

On the dimension of Dolbeault harmonic $(1, 1)$ -forms on almost Hermitian 4-manifolds

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Abstract: We prove that the dimension $h_{\bar{\partial}}^{1,1}$ of the space of Dolbeault harmonic $(1, 1)$ -forms is not necessarily always equal to b^- on a compact almost complex 4-manifold endowed with an almost Hermitian metric which is not locally conformally almost Kähler. Indeed, we provide examples of non integrable, non locally conformally almost Kähler, almost Hermitian structures on compact 4-manifolds with $h_{\bar{\partial}}^{1,1} = b^- + 1$. This gives an answer to [6, Question 3.3] by Holt.

Keywords: Almost complex 4-manifold, Dolbeault Laplacian.

1. Introduction

Let (M, J) be an almost complex manifold of real dimension $2n$. Given, on (M, J) , an almost Hermitian metric g , with fundamental form ω , the triple (M, J, ω) will be called an almost Hermitian manifold. By the map $* : A^{p,q} \rightarrow A^{n-q, n-p}$, we denote the \mathbb{C} -linear extension of the real Hodge $*$ operator. The exterior derivative decomposes as

$$d = \mu + \partial + \bar{\partial} + \bar{\mu},$$

and we set $\partial^* := - * \bar{\partial} *$ and $\bar{\partial}^* := - * \partial *$ to be the formal adjoint operators respectively of ∂ and $\bar{\partial}$. Recall that

$$\Delta_{\bar{\partial}} := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$$

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is the Dolbeault Laplacian, which is a formally self adjoint elliptic operator of order 2. We set

$$\mathcal{H}_{\bar{\partial}}^{p,q} := \ker \Delta_{\bar{\partial}} \cap A^{p,q},$$

the space of Dolbeault harmonic (p, q) -forms. If M is compact, it is well known that the dimensions

$$h_{\bar{\partial}}^{p,q} := \dim_{\mathbb{C}} \mathcal{H}_{\bar{\partial}}^{p,q}$$

are finite. Note that, a priori, $\partial^*, \bar{\partial}^*, \Delta_{\bar{\partial}}, \mathcal{H}_{\bar{\partial}}^{p,q}, h_{\bar{\partial}}^{p,q}$ depend both on the almost complex structure J and on the almost Hermitian metric ω .

If J is integrable, i.e., if (M, J) is a complex manifold, then, assuming the compactness of M , by Hodge theory we know that the space of Dolbeault harmonic forms is isomorphic to the Dolbeault cohomology, i.e.,

$$\mathcal{H}_{\bar{\partial}}^{p,q} \cong H_{\bar{\partial}}^{p,q} := \frac{\ker \bar{\partial} \cap A^{p,q}}{\text{im } \bar{\partial} \cap A^{p,q}}.$$

Since the Dolbeault cohomology is a complex invariant, the numbers $h_{\bar{\partial}}^{p,q}$ do not depend on the choice of the metric in the integrable case.

Conversely, in the almost complex setting, Kodaira and Spencer asked the following question, which appeared as Problem 20 in Hirzebruch’s 1954 problem list [5]: given a compact almost Hermitian manifold (M, J, ω) , does $h_{\bar{\partial}}^{p,q}$ depend on the choice of the almost Hermitian metric ω ? Recently in [7] Holt and Zhang solved this problem, proving that $h_{\bar{\partial}}^{p,q}$ indeed depends on the choice of the metric. They provided an explicit example, building a family of almost Hermitian structures on the Kodaira-Thurston manifold, which has real dimension 4. They also proved that on every 4-dimensional compact almost Hermitian manifold (M, J, ω) , if the metric ω is almost Kähler, i.e., $d\omega = 0$, then $h_{\bar{\partial}}^{1,1} = b^- + 1$. Here b^- is the dimension of the space of anti-self-dual harmonic forms, which is a topological invariant (see, e.g., [3]).

