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Conformally natural extensions of vector fields and applications

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Abstract: In this paper, we study an integral operator L_0 extending tangent vector fields V along the unit circle $\mathbb{S}^1$ to tangent vector fields $L_0(V)$ defined on the closure of the open unit disk $\mathbb{D}$. We first show that L_0 is conformally natural. Then we show: (1) the cross-ratio distortion norm $\|V\|_{cr}$ of V on $\mathbb{S}^1$ is equivalent to $\|\bar{\partial}L_0(V)\|_\infty$; (2) $\bar{\partial}L_0(V)$ is uniformly vanishing near the boundary of $\mathbb{D}$ if and only if V satisfies the little Zygmund bounded condition; (3) for each $0 < \alpha < 1$, $\bar{\partial}L_0(V)(z) = O((1-|z|)^\alpha)$ if and only if V is $C^{1+\alpha}$-smooth. As applications, the collection of V with $||V||_{cr} < \infty$ (resp. being uniformly vanishing near the boundary) recapitulates a known model of the tangent space of the universal Teichmüller space $T(\mathbb{D})$ (resp. the little Teichmüller space $T_0(\mathbb{D})$); the collection of V with $||V||_{cr} < \infty$ and satisfying a group compatible condition characterizes the tangent space of the Teichmüller space $T(\mathcal{R})$ of a hyperbolic Riemann surface $\mathcal{R}$; the collection of the $C^{1+\alpha}$-smooth vector fields V provides a model for the tangent space of the Teichmüller space $T_0^\alpha(\mathbb{D})$ of the $C^{1+\alpha}$ diffeomorphisms of $\mathbb{S}^1$.

Keywords: Conformally natural extension, Zygmund norm, Teichmüller space, little Teichmüller space, Hölder continuity.

1. Introduction

Let $\mathcal{R}$ be a hyperbolic Riemann surface. Using the open unit disk $\mathbb{D}$ as a universal covering space of $\mathcal{R}$, the collection $G(\mathcal{R})$ of the deck/cover transformations of $\mathbb{D}$ over $\mathcal{R}$ is represented by a subgroup of Möb($\mathbb{D}$), where Möb($\mathbb{D}$) is the group of all Möbius transformations preserving $\mathbb{D}$. The Teichmüller

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space $T(\mathcal{R})$ of $\mathcal{R}$ classifies the conformal structures on $\mathcal{R}$ and each point of $T(\mathcal{R})$ is represented by a quasiconformal map f from R to another Riemann surface $f(R)$ of the same topological type. Then f lifts to a quasiconformal map F from $\mathbb{D}$ to itself satisfying that for any $g \in G(\mathcal{R})$, there exist $g' \in G(f(\mathcal{R}))$ such that $F \circ g = g' \circ F$; that is, $F \circ g \circ F^{-1}$ is a Möbius transformation preserving $\mathbb{D}$ for any $g \in G(\mathcal{R})$. When $\mathcal{R}$ has the limit set of $G(\mathcal{R})$ equal to $\mathbb{S}^1$, $T(\mathcal{R})$ is embedded into the universal Teichmüller space $T(\mathbb{D})$. There are two models to describe $T(\mathbb{D})$. One is defined by using the equivalence classes of quasiconformal maps from $\mathbb{D}$ onto itself and the other employs the quasisymmetric homeomorphisms of the unit circle $\mathbb{S}^1$ to itself. Using the Beurling-Ahlfors extension ([5]), one can obtain the equivalence of these two models for $T(\mathbb{D})$ through replacing $\mathbb{D}$ by the upper half plane $\mathbb{H}$. Given a hyperbolic Riemann surface $\mathcal{R}$ with the limit set of $G(\mathcal{R})$ equal to $\mathbb{S}^1$ and a point of $T(\mathcal{R})$ represented by a quasiconformal map f from $\mathcal{R}$ to $f(\mathcal{R})$, let h be the restriction of the lifting F of f to the boundary circle $\mathbb{S}^1$. Then a natural model of using quasisymmetric homeomorphisms of $\mathbb{S}^1$ to describe $T(\mathcal{R})$ is to employ quasisymmstric homeomorphisms of $\mathbb{S}^1$ satisfying the condition that for any $g \in G(\mathcal{R})$, there exists $g' \in \text{Möb}(\mathbb{D})$ such that

$$h \circ g \circ h^{-1}|_{\mathbb{S}^1} = g'|_{\mathbb{S}^1}. \tag{1.1}$$

In order to prove this model to be an alternative description of $T(\mathcal{R})$, one needs to know if a quasisymmetric homeomorphism h satisfies the condition (1.1), then it can be extended to a quasiconformal map of $\mathbb{D}$ extending the condition (1.1) to $\mathbb{D}$. The Douady-Earle extension $DE(h)$ ([9]) of h achieves this goal and such a property of $DE(h)$ is called the conformal naturality of the extension. In fact, the Douady-Earle extension is well defined for any orientation preserving homeomorphism of $\mathbb{S}^1$ and the *conformal naturality* of the extension holds more generally in the sense that for any homeomorphism h of $\mathbb{S}^1$ and any two Möbius transformations g_1 and g_2 preserving $\mathbb{D}$,

$$DE(g_2 \circ h \circ g_1) = g_2 \circ DE(h) \circ g_1.$$

This extension has found many applications. For example, one can find an application in the study of the rigidity of the groups of homeomorphisms of $\mathbb{S}^1$ with uniformly bounded quasisymmetry ([29]), applications in characterizing various subspaces of the universal Teichmüller space in terms of different kinds of quasisymmetric homeomorphisms of $\mathbb{S}^1$ or in the study of the contraction property of these subspaces ([7], [8], [11], [39], [31]), applications in the study of the topological characterizations of Teichmüller spaces and asymptotic Teichmüller spaces in terms of Thurston's earthquake measures ([34] and [13]) or in terms of shears ([37] and [12]).

Let h be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let g_1 and g_2 be two Möbius transformations preserving $\mathbb{D}$. Denote by $||h||_{cr}$ the cross-ratio distortion of h, $K(DE(h))$ the maximal dilatation of $DE(h)$ on $\mathbb{D}$, and $\mu_{DE(h)}$ the Beltrami coefficient of $DE(h)$, which are defined in the next section. Let $0 < \alpha < 1$. Relationships between regularities of h and $DE(h)$ have been studied in [9], [11], [20], [21], [22], [31], and many other papers. The relevant results to this paper are summarized in the following first table.

Properties	*References*
$DE(g_2 \circ h \circ g_1) = g_2 \circ DE(h) \circ g_1$	[9]
$DE(h)$ is quasiconformal if h admits a quasiconformal extension	[9]
$DE(h)$ is quasiconformal iff h is quasisymmetric	[9] & [5] or [20]
$\ln K(DE(h)) \leq C\|\|h\|\|_{cr}$ for a universal constant C	[20]
$DE(h)$ is asymptotically conformal if h admits an asymptotically conformal extension	[11]
$DE(h)$ is asymptotically conformal iff h is symmetric	[11] & [16] or [21]
$\mu_{DE(h)}(z) = O((1-\|z\|)^{\alpha})$ iff h is a $C^{1+\alpha}$ diffeomorphism for each $0 < \alpha < 1$	[31]

In this paper, we investigate an infinitesimal version of the Douady-Earle extension, namely a conformally natural extension of a continuous tangent vector field along the unit circle $\mathbb{S}^1$. In the following, we first provide the motivation to study such extensions.

Given a smooth curve γ in $T(\mathcal{R})$ through the base point, γ is expressed by a curve f^t of quasiconformal maps depending smoothly on t and with $f^0 = id$. Denote by

$$v = \frac{df^t}{dt}|_{t=0}. \tag{1.2}$$

For each value of t, f_t is a quasiconformal map from $\mathcal{R}$ onto $f_t(\mathcal{R})$ and hence it lifts to a quasiconformal map from $\mathbb{D}$ onto itself, which we denote by F^t. Then the curve f^t lifts to a smooth curve F^t of quasiconformal maps of $\mathbb{D}$ with $F^0 = id$ and satisfying the condition that for any $g \in G(\mathcal{R})$, there exists a smooth curve g_t of Möbius transformations preserving $\mathbb{D}$ with $g_0 = g$ such that

$$F^t \circ g = g^t \circ F^t. \tag{1.3}$$

Taking the derivative of both sides to the variable t, we obtain

$$\frac{dF^t}{dt} \circ g = \frac{dg_t}{dt}(F^t) + \frac{dg_t}{dz}\frac{dF^t}{dt}. \tag{1.4}$$

Denote by $\frac{dF^t}{dt}|_{t=0} = V$ and $\frac{dg_t}{dt}|_{t=0} = H$. Note that H is holomorphic on $\mathbb{D}$, V satisfies the so-called bounded Zygmund condition, and both H and V extend continuously to $\overline{\mathbb{D}}$. Evaluating the previous equation (1.4) at $t = 0$, we obtain

$$V(g(z)) = H(z) + g'(z)V(z) \tag{1.5}$$

for any $z \in \mathbb{D}$. The equation (1.5) extends to $\overline{\mathbb{D}}$ with the extensions of H and V. Let $w = g(z)$. Then $g'(z) = \frac{1}{(g^{-1})'(w)}$ and the equation (1.5) is written as

$$V(w) = H(g^{-1}(w)) + \frac{V(g^{-1}(w))}{(g^{-1})'(w)}.$$

Denote by $g^*V = \frac{V\circ g^{-1}}{(g^{-1})'}$, which is called the pushforward of V under g. Then for any $g \in G(\mathcal{R})$, $g^*V(w) - V(w) = -H(g^{-1}(w))$ and hence

(1.6) $g^*V - V$ is holomorphic and extends to a continuous vector field on $\overline{\mathbb{D}}$.

Let $C_T^0(\mathbb{S}^1)$ be the collection of continuous tangent vector fields along $\mathbb{S}^1$. Contrast to the model of $T(\mathcal{R})$ using a collection of quasisymmetric homeomorphisms of $\mathbb{S}^1$, a natural model to characterize the tangent space of $T(\mathcal{R})$ at the base point is the collection of the Zygmund bounded continuous tangent fields V along $\mathbb{S}^1$ satisfying the condition that for each $g \in G(\mathcal{R})$, $g^*V - V$ has a continuous extension $H_{(V,g)}$ to $\overline{\mathbb{D}}$ that is holomorphic on $\mathbb{D}$. To achieve this goal, one needs to know if V can be extended to a continuous vector field $\tilde{V}$ on $\overline{\mathbb{D}}$ such that $\bar{\partial}\tilde{V}$ is essentially bounded and for any $g \in G(\mathcal{R})$, $g^*\tilde{V} - \tilde{V} = H_{(V,g)}$. The first step to fulfill this goal is to find a linear operator L_0 from $C_T^0(\mathbb{S}^1)$ to $C_T^0(\overline{\mathbb{D}})$ that is *conformally natural* in the following sense:

(i) If $V \in C_T^0(\mathbb{S}^1)$ has a continuous extension H to $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$, then $L_0(V) = H$.

(ii) For any Möbius transformation g preserving $\mathbb{D}$ and any $V \in C_T^0(\mathbb{S}^1)$,

$$L_0(g^*V) = g^*(L_0(V)). \tag{1.7}$$

The second step is to show that $\bar{\partial}L_0(V)$ is essentially bounded if V is Zygmund bounded.

In this paper, we first derive an integral formula for the operator L_0. Then we study the relationships between the regularities of V and $L_0(V)$. Denote by $||V||_{cr}$ the cross-ratio distortion norm of V and $||\bar{\partial}L_0(V)||_\infty$ the L^∞-norm of the $\bar{\partial}$-derivative of $L_0(V)$ on $\mathbb{D}$. Properties of L_0 and the work of this paper on the relationship between regularities of V and $L_0(V)$ are summarized in the following second table.

Properties	*References*
$L_0(g_*(V)) = g_*(L_0(V))$	Theorem 1
Uniqueness of the operator L_0	[10]
$\lVert\bar{\partial}L_0(V)\rVert_\infty$ is finite iff V is Zygmund bounded	[35]
$\frac{1}{C}\lVert V\rVert_{cr} \leq \lVert\bar{\partial}L_0(V)\rVert_\infty \leq C\lVert V\rVert_{cr}$ for a universal constant C	Theorem 2
$\bar{\partial}L_0(V)$ is uniformly vanishing near boundary iff V satisfies the little Zygmund bounded condition	Theorem 3
$\bar{\partial}L_0(V)(z) = O((1-\lvert z\rvert)^\alpha)$ iff V is $C^{1+\alpha}$-smooth for each $0 < \alpha < 1$	Theorem 5

The paper is organized as follows. In the next section, we introduce background and give the integral formula for $L_0(V)$ and the precise statements of the results. In the third section, we show how to derive the formula for $L_0(V)$ and prove its conformal naturality. Then we prove the results in Sections 4, 5, 6 and 7.

