1+\cdots+p_{n_2}=p \\ q_1+\cdots+q_{n_2}=n_1-m+p \\ \forall 1\le k\le n_2, q_k\ge p_k\ge 0}} i_1^{q_1}\cdots i_{n_2}^{q_{n_2}} \frac{(-1)^p}{2\pi i} \\
&\cdot \oint_\gamma \frac{\{\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}\}T(v)e^{k_2v}}{v^{n_2+n_1+1-m}}dv \\
=&\sum_{t=0}^{n_1} b_t \sum_{m=t}^{n_1} \sum_{\substack{q_1+\cdots+q_{n_2}=n_1-t \\ \forall 1\le k\le n_2, q_k\ge 0}} \sum_{\substack{p_1+\cdots+p_{n_2}=m-t \\ p_k\le q_k, \forall 1\le k\le n_2}} i_1^{q_1}\cdots i_{n_2}^{q_{n_2}} \\
&\cdot \frac{1}{2\pi i}\oint_\gamma \frac{\{\frac{T^{(p_1)}(v)}{p_1!}\cdots\frac{T^{(p_{n_2})}(v)}{p_{n_2}!}\}(-v)^{m-t}T(v)e^{k_2v}}{v^{n_2+n_1+1-t}}dv \text{ (Let } \ t=m-p) \\
=&\sum_{t=0}^{n_1} \frac{b_t}{2\pi i} \sum_{\substack{\sum_{k=1}^{n_2} q_k=n_1-t \\ \forall 1\le k\le n_2, q_k\ge 0}} i_1^{q_1}\cdots i_{n_2}^{q_{n_2}} \sum_{m=t}^{n_1} \sum_{\substack{\sum p_k=m-t \\ p_k\le q_k}} \\
&\cdot \oint_\gamma \frac{\{\frac{(-v)^{p_1}T^{(p_1)}(v)}{p_1!}\cdots\frac{(-v)^{p_{n_2}}T^{(p_{n_2})}(v)}{p_{n_2}!}\}T(v)e^{k_2v}}{v^{n_2+n_1+1-t}}dv.
\end{aligned}
$$

Note that for fixed integers $q_1,\cdots,q_{n_2}$ with $\sum_{k=1}^{n_2} q_k = n_1-t$, we have

$$
\begin{aligned}
&\sum_{m=t}^{n_1} \sum_{\substack{\sum p_k=m-t \\ p_k\le q_k}} \frac{(-v)^{p_1}T^{(p_1)}(v)}{p_1!}\cdots\frac{(-v)^{p_{n_2}}T^{(p_{n_2})}(v)}{p_{n_2}!} \\
=&\sum_{N=0}^{n_1-t} \sum_{\substack{\sum p_k=N \\ p_k\le q_k}} \frac{(-v)^{p_1}T^{(p_1)}(v)}{p_1!}\cdots\frac{(-v)^{p_{n_2}}T^{(p_{n_2})}(v)}{p_{n_2}!} \\
=&\prod_{1\le k\le n_2}(T(v)-vT'(v)+\cdots+\frac{(-v)^{q_k}T^{(q_k)}(v)}{q_k!}) \\
=&\prod_{1\le k\le n_2}\sum_{m_k=0}^{q_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v).
\end{aligned}
$$

Thus

(3.6)

$$
\begin{aligned}
&\langle Q(u)^{n_1+1}\prod_{j=1}^{n_2}Q(v-i_ju)Q(v)e^{\frac{d_1u+d_2v}{2}},[V]\rangle\\
&=\sum_{t=0}^{n_1}b_t\sum_{\substack{\sum q_k=n_1-t\\ \forall 1\le k\le n_2,q_k\ge 0}} i_1^{q_1}\cdots i_{n_2}^{q_{n_2}}\frac{1}{2\pi i}\oint_\gamma\frac{\prod_{k=1}^{n_2}(\sum_{m_k=0}^{q_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v))}{v^{n_2+1+(n_1-t)}}T(v)e^{k_2v}dv\\
&=\sum_{t=0}^{n_1}b_t\sum_{r=0}^{n_2}\sum_{1\le s_1<s_2<\cdots\ s_r\le n_2} i_{s_1}^{p_1}\cdots i_{s_r}^{p_r}\frac{1}{2\pi i}\\
&\quad\cdot\oint_\gamma\frac{\prod_{k=1}^{r}(\sum_{m_k=0}^{p_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v))}{v^{n_2+1+n_1-t}}T(v)^{n_2-r+1}e^{k_2v}dv,
\end{aligned}
$$

where $\{p_1,\cdots,p_r\}$ be a positive partition of n_1-t.

Lemma 3.4. *Let $\{p_1,\cdots,p_r\}$ be a positive partition of n_1-t. Then*

$$
\frac{1}{2\pi i}\oint_\gamma\frac{\prod_{k=1}^{r}(\sum_{m_k=0}^{p_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v))}{v^{n_2+1+n_1-t}}T(v)^{n_2+1-r}e^{k_2v}dv
$$

$$
=(-1)^{n_1-t}\sum_{\substack{\forall 1\le k\le r,1\le l_k\le p_k\\ l=l_1+\cdots+l_r}}A(p_1,l_1)\cdots A(p_r,l_r)(-1)^l\binom{n_2+l+k_2-r}{n_2+l}.
$$

Proof. By Lemma 3.1,

$$
\sum_{m_k=0}^{p_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v)=(T(v)-vT'(v))(\sum_{l_k=1}^{p_k}A(p_k,l_k)(-v)^{p_k-l_k}T(v)^{l_k-1}).
$$

Thus

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_\gamma\frac{\prod_{k=1}^{r}(\sum_{m_k=0}^{p_k}\frac{(-v)^{m_k}}{m_k!}T^{(m_k)}(v))}{v^{n_2+1+p}}T(v)^{n_2+1-r}e^{k_2v}dv\\
=&\frac{1}{2\pi i}\oint_\gamma\frac{\prod_{k=1}^{r}(\sum_{l_k=1}^{p_k}A(p_k,l_k)(-v)^{p_k-l_k}T(v)^{l_k-1})}{v^{n_2+1+p}}(T(v)-vT'(v))^rT(v)^{n_2+1-r}e^{k_2v}dv
\end{aligned}
$$

$$= \sum_{\substack{\forall 1\le k\le r, 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} \frac{1}{2\pi i}\oint_\gamma \frac{A(p_1,l_1)\cdots A(p_r,l_r)(-v)^{p-l}T(v)^{l-r}}{v^{n_2+1+p}}(T(v)-vT'(v))^r \cdot T(v)^{n_2+1-r}e^{k_2 v}dv$$

$$= \sum_{\substack{\forall 1\le k\le r, 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} A(p_1,l_1)\cdots A(p_r,l_r)\frac{(-1)^{p-l}}{2\pi i}\oint_\gamma \frac{(T(v)-vT'(v))^r T(v)^{n_2+1+l}e^{k_2 v}}{T(v)^{2r}v^{n_2+1+l}}dv$$

$$= \sum_{\substack{\forall 1\le k\le r, 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} A(p_1,l_1)\cdots A(p_r,l_r)\frac{(-1)^{p-l}}{2\pi i}\oint_\gamma \frac{(\frac{v^2e^v}{(e^v-1)^2})^r T(v)^{n_2+1+l}e^{k_2 v}}{(\frac{ve^v}{e^v-1})^{2r}v^{n_2+1+l}}dv$$

$$= \sum_{\substack{\forall 1\le k\le r, 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} A(p_1,l_1)\cdots A(p_r,l_r)(-1)^{p-l}\binom{n_2+l+k_2-r}{n_2+l}. \text{ (By Lemma 3.3)}$$