The study of the number $h_{\bar{\partial}}^{1,1}$ on compact almost Hermitian 4-manifolds (M, J, ω) has been continued by Tardini and the second author in [12]. Following their notation, we say that ω is *strictly locally conformally almost Kähler* if

$$d\omega = \theta \wedge \omega,$$

and $\theta \in A^1$ is d -closed but non d -exact. Conversely, we say that ω is *globally conformally almost Kähler*, if

$$d\omega = \theta \wedge \omega,$$

and $\theta \in A^1$ is d -exact. Indeed, if $\theta = dh$, then the metric $e^{-h}\omega$ is almost Kähler. Also note that, for any given almost Hermitian metric ω , the 1-form θ such that $d\omega = \theta \wedge \omega$ is uniquely determined by the Lefschetz isomorphism. We will also say that ω is *locally conformally almost Kähler* if it is either strictly locally conformally almost Kähler or globally conformally almost Kähler.

Tardini and the second author proved that $h_{\bar{\partial}}^{1,1} = b^-$ on every compact almost complex 4-manifold with a strictly locally conformally almost Kähler metric. They also noted that $h_{\bar{\partial}}^{1,1}$ is a conformal invariant on almost Hermitian 4-manifolds, which implies that $h_{\bar{\partial}}^{1,1} = b^- + 1$ on every compact almost complex 4-manifold with a globally conformally almost Kähler metric, by the previous result by Holt and Zhang in [7]. Moreover, very recently, in [6, Theorem 3.1] Holt proved that $h_{\bar{\partial}}^{1,1} = b^- + 1$ and $h_{\bar{\partial}}^{1,1} = b^-$ are the only two possible options on a compact almost Hermitian 4-manifold. In the remaining case when the almost Hermitian metric is not locally conformally almost Kähler, in general not much is known. For other recent and related results concerning the study of the spaces of harmonic forms on compact almost Hermitian manifolds, see [2], [8] and [13] for Dolbeault harmonic forms, and [6], [10] and [11] for Bott-Chern harmonic forms.

The study of $h_{\bar{\partial}}^{1,1}$ on compact almost Hermitian 4-manifolds is further motivated by what follows. In the integrable case, on a compact complex surface (M, J) , it is well known that the first Betti number b^1 is even if and only if (M, J) admits a Kähler metric, and b^1 is even if and only if $h_{\bar{\partial}}^{1,1} = b^- + 1$, see e.g., [1]. Thanks to the previously mentioned results, it is reasonable to expect $h_{\bar{\partial}}^{1,1}$ to detect almost Kählerness. Motivated by this, Holt asked the following

Question ([6, Question 3.3]). *On a compact almost Hermitian 4-manifold, does the value of $h_{\bar{\partial}}^{1,1}$ gives a full description of whether an almost Hermitian metric is conformally almost Kähler? Specifically, in the case when the metric is not locally conformally almost Kähler, do we always have $h_{\bar{\partial}}^{1,1} = b^-$?*

In [6] Holt also computed the value of $h_{\bar{\partial}}^{1,1} = b^-$ for a large family of almost Hermitian structures on the Kodaira-Thurston manifold and showed that his question is answered positively in that example.

In this note, we prove the following

Theorem 2.6. *Let M be a Hyperelliptic surface in the first isomorphism class. On M , there exist two non integrable almost Hermitian structures (J_1, ω_1) and (J_2, ω_2) which are not locally conformally almost Kähler and which satisfy*

$$h_{\bar{\partial}}^{1,1}(M, J_1, \omega_1) = 2 = b^- + 1, \quad h_{\bar{\partial}}^{1,1}(M, J_2, \omega_2) = 1 = b^-.$$

This gives a negative answer to [6, Question 3.3] by Holt.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 2.6. The structure (J_1, ω_1) is obtained by deforming the standard Kähler structure J_0 on the Hyperelliptic surface M into a non integrable almost complex structure J_t , and then by deforming the diagonal metric via a real parameter ρ to obtain a metric $\omega_{t,\rho}$ which is not locally conformally almost Kähler and satisfies $h_{\frac{\partial}{\partial}}^{1,1} = b^- + 1$. The structure (J_2, ω_2) is obtained by deforming the diagonal metric, again via a real parameter, on a non integrable almost complex structure J on M . The metric ω_2 is not locally conformally almost Kähler and satisfies $h_{\frac{\partial}{\partial}}^{1,1} = b^-$. In Section 3 we consider the 4-dimensional nilmanifold \mathcal{N} which does not admit any integrable almost complex structure. We choose an almost complex structure on \mathcal{N} , and the diagonal metric turns out to be not locally conformally almost Kähler. Again, we prove that $h_{\frac{\partial}{\partial}}^{1,1} = b^- + 1$, see Proposition 3.2.