2. Background and statements of results

In this section, we introduce definitions and background and give the statements of results.

Let $\mathbb{C}$ be the complex plane and $\mathbb{R}$ be the real line, and let $\mathbb{S}^1$ be the unit circle on the complex plane centered at the origin. An orientation-preserving homeomorphism h of $\mathbb{S}^1$ is said to be *quasisymmetric* if there exists a positive constant C such that

$$\frac{1}{C} \leq \frac{|h(e^{2\pi i(s+t)}) - h(e^{2\pi is})|}{|h(e^{2\pi is}) - h(e^{2\pi i(s-t)})|} \leq C \tag{2.1}$$

for any $s \in \mathbb{R}$ and $0 < t < \frac{1}{2}$. Let $\mathrm{QS}(\mathbb{S}^1)$ be the collection of all quasisymmetric homeomorphisms h of $\mathbb{S}^1$ and $\text{Möb}(\mathbb{S}^1)$ be the Möbius transformation preserving $\mathbb{D}$. Two elements h_1 and h_2 of $\mathrm{QS}(\mathbb{S}^1)$ are equivalent if there exists an element $g \in \text{Möb}(\mathbb{S}^1)$ such that $h_2 = g \circ h_1$. The universal Teichmüller space $T(\mathbb{D})$ is defined as the quotient space of $\mathrm{QS}(\mathbb{S}^1)$ under this equivalence relation. It is known that $T(\mathbb{D})$ is an infinitely dimensional complex manifold modelled on a Banach space. The tangent space $\Lambda(\mathbb{S}^1)$ of $T(\mathbb{D})$ at the base point is characterized in [36] by the collection of the continuous tangent vector fields V along $\mathbb{S}^1$ vanishing at the three points -1, $-i$ and 1 and satisfying the following so-called *Zygmund bounded* condition ([41]); that is, there exists a positive constant C such that

$$|\frac{V(e^{2\pi i(s+t)})-2V(e^{2\pi is})+V(e^{2\pi i(s-t)})}{t}|\le C \tag{2.2}$$

for all $s\in\mathbb{R}$ and $0<t<\frac{1}{2}$. For background on quasiconformal mappings and Teichmüller spaces, we refer to the textbooks [3], [28], [27], and [15].

An element h of $\mathrm{QS}(\mathbb{S}^1)$ is said to be *symmetric* if the quotient of (2.1) approaches 0 uniformly on s as t goes to 0. Let $\mathrm{S}(\mathbb{S}^1)$ be the collection of all symmetric elements of $\mathrm{QS}(\mathbb{S}^1)$. Then the so-called little Teichmüller space $T_0(\mathbb{D})$ is defined as the quotient space of $\mathrm{S}(\mathbb{S}^1)$ under the same equivalence relation to define $T(\mathbb{D})$. Gardiner and Sullivan proved in [16] that $\mathrm{S}(\mathbb{S}^1)$ is a normal subgroup of $\mathrm{QS}(\mathbb{S}^1)$ under the composition, $T_0(\mathbb{D})$ is an infinitely dimensional complex manifold modelled on a Banach space, and the coset space $T(\mathbb{D})/T_0(\mathbb{D})$ is also a complex manifold. Furthermore, the tangent space of T_0 at the base point is characterized by the collection $\Lambda_0(\mathbb{S}^1)$ of the elements of $\Lambda(\mathbb{S}^1)$ satisfying that the left hand side of (2.2) approaches 0 uniformly on s as t goes to 0. The elements of $\Lambda_0(\mathbb{S}^1)$ are said to be *little Zygmund bounded* continuous tangent vector fields along $\mathbb{S}^1$ ([41]).

The space $T(\mathbb{D})$ is often defined by using quasiconformal homeomorphisms of $\mathbb{D}$ and the tangent space of $T(\mathbb{D})$ at the base point is defined by the collection of continuous vector fields on $\mathbb{D}$ with essentially bounded $\bar{\partial}$-derivatives. To prove such definitions to be equivalent to the previous definitions, Berling-Ahlfors extensions of quasisymmetric homeomorphisms h of $\mathbb{R}$ are developed and applied ([5]). Correspondingly, the space $T_0(\mathbb{D})$ is often defined by using the quasiconformal homeomorphisms of $\mathbb{D}$ that are uniformly asymptotically conformal towards the boundary $\mathbb{S}^1$ and the tangent space of $T_0(\mathbb{D})$ at the base point is defined by the collection of the continuous vector fields on $\mathbb{D}$ with the $\bar{\partial}$-derivatives essentially bounded and uniformly asymptotically vanishing towards $\mathbb{S}^1$. Berling-Ahlfors extensions of symmetric homeomorphisms of $\mathbb{R}$ and little Zygmund bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are studied and used in [16] to prove these definitions of $T_0(\mathbb{D})$ and its tangent space at the base point to be equivalent to the ones given in the previous paragraph.

Berling-Ahlfors extensions of quasisymmetric homeomorphisms of $\mathbb{R}$ or Zygmund bounded continuous functions V defined on $\mathbb{R}$ are compatible with the affine maps preserving the upper half plane $\mathbb{H}$, but not with all conformal homeomorphisms of $\mathbb{H}$. Therefore, one often needs to pay extra attention (or put extra conditions) to these extensions near ∞, especially when used to study the elements of $T_0(\mathbb{D})$ or the tangent space of $T_0(\mathbb{D})$ at the base point. Douady-Earle (resp. infinitesimal Douady-Earle) extensions of homeomorphisms h of $\mathbb{S}^1$ (resp. continuous tangent vector fields V along $\mathbb{S}^1$) eliminate such inconveniences. The main goal of this paper is two-fold. One is to develop the corresponding results for infinitesimal Douady-Earle extensions that are contrast to some known results for Douady-Earle extensions; the other is to apply the infinitesimal Douady-Earle extensions to prove a model that characterizes the tangent space of the Teichmüller space $T_0^\alpha(\mathbb{D})$ of $C^{1+\alpha}$ diffeomorphisms of $\mathbb{S}^1$ for each $0 < \alpha < 1$.

In the following, we first introduce norms to quantize the quasisymmetry of h and the Zygmund bound of V. The norm of h is invariant under pre- or post-composition by any element of $\text{Möb}(\mathbb{S}^1)$ and the norm of V is invariant under the pushforward by any element of $\text{Möb}(\mathbb{S}^1)$. These properties enable one to show that the norm quantifying the quasiconformality of the Douady-Earle extension of h is controlled by the norm quantifying the quasisymmetry of h on $\mathbb{S}^1$. They also enable us to prove in this paper that the norm to quantify the distortion of the infinitesimal Douady-Earle extension of V is actually equivalent to the norm to quantify the Zygmund bound of V on $\mathbb{S}^1$.

Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points a, b, c and d on the unit circle $\mathbb{S}^1$ arranged in counterclockwise order, we denote the cross ratio $cr(Q)$ of Q by

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

For an orientation-preserving homeomorphism h of $\mathbb{S}^1$, the *cross-ratio distortion norm* of h is defined as

$$\|h\|_{cr} = \sup_{cr(Q)=1} |\ln cr(h(Q))|,$$

where $h(Q)$ be the image quadruple $\{h(a), h(b), h(c), h(d)\}$. We say h is quasisymmetric if $\|h\|_{cr}$ is finite. This definition is equivalent to the definition (2.1) using bounded ratio distortions of h on all symmetric triples on $\mathbb{S}^1$ ([12]).

For any quadruple $Q = \{a, b, c, d\}$ on $\mathbb{S}^1$, the minimal scale $s(Q)$ is defined as

$$s(Q) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

A sequence $\{Q_n = \{a_n, b_n, c_n, d_n\}\}_{n=1}^{\infty}$ of quadruples is said to be *degenerating* if $cr(Q_n) = 1$ for each n and $\lim_{n\to+\infty} s(Q_n) = 0$. It is given in [12] that a quasisymmetric homeomorphism h of $\mathbb{S}^1$ is symmetric if and only if

$$\sup_{\{Q_n\}} \limsup_{n\to\infty} |cr(h(Q_n))| = 0,$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

Let $C^0(\mathbb{S}^1, \mathbb{C})$ be the collection of the continuous functions from $\mathbb{S}^1$ to $\mathbb{C}$ and $V \in C^0(\mathbb{S}^1, \mathbb{C})$. The *cross-ratio distortion norm* of V is defined as

$$\|V\|_{cr} = \sup_{cr(Q)=1} |V[Q]| < +\infty,$$

where

$$V[Q] = \frac{V(b)-V(a)}{b-a} - \frac{V(c)-V(b)}{c-b} + \frac{V(d)-V(c)}{d-c} - \frac{V(a)-V(d)}{a-d}.$$

It is given in [18] that V is Zygmund bounded if and only if $\|V\|_{cr} < +\infty$. Furthermore, $V \in \Lambda_0(\mathbb{S}^1)$ if and only if

$$\sup_{\{Q_n\}} \limsup_{n\to\infty} |V[Q_n])| = 0,$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

For any domain Ω on the complex plane $\mathbb{C}$, let $L^{\infty}(\Omega)$ be the Banach space of essentially bounded measurable complex-valued functions defined on Ω and $M(\Omega)$ be the open unit ball in $L^{\infty}(\Omega)$ centered at the base point represented by the 0-constant function. Given a quasiconformal map f from Ω to $\mathbb{C}$, the *Beltrami coefficient* μ_f is defined as

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z},$$

which is well-defined almost everywhere on Ω with respect to the Lebesgue measure on Ω and $\mu_f \in M(\Omega)$.

Now we give the definition for the Douady-Earle extension of a homeomorphism h of $\mathbb{S}^1$. Given a point $z \in \mathbb{D}$, let η_z be the harmonic measure on

$\mathbb{S}^1$ viewed from z and normalized to have the measure of $\mathbb{S}^1$ equal to 1; that is, for any Borel set $A \subset \mathbb{S}^1$,

$$\eta_z(A) = \frac{1}{2\pi}\int_A \frac{1-|z|^2}{|z-\xi|^2}|d\xi|. \tag{2.3}$$

Now let $h_*(\eta_z)$ be the push-forward of the measure η_z by h; that is,

$$h_*(\eta_z)(A) = \eta_z(h^{-1}(A)) \tag{2.4}$$

for any Borel set $A \subset \mathbb{S}^1$. In [9], the *conformal barycenter* of the measure $h_*(\eta_z)$, denoted by $w = B(h_*(\eta_z))$, is defined to be the unique point $w \in \mathbb{D}$ such that

$$\frac{1}{2\pi}\int \frac{\zeta - w}{1-\bar{w}\zeta} dh_*(\eta_z)(\zeta) = 0. \tag{2.5}$$

Then the *conformal barycentric extension* Φ of h is defined as: $\Phi(z) = B(h_*(\eta_z))$ for each $z \in \mathbb{D}$ and $\Phi(z) = h(z)$ for each $z \in \mathbb{S}^1$. We call Φ the *Douady-Earle extension* of h and denote it by $DE(h)$. This extension is *conformally natural* in the sense that

1. for each element $g \in \text{Möb}(\mathbb{S}^1)$, $DE(g|_{\mathbb{S}^1}) = g$, and
2. for any two elements g_1 and $g_2 \in \text{Möb}(\mathbb{S}^1)$,

$$DE(g_2 \circ h \circ g_1) = g_2 \circ DE(h) \circ g_1.$$

Other properties of $DE(h)$ are summarized in the table 1 in the introduction.

Remark 1. Conformally natural extensions can be introduced for circle maps beyond homeomorphisms. The works in this direction can be found in [1], [2], and [23]. Finally, conformally natural extensions are constructed for arbitrarily continuous maps from $\mathbb{S}^1$ to itself in [24].

In this paper, we pay attention to so-called infinitesimal conformally natural extensions of continuous tangent vector fields V along $\mathbb{S}^1$. In this section, we first give the integral formal for this extension. In the next section, we first show how this integral expression is derived from the conformal naturality condition and then prove it is conformally natural in general.

