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Solution of two-dimensional optimal control problem using Legendre Block-Pulse polynomial basis

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Abstract: In this paper, a numerical method is presented for solving a class of two-dimensional optimal control problems using the Ritz method and orthogonal Legendre block-pulse functions.

The most important reason for using the Ritz method is its high flexibility in boundary and initial conditions. First, the state and control vectors are approximated as a series of hybrid orthogonal Legendre Block-Pulse functions with unknown coefficients. Then, by substituting these approximations into cost functional, we derive an unconstrained optimization problem. By applying optimality conditions for this problem, a system of algebraic equations is obtained. Solving this system, the unknown coefficients and consequently the state and control functions are obtained. At last the convergence of the proposed method is discussed and the accuracy and efficiency of the proposed method is demonstrated in comparison with other methods by providing several examples.

Keywords: Two-dimensional optimal control, Ritz method, Legendre block-pulse, numerical method.

1. Introduction

Optimal control problems, are very important category of optimization problems. A control problem is usually expressed by two types of variables, namely control and state variables. In optimal control problems, both control and state variables are unknown. The purpose of such problems is to determine the control signals as well as the corresponding path, in such a way that it applies to the existing physical constraints while maximizing or minimizing the desired cost functional. Because the structure of most optimal control problems is complex, no exact solution can be obtained for them, so numerical methods play an important role in solving these problems. In recent years,

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researchers have used different methods to obtain approximate solutions to such problems, which will be mentioned herein.

Orthogonal functions have received special attention of researchers for the analysis of optimal control problems [1, 2, 3, 4]. In [5], Marzban has also proposed a method for analyzing and solving optimal control systems using orthogonal hybrid functions. The authors of [6], proposed a method for solving a class of Volterra Fredholm integral equations using the Bernstein operational matrix. In [7], Ebadian investigated the applications of block pulse functions for solving the Volterra integral equations using functional matrices. In nature, many quantities are functions of two independent variables and two-dimensional systems and signals are used to model phenomena that have two independent variables. Accordingly, the importance and necessity of examining two-dimensional optimal control systems is Comprehensible. Many control systems are described by continuous-temporal dynamic equations. Due to the lack of analytical solutions to the two-dimensional optimal control problems of continuous-temporal systems, numerical and semi-analytical methods are suggested to solve a number of such problems. Numerical computational methods which are currently used in science and engineering are very diverse and a specific solution can be provided for each specific problem or special conditions. Since it is very difficult and in some cases impossible to obtain the solution to the analysis of these types of problems, semi-analytical and numerical methods are used for solving them. In the following, some studies in these systems are presented.

Two-dimensional systems were first introduced as part of image processing, and, their spatial models had been proposed by Roesser [8]. In [9], Mamehrashi proposed a method for solving a range of two-dimensional control problems using the Ritz-Galerkin method. Tsai transformed the Roesser type continuous-temporal optimal control problem with a quadratic cost function into a discrete two-dimensional control problem in [10]. In recent years, orthogonal block-pulse functions have been used to solve many optimal control problems, examples of which can be seen in the articles [11, 12, 13, 14, 15]. In [16, 17], Nemati used the Ritz method to solve a series of one-dimensional and two-dimensional control problems, respectively. The two-dimensional fractional optimal control problems have been investigated by Ordokhani in [18].

In this paper, we present a numerical method for solving a class of two-dimensional optimal control problems (2DOCP).

Consider the following controllable and observable two-dimensional system

$$\Im(x, t, u(x, t), z(x, t), z_x(x, t), z_t(x, t), z_{xx}(x, t), z_{xt}(x, t)) = 0, \tag{1}$$

with boundary conditions $z(0,t) = s_1(t)$, $z(x,0) = s_2(x)$.