2. $h_{\frac{\partial}{\partial}}^{1,1}$ of two almost Hermitian structures on a hyperelliptic surface

We start by recalling the description of the first isomorphism class of the Hyperelliptic surfaces as solvmanifolds. Following Hasegawa, [4, Section 3, pp. 754-755], let G be the group \mathbb{C}^2 together with the operation

$$(w^1, w^2) \cdot (z^1, z^2) = (w^1 + e^{i\pi \frac{w^2 + \overline{w^2}}{2}} z^1, w^2 + z^2),$$

and let Γ be the subgroup of G given by $(\mathbb{Z} + i\mathbb{Z})^2$. This corresponds to the hyperelliptic surface with $\eta = \pi$ and $p = q = s = t = 0$ in the notation of Hasegawa. Let M be the solvmanifold $\Gamma \backslash G$, and denote by x^1, y^1, x^2, y^2 the local coordinates of M induced from \mathbb{C}^2 , i.e., $z^1 = x^1 + iy^1$, $z^2 = x^2 + iy^2$. The vector fields

$$\begin{aligned} e_1 &= \cos(\pi x^2) \frac{\partial}{\partial x^1} + \sin(\pi x^2) \frac{\partial}{\partial y^1}, \\ e_2 &= -\sin(\pi x^2) \frac{\partial}{\partial x^1} + \cos(\pi x^2) \frac{\partial}{\partial y^1}, \\ e_3 &= \frac{\partial}{\partial x^2}, \quad e_4 = \frac{\partial}{\partial y^2} \end{aligned}$$

are left invariant and form a basis of TM at each point. The dual left invariant coframe is given by

$$e^1 = \cos(\pi x^2) dx^1 + \sin(\pi x^2) dy^1,$$

$$\begin{aligned} e^2 &= -\sin(\pi x^2)dx^1 + \cos(\pi x^2)dy^1, \\ e^3 &= dx^2, \quad e^4 = dy^2, \end{aligned}$$

with structure equations

$$(1) \quad de^1 = -\pi e^{23}, \quad de^2 = \pi e^{13}, \quad de^3 = 0, \quad de^4 = 0.$$

The de Rham cohomology of M is computed by using left invariant forms, therefore

$$H^1_{dR} = \mathbb{R} \langle e^3, e^4 \rangle, \quad H^2_{dR} = \mathbb{R} \langle e^{12}, e^{34} \rangle.$$

On M , consider the usual left invariant integrable almost complex structure J_0 given by

$$\begin{aligned} \varphi^1 &= e^1 + ie^2, \\ \varphi^2 &= e^3 + ie^4 \end{aligned}$$

being a global coframe of $T^{1,0}(M, J_0)$. The corresponding structure equations are

$$(2) \quad d\varphi^1 = i\frac{\pi}{2}(\varphi^{12} + \varphi^{1\bar{2}}), \quad d\varphi^2 = 0.$$

Endow (M, J_0) with the Kähler metric given by the compatible symplectic form

$$\omega = e^{12} + e^{34} = \frac{i}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}}).$$

Note that $b^- = 1$ and $h_{\frac{\omega}{2}}^{1,1}(M, J_0, \omega) = b^- + 1 = 2$.

Let us consider the following left invariant non integrable almost complex deformation (M, J_t) of the complex manifold (M, J_0) .

Lemma 2.1. *For $t \in \mathbb{C}$, $|t| < 1$, let J_t be the almost complex structure on the Hyperelliptic surface M defined by*

$$\varphi_t^1 = \varphi^1 + t\varphi^{\bar{1}}, \quad \varphi_t^2 = \varphi^2$$

being a coframe of $T^{1,0}(M, J_t)$. Then, the corresponding structure equations are

$$(3) \quad d\varphi_t^1 = \frac{i\pi}{2(1 - |t|^2)} \left((1 + |t|^2)(\varphi_t^{12} + \varphi_t^{1\bar{2}}) + 2t(\varphi_t^{2\bar{1}} - \varphi_t^{\bar{1}2}) \right), \quad d\varphi_t^2 = 0,$$

and for $0 < \rho < 1$, $t \neq 0$ the almost Hermitian metric

$$\omega_{t,\rho} = \frac{i}{2}(\varphi_t^{1\bar{1}} + \varphi_t^{2\bar{2}}) + \frac{\rho}{2}(\varphi_t^{1\bar{2}} - \varphi_t^{2\bar{1}}),$$

is not locally conformally almost Kähler.

