Pure and Applied Mathematics Quarterly
Volume 18, Number 4, 1379–1420, 2022

Singularities of normal quartic surfaces II (char=2)

Fabrizio Catanese* and Matthias Schütt

Dedicated to Herb Clemens on the occasion of his 82nd birthday

Abstract: We show, in this second part, that the maximal number of singular points of a normal quartic surface $X \subset \mathbb{P}^3_K$ defined over an algebraically closed field K of characteristic 2 is at most 14, and that, if we have 14 singularities, these are nodes and moreover the minimal resolution of X is a supersingular K3 surface.

We produce an irreducible component, of dimension 24, of the variety of quartics with 14 nodes.

We also exhibit easy examples of quartics with 7 A_3-singularities.

Keywords: Quartic surface, singularity, Gauss map, genus one fibration, supersingular K3 surface.

arXiv: 2110.03078

Received October 6, 2021.

2010 Mathematics Subject Classification: Primary 14J17, 14J28; secondary 14J25, 14N05.

*The first author acknowledges support of the ERC 2013 Advanced Research Grant – 340258 – TADMICAMT.

1. Introduction

Once upon a time[1] in Cortona there was a Summer school with wonderful courses held by Herb Clemens and Boris Moishezon. The first author had the privilege of attending the Summer school. On that occasion Herb lectured on several beautiful classical topics, and these lectures formed the basis of a lovely book [Clem80]. Even if the course and the book were devoted to complex curves, yet characteristic p appeared on the stage, and was used by Clemens to explain the 'Unity of Mathematics' (section 2.12). In this spirit we are happy to dedicate this 'characteristic 2' paper to Herb.

[1] It was in August-September 1976.

These are our main results. They feature the property of the minimal resolution S of being a supersingular K3 surface (i.e. with Picard number $\rho = 22$).[2] The following is our main theorem:

Theorem 1. *A normal quartic surface $X \subset \mathbb{P}^3_K$ defined over an algebraically closed field K of characteristic 2 contains no more than 14 singular points. If the maximum number of 14 singularities is attained, then all singularities are nodes and the minimal resolution is a supersingular K3 surface. The variety of quartics with 14 nodes contains an irreducible component, of dimension 24.*

If the minimal resolution S of a normal quartic X is not a supersingular K3 surface, then Theorem 1 shows that X has at most 13 singular points. This bound is not sharp; we have examples with 12 nodes, and we will show in a forthcoming paper (part III) that if S is a K3 surface which is not supersingular, then X has at most 12 singular points.

The proof uses mostly classical techniques, notably the Gauss map, but there are some ingredients (notably the main claim in Section 4) which build on the theory of genus one fibrations (see Section 5).

We emphasize that each ingredient has some special feature in characteristic 2; for instance, the Gauss map of a normal surface in $\mathbb{P}^3$ need not be birational, and double points behave differently (this affects the degree formula, see Section 2). The dual surface can be a plane, as we study in Section 3. Elliptic fibrations feature wild ramification (at certain additive fibres), which has surprising consequences for supersingularity (see Section 5). The notion of genus one fibration also encompasses quasi-elliptic fibrations whose properties we exploit in Section 6, especially with a view to the dual surface.

Naturally Theorem 1 leads to the question about what is true for other quasi-polarized K3 surfaces in characteristic 2, which we plan to address in part III as well.

Convention: We work over an algebraically closed field K, mostly of characteristic 2, though many results may also be stated over non-closed fields.

2. The Gauss map

We consider in this section a normal quartic surface

$$X = \{F(x) = 0\} \subset \mathbb{P}^3$$

[2] For the reader who has never seen such a surface, an easy example is provided in Corollary 8 in Section 3, as the resolution of a quartic surface with 7 A_3 singularities, providing 21 independent (-2)-curves on S which together with the hyperplane section H generate a rank 22 finite index sublattice of $\mathrm{Pic}(S)$.

and summarize and extend some considerations made in the first part, [Cat21b] in order to gain control over the (number of) singular points of X.

The Gauss map $\gamma : X \dashrightarrow \mathcal{P} := (\mathbb{P}^3)^\vee$ is the rational map given by

$$\gamma(x) := \nabla F(x), \;\; x \in X^0 := X \setminus \mathrm{Sing}(X).$$

We let $X^\vee := \overline{\gamma(X^0)}$ be the closure of the image of the Gauss map, which is a morphism on X^0, and becomes a morphism $\tilde{\gamma}$ on a suitable blow up $\tilde{S}$ of the minimal resolution S of X. $X^\vee$ is called the dual variety of X.

In order to compute the degree of $X^\vee$ (this is defined to be equal to zero if $X^\vee$ is a curve), we consider a line $\Lambda \subset \mathcal{P}$ such that Λ is transversal to the map $\tilde{\gamma}$, this means:

1) $\Lambda \cap X^\vee = \emptyset$ if $X^\vee$ is a curve;
2) Λ is not tangent to $X^\vee$ at any smooth point, and neither contains any singular point of $X^\vee$, nor any point y where the dimension of the fibre $\tilde{\gamma}^{-1}(y)$ equals 1, so that
3) $\Lambda \cap X^\vee$ is in particular a subscheme consisting of $\deg(X^\vee)$ distinct points, and its inverse image in $\tilde{S}$ is a finite set.

By a suitable choice of the coordinates, we may assume that

$$\gamma^{-1}(\Lambda) \subset X \cap \{F_1 = F_2 = 0\} \qquad (F_i = \partial F / \partial x_i).$$

The latter is a finite set, hence by Bezout's theorem it consists of $4 \cdot 3^2 = 36$ points counted with multiplicity, including the singular points of X.

We have therefore proven the following (probably well known) formula:

$$(DEGREE - FORMULA) \;\;\; \deg(\gamma)\deg(X^\vee) = 36 - \sum_{P \in \mathrm{Sing}(X)} (F, F_1, F_2)_P,$$

where the symbol $(F, F_1, F_2)_P$ denotes the local intersection multiplicity at P, defined by

$$(F, F_1, F_2)_P := \dim_K(\mathcal{O}_{\mathbb{P}^3,P}/(F, F_1, F_2)) = \dim_K(\mathcal{O}_{X,P}/(F_1, F_2)).$$

Under the above assumptions this intersection multiplicity is zero unless P is a singular point, and then we have

$$(F, F_1, F_2)_P \geq 2 \;\; \forall P \in \mathrm{Sing}(X).$$

The integer $(F, F_1, F_2)_P$ shall be called the Gaussian defect.

2.1. Calculation of Gaussian defects

Since quartics with triple points were treated in [Cat21b], we will mostly be concerned with double points, but we will cover all types of singularities in Proposition 4. Double point singularities are divided into three rough types according to the rank of the tangent quadric at P:

i) nodes: here the quadric is smooth, and we have an A_1-singularity, formally equivalent to $xy = z^2$; the nodes give a contribution $(F, F_1, F_2)_P = 2$ to the Gaussian defect;
ii) biplanar double points: here the quadric consists of two planes, and we have an A_n-singularity with $n \geq 2$, formally equivalent to $xy = z^{n+1}$ (see for instance [Cat21a]); the biplanar double points of type A_n give a contribution of $n+1$ to the Gaussian defect;
iii) uniplanar double points: here the quadric consists of a double plane, and we have several types (see [Art66], [Art77], [Roc96]), the Taylor development is of the form $x^2 + \psi = 0$, where ψ has order ≥ 3; the uniplanar double points give a contribution of order at least 8 to the Gaussian defect, since $(F, F_1, F_2)_P = (F, \psi_1, \psi_2) \geq 8$.

Proposition 2. *Let X be a normal quartic surface in $\mathbb{P}^3$.*

(I) If X has ν singular points of multiplicity 2, among them b biplanar double points, and u uniplanar double points, then:

$$36 - \deg(\gamma)\deg(X^\vee) \geq 2\nu + b + 6u. \tag{1}$$

(II) If X contains a node, then the exceptional curve E in S resolving the node maps to a line via an inseparable map of degree two. In particular the Gauss map cannot be birational if $X^\vee$ is a normal surface.

(III) The dual variety $X^\vee$ cannot be a line.

(IV) For $\nu \geq 13$, the dual variety $X^\vee$ is an irreducible surface; in particular $\deg(\gamma)\deg(X^\vee) \geq 2$, and, if $\deg(\gamma) = 1$, $X^\vee$ is non-normal and $\deg(X^\vee) \geq 3$.

(V) For $\nu \geq 14$, if the dual variety $X^\vee$ is not a plane, then the singularities of X are all of type A_n ($u = 0$).

Proof. (I): follows since the nodes give a contribution equal to 2 to the Gaussian defect, the biplanar double points of type A_n give a contribution $n+1 \geq 3$, the uniplanar double points give a contribution at least 8.

(II): given a node P, an A_1-singularity, then the affine Taylor development at P is given by

$$F = xy + z^2 + \psi(x, y, z) = 0$$

and the Gauss map on the exceptional curve E, given as a conic $E = \{xy + z^2 = 0\} \subset \mathbb{P}^2$, is given by $(x, y, 0, 0)$. If $X^\vee$ is a normal surface and γ is birational onto its image, then

$$\tilde{\gamma} : \tilde{S} \to X^\vee$$

is an isomorphism over the complement of a finite number of points of $X^\vee$, a contradiction since E maps 2 to 1 to a line.