The purpose of this paper is to determine the control vector $u(x,t)$ and the corresponding state vector $z(x,t)$ such that the following cost functional is minimized according to constraints (1)

$$J = \int_0^{t_f} \int_0^{x_f} \wp(x,t,u(x,t),z(x,t))dxdt. \tag{2}$$

The proposed method is based on the approximation of state and control variables through a combination of Block-Pulse functions, Legendre polynomials and the Ritz method. Due to the flexibility of the Ritz method in the face of initial and boundary conditions, we used this approach in the proposed method. In calculating the dual integral in the cost functional, we have used the Gaussian quadrature rule. By substituting the approximated functions into constraints of the problem and using the suggested method, the optimal control problem is reduced to an unconstrained optimization problem that can be easily solved. This article is organized as follows: Section 2 introduces the Ritz method and the hybrid Legendre Block-Pulse functions. In Sections 3 and 4, we describe the proposed numerical method and discuss the convergence of the method, respectively. In Section 5, the efficiency and accuracy of the proposed method are examined by providing three examples. Also, a conclusion is given in Section 6.

2. Preliminaries

In this section we briefly explain the Ritz method. Also, we recall the hybrid functions of Block-Pulse and Legendre polynomials and some properties of them.

2.1. Ritz method

The Ritz method is a simple and efficient way to approximate the solution of an optimization problem. In this method, the solution of the functional minimization problem

$$min \quad L[y(x)] = \int_a^b f(x,y,y')dx, \tag{3}$$

with boundary conditions

$$y(a) = a_0\,,\; y(b) = b_0,$$

is considered as follows

$$y_n(x) \approx \sum_{i=1}^{n} c_i \varphi_i(x) + \varphi_0(x). \tag{4}$$

We must select the basic functions $\varphi_i(x)$ that satisfy the following conditions

$$\begin{aligned} \varphi_0(a) &= a_0, \quad \varphi_0(b) = b_0, \\ \varphi_i(a) &= \varphi_i(b) = 0, \quad i = 1, 2, ..., n. \end{aligned} \tag{5}$$

By substituting $y_n(x)$ into the problem (3) and solving it, the unknown coefficients and consequently the solution $y_n(x)$ are obtained.

Now, suppose that $\varphi_i(x) = k(x)p_i(x)$, so Eq. (4) can be written as

$$y_n(x) \approx \sum_{i=1}^{n} k(x) c_i p_i(x) + \varphi_0(x), \tag{6}$$

where, $k(x)$ satisfies the homogeneous conditions and $p_i(x), i = 1, 2, ..., n$ are Legendre polynomials.

Using the Ritz method to approximate the function $z(x,t)$, Eq. (6) is written as

$$z_{mn}(x,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} k(x,t) c_{ij} p_i(x) p_j(t) + w(x,t), \tag{7}$$

where $k(x,t)$ and $w(x,t)$ satisfy the homogeneous and boundary conditions, respectively [20].

2.2. Two-dimensional block-pulse functions

A set of two dimensional Block-Pulse functions (2DBPFs) $\Phi_{i_1,i_2}(x,t)$ for $x \in [0, T_1], t \in [0, T_2]$ is defined as follows

$$\Phi_{i_1,i_2}(x,t) = \begin{cases} 1 & x \in \left[\frac{(i_1-1)T_1}{m_1}, \frac{i_1 T_1}{m_1}\right), t \in \left[\frac{(i_2-1)T_2}{m_2}, \frac{i_2 T_2}{m_2}\right) \\ 0 & \text{otherwise} \end{cases}$$

2DBPFs are disjoined with each other

$$\Phi_{i_1,i_2}(x,t)\Phi_{j_1,j_2}(x,t) = \begin{cases} \Phi_{i_1,i_2}(x,t) & i_1 = j_1 \text{ and } i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

and are orthogonal

$$\int_0^{T_1}\int_0^{T_2}\Phi_{i_1,i_2}(x,t)\Phi_{j_1,j_2}(x,t)dxdt = \begin{cases} h_1h_2 & i_1=j_1 \text{ and } i_2=j_2\\ 0 & \text{otherwise}\end{cases}$$

in the region $x\in[0,T_1)$ and $t\in[0,T_2)$, where

$$i_1,j_1=1,2,...,m_1, i_2,j_2=1,2,...,m_2, \quad h_1=\frac{T_1}{m_1}, h_2=\frac{T_2}{m_2}.$$

Also 2DBPFs are complete when both m_1 and m_2 approach infinity [22].