Proof. The structure equations (3) of (M, J_t) derive from the structure equations (2) of the complex manifold (M, J_0) by elementary computations.

Note that the diagonal metric

$$\omega_t = \frac{i}{2}(\varphi_t^{1\bar{1}} + \varphi_t^{2\bar{2}})$$

is almost Kähler. Conversely, let us show that $\omega_{t,\rho}$ is not locally conformally almost Kähler for $0 < \rho < 1$ and $t \neq 0$. In the following, it will be convenient to set

$$\alpha = 1 + |t|^2 - 2t \in \mathbb{C}, \quad \gamma = 1 - |t|^2 \in \mathbb{R}.$$

We compute

$$d\omega_{t,\rho} = \frac{i\pi\rho}{4\gamma} \left(\bar{\alpha}\varphi_t^{12\bar{2}} - \alpha\varphi_t^{21\bar{2}} \right) = \theta_{t,\rho} \wedge \omega_{t,\rho}$$

with

$$\theta_{t,\rho} = \frac{\pi\rho}{2(1-\rho^2)\gamma} \left(\bar{\alpha}(\varphi_t^1 + i\rho\varphi_t^2) + \alpha(\varphi_t^{\bar{1}} - i\rho\varphi_t^{\bar{2}}) \right).$$

For $0 < \rho < 1$ and $t \neq 0$, the 1-form $\theta_{t,\rho}$ is not closed, i.e., $d\theta_{t,\rho} \neq 0$, therefore $\omega_{t,\rho}$ is not locally conformally almost Kähler. \square

By Lemma 2.1, for $t \in \mathbb{C}$, $0 \neq |t| < 1$, and $0 < \rho < 1$, we have that $(J_t, \omega_{t,\rho})$ is a non integrable almost Hermitian structure on M which is not locally conformally almost Kähler. We prove the following

Theorem 2.2. *For $t \in \mathbb{C}$, $0 \neq |t| < 1$, and $0 < \rho < 1$, let M be the Hyperelliptic surface endowed with the non integrable, non locally conformally almost Kähler structure $(J_t, \omega_{t,\rho})$ of Lemma 2.1. Then,*

$$h_{\bar{\partial}}^{1,1}(M, J_t, \omega_{t,\rho}) = 2 = b^- + 1.$$

Proof. The fundamental form $\omega_{t,\rho}$ can be also rewritten as

$$\omega_{t,\rho} = \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}}),$$

where we set

$$\psi^1 = \varphi_t^1 + i\rho\varphi_t^2, \quad \psi^2 = \sqrt{1 - \rho^2}\varphi_t^2$$

as a unitary global coframe of $T^{1,0}(M, J_t)$. Define the volume form Vol such that

$$\text{Vol} = \frac{\omega_{t,\rho}^2}{2} = \frac{1}{4}\psi^{12\bar{1}\bar{2}}.$$

In the following computations, it will be convenient to set

$$\alpha = 1 + |t|^2 - 2t \in \mathbb{C}, \quad \beta = 1 + |t|^2 \in \mathbb{R}, \quad \gamma = 1 - |t|^2 \in \mathbb{R}.$$

Let $\eta \in A^{1,1}(M, J_t)$ be a left invariant (1, 1)-form on (M, J_t) . We can write

$$\eta = A\psi^{1\bar{1}} + B\psi^{1\bar{2}} + C\psi^{2\bar{1}} + D\psi^{2\bar{2}},$$

where $A, B, C, D \in \mathbb{C}$. Applying the Hodge $*$ operator we get

$$*\eta = D\psi^{1\bar{1}} - B\psi^{1\bar{2}} - C\psi^{2\bar{1}} + A\psi^{2\bar{2}}.$$

We compute

$$\bar{\partial}\eta = \frac{\pi}{2\gamma} \left(-A\rho\alpha + B2it\sqrt{1 - \rho^2} + Ci\beta\sqrt{1 - \rho^2} \right) \varphi_t^{2\bar{1}\bar{2}},$$

and

$$\partial * \eta = \frac{\pi}{2\gamma} \left(D\rho\bar{\alpha} - Bi\beta\sqrt{1 - \rho^2} - C2i\bar{t}\sqrt{1 - \rho^2} \right) \varphi_t^{12\bar{2}}.$$