Given an element $V \in C^0(\mathbb{S}^1, \mathbb{C})$ and any $z \in \mathbb{D}$, $L_0(V)(z)$ is defined as:

$$L_0(V)(z) = \frac{(1-|z|^2)^3}{2\pi i}\int_{\mathbb{S}^1} \frac{V(\zeta)}{(1-\bar{z}\zeta)^3(\zeta - z)}d\zeta. \tag{2.6}$$

In this paper, we prove the following theorems.

Theorem 1. *The operator L_0 is conformally natural in the following sense:*

1. *If $V \in C^0(\mathbb{S}^1, \mathbb{C})$ has a continuous extension H to $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$, then $L_0(V) = H$.*
2. *For any element g in Möb$(\mathbb{S}^1)$ and any $V \in C^0(\mathbb{S}^1, \mathbb{C})$,*

$$L_0(g^*V) = g^*(L_0(V)). \tag{2.7}$$

Remark 2. Any linear operator from $C^0(\mathbb{S}^1, \mathbb{C})$ to $C^0(\mathbb{D}, \mathbb{C})$ satisfying the above condition (2) is equal to L_0 up to multiplication by a constant ([10]). If such a linear operator is further required to either satisfy the above condition (1) or extend the elements of $C^0(\mathbb{S}^1, \mathbb{C})$ to the elements of $C^0(\overline{\mathbb{D}}, \mathbb{C})$ with given boundary maps, then it is equal to L_0. The extension operator L_0 in higher dimensional cases is systematically studied by McMullen in Appendix B of [33], where $L_0(V)$ is called the visual extension of a vector field V.

Remark 3. Let h_t be a smooth curve of diffeomorphisms of $\mathbb{S}^1$ with

$$h_t(\zeta) = \zeta + tV(\zeta) + o(t), \tag{2.8}$$

where $\zeta \in \mathbb{S}^1$, t is a real parameter, and V is a smooth tangent vector field along $\mathbb{S}^1$. It is proved in Theorem 2 of [10] that

$$\frac{dDE(h^t)}{dt}|_{t=0} = L_0(V). \tag{2.9}$$

This means that acting on the smooth tangent vector fields V along $\mathbb{S}^1$, the operator L_0 is the derivative of the Douady-Earle extension operator at the identity map. Therefore, L_0 is called the infinitesimal version of the Douady-Earle extension operator. Here an interested question arises. Suppose V is a continuous tangent vector field along $\mathbb{S}^1$ and a smooth curve h_t of homeomorphisms of $\mathbb{S}^1$ satisfies (2.8). What regularities (weaker than smoothness) on V are sufficient for the equality (2.9) to hold?

Proposition 1. *For any $V \in C^0(\mathbb{S}^1, \mathbb{C})$,*

$$\|\bar{\partial} L_0(V)\|_\infty \leq 3\|V\|_{cr}.$$

Let $\Lambda(\mathbb{S}^1, \mathbb{C})$ be the collection of the continuous functions V from $\mathbb{S}^1$ to $\mathbb{C}$ with $||V||_{cr} < +\infty$, and let $\Lambda(\mathbb{S}^1)$ be the collection of all Zygmund bounded continuous tangent vector fields along $\mathbb{S}^1$. Assume $V \in C^0(\mathbb{S}^1, \mathbb{C})$. Clearly, $V \in \Lambda(\mathbb{S}^1)$ if and only if $V \in \Lambda(\mathbb{S}^1, \mathbb{C})$ and satisfies the following tangency condition

$$Re\ \bar{\zeta} V(\zeta) = 0 \text{ for any } \zeta \in \mathbb{S}^1. \tag{2.10}$$

Proposition 2. *There exists a constant $C > 0$ such that for any $V \in \Lambda(\mathbb{S}^1)$,*

$$\|\bar{\partial} L_0(V)\|_\infty \geq C\|V\|_{cr}. \tag{2.11}$$

From Propositions 1 and 2, we reach the following result.

Theorem 2. *There exists a universal constant $C > 0$ such that for any $V \in \Lambda(\mathbb{S}^1)$,*

$$\frac{1}{C}||V||_{cr} \leq ||\bar{\partial} L_0(V)||_\infty \leq C||V||_{cr}.$$

Theorem 3. *Let $V \in \Lambda(\mathbb{S}^1)$. Then $V \in \Lambda_0(\mathbb{S}^1)$ if and only if*

$$\lim_{|z|\to 1} |\bar{\partial} L_0(V)(z)| = 0.$$

Clearly, Theorem 2 implies the following Theorem A, and it also provides an alternative proof to Theorem B which was obtained in [36] by using the upper half plane and the Berling-Ahlfors extension of V. Theorem 3 gives a different method to show Theorem C, which was reached in [16] by further investigating the Berling-Ahlfors extension of V.

Theorem A ([35]) *Let Let $V \in \Lambda(\mathbb{S}^1)$. Then $||\bar{\partial} L_0(V)||_\infty$ is finite if and only if V is Zygmund bounded.*

Theorem B ([36]) *The tangent space of $T(\mathbb{D})$ at the base point is characterized by the collection of all elements $V \in \Lambda(\mathbb{S}^1)$ vanishing at three points 1, i and -1; that is,*

$$V(1) = V(-1) = V(i) = 0. \tag{2.12}$$

Theorem C ([16]) *The tangent space of $T_0(\mathbb{D})$ at the base point is characterized by the collection of all elements $V \in \Lambda_0(\mathbb{S}^1)$ satisfying the vanishing condition (2.12).*

With the conformal naturality of $L_0(V)$ (Theorem 1) and Theorem 2, we obtain the following theorem.

Theorem 4. *Let $\mathcal{R}$ be a hyperbolic Riemann surface and $G(\mathcal{R})$ be the group of the deck/cover transformations of $\mathbb{D}$ over $\mathcal{R}$. If the limit set of $G(\mathcal{R})$ is $\mathbb{S}^1$, then the tangent space of $T(\mathcal{R})$ at the base point is characterized by the collection of the elements of $V \in \Lambda(\mathbb{S}^1)$ satisfying the group compatible condition*

$$g^*V = V \text{ for any } g \in G(\mathcal{R}). \tag{2.13}$$

The other main goal of this paper is to introduce a characterization of the tangent space of the Teichmüller space of circle diffeomorphisms.

Let $0 < \alpha < 1$. A diffeomorphism h of $\mathbb{S}^1$ belongs to $\mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)$ if its derivative h' is α-Hölder continuous. The space $T_0^\alpha(\mathbb{D}) = \mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)/\mathrm{Möb}(\mathbb{S}^1)$ is introduced and studied by Matsuzaki in [30], [31] and [32]. Let $\rho_{\mathbb{D}}$ be the hyperbolic density on $\mathbb{D}$, it is shown in [31] that $\|\mu_{DE(h)}(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty$ if and only if $h \in \mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)$. A complex structure on $T_0^\alpha(\mathbb{D})$ is introduced in [32].

Let $V \in \Lambda(\mathbb{S}^1)$. By $V \in C^{1+\alpha}$ we mean that V is differentiable and V' is α-Hölder continuous. In this paper, we show that $V \in C^{1+\alpha}$ if and only if

$$\|\bar{\partial} L_0(V)(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty.$$

We achieve this result by introducing an equivalent condition for $V \in C^{1+\alpha}$ and proving the three conditions to be equivalent.

Let $\Lambda_0^\alpha(\mathbb{S}^1)$ be the collection of the elements V of $\Lambda(\mathbb{S}^1)$ satisfying

$$\|V\|_{cr}^\alpha = \sup_{cr(Q)=1, S(Q)\geq\sqrt{2}} |V[Q](\frac{1}{s(Q)})^\alpha| < +\infty, \tag{2.14}$$

where the maximum scale $S(Q)$ is defined as

$$S(Q) = \max\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

We show the following theorem.

Theorem 5. *Let $V \in \Lambda(\mathbb{S}^1)$. The following three conditions are equivalent:*

1. $V \in \Lambda_0^\alpha(\mathbb{S}^1)$.
2. $V \in C^{1+\alpha}$.
3. $\|\bar{\partial} L_0(V)(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty$.

Using Theorem 5, we characterize the tangent space to $T_0^\alpha(\mathbb{D})$ at the base point as follows.

Theorem 6. *The tangent space of $T_0^\alpha(\mathbb{D})$ at the base point is characterized by the collection of all elements of $\Lambda_0^\alpha(\mathbb{S}^1)$ satisfying the vanishing condition (2.12).*

The remaining sections are arranged as follows. We prove Theorem 1 in Section 3, Propositions 1 and 2 in Section 4, Theorem 3 in Section 5, Theorem 5 in Section 6. In Section 7, we provide alternative proofs of Theorems B and C and show Theorems 4 and 6.

3. The conformal naturality of $L_0(V)$

In this section, we first derive the integral formula for $L_0(V)$. Then we prove the conformal naturality of $L_0(V)$.

Given an element $V \in C^0(\mathbb{S}^1, \mathbb{C})$, an extension $L_0(V)$ of V to $\mathbb{D}$ is defined as follows. We define

$$L_0(V)(0) = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})d\theta = \frac{1}{2\pi i}\int_{\mathbb{S}^1} V(\zeta)\frac{d\zeta}{\zeta}. \tag{3.1}$$

For each $a \in \mathbb{D}$, let $w = g_a(\zeta) = \frac{\zeta - a}{1-\bar{a}\zeta}$. Then we apply the condition (1.7) to define $L_0(V)(a)$, which goes as follows:

$$\begin{aligned}
L_0(V)(a) = L_0(V)(g_a^{-1}(0)) &= \frac{L_0(V)(g_a^{-1}(0))}{(g_a^{-1})'(0)}(g_a^{-1})'(0)\\
&= ((g_a)^* L_0(V))(0)(g_a^{-1})'(0) = L_0((g_a)^* V)(0)\frac{1}{g_a'(a)}\\
&= \frac{1}{g_a'(a)}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{V(g_a^{-1}(w))}{(g_a^{-1})'(w)}\frac{dw}{w}\\
&= \frac{1}{g_a'(a)2\pi i}\int_{\mathbb{S}^1} V(\zeta)(g_a'(\zeta))^2\frac{d\zeta}{g_a(\zeta)}\\
&= \frac{(1-|a|^2)^3}{2\pi i}\int_{\mathbb{S}^1}\frac{V(\zeta)}{(1-\bar{a}\zeta)^3(\zeta - a)}d\zeta.
\end{aligned}$$

This means $L_0(V)(a)$ is defined by

$$L_0(V)(a) = \frac{1}{g_a'(a)2\pi i}\int_{\mathbb{S}^1} V(\zeta)(g_a'(\zeta))^2\frac{d\zeta}{g_a(\zeta)}, \tag{3.2}$$

or more explicitly by the integral operator

$$L_0(V)(a) = \frac{(1-|a|^2)^3}{2\pi i}\int_{\mathbb{S}^1}\frac{V(\zeta)}{(1-\bar{a}\zeta)^3(\zeta - a)}d\zeta. \tag{3.3}$$

Clearly, the expression (3.3) of $L_0(V)(a)$ recovers the definition of $L_0(V)(0)$ by substituting a by 0. It is also clear that $L_0(V)$ is a linear operator from $C^0(\mathbb{S}^1, \mathbb{C})$ to $C^0(\overline{\mathbb{D}}, \mathbb{C})$.

Now we prove Theorem 1.

Proof of Theorem 1. From the Riemann integral formula, one can see that if V has a continuous extension H to $\overline{\mathbb{D}}$ that is holomorphic on $\mathbb{D}$, then $L_0(V) = H$.