Since each 2DBPF takes only one value in its subregion, the 2DBPFs can be expressed as

$$\Phi_{i_1,i_2}(x,t)=\varphi_{i_1}(x)\Psi_{i_2}(t),$$

where $\varphi_{i_1}(x)$ and $\Psi_{i_2}(t)$ are one-dimensional Block-Pulse functions related to the variables x and t, respectively.

2.3. Hybrid function of block-pulse and Legendre polynomials

The two-dimensional Legendre Block-Pulse function is defined as follows

(8)

$$\Psi_{i_1j_1i_2j_2}(x,t)= \begin{cases} L_{j_1}(\frac{2N_1x}{x_f}-2i_1+1)L_{j_2}(\frac{2N_2t}{t_f}-2i_2+1) : (x,t)\in[\frac{i_1-1}{N_1}x_f,\frac{i_1}{N_1}x_f]\times[\frac{i_2-1}{N_2}t_f,\frac{i_2}{N_2}t_f]\\ 0 \qquad \text{otherwise}\end{cases}$$

where, $i_1=1,2,...,N_1$, $i_2=1,2,...,N_2$ and $j_1,j_2=0,1,...M-1$ are the order of Block-Pulse functions and Legendre polynomials, respectively and $L_{j_1}(x), L_{j_2}(t)$ are the well known Legendre polynomials.

3. Proposed numerical method

In this section, we present a numerical method based on approximating the state and control variables to solve a class of 2DOCPs.

Consider the following problem:

$$\text{Min } J=\int_0^{t_f}\int_0^{x_f}\wp(x,t,u(x,t),z(x,t))dxdt, \tag{9}$$

subject to the dynamical system,

$$\Im(x,t,u(x,t),z(x,t),z_x(x,t),z_t(x,t),z_{xx}(x,t),z_{xt}(x,t))=0, \tag{10}$$

with the following boundary conditions $z(0,t)=s_1(t)$, $z(x,0)=s_2(x)$.

Suppose that $Q\subset PC^2([0,x_f]\times[0,t_f])$ is a set of all continuous piecewise functions that satisfy the boundary conditions. The cost functional J is a function of $z(x,t)$ and $u(x,t)$, so problem (9)-(10), can be considered as a problem of minimizing the value of J on Q. Suppose $Q_{N_1N_2(M-1)(M-1)}\subset Q$ is a set of the Legendre Block-Pulse hybird functions consisting of N_1N_2 polynomials and the degree of each polynomial is at most $(M-1)(M-1)$. The state variable is approximated using a finite number of the Legendre Block-Pulse hybird functions as

$$z(x,t)=\sum_{i_1=1}^{N_1}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N_2}\sum_{j_2=0}^{M-1} f_{i_1,j_1,i_2,j_2}\Psi_{i_1,j_1,i_2j_2}(x,t). \tag{11}$$

Now by selecting N_1, N_2 as

$$N_1=\begin{cases}\dfrac{x_f}{\tau_1} & \dfrac{x_f}{\tau_1}\ \text{is integer}\\[2ex] \left[\dfrac{x_f}{\tau_1}\right]+1 & \text{otherwise}\end{cases} \tag{12}$$

$$N_2=\begin{cases}\dfrac{t_f}{\tau_2} & \dfrac{t_f}{\tau_2}\ \text{is integer}\\[2ex] \left[\dfrac{t_f}{\tau_2}\right]+1 & \text{otherwise}\end{cases} \tag{13}$$

the interval $[0,x_f]\times[0,t_f]$ is converted to the following $N_1\times N_2$ sub intervals

$$[0,\frac{1}{N_1}x_f]\times[0,\frac{1}{N_2}t_f],[0,\frac{1}{N_1}x_f]\times[\frac{1}{N_2}t_f,\frac{2}{N_2}t_f]\cdots[\frac{N_1-1}{N_1}x_f,x_f]\times[\frac{N_2-1}{N_2}t_f,t_f].$$