(III): if $X^\vee$ is a line, then there are projective coordinates in $\mathbb{P}^3$ such that the partial derivatives with respect to 2 variables, say z, w, are identically zero; hence

$$X = \{az^4 + bw^4 + cz^2w^2 + z^2D(x,y) + w^2E(x,y) + f(x,y) = 0\}.$$

Writing

$$D(x,y) = d_1x^2 + d_2y^2 + dxy, E(x,y) = e_1x^2 + e_2y^2 + exy,$$

$$f(x,y) = q(x,y)^2 + f_1x^3y + f_2xy^3,$$

we see that

$$\mathrm{Sing}(X) = X \cap \{yM = xM = 0\},\ M := dz^2 + ew^2 + f_1x^2 + f_2y^2,$$

hence Sing $(X) \supset X \cap \{M = 0\}$ and X is not normal.

(IV): since for $\nu \geq 13$ there must be a node by the degree formula, $X^\vee$ contains a line; but $X^\vee$ cannot be a line by (III), hence it is a surface; the rest follows from (II) and from the fact that an irreducible quadric is normal.

(V) For $\nu \geq 14$, if the dual variety $X^\vee$ is not a plane, then $\deg(\gamma) \geq 2$, or $\deg(X^\vee) \geq 3$ (since if γ is birational, then $X^\vee$ is not normal by (IV)), hence $2\nu + b + 6u \leq 33$, hence $u = 0$ and the singularities of X are all of type A_n. ☐

Remark 3. The degree formula can be improved substantially by taking the precise types of singularities into account as the proof of (I) shows. For instance, the biplanar double points contribute b in (1) exactly when they all have type A_2.

2.2. Gaussian defect of non-rational double point singularities

In the spirit of Remark 3, we take a closer look at those singularities which are not rational double point. This will enable us to strengthen the results of Proposition 2, and to decide when the minimal resolution S of X is a K3 surface (see Proposition 14).

Proposition 4. *If a given singularity P on X is not a rational double point, we have, for a general (hence any) choice of the affine local coordinates at P,*

$$(F, F_1, F_2)_P \geq 10. \tag{2}$$

Proof. First of all, for a triple point the Gaussian defect is at least 12, since the Gaussian defect is greater or equal to the product of the respective orders of F, F_1, F_2 at P, and a double point which is not a rational double point must be a uniplanar double point.

We can therefore assume that the affine Taylor development of F at the point P is of the form

$$F(x_1, x_2, x_3) = x_1^2 + G(x) + B(x),$$

where G is homogeneous of degree 3 and B of degree 4.

We may take local coordinates such that $x := x_1$, and where y, z are generic linear forms vanishing at P, hence the Gaussian defect will be the intersection number (F, F_y, F_z) at the point P.

This said, we can write

$$F(x, y, z) = x^2(1 + A(x, y, z)) + xg(y, z) + g'(y, z) + B'(x, y, z),$$

and multiplying by $(1 + A(x, y, z))^{-1}$ we get a formal power series equation

$$f = x^2 + xg(y, z) + g'(y, z) + b(x, y, z), \tag{3}$$

where g is a quadratic form, g' is a cubic form, and the power series b has order at least 4.

We consider the blow up of the singular point $P \in X$.

The equation of X is $f(x, y, z) = x^2 + xg(y, z) + g'(y, z) + b(x, y, z) = 0$; set now:

$$x = t\,\xi, \quad y = t\,\eta, \quad z = t\,\zeta$$

(here $t = 0$ is the equation of the exceptional divisor, isomorphic to $\mathbb{P}^2$ and (ξ, η, ζ) are homogeneous coordinates in $\mathbb{P}^2$) so that the equation of the blow up is

$$\xi^2 + t\xi\ g(\eta, \zeta) + t\ g'(\eta, \zeta) + t^2\ b(\xi, \eta, \zeta) = 0.$$

On the exceptional line $\{t = \xi = 0\}$ the singular points are the roots of g'.

Hence either g' is identically zero, or the blow up is normal.

If g' does not have a multiple root, we get 3 nodes, hence P is a singularity of type D_4 and we are done.

Therefore we may assume that g' has a multiple root (or is identically zero) and apply a linear transformation such that $y = 0$ is this root.

We want to show that the length of the Artin algebra

$$\mathcal{A} := \mathcal{O}_P/(f, f_y, f_z)$$

is ≥ 10.

A fortiori it will suffice to replace the algebra $\mathcal{A}$ by the quotient algebra

$$\mathcal{A}_4 := \mathcal{O}_P/((f, f_y, f_z) + \mathfrak{M}_P^4)$$

or by the quotient algebra

$$\mathcal{A}_5 := \mathcal{O}_P/((f, f_y, f_z) + \mathfrak{M}_P^5).$$

Inside the algebra $\mathcal{B}_4 := \mathcal{O}_P/\mathfrak{M}_P^4$, the ideal $\mathcal{I}$ generated by f, f_y, f_z is generated as a vector space by the vectors

$$f, f_y, f_z, xf, xf_y, xf_z, yf, yf_y, yf_z, zf, zf_y, zf_z$$

where the first three have order at least 2, and the latter at least 3.

Since $\mathcal{I} \subset \mathfrak{M}_P^2$, which has colength 4, it suffices to show that

1) $(\mathcal{I} + \mathfrak{M}_P^3)/\mathfrak{M}_P^3$ has codimension at least 3 in $\mathfrak{M}_P^2/\mathfrak{M}_P^3$, which is a 6-dimensional vector space.
2) $((\mathcal{I} \cap \mathfrak{M}_P^3) + \mathfrak{M}_P^4)/\mathfrak{M}_P^4$ has codimension at least 2 in $\mathfrak{M}_P^3/\mathfrak{M}_P^4$, which is a 10-dimensional vector space.
3) if in 2) codimension 2 occurs, then $((\mathcal{I} \cap \mathfrak{M}_P^4) + \mathfrak{M}_P^5)/\mathfrak{M}_P^5$ has codimension at least 2 in $\mathfrak{M}_P^4/\mathfrak{M}_P^5$.

We can now write

$$f = x^2 + xg(y, z) + cy^3 + dy^2 z \pmod{\mathfrak{M}_P^4},$$

$$f_y = axz + cy^2, \quad f_z = axy + dy^2 \pmod{\mathfrak{M}_P^3},$$

where c, d may be equal to zero.

The first assertion is clear, since modulo $\mathfrak{M}_P^3$ we just have three vectors, $x^2, axz + cy^2, axy + dy^2$. In fact, if $a = 0$, then the vectors are linearly dependent modulo $\mathfrak{M}_P^3$, so $(\mathcal{I} + \mathfrak{M}_P^3)/\mathfrak{M}_P^3$ has codimension at least 4 in $\mathfrak{M}_P^2/\mathfrak{M}_P^3$.

As one easily checks that $((\mathcal{I} \cap \mathfrak{M}_P^3) + \mathfrak{M}_P^4)/\mathfrak{M}_P^4$ has codimension at least 3 in $\mathfrak{M}_P^3/\mathfrak{M}_P^4$ in this situation, it follows that $(F, F_1, F_2)_P \geq 11$ as desired. Hence, in what follows, we will assume $a \neq 0$ and thus normalize $a = 1$.

For the second assertion, it suffices to show that, modulo $\mathfrak{M}_P^4$, we get a subspace in degree 3 of dimension at most 8.

From f, in degree 3 we get x^3, x^2y, x^2z, and modulo the subspace generated by the above vectors we get $xf_y \equiv cxy^2, xf_z \equiv dxy^2$: these vectors are all contained in the 4-dimensional subspace V generated by x^3, x^2y, x^2z, xy^2. Since there are only 4 further generators in degree 3 (given below), this already proves the second assertion.

For future use, we further investigate $(\mathcal{I} + \mathfrak{M}_P^4)/\mathfrak{M}_P^4$. Modulo V, we get the 4 vectors,

$$yf_y = y(xz + cy^2),\ zf_y = z(xz + cy^2),\ yf_z \equiv dy^3,\ zf_z = z(xy + dy^2).$$

These are linearly independent if and only if $d \neq 0$; in that case, their span is generated by xyz, xz^2, y^3, y^2z, in agreement with the second assertion.

Note that, if $d = 0$, then $(\mathcal{I} \cap \mathfrak{M}_P^3)/\mathfrak{M}_P^4$ has codimension at least 3 in $\mathfrak{M}_P^3/\mathfrak{M}_P^4$, and the main claim of our proposition follows readily.

Hence we will assume $d \neq 0$ in what follows.

In order to prove the third assertion we observe that the ideal

$$\mathcal{I}' := (x^2, xy^2, xz^2, xyz, y^3, y^2z)$$

arising from the monomials in the above computations contains all monomials of degree 4 except for z^4 and z^3y.

Define W to be the subspace of $\mathfrak{M}_P^4/\mathfrak{M}_P^5$ generated by $\mathcal{I}'$, i.e. by the monomials containing x or divisible by y^2.

Working in

$$U := (\mathfrak{M}_P^4/\mathfrak{M}_P^5)/W \cong Kz^4 \oplus Kz^3y$$

we want to show that $((\mathcal{I} \cap \mathfrak{M}_P^4) + \mathfrak{M}_P^5)/\mathfrak{M}_P^5$ maps to zero in U.

Observe that, in degree 3, $b_y \equiv \lambda y^2z + \mu z^3, b_z \equiv \lambda y^3 + \mu yz^2$ modulo $\mathcal{I}'$.

But in order to get this, we must have some degree 1 relation between the quadratic parts of f, f_y, f_z, giving then rise in degree 4 to some non-zero vector in U.

In fact, one relation is obvious, namely

$$(cy + dz)f + dxf_y + cxf_z \in \mathfrak{M}_P^4,$$

but it only produces multiples of x and y^2 in degree 4, i.e. zero in U.

Direct calculation shows that the above is the only relation in degree 1 occurring, so we are done. □

We will use the proposition soon to derive a criterion for the minimal resolution S of X to be a K3 surface (Proposition 14), but as a preparation we have to discuss the case where the dual surface $X^\vee$ is a plane.