Now we verify that $L_0(V)$ satisfies the condition (1.7), which is called the conformal naturality of the extension. Using the expression (3.2) of $L_0(V)(a)$, we can see that the value of $L_0(V)(a)$ is not changed if we replace g_a by $e^{i\theta}g_a$ for any θ. Let B be a Möbius transformation preserving $\mathbb{D}$. Then $B^*(L_0(V))(B(z)) = B'(z)L(V)(z)$. Thus, it suffices to show that for any $z \in \mathbb{D}$, $L_0(B^*V)(B(z)) = B'(z)L(V)(z)$ as well. This can be proved effectively by using the expression (3.2). Clearly,

$$\begin{aligned} L_0(B^*V)(B(z)) &= \frac{1}{g'_{B(z)}(B(z))2\pi i}\int_{\mathbb{S}^1}(B^*V)(\zeta)(g'_{B(z)}(\zeta))^2\frac{d\zeta}{g_{B(z)}(\zeta)} \\ &= \frac{1}{g'_{B(z)}(B(z))2\pi i}\int_{\mathbb{S}^1}\frac{V(B^{-1}(\zeta))}{(B^{-1})'(\zeta)}(g'_{B(z)}(\zeta))^2\frac{d\zeta}{g_{B(z)}(\zeta)}. \end{aligned}$$

Let $\eta = B^{-1}(\zeta)$. Then

$$L_0(B^*V)(B(z)) = \frac{B'(z)}{g'_{B(z)\circ B}(z)2\pi i}\int_{\mathbb{S}^1}V(\eta)[(g_{B(z)}\circ B)'(\eta)]^2\frac{d\eta}{(g_{B(z)}\circ B)(\eta)}.$$

Since $(g_{B(z)}\circ B)(z) = 0$, it follows that

$$L_0(B^*V)(B(z)) = B'(z)L_0(V)(z).$$

□

4. Conformally natural extensions of the elements in $\Lambda(\mathbb{S}^1)$

In this section, we prove Propositions 1 and 2. We first give a few lemmas.

Let $\Lambda(\mathbb{R},\mathbb{C})$ be the collection of all complex-valued continuous functions V defined on the real line $\mathbb{R}$ satisfying

$$|V(x+t) - 2V(x) + V(x-t)| \le C|t| \tag{4.1}$$

for all $x, t \in \mathbb{R}$ and a positive constant C. Furthermore, denote by $\|V\|_Z$ be the infimum of the constant C in (4.1). The elements of $\Lambda(\mathbb{R},\mathbb{C})$ are called Zygmund bounded complex-valued functions.

We first generalize a lemma of [6] from Zygmund bounded real-valued functions to complex-valued functions. The proof is the same.

Lemma 1 ([6])**.** *Suppose $V \in \Lambda(\mathbb{R},\mathbb{C})$ and $V(0) = V(1) = 0$. Then*

$$M = \max\{|V(x)| : 0 \le x \le 1\} \le \frac{1}{3}\|V\|_Z.$$

Lemma 2 ([19])**.** *Let B be a Möbius transformation preserving $\mathbb{D}$ and $\widetilde{V} = B^*V$. Then for any quadruple Q on $\mathbb{S}^1$,*

$$V[Q] = \widetilde{V}[B(Q)].$$

It follows that $||B^*V||_{cr} = ||V||_{cr}$.

By taking the $\bar{z}$-derivative of the integrand of (2.6), the $\bar{z}$-derivative of $L_0(V)$ is obtained in [35] as

$$\bar{\partial}L_0(V)(z) = \frac{3(1-|z|^2)^2}{2\pi i}\int_{\mathbb{S}^1}\frac{V(\zeta)}{(1-\bar{z}\zeta)^4}d\zeta. \tag{4.2}$$

Lemma 3. *Given a point $z_0 \in \mathbb{D}$, let $B \in \text{Möb}(\mathbb{S}^1)$ with $B(z_0) = 0$. Then*

$$\bar{\partial}(L_0(V))(z_0) = \bar{\partial}(L_0(\widetilde{V}))(0), \tag{4.3}$$

where $\widetilde{V} = B^*V$.

Proof. Denote by $\zeta = B(z) = \frac{z-z_0}{1-\overline{z_0}z}$. Then $B'(z) = \frac{1-|z_0|^2}{(1-\overline{z_0}z)^2}$. Using (4.2), we obtain

$$\begin{aligned}\bar{\partial}(L_0(\widetilde{V}))(0) &= \frac{3}{2\pi i}\int_{\mathbb{S}^1}\widetilde{V}(\zeta)d\zeta = \frac{3}{2\pi i}\int_{\mathbb{S}^1}\frac{V(B^{-1}(\zeta))}{(B^{-1})'(\zeta)}d\zeta \\ &= \frac{3}{2\pi i}\int_{\mathbb{S}^1}B'(B^{-1}(\zeta))V(B^{-1}(\zeta))d\zeta \\ &= \frac{3}{2\pi i}\int_{\mathbb{S}^1}(B'(z))^2V(z)dz = \bar{\partial}(L_0(V))(z_0). \qquad \square\end{aligned}$$

Given any two points a and b on $\mathbb{S}^1$, denote by $\widehat{a,b}$ the arc on $\mathbb{S}^1$ that connects a to b in the counterclockwise direction.

Lemma 4. *Assume that $V \in \Lambda(\mathbb{S}^1, \mathbb{C})$ satisfies the vanishing normalization (2.12). Then*

$$\max_{z\in\mathbb{S}^1}|V(z)| \leq \frac{4}{3}\|V\|_{cr}.$$

More precisely,

$$\max_{z\in\widehat{-1,1}}|V(z)| \leq \frac{4}{3}\|V\|_{cr} \text{ and } \max_{z\in\widehat{1,-1}}|V(z)| \leq \frac{2}{3}\|V\|_{cr}.$$

Proof. We divide the proof into two steps.

Step 1: We show

$$\max_{z\in\widehat{-1,1}} |V(z)| \leq \frac{4}{3}\|V\|_{cr}. \tag{4.4}$$

Let B be the Möbius transformation from $\mathbb{D}$ to the upper half plane $\mathbb{H}$ that maps $-1, 1$ and i to $0, 1$ and ∞ respectively. Then $B(z) = \frac{1-i}{2}\frac{z+1}{z-i}$. Let $\widetilde{V} = B^*V$. Then $\widetilde{V}(x) = \frac{V(B^{-1}(x))}{(B^{-1})'(x)}$, $\widetilde{V}(0) = V(-1) = 0$, and $\widetilde{V}(1) = V(1) = 0$. For any quadruple $Q = (x-t, x, x+t, \infty) \subset \mathbb{R}$, Lemma 2 implies

$$|\frac{\widetilde{V}(x-t) - 2\widetilde{V}(x+t) + \widetilde{V}(x+t)}{t}| = \widetilde{V}[Q] = V[B^{-1}(Q)] \leq \|V\|_{cr}.$$

Using Lemma 1, we obtain

$$\max_{0\leq x\leq 1} |\widetilde{V}(x)| \leq \frac{1}{3}\|\widetilde{V}\|_Z \leq \frac{1}{3}\|V\|_{cr}.$$

Since $\widetilde{V}(x) = \frac{V(B^{-1}(x))}{(B^{-1})'(x)}$ and $B(\widehat{-1,1}) = [0,1]$, it follows that

$$\begin{aligned} \max_{z\in\widehat{-1,1}} |V(z)| &= \max_{z\in\widehat{-1,1}} |\widetilde{V}(B(z))(B^{-1})'(B(z))| \\ &= \max_{0\leq x\leq 1} |\widetilde{V}(x)(B^{-1})'(x)| \\ &\leq \frac{1}{3}\|V\|_{cr} \cdot \max_{0\leq x\leq 1} |(B^{-1})'(x)| \\ &= \frac{1}{3}\|V\|_{cr} \cdot \max_{z\in\widehat{-1,1}} |\frac{1}{B'(z)}|. \end{aligned} \tag{4.5}$$

Clearly, $B'(z) = \frac{-1}{(z-i)^2}$. Thus,

$$\max_{z\in\widehat{-1,1}} |\frac{1}{B'(z)}| \leq 4. \tag{4.6}$$

Then the inequality (4.4) follows from (4.5) and (4.6).

Step 2: We show

$$\max_{z\in\widehat{1,i}} |V(z)| \leq \frac{2}{3}\|V\|_{cr} \text{ and } \max_{z\in\widehat{i,-1}} |V(z)| \leq \frac{2}{3}\|V\|_{cr}. \tag{4.7}$$

Without loss of generality, we show

$$\max_{z\in\widehat{1,i}}|V(z)|\le\frac{2}{3}\|V\|_{cr}.$$

Let B be the Möbius transformation from $\mathbb{D}$ to $\mathbb{H}$ that maps $1, i$ and -1 to $0, 1$ and ∞ respectively. Then $B(z)=-i\frac{z-1}{z+1}$. Let $\widetilde{V}=B^{*}V$. From the same argument in Step 1, it follows that for any quadruple $Q=(x-t,x,x+t,\infty)\subset\mathbb{R}$,

$$|\frac{\widetilde{V}(x-t)-2\widetilde{V}(x+t)+\widetilde{V}(x+t)}{t}|=\widetilde{V}[Q]=V[B^{-1}(Q)]\le\|V\|_{cr}.$$

Clearly, $B'(z)=-i\frac{2}{(z+1)^2}$. Then

$$\max_{z\in\widehat{1,i}}|\frac{1}{B'(z)}|\le 2. \tag{4.8}$$

Using the inequalities (4.5) and (4.8), we obtain

$$\max_{z\in\widehat{1,i}}|V(z)|\le\frac{1}{3}\|V\|_{cr}\cdot\max_{z\in\widehat{1,i}}|\frac{1}{B'(z)}|\le\frac{2}{3}\|V\|_{cr}. \tag{4.9}$$ □

Proof of Proposition 1. Let $P(x)$ be a quadratic polynomial. Note that for any quadruple $Q\in\mathbb{S}^1$, $P[Q]=0$. Thus, for any $V\in C^0(\mathbb{S}^1,\mathbb{C})$,

$$\|V\|_{cr}=\|V+P\|_{cr}. \tag{4.10}$$

Note also that it suffices to prove the conclusion for $V\in C^0(\mathbb{S}^1,\mathbb{C})$ with $||V||_{cr}<+\infty$.

Given a point $z_0\in\mathbb{D}$, we choose $B(z)\in\text{Möb}(\mathbb{S}^1)$ with $B(z_0)=0$ and denote by $\widetilde{V}=B^{*}V$. Applying Lemma 2, we obtain

$$\|V\|_{cr}=\|\widetilde{V}\|_{cr}. \tag{4.11}$$

Now we choose a quadratic polynomial P such that $(\widetilde{V}+P)(1)=(\widetilde{V}+P)(i)=(\widetilde{V}+P)(-1)=0$. Denote by $\widetilde{\widetilde{V}}=\widetilde{V}+P$. Then

$$\|\widetilde{\widetilde{V}}\|_{cr}=\|\widetilde{V}\|_{cr}=\|V\|_{cr}. \tag{4.12}$$

Since $\int_{\mathbb{S}^1} P(z)dz = 0$, it follows from Lemma 3, (4.2) and Lemma 4 that

$$\begin{aligned}
|\bar\partial(L_0(V))(z_0)| &= |\bar\partial(L_0(\widetilde V))(0)| = |\frac{3}{2\pi i}\int_{\mathbb{S}^1}\widetilde V(z)dz| \\
&= |\frac{3}{2\pi i}\int_{\mathbb{S}^1}\widetilde{\widetilde V}(z)dz| \le \frac{3}{2\pi}\int_{\mathbb{S}^1}|\widetilde{\widetilde V}(z)||dz| \\
&= \frac{3}{2\pi}(\int_{\widehat{-1,1}}|\widetilde{\widetilde V}||dz| + \int_{\widehat{1,-1}}|\widetilde{\widetilde V}(z)||dz|) \\
&\le \frac{3}{2\pi}(\pi\cdot\frac{4}{3}\|\widetilde{\widetilde V}\|_{cr} + \pi\cdot\frac{2}{3}\|\widetilde{\widetilde V}\|_{cr}) \\
&= 3\|V\|_{cr}.
\end{aligned}$$

This completes the proof. □

In the next, we prove Proposition 2.

We first recall the holomorphic dependence of the solution of the Beltrami differential equation proved by Ahlfors and Bers [4]. For any $\mu(z) \in L^\infty(\mathbb{C})$, there is a curve of quasiconformal homeomorphisms $f^{t\mu}$ of the extended complex plane $\widehat{\mathbb{C}}$ defined for $|t| < 1/\|\mu\|_\infty$ such that

1) $f^{t\mu}$ is the identity map when $t = 0$,
2) if it is normalized to fix three given points in $\widehat{\mathbb{C}}$, $f^{t\mu}$ is the uniquely quasiconformal mapping with Beltrami coefficient $t\mu$,
3) $f^{t\mu}$ is holomorphic as a function of t and

$$f^{t\mu}(z) = z + tF(z) + O(t^2), \tag{4.13}$$

where $O(t^2)$ is uniform for z on any compact set. If a, b and c are fixed, then

$$F(z) = -\frac{(z-a)(z-b)(z-c)}{\pi}\iint_{\mathbb{C}}\frac{\mu(\zeta)d\xi d\eta}{(\zeta-a)(\zeta-b)(\zeta-c)(\zeta-z)}. \tag{4.14}$$

The function $F(z)$ is uniquely determined by the following two conditions:

a) $F(a) = F(b) = F(c) = 0$, and $F(z)$ has a growth of $|z|^2$ as $z \to \infty$, and
b) $\overline{\partial}F(z) = \mu(z)$ in the generalized sense.