As a result, the state variable (11) is written as

(14)

$$\hat{z}(x,t) = \begin{cases} \hat{z}_{11}(x,t) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,1,j_2} & : (x,t) \in [0, \frac{1}{N_1}x_f]\times[0, \frac{1}{N_2}t_f], \\ \vdots & \\ \hat{z}_{N_1N_2}(x,t) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{N_1,j_1,N_2,j_2}\psi_{N_1,j_1,N_2,j_2} & : (x,t) \in [\frac{N_1-1}{N_1}x_f, x_f] \\ & \times [\frac{N_2-1}{N_2}t_f, t_f], \end{cases}$$

Now, by substituting the values of $z(x,t)$ in (10), the control variable $u(x,t)$ is extracted. Then, substituting the approximations of $z(x,t)$ and $u(x,t)$ into the cost functional (9) an unconstrained optimization problem is obtained as

$$\text{Min } J(\alpha)$$

where,

$$\alpha = [f_{1,0,1,0}, f_{1,0,1,1}, \dots f_{N_1,M-1,N_2,M-1}].$$

By applying the necessary optimality conditions, the following system of algebraic equations is obtained:

$$\frac{\partial J[\alpha]}{\partial f_{i_1j_1i_2j_2}} = 0, \quad i_1 = 1,2,...,N_1, \quad i_2 = 1,2,...,N_2, \quad j_1, j_2 = 1,2,...,M-1.$$

By solving the above algebraic system for α, the unknown coefficients $f_{i_1j_1i_2j_2}$ are achieved and consequently we can find the approximate value of $z(x,t)$ and $u(x,t)$ from (11) and (10), respectively.

4. Convergence analysis

In this section, we recall a theorem and present a lemma which ensure the convergence analysis of suggested method. Here, the approximation convergence of a function is derived with respect to the Legendre Block-Pulse bases.

Consider the restriction of the cost functional J to $Q_{N_1N_2(M-1)(M-1)} \subset Q$ as

$$J[\hat{z}(x,t)] = J[\sum_{i_1=1}^{N_1}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N_2}\sum_{j_2=0}^{M-1} f_{i_1,j_1,i_2,j_2}\Psi_{i_1,j_1,i_2j_2}(x,t)], \tag{15}$$

which is a function of $N_1N_2(M-1)(M-1)$ variables. The coefficients $f_{i_1j_1i_2j_2}$ are chosen in such a way as to minimize (15). Let $\beta_{N_1N_2(M-1)(M-1)}$ indicate the minimum value of J restricted to $Q_{N_1N_2(M-1)(M-1)}$.

The following theorem is a remarkable result of the Weierstrass famous theorem for two-dimensional space.

Theorem 4.1. *For any $\hat{z}(x,t) \in Q \subset PC^2([0,x_f]\times[0,t_f])$, there exists a sequence of polynomials $\{\Psi_{i_1j_1i_2j_2}(x,t)\}_{i_1,j_1,i_2,j_2=0}^{\infty} \in Q$ that converges uniformly to $\hat{z}(x,t)$.*

Proof. See [21]. □

The convergence of the proposed method is provided by the following Lemma. (4.1).

Lemma 4.1. *If*

$$\beta_{N_1N_2(M-1)(M-1)} = \inf_{Q_{N_1N_2(M-1)(M-1)}} J, \qquad \textit{for} \quad N_1, N_2, M = 1,2,3,..,$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a subset of Q including the Legendre Block-Pulse hybird functions involving N_1N_2 polynomials of degree at most $(M-1)(M-1)$, then

$$\lim_{N_1,N_2,M\to\infty} \beta_{N_1N_2(M-1)(M-1)} = \inf_Q J.$$

Proof. Let

$$\beta_{N_1N_2(M-1)(M-1)} = \min_{\alpha_{N_1N_2(M-1)(M-1)}} J(\alpha_{N_1N_2(M-1)(M-1)}),$$

then,

$$\beta_{N_1N_2(M-1)(M-1)} = J(\alpha^*{}_{N_1N_2(M-1)(M-1)}),$$

where

$$\alpha^*{}_{N_1N_2(M-1)(M-1)} \in \\ \text{Argmin}\{J(\alpha_{N_1N_2(M-1)(M-1)}) : \alpha_{N_1N_2(M-1)(M-1)} \in R^{2N_1N_2M}\}.$$