Therefore $\bar{\partial}\eta = \partial * \eta = 0$ iff

$$(4) \quad \begin{cases} -A\rho\alpha + B2it\sqrt{1 - \rho^2} + Ci\beta\sqrt{1 - \rho^2} = 0, \\ D\rho\bar{\alpha} - Bi\beta\sqrt{1 - \rho^2} - C2i\bar{t}\sqrt{1 - \rho^2} = 0. \end{cases}$$

$$(5)$$

We sum (4) multiplied by β and (5) multiplied by $2t$. We derive

$$(6) \quad C = \frac{i\rho}{\gamma^2\sqrt{1 - \rho^2}} (-A\beta\alpha + D2t\bar{\alpha}).$$

We sum (4) multiplied by $2\bar{t}$ and (5) multiplied by β . We derive

$$(7) \quad B = \frac{-i\rho}{\gamma^2\sqrt{1 - \rho^2}} (-A2\bar{t}\alpha + D\beta\bar{\alpha}).$$

Substituting (6) and (7) into the system given by (4) and (5), we find

$$(8) \quad \begin{cases} A\alpha\rho(\gamma^2 + 4|t|^2 - \beta^2) = 0, \\ D\bar{\alpha}\rho(\gamma^2 + 4|t|^2 - \beta^2) = 0. \end{cases}$$

Finally, note that (8) is an identity, since by definition

$$\gamma^2 + 4|t|^2 - \beta^2 = 0.$$

This implies that $\eta \in \mathcal{H}_{\bar{\partial}}^{1,1}(M, J_t, \omega_{t,\rho})$ for every $A, D \in \mathbb{C}$ and for any $C, B \in \mathbb{C}$ given by equations (6) and (7). It follows that $h_{\bar{\partial}}^{1,1}(M, J_t, \omega_{t,\rho}) \geq 2 = b^- + 1$. By [6, Theorem 3.1], $h_{\bar{\partial}}^{1,1}$ can be equal only to b^- or $b^- + 1$, therefore we conclude that $h_{\bar{\partial}}^{1,1}(M, J_t, \omega_{t,\rho}) = 2 = b^- + 1$. \square

In the remainder of this section, we will construct another non integrable almost complex structure J on M admitting an almost Hermitian metric ω which is Gauduchon (that is, in real dimension 4, $\partial\bar{\partial}\omega = 0$ or, equivalently, $dJd\omega = 0$) but not locally conformally almost Kähler. This structure extends the almost Hermitian structure on M considered in [11, Section 6].

Lemma 2.3. *On the Hyperelliptic surface M , define the almost complex structure J given by*

$$\varphi^1 = e^1 + ie^3, \quad \varphi^2 = e^2 + ie^4$$

being a coframe of $T^{1,0}(M, J)$. Then, the corresponding structure equations are

$$(9) \quad d\varphi^1 = i\frac{\pi}{4}(-\varphi^{12} - \varphi^{1\bar{2}} - \varphi^{2\bar{1}} + \varphi^{\bar{1}\bar{2}}), \quad d\varphi^2 = i\frac{\pi}{2}\varphi^{1\bar{1}},$$

and for $0 < \rho < 1$ the almost Hermitian metric

$$\omega_\rho = e^{13} + e^{24} + \rho e^{12} + \rho e^{34} = \frac{i}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}}) + \frac{\rho}{2}(\varphi^{1\bar{2}} - \varphi^{2\bar{1}}),$$

is Gauduchon but not locally conformally almost Kähler.

Proof. The structure equations (9) of (M, J) derive from the real structure equations (1) of M by elementary computations.

Let us show that ω_ρ is not locally conformally almost Kähler for $0 < \rho < 1$. We compute

$$d\omega_\rho = \pi e^{134} = \theta_\rho \wedge \omega_\rho,$$

with

$$\theta_\rho = -\frac{\pi\rho}{1-\rho^2}e^1 + \frac{\pi}{1-\rho^2}e^4.$$

For $0 < \rho < 1$ the 1-form θ_ρ is not closed, i.e., $d\theta_\rho \neq 0$, therefore ω_ρ is not locally conformally almost Kähler. For $\rho = 0$, note that θ_0 is closed but non exact, thus ω_0 is strictly locally conformally almost Kähler.