Now we apply the representation (4.14) of F to prove Proposition 2.

Proof of Proposition 2. We divide the proof into two steps.

Step 1: Find a representation of V in terms of $\bar{\partial}L_0(V)(z)$.

Given an element $V \in \Lambda(\mathbb{S}^1)$, let $H_1(z) = L_0(V)(z)$ in $\mathbb{D}$ and $\mu_1(z) = \bar{\partial}H_1(z)$. By Proposition 1, $\mu_1 \in L^\infty(\mathbb{D})$. When $|z|>1$, let $H_2(z) = -z^2\overline{H_1(\bar{z}^{-1})}$ and $\mu_2(z) = \bar{\partial}H_2(z)$. Then

$$|\mu_2(z)| = |\mu_1(\bar{z}^{-1})|. \tag{4.15}$$

With the normalization (2.10), we know $H_1(z) = H_2(z)$ when $|z| = 1$. Then

$$H(z) = \begin{cases} H_1(z), & |z| \le 1 \\ H_2(z), & |z| > 1 \end{cases}$$

is an extension of V to the complex plane $\mathbb{C}$ with $H(z) = O(z^2)$ as $z \to \infty$. Let $\mu(z) = \bar{\partial}H(z)$. Then

$$\mu(z) = \begin{cases} \mu_1(z), & |z| \le 1, \\ \mu_2(z), & |z| > 1. \end{cases} \tag{4.16}$$

From the unique representation formula (4.14), if follows that when $z \in \mathbb{S}^1$,

$$V(z) = -\frac{(z-1)(z-i)(z+1)}{\pi}\iint_{\mathbb{C}} \frac{\mu(\zeta)d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta-z)}. \tag{4.17}$$

Step 2: Estimate $\|V\|_{cr}$ in terms of $\|\bar{\partial}L_0(V)\|_\infty$.

Given a quadruple $Q = \{a, b, c, d\}$ on $\mathbb{S}^1$ with $cr(Q) = 1$, let $B \in M\ddot{o}b(\mathbb{S}^1)$ mapping a, b, c and d to $1, i, -1$ and $-i$ respectively and let $\widetilde{V} = B^*V$. Now we add a quadratic polynomial P to $\widetilde{V}$ such that $(\widetilde{V}+P)(1) = (\widetilde{V}+P)(i) = (\widetilde{V}+P)(-1) = 0$. Then for $z \in \mathbb{S}^1$,

$$(\widetilde{V}+P)(z) = -\frac{(z-1)(z-i)(z+1)}{\pi}\iint_{\mathbb{C}} \frac{\widetilde{\mu}(\zeta)d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta-z)}, \tag{4.18}$$

where $\widetilde{\mu}(\zeta) = \mu(B^{-1}(\zeta))\frac{\overline{(B^{-1})'(z)}}{(B^{-1})'(z)}$. Note that

$$|\widetilde{\mu}(\zeta)| = |\mu(B^{-1}(\zeta))|. \tag{4.19}$$

Let $Q_0 = \{1, i, -1, -i\}$. Then $V[Q] = \widetilde{V}[Q] = (\widetilde{V}+P)[Q_0] = (\widetilde{V}+P)(-i)$. Thus,

$$V[Q] = (\widetilde{V}+P)(-i) = \frac{4i}{\pi}\iint_{\mathbb{C}} \frac{\widetilde{\mu}(\zeta)d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta+i)}. \tag{4.20}$$

By using (4.19), (4.16), (4.15) and setting

$$\frac{1}{C} = \frac{4}{\pi} \iint_{\mathbb{C}} |\frac{1}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta+i)}| d\xi d\eta < +\infty,$$

we obtain

$$C|V[Q]| \leq \|\bar{\partial}(L_0(V))\|_\infty.$$

It follows that

$$||V||_{cr} \leq \frac{1}{C}\|\bar{\partial}(L_0(V))\|_\infty.$$

We complete the proof of Proposition 2. □

5. Conformally natural extensions of the elements in $\Lambda_0(\mathbb{S}^1)$

In this section, we prove Theorem 3. We first introduce the Farey tessellation of the unit disk. Let $\triangle_0$ be the ideal hyperbolic geodesic triangle in $\mathbb{D}$ with vertices $1, i$ and -1 and let Γ be the group generated by the hyperbolic reflections to the sides of $\triangle_0$. The Farey tessellation $\mathcal{F}$ is the collection of the geodesics on the Γ-orbits of the edges of $\triangle_0$. It is easy to see that the set $\mathcal{P}$ of the endpoints of the geodesics in $\mathcal{F}$ is dense on $\mathbb{S}^1$. For applications of the Farey tessellation in the study of the universal Teichmüller space and the asymptotic Teichmuüller space of the unit disk, we refer to [37] and [12].

Proof of Theorem 3. We divide the proof into two steps.

Step 1: Prove that if $V \in \Lambda_0(\mathbb{S}^1)$, then

$$\lim_{|z|\to 1} |\bar{\partial}(L_0(V)(z)| = 0.$$

Suppose that the conclusion doesn't hold. Then there exist $\epsilon_0 > 0$ and a sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{D}$ such that $|z_n| \to 1$ and

$$|\bar{\partial}(L_0(V))(z_n)| > \epsilon_0 \text{ all } n. \tag{5.1}$$

For each z_n, let $B_n(z) = \frac{z-z_n}{1-\overline{z_n}z}$ and $V_n(x) = \frac{V(B_n^{-1}(x))}{(B_n^{-1})'(x)}$. For each V_n, we choose a quadratic polynomial P_n such that $(V_n+P_n)(1) = (V_n+P_n)(i) = (V_n+P_n)(-1) = 0$. Denote by $\widetilde{V}_n = V_n + P_n$. Then

$$|\bar{\partial}L_0(V)(z_n)| = |\bar{\partial}L_0(V_n)(0)| = |\int_{\mathbb{S}^1} V_n(\xi)d\xi| = |\int_{\mathbb{S}^1} \widetilde{V}_n(\xi)d\xi|. \tag{5.2}$$

Let $\mathcal{P}$ be the collection of the endpoints of the geodesics forming the Farey tessellation $\mathcal{F}$ on $\mathbb{D}$. We claim that

$$\lim_{n\to\infty} \widetilde{V}_n(p) = 0 \text{ for each } p \in \mathcal{P}.$$

Before proving this claim, we first show how this claim implies a contradiction. Since $\|\widetilde{V}_n\|_{cr} = \|V_n\|_{cr} = \|V\| < \infty$, $\widetilde{V}_n$ is α-Hölder continuous for any $0 < \alpha < 1$ and the α-Hölder constant only depends on $||\widetilde{V}_n||_{cr}$ (Lemma 5 in [25]). It follows that $\{\widetilde{V}_n\}_{n=1}^{\infty}$ is a sequence of uniformly α-Hölder continuous vector fields. By Arzela-Ascoli Theorem, $\lim\limits_{n\to\infty} \widetilde{V}_n(x) = 0$ uniformly for $x \in \mathbb{S}^1$. Using the integral formula in (5.2), we obtain $\lim\limits_{n\to\infty} |\bar{\partial} L_0(V)(z_n)| = 0$. This is a contradiction to the assumption (5.1). Therefore, the assumption doesn't hold; that is,

$$\lim_{|z|\to 1} |\bar{\partial}(L_0(V)(z)| = 0.$$

Now we show the claim. We first show that for any quadruple $Q = \{a, b, c, d\}$ of four points on $\mathbb{S}^1$ in the counterclockwise order with $cr(Q) = 1$,

$$\lim_{n\to\infty} s(B_n(Q)) = 0.$$

Let β be the common perpendicular geodesic segment between the two geodesics connecting a to b and c to d respectively. Let D_0 be the hyperbolic disk centered at 0 and of radius 1 (in the hyperbolic metric). Since the hyperbolic distance between the middle point of β and the hyperbolic center of D_0 is preserved under B_n and the Euclidean distance from $B_n(0)$ to $\mathbb{S}^1$ approaches 0 as $n \to \infty$, it follows that both the Euclidean diameter of $B_n(D_0)$ and the Euclidean length of $B_n(\beta)$ approaches 0 as $n \to \infty$. Thus, $\lim_{n\to\infty} s(B_n(Q)) = 0$.

Given two points $a, b \in \mathbb{S}^1$, denote by $\gamma_{a,b}$ the hyperbolic geodesic on $\mathbb{D}$ joining a and b. Clearly, $-i$ is the reflection of i with respect to $\gamma_{-1,1}$. Then $-i \in \mathcal{P}$. Denote by $Q_1 = \{1, i, -1, -i\}$. Clearly, $cr(Q_1) = 1$ and

$$\widetilde{V}_n[Q_1] = \frac{\widetilde{V}_n(-i)}{1-i} + \frac{\widetilde{V}_n(-i)}{1+i}. \tag{5.3}$$

Using the definitions of $\widetilde{V}_n$ and V_n, Lemma 2 and $\lim_{n\to\infty} s(B_n(Q_1)) = 0$, we obtain

$$\lim_{n\to\infty} \widetilde{V}_n[Q_1] = \lim_{n\to\infty} V_n[Q_1] = \lim_{n\to\infty} V[B_n(Q_1)] = 0. \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$\lim_{n\to\infty} \widetilde{V}_n(-i) = 0. \tag{5.5}$$

Let a_1 be the reflection of -1 with respect to $\gamma_{-i,1}$. Then $a_1 \in \mathcal{P}$. Denote by $Q_2 = \{1, -1, -i, a_1\}$. By the same argument to show (5.5), we obtain

$$\lim_{n\to\infty} \widetilde{V}_n(a_1) = 0.$$

Repeating this progress, we conclude that for any $p \in \mathcal{P}$,

$$\lim_{n\to\infty} \widetilde{V}_n(p) = 0.$$

Step 2: Prove that $\lim_{|z|\to 1} |\bar{\partial}(L_0(V)(z)| = 0$ implies $V \in \Lambda_0(\mathbb{S}^1)$.

Given a quadruple $Q = \{a, b, c, d\}$ of four points arranged on $\mathbb{S}^1$ in the counterclockwise direction, let $B \in \text{Möb}(\mathbb{S}^1)$ mapping a, b, c, d to $1, i, -1, -i$ respectively. Denote by $Q_0 = \{1, i, -1, -i\}$.

We apply the same work and the same notation in the proof of Proposition 2 until the expression (4.20), that is,

$$V[Q] = (\widetilde{V} + P)(-i) = \frac{4i}{\pi} \iint_{\mathbb{C}} \frac{\widetilde{\mu}(\zeta) d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta+i)}. \tag{5.6}$$

Given any $\epsilon > 0$, choose $r \in (0, 1)$ such that

$$\iint_{D_{r,\frac{1}{r}}} \left|\frac{d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta+i)}\right| < \epsilon, \tag{5.7}$$

where $D_{r,\frac{1}{r}} = \{z \in \mathbf{C} : |z| \in (r, \frac{1}{r})\}$. Note that

(5.8)
$$|B^{-1}(\zeta)| \longrightarrow 1 \text{ uniformly for } \zeta \in \mathbb{D}_r = \{z \in \mathbf{C} : |z| < r\} \text{ as } s(Q) \to 0.$$

Using the notation introduced in the proof of Proposition 2, we know

$$\widetilde{\mu}(\zeta) = \bar{\partial} L_0(V)(B^{-1}(\zeta)) \frac{\overline{(B^{-1})'(\zeta)}}{(B^{-1})'(\zeta)} \text{ for } \zeta \in \mathbb{D}_r$$

and $|\widetilde{\mu}(\zeta)| = |\widetilde{\mu}(\frac{1}{\zeta})|$ for $\zeta \in \mathbb{D}_{\frac{1}{r}}$. Thus, $\lim\limits_{|z|\to 1} |\bar{\partial}(L_0(V)(z)| = 0$ implies

$$\lim_{s(Q)\to 0} |\iint_{\mathbb{C}\setminus D_{r,\frac{1}{r}}} \frac{\widetilde{\mu}(\zeta)d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta+i)}| = 0. \tag{5.9}$$

Combining (5.6), (5.7) and (5.9), we obtain

$$\lim_{s(Q)\to 0} V[Q] = 0.$$

We complete the proof. □

6. Conformal natural extensions of the elements in $\Lambda_0^\alpha(\mathbb{S}^1)$

In this section, we prove Theorem 5. Relationships between different types of circle diffeomorphisms and vanishing cross-ratio distortions are explored in [14], [17], [18], [25] and [40]. Our Theorem 5 gives a characterization of $C^{1+\alpha}$ tangent vector fields along $\mathbb{S}^1$ in terms of cross-ratio distortion. We decompose the proof of this theorem into proving the following three propositions.