□

To sum up

$$\langle Q(u)^{n_1+1}\prod_{j=1}^{n_2}Q(v-i_ju)Q(v)e^{\frac{d_1u+d_2v}{2}},[V]\rangle$$

$$=\sum_{t=0}^{n_1} b_t \sum_{\substack{0\le r\le n_2\\ p=p_1+\cdots+p_r=n_1-t\\ \forall 1\le k\le r, p_k\ge 1\\ 1\le s_1<s_2<\cdots\ s_r\le n_2}} i_{s_1}^{p_1}\cdots i_{s_r}^{p_r}(-1)^p \cdot \sum_{\substack{\forall 1\le k\le r\\ 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} A(p_1,l_1)\cdots A(p_r,l_r)(-1)^l\binom{n_2+k_2+l-r}{n_2+l}$$

$$= \sum_{\substack{0\le r\le n_2\\ p=p_1+\cdots+p_r\le n_1\\ \forall 1\le k\le r, p_k\ge 1\\ 1\le s_1<s_2<\cdots\ s_r\le n_2}} b_{n_1-p}i_{s_1}^{p_1}\cdots i_{s_r}^{p_r}(-1)^p \cdot \sum_{\substack{\forall 1\le k\le r\\ 1\le l_k\le p_k\\ l=l_1+\cdots+l_r}} A(p_1,l_1)\cdots A(p_r,l_r)(-1)^l\binom{n_2+k_2+l-r}{n_2+l}$$

$$= \sum_{\substack{0\le r\le n_2\\ \forall 1\le k\le r, 1\le l_k\\ l=l_1+\cdots+l_r\le n_1\\ 1\le s_1<s_2<\cdots\ s_r\le n_2}} (-1)^l\binom{n_2+k_2+l-r}{n_2+l}$$

$$\cdot \sum_{\substack{p_1+\cdots+p_r\le n_1\\ \forall 1\le k\le r, p_k\ge l_k}} b_{n_1-p}(-i_{s_1})^{p_1}A(p_1,l_1)\cdots(-i_{s_r})^{p_r}A(p_r,l_r).$$

3.2.2. The second step

Lemma 3.5. *For any fixed* $\{l_1,\cdots,l_r,s_1,\cdots,s_r\}$*, let* $l=l_1+\cdots+l_r$*, we have*

$$\begin{aligned}&\sum_{\substack{p_1+\cdots+p_r\le n_1\\ p_k\ge l_k,\forall 1\le k\le r}} b_{n_1-p}(-i_{s_1})^{p_1}A(p_1,l_1)\cdots(-i_{s_r})^{p_r}A(p_r,l_r)\\ =&\sum_{\substack{1\le j\le r\\ 0\le m_j\le l_r\\ m=m_1+\cdots+m_r}}(-1)^{l-m}\binom{\vec{l}}{\vec{m}}\binom{n_1+k_1-\vec{s}\cdot\vec{m}}{n_1},\end{aligned}$$

where $\vec{l}=(l_1,\cdots,l_r),\vec{m}=(m_1,\cdots,m_r),\vec{s}=(i_{s_1},\cdots,i_{s_r})$ *and* $\binom{\vec{l}}{\vec{m}}:=\binom{l_1}{m_1}\cdots\binom{l_r}{m_r}$*,* $\vec{s}\cdot\vec{m}:=\sum_{j=1}^r m_j\cdot i_{s_j}$*.*

Proof.

$$b_{n_1-p}=\frac{1}{2\pi i}\oint_\gamma\frac{T(u)^{n_1+1}e^{k_1u}}{u^{n_1-p+1}}du=\frac{1}{2\pi i}\oint_\gamma\frac{u^{p_1}\cdots u^{p_r}T(u)^{n_1+1}e^{k_1u}}{u^{n_1+1}}du.$$

Thus

$$\begin{aligned}&\sum_{\substack{p_1+\cdots+p_r\le n_1\\ p_k\ge l_k,\forall 1\le k\le r}} b_{n_1-p}\cdot(-i_{s_1})^{p_1}A(p_1,l_1)\cdots(-i_{s_r})^{p_r}A(p_r,l_r)\\ =&\sum_{\substack{p_1+\cdots+p_r\le n_1\\ p_k\ge l_k,\forall 1\le k\le r}}\frac{1}{2\pi i}\oint_\gamma\frac{(-i_{s_1}u)^{p_1}A(p_1,l_1)\cdots(-i_{s_r}u)^{p_r}A(p_r,l_r)}{u^{n_1+1}}T(u)^{n_1+1}e^{k_1u}du.\end{aligned}$$

Since $(-i_{s_k}u)^{p_k}A(l_k,p_k)=0,\ p_k>n_1$

$$\begin{aligned}=&\sum_{p_k\ge l_k,\forall 1\le k\le r}\frac{1}{2\pi i}\oint_\gamma\frac{(-i_{s_1}u)^{p_1}A(p_1,l_1)\cdots(-i_{s_r}u)^{p_r}A(p_r,l_r)}{u^{n_1+1}}T(u)^{n_1+1}e^{k_1u}du\\ =&\frac{1}{2\pi i}\oint_\gamma\frac{\prod_{j=1}^r\{\sum_{m_j=l_j}^\infty(-i_{s_j}u)^{m_j}A(m_j,l_j)\}}{u^{n_1+1}}T(u)^{n_1+1}e^{k_1u}du.\end{aligned}$$

By Lemma A.1,

$$
\begin{aligned}
&\sum_{m=l}^{\infty} u^m A(m,l) = (e^u - 1)^l \\
=&\frac{1}{2\pi i}\oint_{\gamma} \frac{\prod_{j=1}^{r}(e^{-i_{s_j}u}-1)^{l_j}}{u^{n_1+1}} T(u)^{n_1+1} e^{k_1 u} du \\
=&\frac{1}{2\pi i}\oint_{\gamma} \frac{\prod_{j=1}^{r}\sum_{m_j=0}^{l_j}(-1)^{l_j-m_j}\binom{l_j}{m_j}e^{-m_j i_{s_j}u}}{u^{n_1+1}} T(u)^{n_1+1} e^{k_1 u} du \\
=&\sum_{m_1=0}^{l_1}\cdots\sum_{m_r=0}^{l_r}(-1)^{\sum_{j=1}^{r} l_j - m_j}\frac{\binom{l_1}{m_1}\cdots\binom{l_r}{m_r}}{2\pi i}\oint_{\gamma}\frac{e^{-(\sum_{j=1}^{r} m_j i_{s_j})u}}{u^{n_1+1}} T(u)^{n_1+1} e^{k_1 u} du \\
=&\sum_{\substack{1\le j\le r \\ 0\le m_j\le l_r \\ m=\sum_{j=1}^{r} m_j}} (-1)^{l-m}\binom{\vec{l}}{\vec{m}}\binom{n_1+k_1-\vec{s}\cdot\vec{m}}{n_1}. \text{ (By Lemma 3.3)}
\end{aligned}
$$

□

Combining Lemma 3.4 and Lemma 3.5, we deduce Proposition 3.2.