Finally, since ω_ρ is left invariant, it is Gauduchon. However, let us check it explicitly. Indeed, for $0 \leq \rho < 1$, we have

$$dJd\omega_\rho = dJ\pi e^{134} = -\pi de^{312} = 0. \quad \square$$

By Lemma 2.3, for $0 < \rho < 1$, we have that (J, ω_ρ) is a non integrable almost Hermitian structure on M which is not locally conformally almost Kähler. Let us compute the number $h_{\bar{\partial}}^{1,1}(M, J, \omega_\rho)$. We will need the following result (see [9, Theorem 3.6]). Recall that $d^c = J^{-1}dJ = i(\bar{\partial} - \partial + \mu - \bar{\mu})$.

Theorem 2.4. *Let G be a 4-dimensional Lie group and let Γ be a discrete subgroup such that $M = \Gamma \backslash G$ is compact. Assume that (J, ω) is a left invariant almost Hermitian structure on G . Then $h_{\bar{\partial}}^{1,1}(M, J, \omega) = b^- + 1$ if and only if there exists a left invariant anti-self-dual $(1, 1)$ -form $\gamma \in A^{1,1}(G, J)$ satisfying*

$$id^c\gamma = d\omega.$$

Otherwise, $h_{\bar{\partial}}^{1,1}(M, J, \omega) = b^-$.

We are now ready to prove the following

Theorem 2.5. *For $0 < \rho < 1$, let M be the Hyperelliptic surface endowed with the non integrable, non locally conformally almost Kähler structure (J, ω_ρ) of Lemma 2.3. Then*

$$h_{\bar{\partial}}^{1,1}(M, J, \omega_\rho) = b^- = 1.$$

Proof. The fundamental form ω_ρ can also be rewritten as

$$\omega_\rho = \frac{i}{2}(\psi^{1\bar{1}} + \psi^{2\bar{2}}),$$

where we set

$$\psi^1 = \varphi^1 + i\rho\varphi^2, \quad \psi^2 = \sqrt{1-\rho^2}\varphi^2$$

as a unitary coframe of $T^{1,0}(M, J)$. We have

$$\bar{\partial}\psi^1 = -i\frac{\pi}{4}\varphi^{1\bar{2}} - i\frac{\pi}{4}\varphi^{2\bar{1}} - \rho\frac{\pi}{2}\varphi^{1\bar{1}},$$

$$\begin{aligned} \partial\psi^1 &= -i\frac{\pi}{4}\varphi^{12}, \\ \bar{\partial}\psi^2 &= i\frac{\pi}{2}\sqrt{1-\rho^2}\varphi^{1\bar{1}}, \\ \partial\psi^2 &= 0. \end{aligned}$$

Define the volume form Vol such that

$$\text{Vol} = \frac{\omega_\rho^2}{2} = \frac{1}{4}\psi^{12\bar{1}\bar{2}}.$$

In order to compute $h_{\bar{\partial}}^{1,1}(M, J, \omega_\rho)$, we will apply [9, Theorem 3.6], see Theorem 2.4 above, and therefore it suffices to show that there are no left invariant anti-self-dual (1, 1)-forms γ satisfying

$$id^c\gamma = d\omega_\rho.$$

For bidegree reasons, we have $id^c\gamma = \partial\gamma - \bar{\partial}\gamma$. Any left invariant anti-self-dual (1, 1)-form γ can be written as

$$\gamma = B\psi^{1\bar{1}} + C\psi^{1\bar{2}} + D\psi^{2\bar{1}} - B\psi^{2\bar{2}},$$

with $B, C, D \in \mathbb{C}$. Note that $*\gamma = -\gamma$, i.e., γ is anti-self-dual. We have

$$\begin{aligned} \bar{\partial}\psi^{1\bar{1}} &= i\frac{\pi}{2}\rho^2\varphi^{1\bar{1}\bar{2}}, \\ \bar{\partial}\psi^{1\bar{2}} &= -\frac{\pi}{2}\rho\sqrt{1-\rho^2}\varphi^{1\bar{1}\bar{2}} - i\frac{\pi}{4}\sqrt{1-\rho^2}\varphi^{2\bar{1}\bar{2}}, \\ \bar{\partial}\psi^{2\bar{1}} &= \frac{\pi}{2}\rho\sqrt{1-\rho^2}\varphi^{1\bar{1}\bar{2}} - i\frac{\pi}{4}\sqrt{1-\rho^2}\varphi^{2\bar{1}\bar{2}}, \\ \bar{\partial}\psi^{2\bar{2}} &= i\frac{\pi}{2}(1-\rho^2)\varphi^{1\bar{1}\bar{2}}. \end{aligned}$$