Proposition 3. *If $V \in \Lambda_0^\alpha(\mathbb{S}^1)$, then V is $C^{1+\alpha}$.*

Proposition 4. *If $V \in \Lambda(\mathbb{S}^1)$ is $C^{1+\alpha}$, then*

$$\|\bar{\partial}L_0(V)(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty.$$

Proposition 5. *If $V \in \Lambda(\mathbb{S}^1)$ satisfies*

$$\|\bar{\partial}L_0(V)(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty,$$

then $V \in \Lambda_0^\alpha(\mathbb{S}^1)$.

Let v be a continuous function from the real line $\mathbb{R}$ to itself. For each $0 < \alpha < 1$, define

$$||v||_r^\alpha = \sup_{x\in\mathbb{R},t>0} \frac{|\frac{v(x+t)-v(x)}{t} - \frac{v(x)-v(x-t)}{t}|}{|t|^\alpha}.$$

Clearly, this norm is not changed by adding an affine map to v. Therefore, we may assume that $v(0) = 0$.

Lemma 5. *If $M = ||v||_r^\alpha < \infty$, then v is differentiable and v' is α-Hölder continuous. Furthermore, there exists a positive constant C'' only depending on M and α such that for any two $y_1, y_2 \in \mathbb{R}$,*

$$|v'(y_2) - v'(y_1)| \leq C''|y_2 - y_1|^\alpha.$$

Proof. By subtracting a constant, we may assume that $v(0) = 0$. In the following, we first show that v is differentiable at 0. Note that the method to show the differentiability of v at 0 applies to obtain the same property for v at any other point. We divide the proof into two steps.

Step 1: We show that there exists C (depending on v) and $C_1 > 0$ (only depending on M and α) such that

$$\lim_{k\to\infty} \frac{v(\frac{1}{2^k}) - v(0)}{\frac{1}{2^k}} = C = \lim_{k\to\infty} \frac{v(-\frac{1}{2^k}) - v(0)}{-\frac{1}{2^k}},$$

and for each positive integer k,

$$|v(\pm\frac{1}{2^k}) - C(\pm\frac{1}{2^k})| \leq C_1(\frac{1}{2^k})^{1+\alpha}.$$

For each positive integer k, let $m_k^r = \frac{v(\frac{1}{2^{k-1}})-v(\frac{1}{2^k})}{\frac{1}{2^k}}$ and $m_k^l = \frac{v(\frac{1}{2^k})-v(0)}{\frac{1}{2^k}}$. Clearly,

$$\frac{m_{k+1}^r + m_{k+1}^l}{2} = m_k^l.$$

Then

$$m_{k+1}^l - m_k^l = -\frac{1}{2}(m_{k+1}^r - m_{k+1}^l).$$

Using the definition of $||v||_r^\alpha$, we obtain

$$|m_{k+1}^l - m_k^l| \leq \frac{1}{2}M(\frac{1}{2^{k+1}})^\alpha.$$

Thus, for each positive integer n,

$$|m_{k+n}^l - m_k^l| \leq \sum_{j=1}^{n} |m_{k+j}^l - m_{k+j-1}^l| \leq \frac{M}{2}\sum_{j=1}^{n}(\frac{1}{2^{k+j}})^\alpha \leq M'(\frac{1}{2^{k+1}})^\alpha,$$

where $M' = \frac{M}{2}\frac{1}{1-(\frac{1}{2})^\alpha}$. Therefore, $\{m_k^l\}_{k=1}^\infty$ is a Cauchy sequence and we denote its limit by C; that is,

$$\lim_{k\to\infty}\frac{v(\frac{1}{2^k})-v(0)}{\frac{1}{2^k}}=C.$$

Furthermore, for any positive integer k,

$$|\frac{v(\frac{1}{2^k})-v(0)}{\frac{1}{2^k}}-C|\leq M'(\frac{1}{2^{k+1}})^{\alpha}.$$

Let $C_1=\frac{M'}{2^\alpha}$. Then for any positive integer k,

$$|v(\frac{1}{2^k})-\frac{C}{2^k}|\leq C_1(\frac{1}{2^k})^{1+\alpha}.$$

Similarly, we can show that $\lim_{k\to\infty}\frac{v(-\frac{1}{2^k})-v(0)}{-\frac{1}{2^k}}$ exists, denoted by C', and

$$|v(-\frac{1}{2^k})-C'(-\frac{1}{2^k})|\leq C_1(\frac{1}{2^k})^{1+\alpha}.$$

Applying the condition $||v||_r^\alpha$ to the symmetric triples $-\frac{1}{2^k}$, 0 and $\frac{1}{2^k}$, we conclude that $C'=C$.

Step 2: We show for any $0<|x|<1$,

$$|v(x)-v(0)-Cx|=O(|x|^{1+\alpha}).$$

It suffices to obtain this estimate for any $0<x<1$.

Given any $0<x<1$, let k be the positive integer such that $\frac{1}{2^{k+1}}\leq x<\frac{1}{2^k}$. Let us first introduce some notation. Given an interval I on the real line, let $l(I)$ and $r(I)$ be the left and right endpoints of I respectively, and let $|I|$ be the length of I. Furthermore, we use $m_v(I)$ to denote the difference quotient $\frac{v(r(I))-v(l(I))}{r(I)-l(I)}$.

Now we denote the interval $[\frac{1}{2^{k+1}},\frac{1}{2^k}]$ by I_1^0. Divide the interval I_1^0 into two pieces of equal length and denote them by I_1^1 and I_2^1. Then divide each of I_1^1 and I_2^1 into two pieces of equal length and denote them by I_1^2, I_2^2, I_3^2 and I_4^2 from the left to the right. Inductively, we bisect I_1^0 into 2^n pieces of equal length and denote them by $I_1^n,I_2^n,I_3^n,\cdots,I_{2^n}^n$, where $n=0,1,2,3,\cdots$. For each $j\geq 0$ and each bisection $I_m^j=I_{m_1}^{j+1}\cup I_{m_2}^{j+1}$ from level j to level $j+1$, we obtain

$$\frac{m_v(I_{m_1}^{j+1})+m_v(I_{m_2}^{j+1})}{2}=m_v(I_m^j).$$

It follows the definition of $||v||_r^\alpha$ that

$$|m_v(I_{m_1}^{j+1}) - m_v(I_{m_2}^{j+1})| \leq M(|I_{m_1}^{j+1}|^\alpha) = M(|\frac{I_m^j}{2}|^\alpha).$$

Using the same techniques in Step 1, we first obtain

$$|m_v(I_{m_1}^{j+1}) - m_v(I_m^j)| \leq \frac{M}{2}(|\frac{I_m^j}{2}|^\alpha) \text{ and } |m_v(I_{m_2}^{j+1}) - m_v(I_m^j)| \leq \frac{M}{2}(|\frac{I_m^j}{2}|^\alpha).$$

Then we obtain for each $j \geq 0$ and each interval $I_{m'}^j$ at level j,

$$|m_v(I_{m'}^{j+1}) - m_v(I_0^0)| \leq M'(|\frac{I_0^0}{2}|^\alpha),$$

where M' is the same constant given in Step 1. From the definition of $||v||_r^\alpha$,

$$|m_v(I_0^0) - m_v([0, \frac{1}{2^{k+1}}])| \leq M(\frac{1}{2^{k+1}})^\alpha.$$

From Step 1,

$$|m_v([0, \frac{1}{2^{k+1}}]) - C| \leq C_1(\frac{1}{2^{k+1}})^\alpha.$$

By the previous three inequalities and the triangle inequality, we obtain for each $j \geq 0$ and each interval $I_{m'}^{j+1}$ at level $j+1$,

$$|m_v(I_{m'}^{j+1}) - C| \leq (M' + M + C_1)(\frac{1}{2^{k+1}})^\alpha = C_1'(\frac{1}{2^{k+1}})^\alpha,$$

where $C_1' = M' + M + C_1$. It follows that

$$|v(r(I_{m'}^{j+1})) - v(l(I_{m'}^{j+1})) - C|I_{m'}^{j+1})|| \leq C_1'(\frac{1}{2^{k+1}})^\alpha |I_{m'}^{j+1})|.$$

Since $x \in [\frac{1}{2^{k+1}}, \frac{1}{2^k})$, there exists a sequence $0 \leq j_1 < j_2 < j_3 < \cdots < n$ (n is possibly ∞) and a sequence of interval $\{I_{m_s}^{j_s}\}_{s=1}^n$ with pairwise disjoint interiors such that

$$[\frac{1}{2^{k+1}}, x) = \cup_{s=1}^n I_{m_s}^{j_s}.$$

Then

$$\sum_{s=1}^n [v(r(I_{m_s}^{j_s})) - v(l(I_{m_s}^{j_s})) - C|I_{m_s}^{j_s}|] = v(r(I_{m_n}^{j_n})) - v(\frac{1}{2^{k+1}}) - C| \cup_{s=1}^n I_{m_s}^{j_s}|,$$

where if n is ∞, then we choose n as a finite number and then let n approach ∞. Using the triangle inequality and the previous inequality (if n is ∞, replacing n by a finite integer and letting it approach ∞ and furthermore applying the continuity of v at x), we obtain

$$|v(x) - v(\frac{1}{2^{k+1}}) - C(x - \frac{1}{2^{k+1}})| \leq C_1'(\frac{1}{2^{k+1}})^{\alpha}|x - \frac{1}{2^{k+1}}| \leq C_1'(\frac{1}{2^{k+1}})^{1+\alpha},$$

where $\frac{1}{2^{k+1}} \leq x < \frac{1}{2^k}$. From Step 1,

$$|v(\frac{1}{2^{k+1}}) - v(0) - C(\frac{1}{2^{k+1}} - 0)| \leq C_1(\frac{1}{2^{k+1}})^{1+\alpha}.$$

Thus,

$$|v(x) - v(0) - Cx| \leq (C_1 + C_1')(\frac{1}{2^{k+1}})^{1+\alpha} \leq (C_1 + C_1')x^{1+\alpha} = o(x).$$

It follows that v is differentiable at 0, $v'(0) = C$, and for any $|x| \leq 1$,

$$|v(x) - v(0) - v'(0)x| \leq \frac{C_1''}{2}|x|^{\alpha},$$

where $C_1'' = 2(C_1 + C_1')$.

Similarly, we can prove that v is differentiable at any other point y and for any $|x| \leq 1$,

$$|v(y + x) - v(y) - v'(y)x| \leq \frac{C_1''}{2}|x|^{1+\alpha}.$$

Now given any two points y_1 and y_2 with $y_1 < y_2$ and $y_2 - y_1 < 1$,

$$|v(y_2) - v(y_1) - v'(y_1)(y_2 - y_1)| \leq \frac{C_1''}{2}(y_2 - y_1)^{1+\alpha}$$

and

$$|v(y_1) - v(y_2) - v'(y_2)(y_1 - y_2)| \leq \frac{C_1''}{2}(y_2 - y_1)^{1+\alpha}.$$

It follows from the triangle inequality that

$$|v'(y_2) - v'(y_1)| \leq C_1''(y_2 - y_1)^{\alpha}.$$

Thus, v' is α-Hölder continuous. We complete the proof. $\square$

In fact, the previous lemma has the following local version. Let v be a continuous function from the real line $\mathbb{R}$ to itself and let (a,b) be an open interval on $\mathbb{R}$. For each $0<\alpha<1$, define

$$||v|_{(a,b)}||_r^{\alpha} = \sup_{x-t,x,x+t\in(a,b)} \frac{|\frac{v(x+t)-v(x)}{t} - \frac{v(x)-v(x-t)}{t}|}{|t|^{\alpha}}.$$

Then the previous Lemma 5 has the following corollary.