Now, let

$$(z^*{}_{N_1N_2(M-1)(M-1)}(x,t), u^*{}_{N_1N_2(M-1)}(x,t)) \in \\ \text{Argmin}\{J(z(x,t),u(x,t)) : (z(x,t),u(x,t)) \in Q_{N_1N_2(M-1)(M-1)}\},$$

then

$$J({z^*}_{N_1N_2(M-1)(M-1)}(x,t),{u^*}_{N_1N_2(M-1)(M-1)}(x,t))$$
$$=\min_{(z(x,t),u(x,t))\in Q_{N_1N_2(M-1)(M-1)}} J(z(x,t),u(x,t)),$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a class of combinations of the continuous Legendre Block-Pulse hybird functions involving N_1N_2 polynomials of degree at most $(M-1)(M-1)$, so

$$\beta_{N_1N_2(M-1)(M-1)} = J({z^*}_{N_1N_2(M-1)(M-1)}(x,t),{u^*}_{N_1N_2(M-1)(M-1)}(x,t)).$$

Furthermore, according to $Q_{N_1N_2(M-1)(M-1)} \subset Q_{N_1N_2MM}$, we have

$$\min_{(z(x,t),u(x,t))\in Q_{N_1N_2MM}} J(z(x,t),u(x,t))$$
$$\leq \min_{(z(x,t),u(x,t))\in Q_{N_1N_2(M-1)(M-1)}} J(z(x,t),u(x,t)).$$

Thus, $\beta_{N_1N_2MM} \leq \beta_{N_1N_2(M-1)(M-1)}$ is achieved which means $\beta_{N_1N_2(M-1)(M-1)}$ is a non-increasing sequence. Also, this sequence is lower bounded, so its infimum is the limit. Due to the continuity J and by taking the limit when $N_1, N_2, M \to \infty$, we can write,

$$\lim_{N_1,N_2,M\to\infty} \beta_{N_1N_2(M-1)(M-1)} = \min_{(z(x,t),u(x,t))\in Q} J(z(x,t),u(x,t)).$$

which completes the proof. □

5. Numerical examples

In this section for illustrating the efficiency of our proposed method, three examples are considered.

Example 5.1. Consider the Darbox equation [7],

$$\frac{\partial^2 g(x,t)}{\partial x \partial t} = a_1\frac{\partial g(x,t)}{\partial t} + \frac{\partial g(x,t)}{\partial x} + a_0 g(x,t) + bf(x,t). \tag{16}$$

If considered

$$z^h(x,t) = \frac{\partial g(x,t)}{\partial t} - a_2 g(x,t), \qquad z^v(x,t) = g(x,t), \tag{17}$$

then Eq. (16) can be a continuous two-dimensional linear system in the Roesser's model as follows:

$$\begin{bmatrix} \dfrac{\partial z^h(x,t)}{\partial x} \\ \dfrac{\partial z^v(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} z_c^h(x,t) \\ z_c^v(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u_c(x,t), \tag{18}$$

with boundary conditions

$$z^h(0,t) = \frac{dg_2(t)}{dt} - a_2g_2(t), \qquad z^v(x,0) = g_1(x),$$

Let $a_0 = 0.2$, $a_1 = -3$, $a_2 = -1$, $b = 0.3$, $g_1(x) = e^{-3x}\cos(2\pi x)$, $g_2(t) = e^{-2t}$ and the 2D quadratic cost function

$$J = \int_0^3 \int_0^3 [z_0^T(x,t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z(x,t) + u^T(x,t)u(x,t)]dxdt. \tag{19}$$

After substituting the coefficients in (18) and considering the cost function (19), the problem is converted to

$$\begin{aligned} &\min J = \int_0^3 \int_0^3 \left\{ [z^h(x,t)]^2 + [z^v(x,t)]^2 + [u(x,t)]^2 \right\} dxdt, \\ &s.t \\ &\frac{\partial z^h(x,t)}{\partial x} = -3z^h(x,t) + 3.2z^v(x,t) + 0.3u(x,t), \\ &\frac{\partial z^v(x,t)}{\partial t} = z^h(x,t) - z^v(x,t), \\ &z^v(x,0) = e^{-3x}\cos(2\pi x), \\ &z^h(0,t) = -e^{-2t}. \end{aligned} \tag{20}$$