Moreover, $\partial\psi^{i\bar{j}} = -\overline{\bar{\partial}\psi^{j\bar{i}}}$ for $i, j \in \{1, 2\}$, therefore we obtain

$$\begin{aligned} \bar{\partial}\gamma &= \frac{\pi}{2}\left(i(2\rho^2 - 1)B - \rho\sqrt{1-\rho^2}C + \rho\sqrt{1-\rho^2}D\right)\varphi^{1\bar{1}\bar{2}} + \\ &\quad - i\frac{\pi}{4}\sqrt{1-\rho^2}(C + D)\varphi^{2\bar{1}\bar{2}} \end{aligned}$$

and

$$\partial\gamma = \frac{\pi}{2}\left(i(2\rho^2 - 1)B - \rho\sqrt{1-\rho^2}C + \rho\sqrt{1-\rho^2}D\right)\varphi^{12\bar{1}} +$$

$$-i\frac{\pi}{4}\sqrt{1-\rho^2}(C+D)\varphi^{12\bar{2}}.$$

We have already computed

$$\begin{aligned} d\omega_\rho &= \pi e^{134} \\ &= \pi\frac{i}{2}\varphi^{1\bar{1}} \wedge \frac{1}{2i}(\varphi^2 - \varphi^{\bar{2}}) \\ &= -\frac{\pi}{4}(\varphi^{12\bar{1}} + \varphi^{1\bar{1}\bar{2}}), \end{aligned}$$

therefore $id^c\gamma = d\omega_\rho$ if and only if

$$\begin{cases} \bar{\partial}\gamma = \frac{\pi}{4}\varphi^{1\bar{1}\bar{2}}, \\ \partial\gamma = -\frac{\pi}{4}\varphi^{12\bar{1}}, \end{cases}$$

if and only if

$$\begin{cases} C+D=0, \\ i(2\rho^2-1)B-\rho\sqrt{1-\rho^2}C+\rho\sqrt{1-\rho^2}D=\frac{1}{2}, \\ i(2\rho^2-1)B-\rho\sqrt{1-\rho^2}C+\rho\sqrt{1-\rho^2}D=-\frac{1}{2}, \end{cases}$$

which is a contradiction. It follows that there are no left invariant anti-self-dual $(1, 1)$ -forms γ satisfying $id^c\gamma = d\omega_\rho$ and so $h_{\bar{\partial}}^{1,1}(M, J, \omega_\rho) = b^- = 1$ by Theorem 2.4. \square

Summing up, we have just proved the following

Theorem 2.6. *Let M be a Hyperelliptic surface in the first isomorphism class. On M , there exist two non integrable almost Hermitian structures (J_1, ω_1) and (J_2, ω_2) which are not locally conformally almost Kähler and which satisfy*

$$h_{\bar{\partial}}^{1,1}(M, J_1, \omega_1) = 2 = b^- + 1, \quad h_{\bar{\partial}}^{1,1}(M, J_2, \omega_2) = 1 = b^-.$$

3. $h_{\bar{\partial}}^{1,1}$ on the nilmanifold \mathcal{N}

We start by recalling the description of the nilmanifold \mathcal{N} . Following Hasegawa, [4, Section 4, p. 758], let G be the group \mathbb{R}^4 together with the operation

$$(a, b, c, d) \cdot (x, y, z, t) = (x+a, y+b, z+ay+c, t+\frac{1}{2}a^2y+az+d).$$