3. When the dual surface is a plane

We continue to consider a normal quartic surface $X \subset \mathbb{P}^3$. The term $\deg(\gamma)\deg(X^\vee)$ in (1) deems it essential to study the case where the dual surface $X^\vee$ is a plane.

In this case there are coordinates (x_1, x_2, x_3, z) such that the partial derivative of F with respect to z is identically zero, hence

$$X = \{(x, z) | az^4 + z^2 Q(x) + B(x) = 0\}. \tag{4}$$

We are going to show that for the general such surface X has 14 nodes as singularities (Theorem 11). This will prove part of Theorem 1.

There are a few special equations where $X^\vee$ is a plane which require extra treatment. Two of them were contained in part one of this paper [Cat21b]:

(1) $X = \{(x, z) \mid z^2 Q(x) + B(x) = 0\}$ $(a = 0)$ is the case where there is a singular point P $(x = 0)$ such that projection with centre P is inseparable.
(2) $Q(x)$ is the square of a linear form (see Proposition 9).

One more equation will appear in Proposition 10. Together with Lemma 13, they will suffice to prove the instrumental fact that, with at least 13 singular points, the minimal resolution S of X is a K3 surface (Proposition 14).

The curve $\{B(x) = 0\}$ obtained from equation (4) is a plane quartic curve, and we want now to establish some simple properties of plane quartic curves which will be relevant for our issues.

3.1. The strange points of a plane curve of even degree in characteristic = 2

We define here, as in [Cat21b], the strange points of a plane curve $\{B(x) = 0\} \subset \mathbb{P}^2$ to be the points outside the curve where the gradient ∇B vanishes.

We have seen in Part I ([Cat21b]) for the case of a general plane quartic curve $\{B(x) = 0\} \subset \mathbb{P}^2$:

Proposition 5. *For a homogeneous quartic polynomial $B \in K[x_1, x_2, x_3]_4$ let Σ be the critical locus of B, the locus where the gradient ∇B vanishes. If Σ is a finite set, then it consists of at most 7 points.*

For B general, Σ consists of exactly 7 reduced points.

The above result does indeed nicely extend to the case of a homogeneous polynomial of even degree $B \in K[x_1, x_2, x_3]_{2m}$.

As noticed in [Cat21b], because of the Euler identity

$$x_1 B_1 + x_2 B_2 + x_3 B_3 = 0$$

among partial derivatives, taking coordinates such that the line $\{x_3 = 0\}$ does not intersect Σ, it follows that

$$\Sigma = \{x \mid B_1(x) = B_2(x) = 0, x_3 \neq 0\}.$$

For a polynomial of the Klein form

$$B_0 := x_1^{2m-1} x_2 + x_2^{2m-1} x_3 + x_3^{2m-1} x_1,$$

the critical scheme is defined by

$$x_i^{2m-1} = x_{i+1}^{2m-2} x_{i+2} \implies x = (\epsilon, 1, \epsilon^{2m-1}),\ \epsilon^{(2m-2)(2m-1)+1} = 1.$$

Letting s be the general number of strange points ($s := |\Sigma|$), we have therefore that, setting $d = 2m$, s lies in the interval

$$(d-1)(d-2) + 1 = (d-1)^2 - (d-2) \leq s \leq (d-1)^2.$$

Proposition 6. *The number of strange points $s := |\Sigma|$ ($\Sigma := \{\nabla B(x) = 0\}$) of a general homogeneous polynomial $B(x_1, x_2, x_3)$ of even degree d is equal to $s = (d-1)(d-2) + 1$.*

Whenever the subscheme Σ is finite, its length equals s.

Proof. To show this, two steps suffice:

I) if Σ is finite, then the scheme $\{x \mid B_1(x) = B_2(x) = 0\}$ is finite for general choice of coordinates, and $\Sigma \cap \{x_3 = 0\}$ is empty in general;

II) the subscheme $\{x | B_1(x) = B_2(x) = x_3 = 0\}$, if we write

$$B = x_3 B' + \beta(x_1, x_2) + q(x_1, x_2)^2,\ \beta(x_1, x_2) = \sum_{n \text{ odd}} a_n x_1^n x_2^{2m-n},$$

equals the subscheme

$$\left\{ x \,\middle|\, x_3 = \sum_{n \text{ odd}} a_n x_1^{n-1} x_2^{2m-n} = \sum_{n \text{ odd}} a_n x_1^n x_2^{2m-n-1} = 0 \right\} =$$

$$\left\{ x \,\middle|\, x_3 = x_2 \left(\sum_{n \text{ odd}} a_n x_1^{n-1} x_2^{2m-n-1} \right) = x_1 \left(\sum_{n \text{ odd}} a_{n-1} x_1^{n-1} x_2^{2m-n-1} \right) = 0 \right\}.$$

I) holds since changing variables we get that the new partials are general linear combinations of the old partials: if Σ is finite then we can keep B_1 fixed and vary B_2 so that it has no common factor with B_1; hence the result holds for general choice of linear coordinates, and the rest is obvious.

Our result follows then from I) and II), since then the scheme Σ is disjoint from the length $(d-2)$ scheme

$$\left\{ x_3 = \sum_{n \text{ odd}} a_{n-1} x_1^{n-1} x_2^{2m-n-1} = 0 \right\},$$

and we conclude since their union is the complete intersection subscheme $\{x | B_1(x) = B_2(x) = 0\}$, which has length $(d-1)^2$. $\square$

Remark 7. A more general result (also valid in other characteristics) is contained in Theorem 2.4 of [Lied13], whose formulation, however, does neither mention derivatives nor critical sets.

3.2. Supersingular quartics with 7 A_3-singularities

A first immediate consequence of the previous result is:

Corollary 8. *For B a homogeneous polynomial $B \in K[x_1, x_2, x_3]_4$, a normal quartic surface of the form:*

$$X := \{(x, z) \mid z^4 + B(x) = 0\}$$

has at most 7 singular points.

If B is general, X has 7 A_3-singularities.

Proof. The singular points of X are in bijection with the critical set Σ of B, which consists of 7 reduced points for B general. Hence at these points there are local coordinates u, v such that $B = a^4 + uv$, hence the local equation of X is $(z + a)^4 = uv$, and we have an A_3-singularity. $\square$

3.3. Quartics with 14 nodes blowing up to 7 lines in the plane under the Gauss map

A second immediate consequence concerns the quartics with dual surface equal to a plane, and with a singular point such that the second order term Q of the Taylor development (see formula (4)) is equal to the square of a linear form.

Proposition 9. *Consider a normal quartic of equation*

$$X = \{(x, z) \mid z^4 + z^2 x_1^2 + B(x) = 0\},$$

where B is a homogeneous polynomial of degree 4. Then X has at most 14 singular points, inverse image of the (at most 7) points in the plane where $\nabla B(x) = 0$.

For general choice of B, X has exactly 14 nodes as singularities.

The Gauss map γ is inseparable, it factors through the projection $(x, z) \mapsto x$ and a degree two map $\mathbb{P}^2 \to \mathbb{P}^2$, $x \mapsto \nabla B(x)$.

In particular, if X has 14 singular points, these are nodes.

Proof. The Gauss map is given by

$$\gamma(x, z) = (\nabla B(x), 0).$$

The singular points are the inverse image of the critical locus of B, $\Sigma = \{x \mid \nabla B(x) = 0\}$. Σ consists of at most 7 points by Proposition 5.

For general B we get 7 reduced points, and since z^2 is the root of a quadratic polynomial with derivative x_1^2, if the line $\{x_1 = 0\}$ does not meet the locus Σ, we get 14 nodes as singularities.

Observe finally that $x \mapsto \nabla B(x)$ has degree 2 since the base locus consists of the length 7 subscheme Σ.

The last assertion follows now easily from the fact that the Gauss map has degree 8, and from the Gauss estimate (1) of Proposition 2. ☐

3.4. Quartics with inseparable projection from one node

This is another specialization, corresponding to the case $a = 0$ in (4), but the conic Q is a smooth one.

Proposition 10. *Consider a quartic of equation*

$$X = \{(x, z) \mid z^2(x_1 x_2 + x_3^2) + B(x) = 0\},$$

where B is a homogeneous polynomial of degree 4*.*

The Gauss map γ is inseparable, it factors through the degree two projection $(x,z) \mapsto x$ and a degree four map $\mathbb{P}^2 \to \mathbb{P}^2$.

X has at most 14 *singularities, the node $P = \{x = 0\}$, and the inverse image of a* 0*-dimensional subscheme of the plane of length* 13*. If X has* 14 *singularities, these are nodes; and, for general choice of B, X has* 14 *nodes as singularities.*

Proof. The Gauss map is given by

$$\gamma(x,z) = (z^2 x_2 + B_1, z^2 x_1 + B_2, B_3).$$

Multiplying by $Q = (x_1 x_2 + x_3^2)$ and using the equation of X, we get that

$$\gamma(x,z) = \gamma'(x) := (Bx_2 + B_1 Q, Bx_1 + B_2 Q, B_3 Q).$$

The base point scheme of γ' in the plane consists of $\{Q = B = 0\}$, which is a length 8 subscheme which in general consists of 8 reduced points, and, since outside of this subscheme we may assume that $Q(x) \neq 0$ (since $x_1 = x_2 = Q = 0$ has no solutions), of the locus

$$\mathcal{S} := \{Bx_2 + B_1 Q = Bx_1 + B_2 Q = B_3 = 0\}.$$

We observe now that every quartic polynomial can be uniquely written as the sum of a square q^2 plus a polynomial of the special form below

$$B' := \sum_{i \neq j} b_{ij} x_1^3 x_j + \sum_i c_i x_i x_1 x_2 x_3.$$

Then, working modulo (x_3), we get:

$$B' \equiv b_{12} x_1^3 x_2 + b_{21} x_2^3 x_1, \quad B_2' \equiv b_{12} x_1^3 + b_{21} x_2^2 x_1,$$

$$B_1' \equiv b_{12} x_1^2 x_2 + b_{21} x_2^3.$$

Hence $x_1 B_1' \equiv x_2 B_2' \equiv B' \pmod{x_3}$.