According to the above dynamical equations, we can write

$$\begin{aligned} &z^h(x,t) = \frac{\partial z^v(x,t)}{\partial t} + z^v(x,t), \\ &u(x,t) = \frac{10}{3}\left[\frac{\partial z^h(x,t)}{\partial x} + 3z^h(x,t) - 3.2z^v(x,t)\right]. \end{aligned} \tag{21}$$

Using the method introduced in section 3, we first approximate $z^v(x,t)$ based on the Ritz method and using the two-dimensional Legendre Block-Pulse functions as follows

(22)
$$z^v(x,t) = \sum_{i_1=1}^{N}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N}\sum_{j_2=0}^{M-1} tx(f_{i_1,j_1,i_2,j_2}\Psi_{i_1,j_1,i_2,j_2}(x,t) + e^{-3x-2t}\cos(2\pi x),$$

where N is the number of blocks and M is the number of expressions used in Legendre polynomials. $z^v(x,t)$ for $(x,t) \in \left[\frac{i_1-1}{N}, \frac{i_1}{N}\right) \times \left[\frac{i_2-1}{N}, \frac{i_2}{N}\right)$, can be written as follow:

$$z^v[i_1,i_2] = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} L_{j_1}(\frac{2Nx}{x_f} - 2i_1 + 1)L_{j_2}(\frac{2Nt}{t_f} - 2i_2 + 1)f_{i_1,j_1,i_2,j_2},$$

where, $i_1, i_2 = 1, \ldots, N$, $j_1, j_2 = 0, \ldots, M-1$.

According to Eq. (18), $z^h[i_1,i_2]$ and $u[i_1,i_2]$ can can be easily obtained, in any corresponding sub-interval.

By substituting the value of $z^v[i_1,i_2]$ and the corresponding values obtained from (18) in the cost functional J, problem (16) reduces to an unconstrained optimization problem that can be easily solved with existing optimization methods.

The vertical, horizontal and control variables obtained for this example are shown in Fig. 1, 2 and 3, respectively. The cost functional J by the present method for different values of M and N and its comparison with the result evaluated by other methods are shown in Tables 1, in which verifies the superiority of the proposed method.

Example 5.2. Consider the 2-D optimal control problem [20]

$$\min J = \frac{1}{2}\int_0^1\int_0^1 x(z^2(x,t) + u^2(x,t))dxdt,$$
$$s.t$$
$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial z(x,t)}{\partial t} + u(x,t),$$

with the boundary conditions

$$z(x,0) = 1 - x^2, \qquad z(1,t) = 0.$$

We use our proposed method for solving this example. The approximated state and control variables obtained for this example are shown in Fig. 4 and 5,

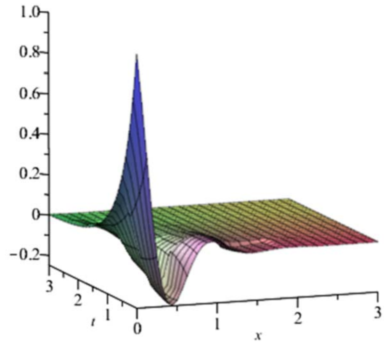


Figure 1: Approximated vertical component of state function $z^v(x,t)$ for example 1.

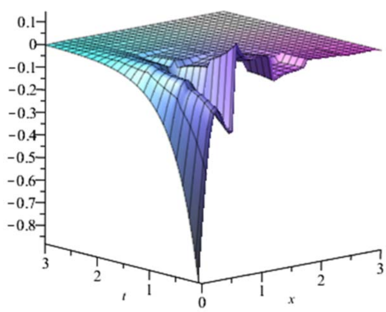


Figure 2: Approximated horizontal component of state function $z^h(x,t)$ for example 1.