3.3. $\widehat{A}$-genus and α-invariant

Recall that in Section 2, we have shown that

$$c(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) = i^*(c(V)(1+d_1u+d_2v)^{-1}),$$

$$c(V) = (1+u)^{n_1+1}(1+v)\prod_{j=1}^{n_2}(1+v-i_ju).$$

Therefore

$$p(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) = i^*\{(1+u^2)^{n_1+1}(1+v^2)\prod_{j=1}^{n_2}(1+(v-i_ju)^2)(1+(d_1u+d_2v)^2)^{-1}\}$$

The characteristic power series of the $\widehat{A}$-genus is just $\frac{\sqrt{z}/2}{sinh(\sqrt{z}/2)} = Q(x)$ for $z = x^2$.

For $n_1+n_2 \equiv 1 \bmod 2$, by Poincaré duality and Proposition 3.2, one has

(3.7)
$$
\begin{aligned}
&\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))\\
=&\langle Q(u)^{n_1+1}Q(v)\prod_{j=1}^{n_2} Q(v-i_ju)\cdot Q(d_1u+d_2v)^{-1}, [V]\cap(d_1u+d_2v)\rangle\\
=&\langle Q(u)^{n_1+1}Q(v)\prod_{j=1}^{n_2} Q(v-i_ju)\cdot \frac{e^{d_1u+d_2v}-1}{(d_1u+d_2v)e^{d_1u+d_2v}}e^{\frac{d_1u+d_2v}{2}}(d_1u+d_2v), [V]\rangle\\
=&\langle Q(u)^{n_1+1}Q(v)\prod_{j=1}^{n_2} Q(v-i_ju)\cdot (e^{\frac{d_1u+d_2v}{2}}-e^{\frac{-d_1u-d_2v}{2}}), [V]\rangle\\
=&F_{n_1,n_2,\mathbf{I}}(d_1,d_2)-F_{n_1,n_2,\mathbf{I}}(-d_1,-d_2).
\end{aligned}
$$

So we have

Theorem 3.6. *For $n_1+n_2 \equiv 1 \bmod 2$ and $d_1, d_2 \in \mathbb{Z}$.*

$$\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) = F_{n_1,n_2,\mathbf{I}}(d_1,d_2)-F_{n_1,n_2,\mathbf{I}}(-d_1,-d_2).$$

The α-invariant (§6, Chapter V in [5]) is a ring homomorphism

$$\alpha : \Omega^{Spin}_* \to KO_*(pt).$$

As is well known, KO_* is 8 periodic,

$$KO_n(pt) = \begin{cases} 0, & \text{for } n \equiv 3,5,6,7 \bmod 8,\\ \mathbb{Z}, & \text{for } n \equiv 0,4 \bmod 8,\\ \mathbb{Z}_2, & \text{for } n \equiv 1,2 \bmod 8,\end{cases}$$

and

$$\alpha(M) = \begin{cases} \widehat{A}(M), & \text{for } n \equiv 0 \bmod 8,\\ \frac{1}{2}\widehat{A}(M) & \text{for } n \equiv 4 \bmod 8.\end{cases}$$

The α-invariant of an $8k+2$ dimensional manifold is the mod 2 index of Atiyah-Singer Dirac operator, which is usually difficult to compute. To perform the computation, we need to use the following analytic Rokhlin congruence formula due to Zhang.

Theorem 3.7 (Zhang [29, 30, 31], c.f. [32])**.** *Let M be a compact connected spin^c-manifold of dimension $8k+4$, ξ is a complex line bundle on M with*

$c_1(\xi) \equiv \omega_2(TM)$. B is the spin submanifold of M Poincaré dual to $c_1(\xi) \in H^2(M;\mathbb{Z})$, then B carries an induced spin structure and

$$\alpha(B) \equiv \langle \widehat{A}(M)e^{\frac{c_1(\xi)}{2}}, [M]\rangle \bmod 2.$$

Thus when $\dim_{\mathbb{R}} H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2) \equiv 2 \bmod 8$, applying Zhang's theorem, we have

$$\begin{aligned} &\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \\ \equiv&\langle \widehat{A}(V)e^{\frac{d_1u+d_2v}{2}}, [V]\rangle \\ \equiv&\langle Q(u)^{n_1+1}\prod_{j=1}^{n_2} Q(v-i_ju)Q(v)e^{\frac{d_1u+d_2v}{2}}, [V]\rangle \bmod 2. \end{aligned} \tag{3.8}$$

By Proposition 3.2, we have

Theorem 3.8. *If $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ is spin (i.e $k_1,k_2 \in \mathbb{Z}$ see Convention 1.3) and $n_1+n_2 \equiv 2 \bmod 4$, then*

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv F_{n_1,n_2,\mathbf{I}}(d_1,d_2) \bmod 2.$$

3.4. Some examples

In this subsection, we assume $n_1+n_2 \equiv 1 \bmod 2$ when discussing $\widehat{A}$-genus and $n_1+n_2 \equiv 2 \bmod 4$ when discussing α-invariant.

Observation: From formula (3.6), we can see that terms with r greater than the number of non-zero elements in $\mathbf{I}$ vanish. In particular, if $\mathbf{I} = \mathbf{0}$ or $\mathbf{I} = (j,0,\cdots,0)$, $F_{n_1,n_2,\mathbf{I}}(d_1,d_2)$ has a simpler expression. More precisely, if $\mathbf{I} = \mathbf{0}$, non-vanishing terms appears only when $r=0$, namely 3.9; if $\mathbf{I} = (j,0,\cdots,0)$, we will see that non-vanishing terms happen only when $r=0,1$. See Example 3.3.

By above observation, we have

$$F_{n_1,n_2,\mathbf{0}}(d_1,d_2) = \binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}. \tag{3.9}$$

Example 3.1. *When* $\mathbf{I} = \mathbf{0}$, $V = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$, $H^{\mathbf{0}}_{n_1,n_2}(1,1)$ *is the usual Milnor hypersurface. We have*

$$\begin{aligned}\widehat{A}(H^{\mathbf{0}}_{n_1,n_2}(1,1)) &= F_{n_1,n_2,\mathbf{0}}(1,1) - F_{n_1,n_2,\mathbf{0}}(-1,-1)\\ &= \binom{\frac{n_1}{2}}{n_1}\binom{\frac{n_2}{2}}{n_2} - \binom{\frac{n_1}{2}-1}{n_1}\binom{\frac{n_2}{2}-1}{n_2}\\ &= \binom{\frac{n_1}{2}}{n_1}\binom{\frac{n_2}{2}}{n_2} - (-1)\binom{\frac{n_1}{2}}{n_1}(-1)\binom{\frac{n_2}{2}}{n_2}\\ &= 0\end{aligned} \tag{3.10}$$

and

$$\alpha(H^{\mathbf{0}}_{n_1,n_2}(1,1)) \equiv \binom{\frac{n_1}{2}}{n_1}\binom{\frac{n_2}{2}}{n_2} \equiv 0 \bmod 2, \tag{3.11}$$

which coincides with the result in [16, p. 40] that the $\widehat{A}$*-genus of Milnor hypersurface always vanishes.*

Example 3.2. *When* $\mathbf{I} = \mathbf{0}$, $V = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$, *we have*

$$\begin{aligned}\widehat{A}(H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)) &= F_{n_1,n_2,\mathbf{I}=\mathbf{0}}(d_1,d_2) - F_{n_1,n_2,\mathbf{I}=\mathbf{0}}(-d_1,-d_2)\\ &= \binom{\frac{d_1+n_1-1}{2}}{n_1}\binom{\frac{d_2+n_2-1}{2}}{n_2} - \binom{\frac{-d_1+n_1-1}{2}}{n_1}\binom{\frac{-d_2+n_2-1}{2}}{n_2}\\ &= \binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2} - \binom{-k_1-1}{n_1}\binom{-k_2-1}{n_2}\\ &= (1-(-1)^{n_1+n_2})\binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}\\ &= 2\binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}\end{aligned}$$

When $k_1 = -\frac{n_1+1-d_1}{2}, k_2 = -\frac{n_2+1-d_2}{2}$ *are integers,* $H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)$ *is spin and*

1. $\widehat{A}(H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)) = 0 \iff -n_1 < k_1 < 0$ *or* $-n_2 < k_2 < 0$;
2. $\alpha(H^{\mathbf{0}}_{n_1,n_2}(d_1,d_2)) \equiv \binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2} \bmod 2$.