Consider the subscheme

$$\mathcal{L} := \{x_3 = Bx_2 + B_1 Q = Bx_1 + B_2 Q = 0\}.$$

For $B = B'$ we get $\mathcal{L} = \{x_3 = 0\}$, because $Q = x_1 x_2 + x_3^2$.

If we now add to B' the square of a quadratic form q^2, $\mathcal{L}$ coincides with

$$\{x_3 = q^2 x_2 = q^2 x_1 = 0\} = \{x_3 = q^2 = 0\}.$$

This is a subscheme of length equal to 4.

Our subscheme $\mathcal{S}$ is the residual scheme with respect to the above length 4 scheme $\mathcal{L}$ of the scheme

$$\mathcal{HB} := \{Bx_2 + B_1 Q = Bx_1 + B_2 Q = x_3 B_3 = 0\}.$$

$\mathcal{HB}$ is a Hilbert-Burch Cohen-Macaulay subscheme of codimension 2, corresponding to the 2×3 matrix with rows (x_1, x_2, Q) and (B_2, B_1, B).

Since the ideal $\mathcal{I}$ of the subscheme has a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-6) \to \mathcal{O}_{\mathbb{P}^2}(-5)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{I} \to 0,$$

an easy Chern class computation shows that the length of $\mathcal{HB}$ is 17. Moving q, the scheme $\mathcal{L}$ is disjoint from $\mathcal{S}$, hence we conclude that the length of $\mathcal{S}$ is 13.

The degree of the Gauss map is then $2(25 - 8 - 13) = 8$; hence, by the Gauss estimate (1) of Proposition 2 we get $28 \geq 2\nu + b + u$, hence for $\nu = 14$ we obtain 14 nodes.

That the subscheme $\mathcal{S}$ consists in general of 13 distinct points follows from the examples given in [Cat21b], step IV of proposition 3. □

3.5. A 24-dimensional family of quartics with 14 nodes

We pass now to the general case, where $a \neq 0$, and Q is not a double line.

Theorem 11. *Let K be an algebraically closed field of characteristic 2, and let $X \subset \mathbb{P}^3_K$ be a general quartic hypersurface such that the dual variety is a plane.*

Then X has 14 nodes as singularities and is unirational, hence supersingular.

These quartic surfaces form an irreducible component, of dimension 24, of the variety of quartics with 14 nodes.

Proof. The condition that the dual variety is a plane is equivalent to the existence of coordinates (x, z) $(x = (x_1, x_2, x_3))$ such that

$$X = \{(x, z) \mid az^4 + z^2 Q(x) + B(x) = 0\}.$$

Since we have already dealt with the special case $a = 0$, and with the cases $Q = 0$ or the square of a linear form, let us assume that

$$a = 1, \;\; Q(x) = x_1 x_2 + \lambda x_3^2.$$

The Gauss map is given by

$$\gamma(x, z) = z^2 \nabla Q + \nabla B = (z^2 x_2 + B_1, z^2 x_1 + B_2, B_3).$$

Hence for the singular points $B_3(x) = 0$, which implies $x_1 B_1 + x_2 B_2 = 0$. Therefore, for the singular points we have

$$z^2 = \frac{B_1}{x_2} = \frac{B_2}{x_1}.$$

More precisely, if we have a point $x \in \mathbb{P}^2$ such that $B_3(x) = 0$, and $x_2 \neq 0$, necessarily we have $z^2 = \frac{B_1}{x_2}$ and we have a singular point if the equation of X is satisfied, namely if

$$z^4 + z^2 Q(x) + B(x) = 0 \iff B_1^2 + B_1 x_2 Q + B x_2^2 = 0.$$

An easy calculation shows that

$$B_3 = b_{32} x_3^2 x_2 + b_{31} x_3^2 x_1 + b_{13} x_1^3 + b_{23} x_2^3 + c_1 x_1^2 x_2 + c_2 x_2^2 x_1,$$

which is in the ideal (x_1, x_2) but does not in general vanish neither on $x_1 = x_3 = 0$ nor on $x_2 = x_3 = 0$.

We look now at the points x where

$$B_3 = B_1^2 + B_1 x_2 Q + B x_2^2 = x_2 = 0 \iff B_3 = x_2 = B_1 = 0 :$$

these are contained in the set $\{x_2 = b_{31} x_3^2 x_1 + b_{13} x_1^3 = 0\}$, which consists of the point $P'' := \{x_2 = x_1 = 0\}$, and the point $P' := \{x_2 = b_{31} x_3^2 + b_{13} x_1^2 = 0\}$.

Since

$$B_1 = b_{12} x_1^2 x_2 + b_{13} x_1^2 x_3 + b_{31} x_3^3 + b_{21} x_2^3 + c_3 x_3^2 x_2 + c_2 x_2^2 x_3,$$

B_1 does not in general vanish in P'', but it vanishes indeed in P'.

At the point P', for general choice of B, $x_1 \neq 0$, hence if P' were to correspond to a singular point of X, we would have

$$z^2 = \frac{B_2}{x_1} \Rightarrow B_2^2 + B_2 x_1 Q + B x_1^2.$$

But the right hand side does not in general vanish at P', since $x_1 \neq 0$, and since we can add to B the square of a quadratic form $q(x)$ without affecting the partial derivatives.

The number of singular points of X is then equal, by the Bézout theorem, to the difference between 18 and the intersection multiplicity of B_3 and $B_1^2 + B_1 x_2 Q + B x_2^2$ at the point P'.

In the special case $B = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 + q^2$, we get the point $P' = \{x_2 = x_3 = 0\}$, and

$$B_3 = x_3^2 x_1 + x_2^3, \quad B_1 = x_1^2 x_2 + x_3^3.$$

The curve $\{B_3 = 0\}$ has a cusp with tangent $\{x_2 = 0\}$, so that x_3 has order 3, x_2 has order 2, hence $B_1^2 + B_1 x_2 Q + B x_2^2$ has order equal to 4 for general choice of q.

By semicontinuity the intersection multiplicity is in general at most 4, hence the ‘number’ of singular points of X is at least 14. But in the special case of proposition 9 we have exactly 14 nodes, so 14 points counted with multiplicity 1; hence by semicontinuity in the other direction we have in general exactly 14 nodes.

That X is unirational, hence supersingular by [Shio74b], follows since X is an inseparable double cover of the surface

$$Y = \{(x, w) \mid a w^2 + w Q(x) + B(x) = 0\} \subset \mathbb{P}(1, 1, 1, 2).$$

Y has degree 4, hence $\omega_Y = \mathcal{O}_Y(-1)$ and Y is a Del Pezzo surface, hence rational.

The dimensionality assertion follows by a simple parameter counting, $1 + 6 + 15 - 1 = 21$ parameters for the above polynomial equations, plus 3 parameters for the plane $X^\vee$, which in the chosen equations is the plane $\{z = 0\}$.

Finally, consider the surface

$$X_0 := \{(x, z) \mid z^4 + z^2 l(x)^2 + B_0(x) = 0\}, \quad B_0 = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1,$$

and consider the deformations obtained by adding to the equation of X_0 a polynomial

$$f := z \sum_{i=1}^{7} \lambda_i G(i)(x) + z^3 \sum_{j=1}^{3} \mu_j L_j(x),$$

where $G(1), \dots, G(7)$ are polynomials of degree 3 such that $G(i)$ is vanishing at exactly all the critical points of B_0 except the i-th point P_i, and the linear forms $L_j(x)$ vanish on the points $P_i, 1 \leq i \leq 3,\ i \neq j$.

The polynomial f belongs to a 10-dimensional vector subspace, and we shall show now that we get independent smoothings of ten of the nodes: one for each of the pairs of singular points P_i', P_i'' lying over P_i, for $i = 4, 5, 6, 7$, and two over each P_i for $i = 1, 2, 3$.

Then, if we choose one of the two singular points P_i', P_i'' lying over P_i, for $i = 4, 5, 6, 7$, say P_i', the map to the local deformation space of the singularity is of the form (in local coordinates $u, v, \zeta := (z + z_i')$ such that $B = uv + \text{constant}$)

$$uv + (z + z_i')^2 + \lambda_i z + z \sum_{j=1}^{3} \mu_j L_j(P_i)(z_i')^2,$$

since $z^3 = (\zeta + z_i')^3 \equiv z(z_i')^2 \pmod{\zeta^2}$; whereas for $j = 1, 2, 3$ the map is given by

$$uv + (z + z_j')^2 + \lambda_j z + z\mu_j(z_j')^2,$$

respectively by

$$uv + (z + z_j'')^2 + \lambda_j z + z\mu_j(z_j'')^2.$$

Observe that, if we have a node of equation $uv + \zeta^2 = 0$, the local deformations are of the form

$$uv + \zeta^2 + c_0 + c_1\zeta = 0,$$

and we have a smoothing iff $c_1 \neq 0$.

It is easily seen that the deformation yields ten independent smoothings of the ten nodes $P_1', \dots, P_7', P_1'', P_2'', P_3''$, hence it follows that the variety of quartics with 14 nodes, at the point X_0, has Zariski tangent space of codimension at least 10 in the space of all quartics. Since the space of all quartics has dimension 34, and our family is irreducible of codimension 10, it follows that at the point X_0 our family coincides with the variety of quartics with 14 nodes, and our family is a component of this variety. $\square$

Remark 12. Since our family yields a dimension 9 locus in the moduli space, we have found an irreducible component of the moduli space of supersingular K3 surface with a quasi-polarization of degree 4. This may be compared to Shimada's results on double sextics where there is an irreducible component with 21 nodes [Shim04].