Then the following 1-forms

$$\begin{aligned} e^1 &= dx, \\ e^2 &= dy, \\ e^3 &= dz - xdy, \\ e^4 &= dt + \frac{1}{2}x^2dy - xdz, \end{aligned}$$

on G form a basis for $A^1(G)$, and they are left invariant. Let Γ be the subgroup of G given by $(2\mathbb{Z})^4$ and call \mathcal{N} the compact quotient $\Gamma \backslash G$. It follows that e^1, e^2, e^3, e^4 are still 1-forms on \mathcal{N} and form a basis of $A^1(\mathcal{N})$. The structure equations for \mathcal{N} are

$$de^1 = 0, \quad de^2 = 0, \quad de^3 = -e^{12}, \quad de^4 = -e^{13}.$$

Remark 3.1. *As \mathcal{N} is a quotient of \mathbb{R}^4 by the action of Γ , we can see a smooth function $f : \mathcal{N} \rightarrow \mathbb{R}$ as a smooth function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ which satisfies the following invariance property*

$$f(x, y, z, t) = f(x + 2\alpha, y + 2\beta, z + 2\alpha y + 2\gamma, t + 2\alpha^2 y + 2\alpha z + 2\delta)$$

for every $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$.

We endow \mathcal{N} with the non integrable almost complex structure J defined by taking

$$\varphi^1 = e^1 + ie^2, \quad \varphi^2 = e^3 + ie^4$$

as a basis of $A^{1,0}(\mathcal{N}, J)$. The structure equations are

$$d\varphi^1 = 0, \quad d\varphi^2 = -\frac{i}{4}(\varphi^{12} + 2\varphi^{1\bar{1}} + \varphi^{1\bar{2}} - \varphi^{2\bar{1}} + \varphi^{\bar{1}\bar{2}}).$$

We consider, on (\mathcal{N}, J) , the almost Hermitian metric given by the fundamental form

$$\omega = e^{12} + e^{34} = \frac{i}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}}).$$

Take

$$\text{Vol} = \frac{\omega^2}{2} = e^{1234} = \frac{1}{4}\varphi^{12\bar{1}\bar{2}}$$

as the volume form. The de Rham cohomology can be computed by looking only at the invariant forms, therefore

$$H_{dR}^2 = \mathbb{R} \langle e^{14}, e^{23} \rangle$$

and $b^- = 1$. Since

$$d\omega = -e^{124} = \theta \wedge \omega$$

with $\theta = -e^4$, which is not closed, it follows that ω is not locally conformally almost Kähler. We are going to show that $h_{\bar{\partial}}^{1,1}(\mathcal{N}, J, \omega) = b^- + 1$.

Let $\eta \in A^{1,1}(\mathcal{N}, J)$ be a left invariant $(1, 1)$ -form. We can write

$$\eta = A\varphi^{1\bar{1}} + B\varphi^{1\bar{2}} + C\varphi^{2\bar{1}} + D\varphi^{2\bar{2}},$$

where $A, B, C, D \in \mathbb{C}$. Applying the Hodge $*$ operator we get

$$*\eta = D\varphi^{1\bar{1}} - B\varphi^{1\bar{2}} - C\varphi^{2\bar{1}} + A\varphi^{2\bar{2}}.$$

Recall that $\eta \in \mathcal{H}_{\bar{\partial}}^{1,1}$ iff $\bar{\partial}\eta = \partial * \eta = 0$. We compute

$$\bar{\partial}\eta = \frac{i}{4}(-B + C - 2D)\varphi^{1\bar{1}\bar{2}},$$

and

$$\partial * \eta = \frac{i}{4}(-B + C - 2A)\varphi^{1\bar{2}\bar{1}}.$$

It follows that the dimension of the space of invariant Dolbeault harmonic $(1, 1)$ -forms is $2 = b^- + 1$. By [6, Theorem 3.1], $h_{\bar{\partial}}^{1,1}$ can be equal only to b^- or $b^- + 1$, therefore we conclude that $h_{\bar{\partial}}^{1,1}(\mathcal{N}, J, \omega) = b^- + 1$. Explicitly,

$$\mathcal{H}_{\bar{\partial}}^{1,1}(\mathcal{N}, J, \omega) = \mathbb{C} \langle 2\varphi^{1\bar{2}} - \varphi^{1\bar{1}} - \varphi^{2\bar{2}}, 2\varphi^{2\bar{1}} + \varphi^{1\bar{1}} + \varphi^{2\bar{2}} \rangle.$$

We have just proved the following

Proposition 3.2. *Let \mathcal{N} be the 4-manifold previously defined. On \mathcal{N} , there exists a non integrable almost Hermitian structure (J, ω) which is not locally conformally almost Kähler and which satisfies $h_{\bar{\partial}}^{1,1}(\mathcal{N}, J, \omega) = 2 = b^- + 1$.*

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