Example 3.3. *When* $\mathbf{I} = (j,0,\cdots,0)$, $j \in \mathbb{Z}$, *By the previous observation,*

$$\widehat{A}(H^{(j,0,\cdots,0)}_{n_1,n_2}(d_1,d_2)) =$$

$$\binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}-\binom{-k_1-1}{n_1}\binom{-k_2-1}{n_2}+\sum_{l=1}^{n_1}\sum_{m=0}^{l}(-1)^m\binom{l}{m}$$
$$\cdot\left\{\binom{n_1+k_1-mj}{n_1}\binom{n_2+k_2+l-1}{n_2+l}\right.$$
$$\left.-\binom{-k_1-1+(1-m)j}{n_1}\binom{-k_2-1+l-1}{n_2+l}\right\}$$

and

$$\alpha(H^{(j,0,\cdots,0)}_{n_1,n_2}(d_1,d_2))$$
$$\equiv\binom{n_1+k_1}{n_1}\binom{n_2+k_2}{n_2}$$
$$+\sum_{l=1}^{n_1}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\binom{n_1+k_1-mj}{n_1}\binom{n_2+k_2+l-1}{n_2+l}.$$

Assume $n_1=2, n_1+n_2\equiv 1 \bmod 2,\ d_1=1, d_2=n_2+1$ *(i.e.* $k_1=-1+\frac{j}{2}$, $k_2=0$*). Therefore*

$$\widehat{A}(H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1))$$
$$=\binom{1+\frac{j}{2}}{2}-\binom{-\frac{j}{2}}{2}\binom{-1}{n_2}+\sum_{m=0}^{l=1}(-1)^{m+1}\binom{(\frac{1}{2}-m)j}{2}\binom{-1}{n_2+1}$$
$$=\frac{j}{2}(\frac{j}{2}+1).$$

Thus

$$\widehat{A}(H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1)\neq 0 \iff j\neq 0\ or\ -2.$$

Note that if $j\equiv 0 \bmod 2$*, then* $H^{(j,0,\cdots,0)}_{2,n_2}(1,n_2+1)$ *is spin.*

Since $\widehat{A}(H^{\mathbf{0}}_{2,n_2}(1,n_2+1)=0$*, this provides a good example to illustrate the difference between twisted Milnor hypersurface and non twisted one.*

Example 3.4. *Assume* $d_1=d_2=1$ *and* $k_1=-\frac{n_1-\sigma_1}{2}, k_2=-\frac{n_2}{2}$ *are integers,* *then* $H^{\mathbf{I}}_{n_1,n_2}(1,1)$ *is spin and*

(3.12)

$$\begin{aligned} &F_{n_1,n_2,\mathbf{I}}(1,1)\\ =&\sum_{\substack{0\le r\le n_2\\ \forall 1\le j\le r\\ l_j\ge 1, l\le n_1, 0\le m_j\le l_j\\ 1\le s_1<s_2<\cdots\ s_r\le n_2}} (-1)^{\sum_{j=1}^r m_j}\binom{\vec{l}}{\vec{m}}\binom{n_1+\frac{\sigma_1-n_1}{2}-\vec{s}\cdot\vec{m}}{n_1}\binom{n_2+\frac{-n_2}{2}+\sum_{j=1}^r l_j-r}{n_2+\sum_{j=1}^r l_j}, \end{aligned}$$

Claim:

$$\binom{\frac{n_2}{2}+\sum_{j=1}^r l_j-r}{n_2+\sum_{j=1}^r l_j}=0.$$

In fact, let $m=\frac{n_2}{2}+\sum_{j=1}^r l_j-r,\ n=n_2+\sum_{j=1}^r l_j$, *we have*

- m *is a positive integer, since* $,l_j\ge 1$, $\sum_{j=1}^r l_j-r\ge 0$ *and* $k_2=\frac{-n_2}{2}$ *integer.*
- $m<n$, *since* $\frac{n_2}{2}+\sum_{j=1}^r l_j-r<n_2+\sum_{j=1}^r l_j \iff -r<\frac{n_2}{2}$.

Thus $\binom{m}{n}=0$.

Therefore $F_{n_1,n_2,\mathbf{I}}(1,1)=0$. *Similarly,* $F_{n_1,n_2,\mathbf{I}}(-1,-1)=0$. *Thus*

$$\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(1,1))=0,\ \alpha(H^{\mathbf{I}}_{n_1,n_2}(1,1))=0.$$

Remark 5. *More generally, as long as* n_2 *is even and* $d_2=1$, *we have* $\widehat{A}(H^{\mathbf{I}}_{n_1,n_2}(d_1,1))=0$, $\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,1))=0$.

Example 3.5. *For* $n_1=1$ *and* $n_2\equiv 1 \bmod 4$, *we have*

$$\begin{aligned} &\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))\\ \equiv&\sum_{\substack{l=1\\ 1\le s\le n_2}}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\binom{n_1+k_1-i_s m}{n_1}\binom{n_2+k_2+l-1}{n_2+l}\\ &+\binom{k_1+1}{1}\binom{n_2+k_2}{n_2} \bmod 2.\\ \equiv&\sum_{1\le s\le n_2}\binom{n_2+k_2}{n_2+1}\sum_{m=0}^{1}(-1)^m(1+k_1-i_s m)+\binom{k_1+1}{1}\binom{n_2+k_2}{n_2} \bmod 2.\\ \equiv&\binom{n_2+k_2}{n_2+1}\sum_{1\le s\le n_2}(1+k_1-(1+k_1-i_s))+\binom{k_1+1}{1}\binom{n_2+k_2}{n_2} \bmod 2. \end{aligned}$$

$$\equiv \sigma_1 \binom{n_2+k_2}{n_2+1} + \binom{k_1+1}{1}\binom{n_2+k_2}{n_2} \mod 2.$$

Example 3.6. *For $n_1 = 2$ and $n_2 \equiv 0 \mod 4$, we have*

$$\begin{aligned}
&\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2))\\
\equiv&\binom{k_1+2}{2}\binom{n_2+k_2}{n_2}\\
&+\sum_{\substack{1\le l\le 2\\ 1\le s\le n_2}}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\binom{n_1+k_1-i_s m}{n_1}\binom{n_2+k_2+l-1}{n_2+l}+\\
&\sum_{\substack{1\le l_1\le 2\\ 1\le l_2\le 2\\ 1\le s_1<s_2\le n_2}}\sum_{\substack{0\le m_1\le l_1\\ 0\le m_2\le l_2}}(-1)^{m_1+m_2}\binom{l_1}{m_1}\binom{l_2}{m_2}\binom{n_1+k_1-i_{s_1}m_1-i_{s_2}m_2}{n_1}\\
&\cdot\binom{n_2+k_2+l_1+l_2-2}{n_2+l_1+l_2}\\
\equiv&\binom{k_1+2}{2}\binom{n_2+k_2}{n_2}+\left(-\frac{\sigma_1^2-2\sigma_2}{2}+\frac{(2k_1+3)\sigma_1}{2}\right)\binom{n_2+k_2}{n_2+1}\\
&+(\sigma_1^2-2\sigma_2)\binom{n_2+k_2+1}{n_2+2}+\sigma_2\binom{n_2+k_2}{n_2+2} \mod 2.
\end{aligned}$$

4. Applications

In this section, we give some applications of $\widehat{A}$-genus and α-invariant of twisted Milnor hypersurfaces.