3.6. When the minimal resolution is a K3 surface

Concerning the degree of the Gauss map, which is in the above situation generally equal to 8, we have a weaker result, which is sufficient, as we will see in Proposition 14, for the purpose of showing that the minimal resolution of X is always a K3 surface if the number of singular points is at least 13. Equivalently, all singularities are rational double points.

Lemma 13. *Assume that the normal quartic X has the following equation*

$$X = \{(x,z) \mid z^4 + z^2 Q(x) + B(x) = 0\},$$

where the quadratic form Q is not the square of a linear form.

Then the degree of the Gauss map is at least 4 or X has at most 12 singular points.

Proof. We use again the normal form where $Q(x) = x_1x_2 + \lambda x_3^2,\ \lambda \in \{0,1\}$.

The Gauss map factors through the inseparable double cover (setting $w := z^2$) of the Del Pezzo surface Y of degree 2 in $\mathbb{P}(1,1,1,2)$, such that $\omega_Y = \mathcal{O}_Y(-1)$.

The projection to the $\mathbb{P}^2$ with coordinates x and the Gauss map to the plane with coordinates y induce a birational embedding of Y in $\mathbb{P}^2 \times \mathbb{P}^2$, since $y = \gamma(x,w) = (wx_2 + B_1, wx_1 + B_2, B_3)$, hence

$$y_1/y_3 = (wx_2 + B_1)/B_3 \Rightarrow w = (B_3/x_2)(y_1/y_3 + B_1/B_3).$$

The image lands, as it is immediate to verify, in the flag manifold $\mathbb{F}$, a smooth divisor of bitype (1,1)

$$\mathbb{F} = \left\{ (x,y) \,\middle|\, \sum_i x_i y_i = 0 \right\},$$

and inside $\mathbb{F}$ the image Z of Y is a divisor of bitype $(d,2)$ where $2d$ is the degree of the Gauss map.

We want to show that $d > 1$.

By adjunction the dualizing sheaf ω_Z of Z is a divisor of bitype $(d-2,0)$. whereas the canonical system of Y corresponds to a divisor of bitype $(-1,\, 0)$. The crucial observation is that, if $d = 1$, then these two divisors coincide.

Y has a rational map to Z and composing with the first projection we get a morphism, while composing with the second projection we get the blow up of some points.

Let Y' be the blow up of Y, such that $\pi : Y' \to Z$ is a birational morphism. Also the second projection $p : Z \to \mathbb{P}^2$ is a birational morphism, moreover the fibres of p are contained in the fibres of $p : \mathbb{F} \to \mathbb{P}^2$, which are isomorphic to $\mathbb{P}^1$. We blow up the points of $\mathbb{P}^2$ where the fibre of $p : Z \to \mathbb{P}^2$ has dimension 1, obtaining Z'. Then we get a factorization $Z \to Z' \to \mathbb{P}^2$.

Since Z' is smooth, and $Z \to Z'$ is finite and birational, follows that $Z \cong Z'$ and Z is smooth.

Now Z and Y are birational normal Del Pezzo surfaces, and for both the anticanonical divisor is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the first projection (to the $\mathbb{P}^2$ with coordinates (x)).

The first projection $\phi : Z \to \mathbb{P}^2$ has degree two and is either finite, or its fibres are isomorphic to $\mathbb{P}^1$. By normality we have a birational morphism $\psi : Z \to Y$. In the first case ψ is an isomorphism, in the second case it is a minimal resolution of singularites. And since the fibres are smooth rational curves with normal bundle of degree -2, then the corresponding singularities of Y are nodes.

This shows that $d = 1$ is only possible if there are no singular points of X which do not map to singular points of Y, and the latter are nodes.

Since the singularities of Y correspond to the singularities of X for which $Q(x) = 0$, we see that all the singular points of X satisfy $Q(x) = 0$. Since the singular points of Y are defined by $Q(x) = 0$ and by 3 equations of degree 3, it follows that there is a linear combination $B'(w, x)$ of these 3 equations such that the singular points of Y are contained in the finite set defined by $Q(x) = B'(w, x) = 0$.

Since $\mathcal{O}_Y(1)$ has self-intersection equal to 2, Y has at most 12 singular points. □

Proposition 14. *If $2\nu > 28 - \deg(\gamma)\deg(X^\vee)$, then all singularities of X are rational double points, and the minimal resolution S is a K3 surface. In particular, this holds for $\nu \geq 13$.*

Proof. The first statement follows directly from combining Propositions 2 and 4.

Let's deal with the second assertion.

If $X^\vee$ is not a plane, then, by Proposition 2 (IV), $\deg(\gamma)\deg(X^\vee) \geq 3$ and we are done.

Hence we may assume that $X^\vee$ is a plane.

By Lemma 13, Propositions 9 and 10, either the number of singular points is at most 12, or $\deg(\gamma)\deg(X^\vee) \geq 4$, or we are in the cases where

$$X = \{(x, z) | z^2 x_1^2 + B(x) = 0\},$$

or

$$X = \{(x,z)|z^2x_1x_2 + B(x) = 0\}.$$

The former case was dealt in Step I of Proposition 3 of Part I, showing that X has at most 8 singular points, and in this case Example 10 ibidem shows that $\deg(\gamma) \geq 4$.

In the latter case Step II of Proposition 3 of Part I shows that X has at most 13 singular points; and that it has exactly 13 points only if it has 12 nodes (corresponding to the points of the plane where $B_3 = B_1x_1 + B = 0$), and a biplanar singular point (at $x = 0$): hence also in this case the minimal resolution is a K3 surface. □

The following result improves upon part (V) of Proposition 2.

Corollary 15. *If $\nu \geq 14$ all the singularities are either nodes or biplanar double points.*

Proof. Recall the basic inequality

$$36 - \deg(\gamma)\deg(X^\vee) \geq 2\nu + b + 6u.$$

We are claiming $u = 0$ if $\nu \geq 14$, hence it suffices to recall that we saw in the previous proposition that $\deg(\gamma)\deg(X^\vee) \geq 3$. □

4. Proof of the Main Theorem 1 – general bound

Throughout this section until 4.4, we assume that X is a normal quartic surface with $\nu \geq 15$ singular points in order to establish a contradiction and prove the general bound of Theorem 1. We use the following result which will follow from Propositions 29 and 30 (to be proven in Section 6 using the theory of elliptic and quasi-elliptic fibrations on K3 surfaces).

Main Claim 16. *If X has $\nu \geq 15$ singular points, then, for each pair P_i, P_j of singular points of X, the line $L_{ij}^\vee$ dual to $L_{ij} := \overline{P_iP_j}$ is contained in the dual surface $X^\vee$.*

4.1. The main claim implies the general bound of Theorem 1

It will suffice to show that:

Claim 17. *In the above setting, $X^\vee$ contains two skew lines and 7 distinct coplanar lines.*

Indeed the claim implies that $X^\vee$ is a surface of degree ≥ 7, and by the Gauss map estimate (1) of Proposition 2 we have

$$36 - 7\deg(\gamma) \geq 2\nu + b + 6u,$$

hence $\nu \leq 14$ as announced. □

4.2. Proof of Claim 17

We observe first that if a line L_{ij} passes through a third singular point P of X, then it is contained in X, and the planes $H \supset L_{ij}$ cut X in the line L_{ij} plus a cubic C meeting L_{ij} in the three points P, P_i, P_j.

Hence there cannot be 4 collinear singular points: because then C would contain L_{ij} and $L_{ij} \subset \mathrm{Sing}(X)$, contradicting the normality of X.

We show now that each plane contains at most 6 singular points of X.

In fact, if the plane is the plane $z = 0$, and the equation of X is

$$B(x) + zG(x) \mod (z^2),$$

the singular points on the plane are the solutions of

$$z = \nabla B(x) = B(x) = G(x) = 0.$$

A reduced plane quartic has at most 6 singular points. If the quartic is non-reduced, and $B(x) = q(x)^2$, then the singular points are the solutions of $z = q(x) = G(x) = 0$ and they are at most 6 by the theorem of Bézout and since X is normal.

The case where $\{x \mid B(x) = 0\}$ consists of a double line and a reduced conic leads to at most one singular point outside the line, hence at most 4 singular points in the plane.

Whence, if $\nu \geq 7$, there are 4 linearly independent singular points of X, and we have found two skew lines L_{ij}, L_{hk}: likewise the dual lines are skew.

Assume now that $\nu \geq 15$ and consider all the lines of the form L_{1j}: these are at least 7, since at most 3 singular points are collinear, and the dual lines are contained in the plane dual to the point P_1. □

4.3. Propositions 29 and 30 imply the Main Claim 16

Since we assume $\nu \geq 15$, we can apply Proposition 29 to show that each pair (P_i, P_j) induces a quasi-elliptic fibration. By the degree estimate in Proposition 2, all singularities are nodes or biplanar double points, so Proposition 30 proves that the pencil of planes containing L_{ij} yields a line $L_{ij}^\vee$ contained in $X^\vee$. □

4.4. Auxiliary results

We establish here, with similar arguments, two easy results for later use. To this end, we distinguish whether two given singular points P_1, P_2 are collinear with a third singularity or not (in the latter case we call P_1, P_2 companions). Recall that in the first case, the line $L = \overline{P_1 P_2}$ is contained in X, and each plane containing L contains at most 6 singularities.