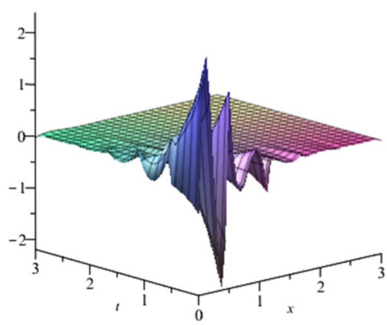


Figure 3: Approximated control function $u(x,t)$ for example 1.

Table 1: Comparison of estimated value of J by different methods for example 1

Methods	J	CPU Time
Method of Tsai et al. [9]		
$X = 0.1, T = 0.1$	0.7348	
$X = 0.05, T = 0.05$	0.5510	
$X = 0.03, T = 0.03$	0.4760	
Method of Mamehrashi and Yousefi [8]		
$m = 7, n = 8$	0.6202	1.802
$m = 8, n = 3$	0.2792	1.560
$m = 8, n = 8$	0.2026	1.986
Method of Nemati and Yousefi [15]		
$n = 9, m = 5$	0.1770	
$n = 10, m = 5$	0.0951	
$n = 10, m = 6$	0.0947	
Method of Nemati [23]		
$m = 7, n = 6$	0.3094	
$m = 9, n = 8$	0.0951	
$m = 10, n = 8$	0.0608	
Present Method		
$N = 5, M = 3$	0.7020	0.079
$N = 5, M = 4$	0.0665	0.191
$N = 5, M = 5$	0.0551	0.438
$N = 7, M = 5$	0.0437	0.983
$N = 7, M = 7$	0.0088	1.801

respectively. Also, the cost functional J for different values M and N and its comparison with the obtained results via some other methods are shown in Table 2. As the previous example, the results of this example confirm the superiority of the proposed method.

Example 5.3. Consider the following optimal control problem [25]

$$\min J[z,u] = \int_0^1 \int_0^\pi ((z(x,t) - e^{-t}\sin(x))^2 + e^{x-t}(u(x,t) + e^{-t}\sin(2x))^2)dxdt,$$

s.t

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + 2\cos(x)z(x,t) + u(x,t),$$

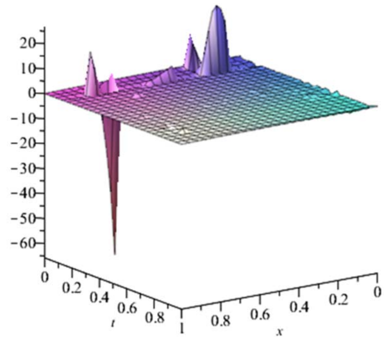


Figure 4: Approximated state function $z(x,t)$ for example 2.

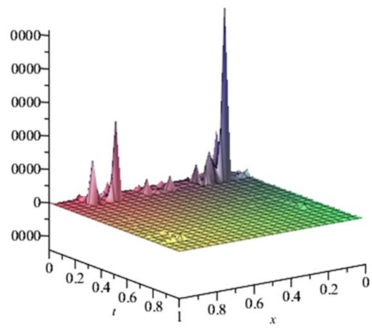


Figure 5: Approximated control function $u(x,t)$ for example 2.

with the boundary conditions

$$z(x,0) = \sin x, \qquad z(0,t) = 0.$$

The exact solutions are $z(x,t) = e^{-t}\sin(x)$, $u(x,t) = -e^{-t}\sin(2x)$ and its minimum value is $J^* = 0$.

We have solve this problem by applying the method presented in section 3 with different values of M and N. Using the proposed method, we first approximate $z(x,t)$ using the hybrid Legendre Block-Pulse functions, and then use the given equations to approximate $u(x,t)$, where by solving the problem of unknown coefficient optimization, the approximations of the state and control variables are obtained.