First, we investigate the existence of circle actions on twisted Milnor hypersurfaces.

Theorem 4.1 (Atiyah-Hirzebruch [1])**.** *If the circle S^1 acts nontrivially on a connected spin manifold M, then $\widehat{A}(M) = 0$.*

Since we have given the sufficient and necessary conditions for non vanishing $\widehat{A}$-genus about spin twisted Milnor hypersurfaces in Example 3.2 and 3.3, we have

Corollary 4.2.

(1) *Assume $\mathbf{I} = \mathbf{0}$, $n_1 + n_2$ is odd and $k_i \in \mathbb{Z}, k_i \le -n_i$ or $k_i \ge 0, i = 1, 2$. Then there does not exist non-trivial circle actions on $H^{\mathbf{0}}_{n_1,n_2}(d_1, d_2)$;*

(2) *Assume n_2 is odd and $j \neq 0, -2$ is even. Then there does not exist non-trivial circle action on the twisted spin Milnor hypersurface $H_{2,n_2}^{(j,0,\cdots,0)}(1, n_2+1)$.*

Let $a_i(n)$ be the coefficient in the dyadic expansion of $n \in \mathbb{Z}_+$:

$$n = a_0(n) + a_1(n)2^1 + a_2(n)2^2 + \cdots + a_m(n)2^m.$$

Theorem 4.3 (Lucas Theorem [24])**.** *For any $k \in \mathbb{Z}$,*

$$\binom{n+k}{n} \equiv \prod_{i=0}^{m} \binom{a_i(n+k)}{a_i(n)} \bmod 2.$$

The following properties are based on Lucas Theorem, where the case $k \geq 0$ is due to Zhang [32], and a proof is included for local completeness.

Proposition 4.4. $\binom{n+k}{n} \equiv 1 \bmod 2, n \in \mathbb{N}, k \in \mathbb{Z}$ *iff*

$$(4.1) \qquad \begin{cases} a_i(n) + a_i(k) \leq 1, \forall\ i \geq 0, & \text{for } k \geq 0; \\ a_i(-k-1-n) + a_i(n) \leq 1, \forall\ i \geq 0, & \text{for } k < -n-1. \end{cases}$$

Proof. If $k \geq 0$, by Lucas Theorem,

$$\binom{n+k}{n} \equiv 1 \mod 2 \iff \prod_{i=0}^{m} \binom{a_i(n+k)}{a_i(n)} \equiv 1 \mod 2,$$

which equals to

$$\forall\ 0 \leq i \leq m, \binom{a_i(n+k)}{a_i(n)} = 1, i.e.\ \forall\ 0 \leq i \leq m, a_i(n+k) \geq a_i(n).$$

Since $k = (n+k) - n$,

$$(n+k) = a_0(n+k)2^0 + a_1(n+k)2^1 + \cdots + a_m(n+k)2^m + \cdots,$$

we have

$$\sum_{i=0}^{m} a_i(k)2^i = \sum_{i=0}^{m} (a_i(n+k) - a_i(n))2^i.$$

In general, $a_i(k) \neq a_i(n+k) - a_i(n)$, but if $a_i(n+k) \geq a_i(n)$, and since $a_i(n+k) \leq 1$, we have $0 \leq a_i(n+k) - a_i(n) \leq 1$. And since the factorization is unique, therefore

$$0 \leq a_i(n+k) - a_i(n) \leq 1 \iff a_i(k) = a_i(n+k) - a_i(n).$$

Thus

$$\binom{n+k}{n} \equiv 1 \bmod 2 \iff \forall 0 \leq i \leq m, a_i(k) + a_i(n) = a_i(n+k) \leq 1.$$

If $k < -n$,

$$\binom{n+k}{n} = \frac{(-1)^n(-k-1)\cdots(-n-k)}{n!} = (-1)^n\binom{-k-1}{n}.$$

Thus

$$\binom{n+k}{n} \equiv \binom{-k-1}{n} \equiv 1 \bmod 2 \iff \forall\, 0 \leq i \leq k, a_i(-k-1-n)+a_i(n) \leq 1.$$

□

We give a useful tool to simplify our computation.

Proposition 4.5 (Appendix B.3)**.** *For any nonnegative integers m, n, we have*

$$\binom{m+n}{n} \equiv \begin{cases} 0 \quad \bmod 2, & n \cdot m \text{ or } [\frac{n}{2}] \cdot [\frac{m}{2}] \text{is odd;} \\ \binom{[\frac{n}{4}]+[\frac{m}{4}]}{[\frac{n}{4}]} \quad \bmod 2, & \text{otherwise.} \end{cases} \tag{4.2}$$

Theorem 4.6 (Stolz [27])**.** *Let M be a simply connected, closed manifold of dimension≥ 5. Then M does not admit a Riemannian metric of PSC iff M is a spin manifold and $\alpha(M) \neq 0$.*

By [19, Theorem A], for any oriented manifold M^{2n}, any codimension 2 homology class is represented by a submanifold $K \subset M$, and (M, K) is n-connected. In the following discussion, let us assume $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ to be simply connected.

Thus a spin $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ with dimension no less than than 5 does not admit a Riemannian metric of PSC iff $\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)) \neq 0$.

Corollary 4.7. *$H^{\mathbf{I}}_{n_1,n_2}(1, 1)$ always admits a Riemannian metric of PSC (Example 3.4). In particular, Milnor hypersurfaces always admit a Riemannian metric of PSC (Example 3.1).*

Corollary 4.8. *When $k_1 = -\frac{n_1+1-d_1}{2}, k_2 = -\frac{n_2+1-d_2}{2}$ are integers, The spin hypersurface $H^{\mathbf{0}}_{n_1,n_2}(d_1, d_2)$ of $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$ does not admit a Riemannian metric of PSC if and only if $\forall\ i,\ a_i(k_l) + a_i(n_l) \leq 1, l = 1, 2$ (Example 3.2)*

Motivated by Zhang's work in [32], we give the sufficient and necessary conditions for the existence of Riemannian metric of positive scalar curvature on the $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$, which are described in Example 3.5 and 3.6.