Lemma 18. *If $\nu \geq 9$, then there is a singular point with two companions.*

Proof. Assume to the contrary that each singularity has at most one companion. Take a point P_1 and three collinear pairs, say $P_1, P_2, P_3 \in L$, $P_1, P_4, P_5 \in L_1,\ P_1, P_7, P_8 \in L'$.

Let H be the plane containing P_1, P_2, P_3, P_4, P_5, and let H' be the plane containing P_1, P_2, P_3, P_7, P_8. These are different, since each plane contains at most 6 singular points.

By assumption, we may assume without loss of generality that P_4 is not companion of P_2, hence there is P_6 collinear with P_2, P_4, so that $P_2, P_4, P_6 \in L_2 \subset X$. At this stage we have obtained 6 singular points (the maximum) in the plane H, and we observe that P_3 is not companion of P_4 or P_5.

Hence we get 4 lines

$$X \cap H = L + L_1 + L_2 + L_3,$$

where L_3 must contain the singular points P_3, P_6 and thus also P_5. Thereby we reach the conclusion that P_1 is companion of P_6.

We establish now a contradiction as follows. Playing the same game for the other plane H', we find another companion of P_1, call it P_9.

Since $P_9 \in H'$, while $P_6 \notin H'$ (since $H \cap H' = L$) we have found two different companions for P_1, and we have reached a contradiction. □

Proposition 19. *If X has $\nu = 14$ singular points and, for each pair P_i, P_j of singular points of X, the pencil of planes containing the line $L_{ij} = \overline{P_i P_j}$ yields a line $L_{ij}^\vee$ contained in the dual surface $X^\vee$, then the degree of the dual surface is at least 8. In particular, the singular points are just 14 nodes.*

Proof. By the Gauss estimate it suffices to show the first assertion, and since $X^\vee$ contains at least two skew lines, it suffices to show that it contains at least 8 coplanar lines. But this follows from Lemma 18 as there is a singular point P_1 on X collinear with at most 5 pairs of singularities, thus companion to at least 3, yielding a total number of at least 8 coplanar lines on $X^\vee$. □

We will use Proposition 19 later, and we observe that a weaker form suffices, where there is one point P_1 with the property of the proposition holding for all lines $\overline{P_1 P_j}$, if $X^\vee$ contains a line skew to (one of) the 8 dual lines from P_1.

5. Genus one fibrations

We shall now invoke some results from the theory of genus one fibrations on K3 surfaces in order to achieve the proof of Propositions 29 and 30.

These will also be used for the proof of the other parts of Theorem 1.

Let X be a projective K3 surface. Let $L \in \mathrm{Pic}(X)$ be a divisor class with $L^2 \geq -2$; then, by Riemann-Roch, $\chi(L) \geq 1$ hence L or $-L$ is effective. Hence let us assume that L is linearly equivalent to an effective divisor D. If $D^2 = 0$, then the linear system $|D|$ has dimension ≥ 1, and we can write $|D| = |M| + \Psi$, where Ψ is the fixed part. Clearly then $\Psi = \sum_i E_i$ where each E_i is an irreducible curve with $E_i^2 = -2$.

Since

$$0 = D^2 = M^2 + D\Psi + M\Psi, \; M^2 \geq 0, \; M\Psi \geq 0, \tag{5}$$

we have $D\Psi < 0$, or $\Psi = 0$. Because, if $D\Psi \geq 0$ and $\Psi > 0$, then (5) implies $M^2 = D\Psi = M\Psi = 0$, hence $\Psi^2 = 0 \Rightarrow \Psi = 0$, the intersection form being negative definite by Zariski's lemma on the divisor Ψ: because Ψ is contained in the fibres of the fibration associated to $|M|$, there are no multiple fibres, and Ψ does not contain any full fibre (else, it would not be the fixed part).

The conclusion is that either $|D|$ has no fixed part or there is E_1 such that $DE_1 < 0$, hence reflection in the (-2)-curve E_1 produces a new divisor class

$$D'' := D + (DE_1)E_1$$

such that $(D'')^2 = 0$. The system $|D''|$ has dimension ≥ 1, and since the degree of D'' is smaller than the degree of D, the process terminates producing a base point free system $|D'|$, with $(D')^2 = 0$, hence $|D'|$ is a pencil of genus 1 curves. We may also assume that D' is primitive, so that D' is indeed a fibre of a fibration $f : X \to \mathbb{P}^1$.

If the general fibre is smooth, we call the fibration elliptic and we may further distinguish whether the fibration admits a section or not. In characteristics 2 and 3, however, the general fibre may also be a cuspidal cubic curve whence the fibration is called quasi-elliptic.

Examples are given by sparse Weierstrass forms; more precisely, in terms of the general equation (8) which shall be recalled later, those forms which do not contain terms linear in y (in characteristic 2) or all of whose terms have degree 0 or 3 in x (in characteristic 3).

In particular, quasi-elliptic surfaces over $\mathbb{P}^1$ are unirational and thus supersingular ($\rho = b_2$) by [Shio74b] which makes them quite special (see [Ru-Sha79], for instance).

Remark 20. Any (-2) curve C on X which is perpendicular to D' features as a fibre component of $|D'|$ (but the analogous statement for (-2)-curves orthogonal to D is surprisingly subtle in case there is some base locus involved. We will come back to this problem in part III.

Remark 21. In general, given an effective divisor D with $D^2 = 0$, $|D|$ need not be a pencil, the easiest example being the one where D consists of a genus 2 curve and a disjoint (-2)-curve, that is,

$$D = M + E, M^2 = 2, E^2 = -2, ME = 0,$$

here $M - E$ gives the desired pencil.

A sufficient condition for $\dim |D| = 1$ is that the divisor D is numerically connected, that is, any decomposition $D = A + B$, where A, B are effective, satisfies $AB \geq 1$.

Because, by the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$$

we have $H^1(\mathcal{O}_X(D)) = 0$ unless $h^1(\mathcal{O}_D(D)) \geq 2$. Since $h^1(\mathcal{O}_D(D)) = h^0(\mathcal{O}_D)$, and $h^0(\mathcal{O}_D) = 1$ if D is numerically connected, [Fran49], [Ram72], our claim follows.

In this case, $M^2 = 0$, and D could, for instance, consist of a fibre plus a (-2)-curve which is a section,

$$D = M + E, \quad M^2 = 0, \quad E^2 = -2, \quad ME = 1.$$

5.1. Disjoint smooth rational fibre components

For later use, let us record some rather special features of elliptic fibrations in characteristic 2.

Proposition 22. *In characteristic 2, on an elliptic K3 surface the singular fibres contain at most 12 disjoint (-2)-curves.*

At first, this result may seem rather surprising, since usually, i.e. outside characteristic 2, elliptic fibrations allow for as many as 16 disjoint (-2)-curves. This happens in the case of 4 fibres of Kodaira type I_0^*, each containing 4 disjoint (-2)-curves – for instance, on the Kummer surface of a product of two elliptic curves.

Proof. What prevents the same as above to happen in characteristic 2 is the fact that all additive fibres, except for Kodaira types IV, IV*, come with wild ramification by [SS13].

More precisely, there still is a representation of the Euler-Poincaré characteristic of the elliptic K3 surface X as a sum over the fibres:

$$24 = e(X) = \sum_v (e(F_v) + \delta_v).$$

Here δ_v denotes the index of wild ramification, studied in more generality in [Del73]. On an elliptic surface, it can be computed as the difference of the Euler number $e(F_v)$ and the local multiplicity of the discriminant which is the equation for the singular fibres and may be computed on the Jacobian by [CDL21, p.348]. The bounds for δ_v in the next table have been taken from [SS13, Prop. 5.1]. Note that the number of components m_v is the index of the Dynkin type plus one, while, except in the first case, the Euler number is $m_v + 1$. The table also collects the maximal number N_v disjoint (-2)-fibre components, to be computed below.

fibre type	I_n	II	III	IV	$\mathrm{I}_n^*(n \neq 1)$	I_1^*	IV*	III*	II*
Dynkin type	A_{n-1}	A_0	A_1	A_2	D_{n+4}	D_5	E_6	E_7	E_8
m_v	n	1	2	3	$n+5$	6	7	8	9
δ_v	0	≥ 2	≥ 1	0	≥ 2	1	0	≥ 1	≥ 1
$e(F_v)$	n	2	3	4	$n+6$	7	8	9	10
N_v	$\lfloor \frac{n}{2} \rfloor$	0	1	1	$4 + \lfloor \frac{n}{2} \rfloor$	4	4	5	5

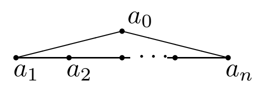
For the convenience of the reader, we also include the dual graphs of the fibres in terms of the extended Dynkin diagrams $\tilde{A}_n, \tilde{D}_k\,(k \geq 4)$. For fibre types IV*, III*, II*, we only give the Dynkin diagram $E_l\,(l = 6, 7, 8)$ for sake of a unified presentation. For these types the fibre is obtained by adding another fibre component e_0 adjacent to the vertex e_1 in case E_6, resp. e_2 in case E_7, resp. e_8 in case E_8.

In total, the simple fibre components (i.e. those having multiplicity 1 in the fibre) are, depending on the fibre type:

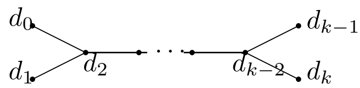
$\tilde{A}_n$ all components,

$\tilde{D}_k$ the exterior components,
$\tilde{E}_l$ e_0, e_2, e_6 $(l = 6)$ resp. e_0, e_7 $(l = 7)$, resp. e_0 $(l = 8)$.

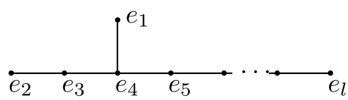
$(\tilde{A}_n)$



$(\tilde{D}_k)$



(E_l)



Case by case, this allows us to compare the maximal number N_v of disjoint (-2)-fibre components with the contribution to the Euler-Poincaré characteristic, see the above table.