Corollary 1. *If $M = ||v|_{(a,b)}||_r^{\alpha} < \infty$, then v is differentiable on (a,b) and v' is α-Hölder continuous on (a,b). Furthermore, there exists a positive constant C'' only depending on M and α such that for any two $y_1, y_2 \in (a,b)$,*

$$|v'(y_2) - v'(y_1)| \leq C''|y_2 - y_1|^{\alpha}.$$

Proof of Proposition 3. Given an element $V \in \Lambda_0^{\alpha}(\mathbb{S}^1)$, we may assume that $V(1)=0$ (by subtracting a constant). Let B be the Möbius transformation from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$ such that $B(-i)=-1$, $B(1)=0$ and $B(i)=1$. Let $v(x) = \frac{V(B^{-1}(x))}{(B^{-1})'(x)}$, where $x\in\mathbb{R}$. Clearly, $v(0)=0$ and $V(z) = \frac{v(B(z))}{B'(z)}$. Furthermore, $V \in \Lambda_0^{\alpha}(\mathbb{S}^1)$ implies that $||v|_{(-1,1)}||_r^{\alpha}$ is finite. Using Corollary 1, we know that v is differentiable on $(-1,1)$ and v' is α-Hölder continuous on $(-1,1)$. Let C'' be the constant in Corollary 1 such that for any $x_1, x_2 \in (-1,1)$,

$$|v'(x_2) - v'(x_1)| \leq C''|x_2 - x_1|^{\alpha}.$$

Now we consider $V(z) = \frac{v(B(z))}{B'(z)}$. Clearly, V is differentiable at any point z on the right half circle on $\mathbb{D}$ and

$$V'(z) = v'(B(z)) - v(B(z))\frac{B''(z)}{(B'(z))^2}. \tag{6.1}$$

It remains to show that V' is α-Hölder continuous on the right half circle. Clearly, $B(z) = \frac{z-1}{z+1}$ is univalent on the disk $\{z : |z-1| < 2\}$. Using the Koebe distortion theorem, there exists a universal constant $M>0$ such that for any $z \in \{z : |z-1| < \sqrt{2}\}$,

$$\frac{1}{M} \leq |\frac{B'(z)}{B'(1)}| \leq M.$$

It follows that $|B'(z)| \leq M|B'(1)|$ for any $z \in \{z : |z-1| < \sqrt{2}\}$. Then for any two points z_1 and z_2 on the right half circle of $\mathbb{S}^1$,

(6.2)
$$|V'(B(z_1)) - V'(B(z_2))| \leq C''|B(z_1) - B(z_2)|^\alpha \leq C''(M|B'(1)|)^\alpha |z_1 - z_2|^\alpha.$$

In the mean time, $w(z) = v(B(z))\frac{B''(z)}{(B'(z))^2}$ is differentiable at any point z on the right half circle of $\mathbb{S}^1$, and its derivative

$$w'(z) = v'(B(z))\frac{B''(z)}{B'(z)} + v(B(z))[\frac{B'''(z)}{(B'(z))^2} - \frac{2B''(z)}{(B'(z))^3}] \tag{6.3}$$

is bounded on the right half circle of $\mathbb{S}^1$ by a positive constant M' only depending on C'', M, $|B'(1)|$ and $v'(0)$. From (6.1), $V'(1) = v'(0)$ since we assume $V(1) = 0$ (hence $v(0) = 0$).

Therefore, using the mean value inequality we can find a positive constant C''', only depending on C'', M, $|B'(1)|$ and $|V'(1)|$, such that for any z_1 and z_2 on the right half circle of $\mathbb{S}^1$,

$$|w'(z_2) - w'(z_1)| \leq C'''|z_2 - z_1|. \tag{6.4}$$

Combining the estimates in (6.2) and (6.4), we can see that V is $C^{1+\alpha}$ on the right half circle of $\mathbb{S}^1$. Similarly, we can prove that V is $C^{1+\alpha}$ on the left half, upper half or lower half circle of $\mathbb{S}^1$ respectively. These four half circles form a finite open cover of $\mathbb{S}^1$. Hence, V is $C^{1+\alpha}$ on $\mathbb{S}^1$. We complete the proof. $\square$

Now we start to prove Proposition 4. Let us first show one lemma.

Lemma 6. *There exists a positive constant C_2 such that for any $\frac{1}{2} \leq t < 1$ and for any $\xi = e^{i\theta}$ with $1 - t \leq |\theta| \leq \pi$,*

$$\frac{|\xi - 1|^{1+\alpha}}{|\xi - t|^4} \leq C_2|\theta|^{-3+\alpha}.$$

Proof. Clearly,

$$|\xi - 1| = |e^{i\theta} - e^{i0}| \leq |\theta|.$$

Secondly,

$$\begin{aligned} |\xi - 1| = \sqrt{|\cos\theta + i\sin\theta - 1|} &= \sqrt{(\cos\theta - 1)^2 + \sin^2\theta} \\ &= \sqrt{2 - 2\cos\theta} = 2|\sin\frac{\theta}{2}|. \end{aligned}$$

Let $\beta = \frac{\theta}{2}$. Clearly, $0 \le |\beta| \le \frac{\pi}{2}$. Then

$$\frac{|\xi - 1|}{|\theta|} = |\frac{2\sin\frac{\theta}{2}}{\theta}| = |\frac{\sin\beta}{\beta}| = \frac{\sin\beta}{\beta}.$$

Let $C_0 = \min_{\beta\in[0,\frac{\pi}{2}]} \frac{\sin\beta}{\beta}$. One can easily check that $C_0 > 0$. Therefore, we first obtain that for any $0 \le |\theta| \le \pi$,

$$C_0\theta \le |\xi - 1| \le |\theta|. \tag{6.5}$$

Now let $\epsilon = 1 - t$. Then $\epsilon > 0$, $t = 1 - \epsilon$, and

$$\begin{aligned}
|\xi - t| &= |\cos\theta + i\sin\theta - t| = \sqrt{(\cos\theta - t)^2 + \sin^2\theta} \\
&= \sqrt{1 - 2t\cos\theta + t^2} = \sqrt{1 - 2(1-\epsilon)\cos\theta + (1-\epsilon)^2} \\
&= \sqrt{2 - 2\cos\theta + 2\epsilon\cos\theta - 2\epsilon + \epsilon^2} = \sqrt{4\sin^2\frac{\theta}{2} - 4\epsilon\sin^2\frac{\theta}{2} + \epsilon^2} \\
&\sqrt{(1-\epsilon)4\sin^2\frac{\theta}{2} + \epsilon^2} \ge \sqrt{(1-\epsilon)4\sin^2\frac{\theta}{2}} = \sqrt{(1-\epsilon)}(2|\sin\frac{\theta}{2}|).
\end{aligned}$$

Thus,

$$\frac{|\xi - t|}{|\theta|} \ge \sqrt{1-\epsilon}\frac{\sin\beta}{\beta}.$$

Since $1 - \epsilon = t \ge \frac{1}{2}$ and $1 - t \le |\theta| \le \pi$, we obtain

$$\frac{|\xi - t|}{|\theta|} = \sqrt{t}\frac{\sin\beta}{\beta} \ge C_0\sqrt{t} \ge \frac{C_0}{\sqrt{2}}.$$

Then for any $\frac{1}{2} \le t < 1$ and $1 - t \le |\theta| \le \pi$,

$$|\xi - t| \ge \frac{C_0}{\sqrt{2}}|\theta|. \tag{6.6}$$

Combining the inequalities (6.5) and (6.6), we obtain for any $\frac{1}{2} \le t \le \pi$ and any $\xi = e^{i\theta}$ with $1 - t \le |\theta| \le \pi$,

$$\frac{|\xi - 1|^{1+\alpha}}{|\xi - t|^4} \le \frac{|\theta|^{1+\alpha}}{(\frac{C_0}{\sqrt{2}}|\theta|)^4} = C_2|\theta|^{-3+\alpha},$$

where $C_2 = \frac{4}{C_0^4}$. We reach the conclusion. □

Proof of Proposition 4. Given a point $z \in \mathbb{D}$, we obtain from (4.2) that

$$\mu(z) = \overline{\partial}(L_0(V))(z) = \frac{3(1-|z|^2)^2}{2\pi i}\int_{\mathbb{S}^1} \frac{V(\xi)}{(\xi - z)^4\xi^4}d\xi. \tag{6.7}$$

Through pre-composing μ and V by a rotation around the origin, we may assume that $z = t$ with $0 \le t < 1$. To prove this proposition, it is sufficient to handle the case when t is close to 1. By Proposition 1, we know that for any constant tangent vector field V to $\mathbb{S}^1$, $\bar{\partial}L_0(V)(z) = 0$ all $z \in \mathbb{D}$. It follows that we may assume that $V(1) = 0$. In the following we explain why we may assume that $V'(1) = 0$. Let B be the Möbius transformation from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$ such that $B(-i) = -1$, $B(1) = 0$ and $B(i) = 1$. Let $v(x) = \frac{V(B^{-1}(x))}{(B^{-1})'(x)}$, where $x \in \mathbb{R}$. Clearly, $v(0) = 0$ and v is differentiable. Now let $C = v'(0)$, $\tilde{v}(x) = v(x) - Cx$, and $\tilde{V}(z) = \frac{\tilde{v}(B(z))}{B'(z)}$. Then $\tilde{v}(0) = 0$ and $\tilde{v}'(0) = 0$. Correspondingly, $\tilde{V}(1) = 0$ and $\tilde{V}'(1) = 0$, and $\tilde{V}(z) = V(z) - C\frac{B(z)}{B'(z)}$ is also differentiable at any point $z \in \mathbb{S}^1 \setminus \{-1\}$ and $C^{1+\alpha}$ on the right half circle of $\mathbb{S}^1$. Since $\frac{B}{B'}$ is a quadratic polynomial, we can easily check that $||\frac{B}{B'}||_{cr} = 0$. From Proposition 1, we know $\bar{\partial}L_0(\frac{B}{B'})(z) = 0$ for any $z \in \mathbb{D}$. Therefore, we may assume that $z = t$ with $0 < t < 1$ and close to 1 and assume that $V(0) = V'(0) = 0$ for z and V in the expression of (6.7). Then we can complete the proof as follows.

Let $\xi = e^{i\theta}$. Then

$$\mu(t) = \frac{3(1-t^2)^2}{2\pi i}\int_{\mathbb{S}^1} \frac{V(\xi)}{(\xi - t)^4}\frac{d\xi}{\xi^4}.$$

Thus,

$$\begin{aligned} |\mu(t)| &\le \frac{6}{\pi}(1-t)^2\int_{-\pi}^{\pi}\frac{|V(\xi)|}{|\xi - t|^4}d\theta \\ &\le \frac{6}{\pi}[(1-t)^2\int_{-(1-t)}^{1-t}\frac{|V(\xi)|}{|\xi-t|^4}d\theta + \frac{6}{\pi}(1-t)^2\int_{|\theta|\ge 1-t}\frac{|V(\xi)|}{|\xi-t|^4}d\theta] \\ &= \frac{6}{\pi}(I_1 + I_2), \end{aligned}$$

where I_1 and I_2 denote respectively the two summands in the bracket.

As explained in the above, $V(0) = V'(0) = 0$ and V is $C^{1+\alpha}$ on the right half circle of $\mathbb{S}^1$. It follows that for any ξ on the right half circle of $\mathbb{S}^1$,

$$|V(\xi) = |V(\xi) - V(1)| \le C_1|\xi - 1|^{1+\alpha}.$$

Furthermore, if $\xi = e^{i\theta}$ and $|\theta| \le 1-t$, then $|\xi - 1| \le 1-t$. Therefore,

$$|V(\xi)| \le C_1(1-t)^{1+\alpha}$$

for any $\xi = e^{i\theta}$ with $|\theta| \le 1-t$. Clearly, $|\xi - t| \ge |1-t|$ for any $\xi = e^{i\theta}$. Then

$$\begin{aligned} I_1 &= (1-t)^2 \int_{-(1-t)}^{1-t} \frac{|V(\xi)|}{|\xi - t|^4} d\theta \le (1-t)^2 \int_{-(1-t)}^{1-t} \frac{C_1|1-t|^{1+\alpha}}{|1-t|^4} d\theta \\ &= 2C_1(1-t)^2(1-t)\frac{(1-t)^{1+\alpha}}{(1-t)^4} = 2C_1(1-t)^\alpha. \end{aligned}$$

Now we apply the previous Lemma 6 to I_2. One can see that if $\frac{1}{2} < t < 1$, then

$$\begin{aligned} I_2 &\le (1-t)^2 \int_{|\theta| \ge 1-t} C_2 \theta^{-3+\alpha} d\theta = (1-t)^2[C_3 + C_4(1-t)^{-2+\alpha}] \\ &= [C_3(1-t)^{2-\alpha} + C_4](1-t)^\alpha = O((1-t)^\alpha), \end{aligned}$$

where C_3 and C_4 are two positive constants depending on α only as soon as $\frac{1}{2} \le t < 1$ and $1-t \le |\theta| \le \pi$.