Corollary 4.9. *Assume $n_1 + n_2 \equiv 2 \mod 4$, $n_1 = 1$ and $k_1 = -\frac{n_1+1-d_1-\sigma_1}{2}$, $k_2 = -\frac{n_2+1-d_2}{2}$ are integers. A spin $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ does not admit a Riemannian metric of PSC if and only if one of the following conditions satisfied*

- $k_2 \geq 0,\ k_2 \equiv 0 \mod 4,\ k_1 \equiv 0 \mod 2,\ \forall\ i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1;$
- $k_2 \geq 0,\ k_2 \equiv 1 \mod 4,\ \sigma_1 \equiv 1 \mod 2,\ \forall\ i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1;$
- $k_2 \geq 0,\ k_2 \equiv 2 \mod 4,\ k_1 + \sigma_1 \equiv 0 \mod 2,\ \forall\ i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1;$
- $k_2 \leq -n_2 - 1,\ -k_2 \equiv 0 \mod 4,\ k_1 \equiv 0 \mod 2,\ \forall\ i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1;$
- $k_2 \leq -n_2 - 1,\ -k_2 \equiv 2 \mod 4,\ k_1 + \sigma_1 \equiv 0 \mod 2,\ \forall\ i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1;$
- $k_2 \leq -n_2 - 1,\ -k_2 \equiv 3 \mod 4,\ \sigma_1 \equiv 1 \mod 2, \forall\ i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1.$

Proof. In the following we assume $n_2 = 4m + 1$ for some $m \in \mathbb{Z}$. And since $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ is spin, $k_1, k_2 \in \mathbb{Z}$.

By Example 3.5, we know that

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)) \equiv \binom{k_1 + 1}{1}\binom{n_2 + k_2}{n_2} + \sigma_1 \binom{n_2 + k_2}{n_2 + 1} \mod 2.$$

1 If $-n_2 \leq k_2 < 0$, clearly $\alpha = 0$. Thus $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ admits a Riemannian metric of PSC.

2 If $k_2 \geq 0$, $\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)) \equiv (k_1 + 1)\binom{n_2+k_2}{n_2} + \sigma_1\binom{n_2+k_2}{n_2+1} \mod 2$.

(1) If $k_2 \equiv 0 \mod 4$, assume $k_2 = 4k$, for some $k \in \mathbb{Z}$. Then
$\binom{n_2+k_2}{n_2} = \binom{4m+1+4k}{4m+1} \equiv \binom{m+k}{k} \mod 2$,
$\binom{n_2+k_2}{n_2+1} = \binom{4m+1+4k}{4m+2} = \binom{(4m+2)+(4(k-1)+3)}{4m+2} \equiv 0 \mod 2$.
Thus

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)) \equiv (k_1 + 1)\binom{m + k}{k} \mod 2.$$

So $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ does not admit a Riemannian metric of PSC iff

$$k_1 \equiv 0 \mod 2, \forall\ i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1.$$

(2) If $k_2 \equiv 1 \bmod 4$, assume $k_2 = 4k+1$, for some $k \in \mathbb{Z}$. Thus

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv \sigma_1 \binom{m+k}{k} \bmod 2.$$

So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC iff

$$\sigma_1 \equiv 1 \bmod 2, \forall i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1.$$

(3) If $k_2 \equiv 2 \bmod 4$, assume $k_2 = 4k+2$, for some $k \in \mathbb{Z}$. Thus

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv (k_1+1+\sigma_1) \binom{m+k}{k} \bmod 2.$$

So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC iff

$$(k_1+\sigma_1) \equiv 0 \bmod 2, \forall i, a_i([\frac{k_2}{4}]) + a_i([\frac{n_2}{4}]) \leq 1.$$

(4) If $k_2 \equiv 3 \bmod 4$, assume $k_2 = 4k+3$, for some $k \in \mathbb{Z}$. Thus

$$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv 0 \bmod 2.$$

So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ always admits a Riemannian metric of PSC.

3 If $k_2 \leq -n_2-1$, the analysis is similar to the case $k_2 \geq 0$. □

Corollary 4.10. *Assume $n_1+n_2 \equiv 2 \bmod 4$, $n_1 = 2$ and $k_1 = -\frac{n_1+1-d_1-\sigma_1}{2}$, $k_2 = -\frac{n_2+1-d_2}{2}$ are integers. A spin $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC if and only if one of the following condition satisfied*

- $k_2 \geq 0$, $k_2 \equiv 0 \bmod 4$, $k_1 \equiv 0$ *or* $1 \bmod 4$, $\forall\, i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \geq 0$, $k_2 \equiv 1 \bmod 4$, $\binom{k_1+2}{2} + \frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2} \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \geq 0$, $k_2 \equiv 2 \bmod 4$, $\binom{k_1+2}{2} + \sigma_1^2 - \sigma_2 \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \geq 0$, $k_2 \equiv 3 \bmod 4$, $\binom{k_1+2}{2} + \frac{\sigma_1(2k_1+3-\sigma_1)}{2} \equiv 1 \bmod 2, \forall\, i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 = -n_2-1$, $\frac{(k_1+1)(k_1+2)}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \bmod 2$;
- $k_2 \leq -n_2-2$, $-k_2 \equiv 0 \bmod 4$, $k_1 \equiv 0$ or $1 \bmod 4$, $\forall\, i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;

- $k_2 \leq -n_2-2$, $-k_2 \equiv 1 \bmod 4$, $\binom{k_1+2}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \bmod 2$, $\forall\ i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \leq -n_2-2$, $-k_2 \equiv 2 \bmod 4$, $\binom{k_1+2}{2} + \sigma_1^2 - \sigma_2 \equiv 1 \bmod 2, \forall\ i$, $a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;
- $k_2 \leq -n_2-2$, $-k_2 \equiv 3 \bmod 4$, $\binom{k_1+2}{2} + \frac{(2k_1+3+3\sigma_1)\sigma_1}{2} - \sigma_2 \equiv 1 \bmod 2$, $\forall\ i, a_i([\frac{-k_2-1-n_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1$;

Proof. Assume $n_2 = 4m$, for some $m \in \mathbb{Z}$. The following discussion is similar to Corollary 4.9.

1 For $k_2 \geq 0$:

$$\begin{aligned}
&\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \\
\equiv&\binom{k_1+2}{2}\binom{4m+k_2}{4m} + \left(-\frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\right)\binom{4m+k_2}{4m+1} \\
&+ (\sigma_1^2-2\sigma_2)\binom{4m+k_2+1}{4m+2} + \sigma_2\binom{4m+k_2}{4m+2} \bmod 2
\end{aligned}$$

(1) If $k_2 \equiv 0 \bmod 4$, assume $k_2 = 4k$, then

$$\begin{aligned}
&\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \\
\equiv&\binom{k_1+2}{2}\binom{4m+4k}{4m} + \left(-\frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\right)\binom{4m+4k}{4m+1} \\
&+ (\sigma_1^2-2\sigma_2)\binom{4m+4k+1}{4m+2} + \sigma_2\binom{4m+4k}{4m+2} \bmod 2 \\
\equiv&\binom{k_1+2}{2}\binom{m+k}{m} + \left(-\frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\right)\binom{4m+1+4k-1}{4m+1} \\
&+ (\sigma_1^2-2\sigma_2)\binom{4m+2+4(k-1)+3}{4m+2} \\
&+ \sigma_2\binom{4m+2+4k-2}{4m+2} \bmod 2 \\
\equiv&\binom{k_1+2}{2}\binom{m+k}{m} \bmod 2.
\end{aligned}$$

And $\binom{2+k_1}{2} \equiv 1 \bmod 2$ equals to $a_1(k_1) = 0$, more precisely, $k_1 \equiv 0$ or $1 \bmod 4$.