Overall, we find

$$N_v \leq \frac{1}{2}\lfloor e(F_v) + \delta_v \rfloor \tag{6}$$

and thus

$$\sum_v N_v \leq \sum_v \frac{1}{2}\lfloor e(F_v) + \delta_v \rfloor \leq \frac{1}{2}\sum_v (e(F_v) + \delta_v) = 12. \tag{7}$$

This yields the desired inequality and proves our assertion. □

Remark 23. If equality holds at each step of the chain of inequalities

$$N_v \leq \frac{1}{2}\lfloor e(F_v) + \delta_v \rfloor \leq \frac{1}{2}(e(F_v) + \delta_v),$$

then δ_v attains its minimum value, and the multiplicity $(e(F_v) + \delta_v)$ is an even number, hence we get only the types

$$\mathrm{I}_{2n}\ (n > 0),\ \ \mathrm{I}_{2n}^*\ (n \geq 0),\ \ \mathrm{I}_1^*,\ \ \mathrm{IV}^*,\ \ \mathrm{III}^*.$$

Corollary 24. *If the fibres of an elliptic K3 surface in characteristic* 2 *contain 12 disjoint* (-2)*-curves, then the only possible singular fibre types are (with minimum possible* δ_v *each)*

$$\mathrm{I}_{2n}\ (n > 0),\ \ \mathrm{I}_{2n}^*\ (n \geq 0),\ \ \mathrm{I}_1^*,\ \ \mathrm{IV}^*,\ \ \mathrm{III}^*.$$

Proof. This is a direct consequence of the proof of Proposition 22 since all the inequalities in (7) actually have to be equalities (in particular the same must hold for (6) at each v, as in Remark 23). ☐

A close inspection of the fibres in the proof of Proposition 22 allows us even to rule out higher ADE-types:

Corollary 25. *If the fibres of an elliptic K3 surface in characteristic 2 support 12 disjoint ADE-configurations of* (-2)*-curves, then each has type* A_1*.*

Proof. These 12 disjoint ADE-configurations produce at least 12 disjoint (-2)-curves, hence we may apply the previous corollary and check directly. ☐

5.2. Connection with supersingularity

To relate with Theorem 1, especially with the statement about supersingular K3 surfaces, we provide the next result which concerns the case of exact equality in Proposition 22.

Proposition 26. *Let* X *be an elliptic K3 surface such that there are 12 disjoint* (-2)*-curves contained in the fibres. Then* X *is supersingular or there are two additive fibres.*

Note that the fibres in Proposition 26 are the fibres of Corollary 24: either I_{2n}, or additive fibres which are non-reduced. This will be of great use in what follows.

Remark 27. (i) It is easy to see that both cases of Proposition 26 can occur: the first one via inseparable base change from rational elliptic surfaces (see [SS19, p. 342], Proposition 12.32) the other one (as in characteristic zero!) by taking the Kummer surface of the product of two elliptic curves (both not supersingular): here there are two singular fibres of Kodaira type I_4^* by [Shio74a].

(ii) The second case of Proposition 26 encompasses the case where there are 12 disjoint (-2)-curves contained in the fibres and the j-invariant is constant, since then every reducible fibre is additive, and if there were a single reducible fibre, it would have type I_{16}^*, which is impossible by [Sch06].

Proof of Proposition 26. If the singular fibres contain 12 disjoint (-2)-curves, then by the proof of Proposition 22, both inequalities in (7) are in fact equalities, with fibre types given in Corollary 24.

Hence δ_v attains the minimal possible value $\delta_v(min)$ and $e(F_v) + \delta_v = e(F_v) + \delta_v(min)$ is always even.

Since $e(F_v) + \delta_v$ is exactly the vanishing order of the discriminant Δ at v by [Ogg67], we find that Δ is a square in $k(t)$.

5.2.1. The Jacobian fibration We now switch to the Jacobian J of X – another elliptic K3 surface, since it shares the same invariants of X by [CD89, Cor. 5.3.5]. Note that J also has the same Picard number as X, but, by definition, J has a section while X may not.

By [CD89, Theorem 5.3.1] J and X share the same singular fibres (and by [CDL21, p.348] also the same Δ and δ_v (minimal!)) since, by virtue of the canonical bundle formula (Theorem 2 of [Bom-Mum77]), there are no multiple fibres.

In terms of a minimal Weierstrass equation for J,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in k[t], \deg(a_i) \le 2i, \tag{8}$$

there are essentially two options for a_1 (since $a_1 \equiv 0$ forces all singular fibres to be additive and is thus covered by the second alternative of Proposition 26, cf. Remark 27 (ii)), up to Möbius transformations:

$$a_1 = t \quad \text{or} \quad a_1 = t^2.$$

In the first case, we can argue directly with the general expression of the discriminant,

$$\Delta = a_3^4 + a_1^3a_3^3 + a_1^4a_4^2 + a_1^4a_2a_3^2 + a_1^5a_3a_4 + a_1^6a_6. \tag{9}$$

Notably, if $a_1 = t$, then this reads modulo t^4

$$\Delta \equiv a_3(0)^4 + a_3(0)^3t^3 \mod t^4,$$

so Δ can only be a square if $a_3(0) = 0$ which makes the fibre at $t = 0$ singular and in fact additive. By symmetry, the same reasoning applies at $t = \infty$, so there are two additive fibres and we reach the second alternative of this proposition.

5.2.2. Normal forms for additive fibre types There remains to study the case $a_1 = t^2$. We start arguing with the minimality of δ_0 to reduce to just 3 cases.

If there is a singular fibre at $t = 0$ (then a_3 vanishes at $t = 0$ and we have an additive fibre), then we can use Tate's algorithm to develop a normal form for the fibre [Tat75], [Sil94, IV.9].

For fibres of type I_{2n}^*, the normal form is

$$y^2 + t^2xy + t^{n+2}a_3'y \quad = \quad x^3 + ta_2'x^2 + t^{n+2}a_4'x + t^{2n+4}a_6' \tag{10}$$

with $t \nmid a_2'a_4'$; here we have used Steps 6 and Step 7 in [Sil94, IV.9], pages 367-368. For $n = 0$ we use indeed Step 6, and the fact that the auxiliary polynomial $P(T)$ in loc. cit. has three distinct roots to infer that $t \nmid a_2'a_4'$ after locating one root at $T = 0$. For $n = 1$ the assertions are proven in Step 7, page 367; for higher n one proceeds by induction on n, see line 8 of page 368 concerning the assertion on the divisibility of a_3, a_4, a_6 going up in each induction step. Note that by the argument in loc. cit., the divisibility of a_6 grows in fact by two in each of our steps. This shows that $t^{n+2} \mid a_3, a_4$ and $t^{2n+3} \mid a_6$ and then a translation in x ensures that indeed $t^{2n+4} \mid a_6$ as claimed.

Substituting into (9) gives

$$\Delta = t^{4n+8}a_3'^4 + t^{3n+12}a_3'^3 + t^{2n+12}a_4'^2 + h.o.t.,$$

whence, for the wild ramification

$$\delta_0 = \mathrm{ord}(\Delta) - e(F) = \mathrm{ord}(\Delta) - (2n+6) \geq 2n+2,$$

we have $\delta_0 \geq 4$ for $n > 0$. Since Corollary 24 requires minimal wild ramification $\delta_0 = 2$, this leaves only fibres of type I_0^* among all fibre types I_{2m}^*.

For a fibre of type I_1^*, the normal form arises from an additional vanishing condition at a_4 compared to (10), again by [Sil94, IV.9, Step 7]:

$$y^2 + t^2xy + t^2a_3'y = x^3 + ta_2'x^2 + t^3a_4'x + t^4a_6' \quad \text{with} \quad t \nmid a_2'a_3'.$$

Then fibre type IV^* is given by further imposing $t^2 \mid a_2$ by [Sil94, IV.9, Step 8], still with $t \nmid a_3'$ (in agreement with $\delta_v = 0$). Meanwhile a fibre of type III^* imposes additional vanishing conditions $t^3 \mid a_3$, $t^5 \mid a_6$, but $t^4 \nmid a_4$ [Sil94, IV.9, Step 9]. Substituting into (9) gives

$$\Delta = t^{12}a_3'^4 + t^{14}a_4'^2 + t^{15}a_3'^3 + h.o.t.,$$

so in particular $\delta_0 \geq 3$, ruling out fibre type III^* by Corollary 24 again.

To sum it up, the only additive fibre types remaining from Corollary 24 are I_0^*, I_1^* and IV^*. In each case, one can easily parametrize all K3 surfaces with such a given fibre and square discriminant, starting from the above normal form. It should be noted that for these types the normal form can be derived by means of a linear transformation

$$(x, y) \mapsto (x + \alpha_4, y + \alpha_2 x + \alpha_6) \tag{11}$$

with $\alpha_i \in k[t]$ of degree at most i; in particular, the degree bounds of (8) are preserved.

5.2.3. Conditions for the discriminant to be a square For type I_0^*, (10) leads to the discriminant

$$\Delta = t^8(a_3'^4 + t^4 a_3'^3 + t^4 a_4'^2 + t^5 a_2' a_3'^2 + t^6 a_3' a_4' + t^8 a_6')$$

where, by the minimality of wild ramification, $t \nmid a_3'$. Modulo square summands, this simplifies as

$$\Delta \equiv t^{12}(a_3'^3 + t a_2' a_3'^2 + t^2 a_3' a_4' + t^4 a_6') \mod k[t]^2.$$

Write $a_i' = \sum_j a_{i,j}' t^j$. Then the condition that Δ is a square, i.e. that all odd degree coefficients vanish, determines

- the odd degree coefficients of a_6' in terms of the coefficients of the other forms a_m' (looking at the coefficients of Δ at $t^{17}, \dots, t^{23}$).
- $a_{2,0}' = a_{3,1}'$ (from the t^{13}-coefficient);
- $a_{4,1}' = (a_{2,2}' a_{3,0}'^2 + a_{3,0}'^2 a_{3,3}' + a_{3,1}' a_{4,0})/a_{3,0}'$ (from the t^{15}-coefficient).