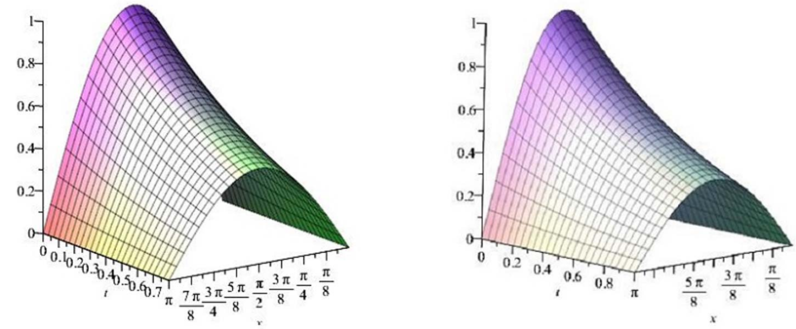
Table 2: Comparison of estimated value of J for different methods for example 2

Methods	J	CPU Time
Method of Mamehrashi and Yousefi [18]		
$m = 1, n = 4$	8.1044	0.141
$m = 2, n = 4$	2.8790	0.162
$m = 2, n = 6$	1.8283	0.563
$m = 3, n = 9$	1.0405	1.610
$m = 3, n = 10$	0.0075	1.902
Method of Hassani and Avazzadeh [24]		
$m_1 = 2, m_2 = 2, n_1 = 2, n_2 = 2$	1.8525	
$m_1 = 2, m_2 = 3, n_1 = 2, n_2 = 2$	7.9475	
$m_1 = 2, m_2 = 3, n_1 = 3, n_2 = 2$	4.4856	
$m_1 = 2, m_2 = 3, n_1 = 3, n_2 = 3$	5.5123	
$m_1 = 3, m_2 = 3, n_1 = 3, n_2 = 3$	1.4743	
Present Method		
$N = 2, M = 2$	0.084615	0.004
$N = 2, M = 3$	0.029328	0.009
$N = 3, M = 3$	0.016930	0.018
$N = 7, M = 7$	0.000007	1.759

The approximated solutions obtained from solving the above example and the absolute error functions for the state and control variables are shown in Fig. 6 and 7, respectively. Also the values of J and CPU time for different values of M and N are shown in Table 3.

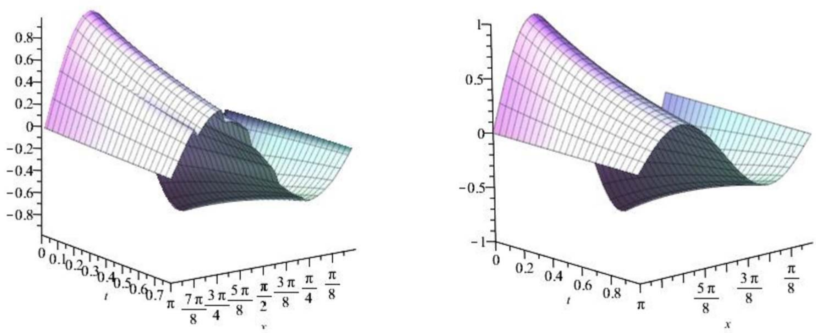
6. Conclusion

This paper presents an efficient numerical method for solving a category of 2DOCPs. Using the Ritz method and the hybrid Legendre Block-Pulse functions, an approximation of state and control variable is obtained. By applying the proposed method, without using derivative and multiplicative functional matrices, the 2D optimal control problem reduces to an unconstrained optimization problem that can be easily solved. We described the proposed numerical method and discussed the convergence of the method. The obtained results showed that our method gives the satisfactory results with only a few number of basis functions. Also, the simulations results in comparison with



(a) approximate solution (b) Exact solution

Figure 6: Graphs for the state variable for example 3.



(a) approximate solution (b) Exact solution

Figure 7: Graphs for the control variable for example 3.

Table 3: Estimated values of J and CPU time for various values of N, M for example 3

N, M	J	CPU Time
$N = 3, M = 3$	2.577832×10^{-4}	0.016
$N = 3, M = 5$	5.488613×10^{-8}	0.021
$N = 5, M = 5$	2.037223×10^{-9}	0.752
$N = 7, M = 7$	1.296438×10^{-9}	1.021

the exact solution and the results obtained by some other available methods, confirm the effectiveness and accuracy of the current approach.

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