So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC iff

$$k_1 \equiv 0 \text{ or } 1 \bmod 4, \forall\ i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.$$

(2) If $k_2 \equiv 1 \bmod 4$, assume $k_2 = 4k+1$, then
$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv \{\binom{k_1+2}{2} + \frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\}\binom{m+k}{k} \bmod 2$.
So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC iff

$$\begin{aligned}\binom{k_1+2}{2} + \frac{\sigma_1^2-2\sigma_2}{2} + \frac{(2k_1+3)\sigma_1}{2}\\ \equiv 1 \bmod 2, \forall i, a_i([\frac{k_2}{4}]) + a_i(\frac{n_2}{4}) \leq 1.\end{aligned}$$

(3) If $k_2 \equiv 2 \bmod 4$, assume $k_2 = 4k+2$, then
$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv (\binom{k_1+2}{2} + \sigma_1^2 - \sigma_2)\binom{m+k}{m} \bmod 2$.

(4) If $k_2 \equiv 3 \bmod 4$, assume $k_2 = 4k+3$, then
$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv (\binom{k_1+2}{2} + \frac{\sigma_1(2k_1+3-\sigma_1)}{2})\binom{m+k}{m} \bmod 2$.

2 If $-n_2 \leq k_2 < 0$, $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ always admits a Riemannian metric of PSC.

3 If $k_2 = -n_2 - 1$, then
$\alpha(H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)) \equiv \frac{(k_1+1)(k_1+2)}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \bmod 2$.
So $H^{\mathbf{I}}_{n_1,n_2}(d_1,d_2)$ does not admit a Riemannian metric of PSC iff

$$\frac{(k_1+1)(k_1+2)}{2} + \frac{\sigma_1(\sigma_1-2k_1-3)}{2} \equiv 1 \bmod 2.$$

4 If $k_2 \leq -n_2 - 2$, the argument is similar to the case $k_2 \geq 0$. □

Appendix A. Relations between $A(n,l)$ and some classical numbers

The number

$$A(n,l) = \begin{cases} \frac{1}{n!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^n, & 0 \leq l \leq n; \\ 0, & \text{otherwise} \end{cases}$$

appearing in Section 3.1 has a generating function

$$e^{y(e^x-1)} = \sum_{l,n\geq 0} A(n,l)x^n \frac{y^l}{l!}$$

and $A(n,l)$ can be obtained recursively by $A(n,l) = \frac{l}{n}(A(n-1,l) + A(n-1, l-1))$ and $A(0,0) = 1$.

Lemma A.1. *$A(n,l)$ has the following relations with the Stirling number, Bell number, Bernoulli number (c.f. [10]), and divided difference (c.f. [8]).*

(1) *Let $S(n,l)$ be the Stirling number of the second kind which is generated by*

$$e^{y(e^x-1)} = \sum_{l,n\geq 0} S(n,l)y^l \frac{x^n}{n!},$$

we have

$$S(n,l) = \frac{n!}{l!}A(n,l)$$

and

$$\sum_{n=l}^{\infty} A(n,l)u^n = (e^u - 1)^l.$$

(2) *Let B_n be the Bell number, we have*

$$B_n = \sum_{l=0}^{n} \frac{n!}{l!}A(n,l).$$

(3) *Let $B_n(0)$ be the Bernoulli number which is generated by*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n(0)\frac{x^n}{n!},$$

we have

$$B_n(0) = \sum_{l=0}^{n} \frac{n!}{l+1}A(n,l).$$

(4) *Let $P_n[0,1,2,\cdots,l]$ be the l-th divided difference for $P_n(x) = x^n$, we have*

$$P_n[0,1,2,\cdots,l] = \frac{n!}{l!}A(n,l).$$

Proof. (1): Stirling number of second kind is of recurrence relation:

$$S(n,l) = lS(n-1,l) + S(n-1,l-1).$$

Thus

$$\frac{l!}{n!}S(n,l) = \frac{l}{n}\{\frac{l!}{(n-1)!}S(n-1,l) + \frac{(l-1)!}{(n-1)!}S(n-1,l-1)\}.$$

Since $A(n,l)$ is of the same recurrence with $\frac{l!}{n!}S(n,l)$,

$$A(n,l) = \frac{l!}{n!}S(n,l).$$

By definition,

$$A(n,l) = \frac{l!}{n!}S(n,l) = \frac{l!}{n!}\frac{1}{l!}\sum_{m=0}^{l}(-1)^m\binom{l}{m}(l-m)^n = \frac{1}{n!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^n.$$

So

$$\begin{aligned}
&\sum_{n=l}^{\infty}A(n,l)u^n\\
=&\sum_{n=l}^{\infty}\frac{1}{n!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^n u^n\\
=&\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\sum_{n=l}^{\infty}\frac{(mu)^n}{n!}\\
=&\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}(e^{mu} - \sum_{n=0}^{l-1}\frac{(mu)^n}{n!})\\
=&\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}(e^u)^m - \sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\sum_{n=0}^{l-1}\frac{(mu)^n}{n!}\\
=&(e^u-1)^l - \sum_{n=0}^{l-1}\frac{u^n}{n!}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}m^n\\
=&(e^u-1)^l - \sum_{n=0}^{l-1}\frac{u^n}{n!}S(n,l)\\
=&(e^u-1)^l.
\end{aligned}$$

Note that $S(n,l)=0$ for $n<l$.
(2): A simple consequence of (1),

$$B_n=\sum_{l=0}^{n}S(n,l)=\sum_{l=0}^{n}\frac{n!}{l!}A(n,l).$$

(3):

$$\sum_{l,n\geq 0}\frac{(-1)^l}{l+1}A(n,l)x^n=\sum_{l=0}^{\infty}\frac{(-1)^l}{l+1}\sum_{n=l}^{\infty}A(n,l)x^n=\sum_{l=0}^{\infty}\frac{(-1)^l}{l+1}(e^x-1)^l=\frac{x}{e^x-1}.$$

(4): The divided difference of $P_n(x)=x^n$ is

$$P_n[x_0,x_1,\cdots,x_l]=\frac{1}{l!}\sum_{m=0}^{l}(-1)^n\binom{l}{m}x_{l-m}^n,$$

so

$$P_n[x_0,x_1,\cdots,x_l]=\begin{cases}0 & l>n\\ 1 & l=n\\ \sum\limits_{0\leq t_1\leq t_2\leq\cdots\leq t_{n-l}\leq l}x_{t_1}\cdots x_{t_{n-l}} & l<n\end{cases}$$

By induction:

- Claim 1: $A(l,l)=\frac{l!}{l!}P_l[0,1,2,\cdots,l]=1$ is true.
  By the recurrence, $A(1,1)=1=\frac{1!}{1!}P_1[0,1]$ is true and $A(n,l)=0$ for $l>n$ or $l<0$. Assume $A(l,l)=1$ is true for any integer less than n.
  $A(n,n)=\frac{n}{n}(A(n-1,n)+A(n-1,n-1))=A(n-1,n-1)=\cdots=A(1,1)$.
  So the claim is proved.
- Claim 2: $A(n,l)=\frac{l!}{n!}P_n[0,1,2,\cdots,l]$ is true.
  By claim 1, $A(l,l)$ is true, assume $A(n,l)$ is true for all integers l and n less than $l+1$ and $n+1$, respectively.