Based on the previous estimates of I_1 and I_2, we obtain

$$|\mu(t)| = O((1-t)^\alpha)$$

as soon as $\frac{1}{2} \le t < 1$. This means that for any $\frac{1}{2} \le t < 1$, $|\bar{\partial}(L_0(V))(t)\rho_{\mathbb{D}}^\alpha(t)|$ is bounded by a constant. It follows that for any point z of $\mathbb{D}$ with $\frac{1}{2} \le |z| < 1$, $|\bar{\partial}(L_0(V))(z)\rho_{\mathbb{D}}^\alpha(z)|$ is bounded by the same constant. Therefore,

$$\|\bar{\partial}(L_0(V))(z)\rho_{\mathbb{D}}^\alpha(z)\|_\infty < +\infty. \qquad \square$$

In the remaining part of this section, we prove Proposition 5.

Note that if the three fixed points of the maps on the curve in (4.13) are arranged at 0, 1 and ∞, then the function F in (4.14) is equal to

$$F(z) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)}, \tag{6.8}$$

which is rewritten as

$$F(z) = -\frac{1}{2\pi} \iint_{\mathbb{C}} (\frac{1}{\zeta - z} + \frac{z-1}{\zeta} - \frac{z}{\zeta - 1}) \mu(\zeta) d\xi d\eta. \tag{6.9}$$

Then one can easily obtain

$$\frac{F(z+t)+F(z-t)-2F(z)}{t} = -\frac{1}{2t\pi}\iint_{\mathbb{C}}\left(\frac{1}{\zeta-(z+t)}+\frac{1}{\zeta-(z-t)}-\frac{2}{\zeta-z}\right)\mu(\zeta)d\xi d\eta. \tag{6.10}$$

Thus,

$$\frac{F(z+t)+F(z-t)-2F(z)}{t} = -\frac{t}{\pi}\iint_{\mathbb{C}}\frac{\mu(\zeta)d\xi d\eta}{(\zeta-(z+t))(\zeta-(z-t))(\zeta-z)}. \tag{6.11}$$

Using a substitution of ζ by $z+t\zeta$, we obtain

$$\frac{F(z+t)+F(z-t)-2F(z)}{t} = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\mu(z+t\zeta)d\xi d\eta}{(\zeta-1))(\zeta-1)\zeta}. \tag{6.12}$$

Proof of Proposition 5. Let $Q = \{a,b,c,d\}$ be a quadruple of four points a,b,c,d on $\mathbb{S}^1$ in the counterclockwise order with $cr(Q)=1$ and $S(Q)\geq\sqrt{2}$. Without loss of generality, we may assume that $s(Q)$ is quite small and we may assume that the distance between b and d is greater than or equal to $\sqrt{2}$ and the distance between a and b or b and c is the shortest. Now let $B(z) = -i\frac{z-b}{z-d}$. Then B maps $\mathbb{D}$ to the upper half plane $\mathbb{H}$, b to 0, and d to ∞. Let $B(a)=-t$. Then $B(c)=t$.

Now let $V\in\Lambda(\mathbb{S}^1)$ with $\|\bar\partial L_0(V)(z)\rho_{\mathbb{D}}^{\alpha}(z)\|_\infty < +\infty$. By subtracting a quadratic polynomial, we may assume that $V(d)=0$, $V(b)=0$ and $V(B^{-1}(1)=0$. Let $\mu(z)=\bar\partial L_0(V)(z)$. Define

$$\widetilde{V}(x) = \frac{V(B^{-1}(x))}{(B^{-1})'(x)} \quad\text{and}\quad \widetilde{\mu}(\zeta) = \mu(B^{-1}(\zeta))\frac{\overline{(B^{-1})'(z)}}{(B^{-1})'(z)}.$$

Let $\widetilde{Q} = B(Q) = \{-t,0,t,\infty\}$. Then $V[Q]=\widetilde{V}[\widetilde{Q}]$. Using the expression in (6.12), we obtain

$$\widetilde{V}[\widetilde{Q}] = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\widetilde{\mu}(t\zeta)d\xi d\eta}{(\zeta-1)(\zeta-1)\zeta}. \tag{6.13}$$

Clearly, B^{-1} maps a thin horizontal strip with the real line as one side to a region Ω between the unit circle and a holocycle circle Γ intersecting $\mathbb{S}^1$ at the point d. Let

$$d(\Gamma,\mathbb{S}^1) = \sup_{z\in\Gamma} d(z,\mathbb{S}^1).$$

Because of $S(Q) \geq \sqrt{2}$ and the Koebe distortion theorem, we know that when the width of the horizontal strip is small enough, only depending on $S(Q)$ (and hence it is universal), its width λ is commeasurable with $d(\Gamma, \mathbb{S}^1)$. Clearly, the region Ω is contained in the round annulus between $\mathbb{S}^1$ and the circle of radius $1 \pm d(\Gamma, \mathbb{S}^1)$. Now for any $t\zeta$ with $|t\eta|$ small, we let the width of the thin strip equal to $|t\eta|$. Then all previous reasons together imply

$$|\widetilde{\mu}(t\zeta)| = |\mu(B^{-1}(t\zeta))| = O((1 - |B^{-1}(t\zeta)|)^\alpha) = O(d(\Gamma, \mathbb{S}^1)^\alpha) = O(|t\eta|^\alpha).$$

Since $\widetilde{\mu}$ is bounded on $\mathbb{C}$, it follows that there exists a positive constant C such that

$$|\widetilde{\mu}(t\zeta)| \leq C|t\eta|^\alpha \leq C|t\zeta|^\alpha.$$

From the expression (6.13), we know

$$|\widetilde{V}[\widetilde{Q}]| \leq \frac{|t|^\alpha}{\pi} \iint_{\mathbb{C}} \frac{|\zeta|^\alpha d\xi d\eta}{|(\zeta - 1)(\zeta - 1)\zeta|}.$$

Because of $0 < \alpha < 1$, the integral $\iint_{\mathbb{C}} \frac{|\zeta|^\alpha d\xi d\eta}{|(\zeta-1)(\zeta-1)\zeta|}$ is finite. Therefore,

$$|V[Q]| = |\widetilde{V}[\widetilde{Q}]| = O(|t|^\alpha) = O(s(Q)^\alpha).$$

This estimate holds for any quadruple Q with $cr(Q) = 1$, $S(Q) \geq \sqrt{2}$ and $s(Q)$ small. Therefore, $V \in \Lambda_0^\alpha(\mathbb{S}^1)$. We complete the proof. $\square$

7. Tangent spaces of $T(\mathbb{D})$, $T_0(\mathbb{D})$, $T(\mathcal{R})$ and $T_0^\alpha(\mathbb{D})$ at the base point

In this section, we apply Theorems 2, 3, 1 and 5 to provide alternative proofs of Theorems B and C and show Theorems 4 and 6 respectively.

An alternative proof of Theorem B. Assume that $T(\mathbb{D})$ is modelled by the collection of the quasisymmetric homeomorphisms of $\mathbb{S}^1$ fixing three points ± 1 and i. Let h_t $(t \geq 0)$ be a smooth curve emanating from the base point. Then h_t is a smooth curve of quasisymmetric homeomorphisms of $\mathbb{S}^1$ fixing ± 1 and i, and

$$h_t(z) = z + tV(z) + o(t),\ t \longrightarrow 0 \text{ uniformly for } z \in \mathbb{S}^1.$$

Note that $T(\mathbb{D})$ has a complex structure and a holomorphic split submersion from $M(\mathbb{D})$ to $T(\mathbb{D})$. We consider the real model for $T(\mathbb{D})$ and the real model for the tangent space of $T(\mathbb{D})$ at the base point. Since the Hilbert transformation $H(V)$ of V preserves the set $\Lambda(\mathbb{S}^1)$ (Section 16.7 of [15]), $V + iH(V)$ characterizes a point in the complex model for the tangent space of the complex model of $T(\mathbb{D})$ at the base point.

It follows that there is a differentiable curve $\mu_t \in M(\hat{\mathbb{C}})$ with $\mu_t(z) = \mu_t(\frac{1}{\bar{z}})$ for each z such that $h_t = f_{\mu_t}|_{\mathbb{S}^1}$, where f_{μ_t} is the quasiconformal homeomorphism of $\hat{\mathbb{C}}$ with $\mu_{f_{\mu_t}} = \mu_t$ fixing ± 1 and i, where $\hat{\mathbb{C}}$ stands for the extended complex plane. Furthermore, there exists $\mu \in M(\hat{\mathbb{C}})$ with $\mu(z) = \mu(\frac{1}{\bar{z}})$ such that

$$\mu_t = t\mu + o(t).$$

By the holomorphic dependence of the solution of the Beltrami differential equation ([4]),

$$f_{\mu_t}(z) = z + tF(z) + o(t),\ t \longrightarrow 0,$$

where

$$F(z) = -\frac{(z-1)(z-i)(z+1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta)d\xi d\eta}{(\zeta-1)(\zeta-i)(\zeta+1)(\zeta-z)}. \tag{7.1}$$

Then for each $z \in \mathbb{S}^1$, $F(z) = \frac{d}{dt} f_{\mu_t}(z)|_{t=0} = V(z)$. So F is an extension of V to $\hat{\mathbb{C}}$ with $\bar{\partial} F = \mu(f_{\mu_t})|_{t=0} = \mu$ and satisfies the tangency condition (2.10) and the normalization (2.12). Then the work of Step 2 in the proof of Proposition 2 shows $||V||_{cr} \le \frac{1}{C}||\mu||_\infty$. Thus, $V \in \Lambda(\mathbb{S}^1)$.

Conversely, given an element $V \in \Lambda(\mathbb{S}^1)$ with the normalization (2.12), from Proposition 1 we obtain an extension $L_0(V)$ of V to $\mathbb{D}$ with $\mu = \bar{\partial} L_0(V) < +\infty$. Extend μ to $\hat{\mathbb{C}}$ by letting $\mu(z) = \mu(\frac{1}{\bar{z}})$ for each z outside $\mathbb{D}$. We continue to denote this extension by μ. Let $\mu_t = tu$ for $t \ge 0$. When $0 \le t < \frac{1}{||\mu||_\infty}$, f_{μ_t} is a differentiable curve of quasisymmetric homeomorphisms of $\mathbb{S}^1$ and

$$f_{\mu_t}(z) = z + tF(z) + o(t), \tag{7.2}$$

where $F(z)$ is given by (7.1).

Both F and $L_0(V)$ satisfy the tangency condition (2.10) and the normalization (2.12) and their $\bar{\partial}$ derivatives are equal to μ. Thus, $F = L_0(V)$ and $F|_{\mathbb{S}^1} = V$. Then the restriction of the equation (7.2) shows that V represents a vector in the tangent space of the real model of $T(\mathbb{D})$ at the base point. □

Note that the Hilbert transformation $H(V)$ of V preserves $\Lambda_0(\mathbb{S}^1)$ and $\Lambda_0^\alpha(\mathbb{S}^1)$ respectively, where $0 < \alpha < 1$ (Section 16.7 of [15]).

Using the work of Step 2 in the proof of Theorem 3 and the conclusion of this theorem, one can easily modify the proof of Theorem B to reach a proof of Theorem C.

Using the result of [31] stated in the first table, Theorem 5 and the work to prove Proposition 5, one can also modify the proof of Theorem B to reach a proof of Theorem 6.

It is clear that by applying Theorem 1 into the proof of Theorem B, we obtain a proof for Theorem 4.

Remark 4. The extension $L_0(V)$ of V is used in [26] to characterize the tangent space of the universal Weil-Petersson Teichmüller space and in [38] to describe the tangent spaces of the BMO and VMOA Teichmüller spaces at the base point in terms of different subspaces of $\Lambda(\mathbb{S}^1)$.

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