$$\begin{aligned}&A(n+1,l)\\=&\frac{l}{n+1}(A(n,l)+A(n,l-1))\\=&\frac{l}{n+1}(\frac{l!}{n!}P_n[0,1,2,\cdots,l]+\frac{(l-1)!}{n!}P_n[0,1,2,\cdots,l-1])\end{aligned}$$

$$=\frac{l!}{(n+1)!}\sum_{\substack{1\le i\le n-l\\ t_i\in\{0,1,2\cdots,l\}\\ t_1\le t_2\le\cdots\le t_{n-l}}} t_1\cdots t_{n-l}l+\frac{l!}{(n+1)!}\sum_{\substack{1\le i\le n+1-l\\ t_i\in\{0,1,2\cdots,l-1\}\\ t_1\le t_2\le\cdots\le t_{n+1-l}}} t_1\cdots t_{n+1-l}$$

$$=\frac{l!}{(n+1)!}\{\sum_{\substack{1\le i\le n-l\\ t_i\in\{0,1,2\cdots,l\}\\ t_1\le t_2\le\cdots\le t_{n+1-l}=l}} t_1\cdots t_{n-l}t_{n+1-l}$$

$$+\sum_{\substack{1\le i\le n+1-l\\ t_i\in\{0,1,2\cdots,l\}\\ t_1\le t_2\le\cdots\le t_{n+1-l}<l}} t_1\cdots t_{n+1-l}\}$$

$$=\frac{l!}{(n+1)!}\sum_{\substack{t_i\in\{1,2\cdots,l\}\\ t_1\le t_2\le\cdots\le t_{n+1-l}\le l}} t_1\cdots t_{n-l}t_{n+1-l}$$

$$=\frac{l!}{(n+1)!}P_{n+1}[0,1,2,\cdots,l].$$ □

Appendix B. Some combinatorial properties

Let $a_i(n)$ denote the coefficient in dyadic expansion of n:

$$n=a_0(n)+a_1(n)2^1+a_2(n)2^2+\cdots+a_i(n)2^i,$$

where $i\in\mathbb{Z}$ is a finite integer.

The following properties are based on the Lucas Theorem and Proposition 4.4.

Proposition B.1. *For $m,n\in\mathbb{N}$,*

$$\binom{4m+n}{4m}\equiv\binom{m+[\frac{n}{4}]}{m} \bmod 2.$$

Proof. If $n\equiv 0 \bmod 4$, assume $n=4k$,

$$\begin{aligned}\binom{4m+4k}{4m}&=\frac{(4m+4k-1)!!}{(4m-1)!!(4k-1)!!}\frac{(4m+4k)!!}{(4k)!!(4m)!!}\\&=\frac{(4m+4k-1)!!}{(4m-1)!!(4k-1)!!}\frac{(2m+2k)!}{(2k)!(2m)!}\\&=\frac{(4m+4k-1)!!}{(4m-1)!!(4k-1)!!}\frac{(2m+2k-1)!!}{(2m-1)!!(2k-1)!!}\frac{(2m+2k)!!}{(2k)!!(2m)!!}\end{aligned}$$

$$= \frac{(4m+4k-1)!!}{(4m-1)!!(4k-1)!!} \frac{(2m+2k-1)!!}{(2m-1)!!(2k-1)!!} \frac{(m+k)!}{k!m!}$$
$$\equiv \binom{m+k}{m} \mod 2.$$

For the other dimensions, the analysis is similar. □

Proposition B.2. *For $m, n \in \mathbb{N}$,*

$$\binom{4m+1+n}{4m+1} \equiv \begin{cases} 0 \mod 2, & n \text{ is odd;} \\ \binom{m+[\frac{n}{4}]}{m} \mod 2, & n \text{ is even.} \end{cases}$$

Proof. When $n, 4m+1$ are both odd, we have $a_0(4m+1) = 1, a_0(n) = 1$. By Proposition 4.4, $\binom{4m+1+n}{n}$ is even.

When $n = 4k$,

$$\binom{4m+1+n}{4m+1} = \binom{4k+4m+1}{4k} \equiv \binom{m+k}{k} \equiv \binom{m+k}{m} \mod 2.$$

When $n = 4k+2$,

$$\begin{aligned}\binom{4m+1+4k+2}{4m+1} &= \frac{(4m+1+4k+2)!!}{(4m+1)!!(4k+1)!!} \frac{(4m+4k+2)!!}{(4m)!!(4k+2)!!} \\ &= \frac{(4m+1+4k+2)!!}{(4m+1)!!(4k+1)!!} \frac{(2m+2k+1)!}{(2m)!(2k+1)!} \\ &= \frac{(4m+1+4k+2)!!}{(4m+1)!!(4k+1)!!} \frac{(2m+2k+1)!!}{(2m-1)!!(2k+1)!!} \frac{(m+k)!}{(m)!(k)!} \\ &\equiv \binom{m+k}{m} \mod 2.\end{aligned}$$

□

Corollary B.3. *For $m, n \in \mathbb{N}$,*

$$\binom{m+n}{n} \equiv \begin{cases} 0 \mod 2, & n \cdot m \text{ or } [\frac{n}{2}] \cdot [\frac{m}{2}] \text{ is odd;} \\ \binom{[\frac{n}{4}]+[\frac{m}{4}]}{[\frac{n}{4}]} \mod 2, & \text{otherwise.} \end{cases}$$

Proof. When $n \cdot m$ is odd, n, m are both odd, we have $a_0(m) = 1, a_0(n) = 1$ and by Proposition 4.4, $\binom{m+n}{n}$ is even.

For any $l \in \mathbb{N}$

$$4l+2 = a_0(4l+2)+a_1(4l+2)2^1+\cdots = 2(2l+1) \Longrightarrow a_1(4l+2) = a_0(2l+1) = 1;$$

Similarly,

$$4l+3 = 1+(4l+2) \Longrightarrow a_1(4l+3) = 1.$$

When $[\frac{n}{2}]\cdot[\frac{m}{2}]$ is odd, then $n,m \in \{4l_1+2, 4l_2+3|l_1,l_2 \in \mathbb{N}\}$. Thus $a_1(n) = a_1(m) = 1$. By Proposition 4.4, $\binom{m+n}{n}$ is even.

When $[\frac{n}{2}]\cdot[\frac{m}{2}]$ is even, we have

- if $n = 4k$, then $\binom{m+n}{n} = \binom{4k+m}{m} \equiv \binom{k+[\frac{m}{4}]}{[\frac{m}{4}]} \equiv \binom{[\frac{m}{4}]+[\frac{n}{4}]}{[\frac{n}{4}]} \mod 2$;
- if $n = 4k+1$, we only need to consider that m is an even number, since the case $n,m \in \{2k+1|k \in \mathbb{Z}\}$ has been discussed above. By Proposition B.2.
  $\binom{m+n}{n} = \binom{4k+1+m}{m} \equiv \binom{[\frac{m}{4}]+[\frac{n}{4}]}{[\frac{n}{4}]} \mod 2$;
- if $n = 4k+2$, we only need to consider $m = 4l, 4l+1$, since the case $n,m \in \{4l_1+2, 4l_2+3|l_1,l_2 \in \mathbb{N}\}$ has been discussed above. By Proposition B.2 and B.1.
  $$\binom{m+n}{n} = \binom{m+n}{m} = \binom{4k+2+4l}{4l} \equiv \binom{k+l}{k} \mod 2,$$
  $$\binom{m+n}{n} = \binom{m+n}{m} = \binom{4k+2+4l+1}{4l+1} \equiv \binom{k+l}{k} \mod 2;$$
- if $n = 4k+3$, we only need to consider $m = 4l$. By Proposition B.2,
  $$\binom{m+n}{n} = \binom{m+n}{m} = \binom{4k+3+4l}{4l} \equiv \binom{k+l}{k} \mod 2. \qquad \square$$

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