In particular, we find that the family of elliptic K3 surfaces with a fibre of type I_0^* with wild ramification of index 2 and all other singular fibres of type I_{2n} (generically 8 I_2's) is irreducible.

Its moduli dimension, equal to 7, is obtained by comparing the degrees

$$\deg(a_3') \leq 4, \ \ \deg(a_2') \leq 3, \ \ \deg(a_4') \leq 6, \ \ \deg(a_6') \leq 8$$

(these bounds follow from the degree bounds in (8) and from (10)), against Möbius transformations $t \mapsto ut/(\varepsilon t + 1)$ ($u \in k^\times, \varepsilon \in k$) and the following variable transformations preserving the shape of (10) (since Δ being a square is automatically preserved):

$$(x, y) \mapsto (u^4 x + t^2 \beta_2, u^6 y + t\beta_1 x + t^2 \beta_4) \tag{12}$$

where the degree of each polynomial $\beta_i \in k[t]$ is at most i.

5.2.4. Conclusion of proof using number of moduli Any smooth K3 surface arising from a member of the above family satisfies

$$\rho \geq 2 + 8 + 4 = 14$$

by the Shioda–Tate formula where the first entry comes from the zero section and the fibre, the second from the semi-stable fibres (each of type I_{2n} for some $n \in \mathbb{N}$, hence contributing $2n$ to the Euler–Poincaré characteristic and $2n-1$ to the Shioda–Tate formula) and the third from the fibre at $t = 0$ (contributing 8 to the Euler–Poincaré characteristic, including wild ramification, and 4 to the Shioda–Tate formula). If a very general member were not supersingular, then it would deform in a 6-dimensional family as in [LM18, Prop. 4.1] (based on [Del81]) but this is exceeded by our moduli count. Hence the whole family is supersingular as claimed.

We pass now to the case of a fibre of type I_1^* or of type IV^*. As explained before, the K3 surfaces with a fibre of type I_1^* are contained in the subfamily where $t^3 \mid a_4$ (while for I_0^* we simply had $t^2 \mid a_4$) and type IV^* additionally requires $t^2 \mid a_2$. Each family allows the same transformations, so the moduli dimension is 6, resp. 5. But ρ generically goes up by 1 each time (promoting the root lattice at the special fibre from D_4 through D_5 to E_6), so the whole family is supersingular again by [LM18, Prop. 4.1].

If there is no additive fibre, then the condition that Δ is a square gives 9 moduli for Δ: moreover the condition that $a_1 = t^2$ reduces the number of moduli to 8, and one can show by the same kind of arguments as above that we have an irreducible 8-dimensional family of semi-stable elliptic K3 surfaces with 12 disjoint A_1's embedding into the singular fibres; since $\rho \geq 2+12 = 14$, again by the formula of [LM18, Prop. 4.1] the family is supersingular. $\square$

Remark 28. Another possible argument of proof is as follows: in each case we have an irreducible family of a certain dimension k, and inside it we can construct a family of the same dimension k of surfaces arising via an inseparable base change from a rational elliptic surface. The surfaces are thus unirational, hence supersingular, and this shows directly that the original family is a family of supersingular surfaces.

Indeed, starting from rational elliptic surfaces with singular fibre at $t = 0$ of type I_0 (smooth supersingular), $\mathrm{II}, \mathrm{III}, \mathrm{IV}$, respectively, inseparable base change exactly results in a family of supersingular K3 surfaces of the expected type and dimension. Note that, since the elliptic fibrations admit a 2-torsion section by [SS19, p.342], the Artin invariants [Art74] satisfy $\sigma \leq 9$.

6. Proof of the main claim: there cannot be at least 15 singularities

In order to bound the number of singularities on a normal quartic $X \subset \mathbb{P}^3$, we shall use the theory of genus 1 fibrations laid out in the previous section.

By Proposition 14, if X has at least 13 singular points ($\nu \geq 13$), then the singularities are rational double points and the minimal resolution S is a K3 surface.

S is endowed with the following divisors: the pull-back H of a plane section and, for each pair of singular points, say P_1, P_2, the respective fundamental cycles D_1, D_2 (see [Art66]), consisting of the exceptional curves with suitable multiplicities, and equal to the pull back of the maximal ideal at the singular point.

Then, since $D_i^2 = -2$,

$$E := H - D_1 - D_2$$

gives an effective isotropic class in $\mathrm{Pic}(S)$.

We have that the linear system $|E|$ is base point free if and only if the line $L = \overline{P_1 P_2}$ is not contained in X: this is clear for the points of S not lying over P_1, P_2; moreover, since for each exceptional curve C the intersection number $D_i C \leq 0$, E has no fixed part (it was observed at the beginning of the previous section that the fixed part Ψ satisfies, if non empty, $E\Psi < 0$) hence it has no base points since $E^2 = 0$.

If instead the line L is contained in X, denote still by L the strict transform of the line and replace E by $E - L$, observing that $EL = -1$, hence $(E - L)^2 = 0$, and continue until we get a base point free pencil $|E'|$, which gives a morphism $S \to \mathbb{P}^1$ whose fibres correspond to the planes through P_1, P_2.

Proposition 29. *Let $X \subset \mathbb{P}^3$ be a normal quartic with at least 15 singularities. Then every genus one pencil $|E'|$ arising from two singularities on X as above is quasi-elliptic.*

Proof. Let $\nu \geq 15$ denote the number of singularities, $P_1, \dots, P_\nu$, and let $C_i^j, j = 1, \dots, n(i)$ be the irreducible exceptional curves lying above the point P_i.

Let us first assume that no P_i ($i > 2$) lies on L, so that each lies on a unique plane through P_1, P_2; hence the C_i^j's are components of the corresponding fibre of $|E'|$ (as in Remark 20). Then the fibration $|E'|$ has $\nu - 2 > 12$ disjoint smooth rational fibre components (the C_i^j).

If, on the other hand, there is a third singularity on L, say P_3, then this implies not only that $L \subset X$, but also that L appears as a multiple component of $X \cap H$ for a unique plane H (just take a point $P \in L$ which is a smooth point of X, and let H be the tangent plane to X at P: then $H \cap X \geq 2L$).

This implies that L is a component of the fibre corresponding to H, and, together with $C^1_4, \dots, C^1_\nu$, we obtain $\nu - 2 > 12$ disjoint smooth rational fibre components as before.

In both cases the proposition follows then from Proposition 22. □

Proposition 30. *Let $X \subset \mathbb{P}^3$ be a normal quartic. Let L be a line through two singular points of X such that $X \cap L$ consists of nodes and biplanar double points (and smooth points if $L \subset X$). If the fibration induced by L is quasi-elliptic, then the line dual to L is contained in the dual surface $X^\vee$.*

Proof. We consider the curve $\Sigma_0 \subset S$ (S is the minimal resolution of X as usual) consisting of the horizontal divisorial part of the set of singular points of the fibres, the so-called curve of cusps.

The first case is when this curve is not exceptional for the map

$$\Phi : S \to X \subset \mathbb{P}^3;$$

then we get a curve on X consisting of singular points of the intersections $H \cap X$, where H is a plane of the pencil through $L = \overline{PP'}$. Therefore the dual line $L^\vee$ is contained in $X^\vee$.

The second case is where Σ_0 is exceptional: we use for this Proposition 1, page 199 of [Bom-Mum76], and denote as in loc. cit. $f : S \to B$ the quasi-elliptic fibration. At a general point $Q' \in \Sigma_0$, the fibre $F := F_{f(Q')}$ has a cusp and, if t is a local parameter for B at $f(Q')$, the map is given by $t = u(x^2 + y^3)$ where u is a unit in the formal power series ring which is the completion of the local ring $\mathcal{O}_{S,Q'}$.

Bombieri and Mumford show that there is a local parameter σ such that $\Sigma_0 = \{\sigma = 0\}$, and that $(\Sigma_0 \cdot F)_{Q'} = 2$, so that x, σ are local parameters for S at Q', and we can write $y = \sigma + \lambda x$ plus higher order terms.

Since we assume that the curve Σ_0 is contracted by the map Φ, it follows that Φ has a local Taylor development which contains only terms in the ideal generated by y.

Hence we are left only with monomials $y, y^2, xy, \dots$ whose respective orders on the normalization of the fibre F are: $2, 4, 5$.

We conclude that the image of F under Φ has a higher order cusp at a singular point P'' lying in L.

By assumption, we can write the equation of X at P'' in local affine coordinates as

$$h := xy + \lambda z^2 + g(x, y, z) = 0 \quad (\lambda \in K),$$

where g has order ≥ 3.

Since we want that the planes of the pencil cut a cusp at P'', the quadratic part of the restriction of the equation h to the planes must be the square of a linear form, hence in the projectivized tangent space we get lines intersecting the exceptional conic C with multiplicity two, hence lines tangent to the conic; from the equation $xy + \lambda z^2$ of the quadratic part follows that this pencil is generated by the linear forms x, y.

We claim now that, as in the first case, the pencil of planes through L yields a line in the dual surface $X^\vee$.

Because the Gauss map is given by $(y, x, 0, 0) + h.o.t$, and the image of the exceptional divisor in the dual surface is the pencil of planes $\mu_0 x + \mu_1 y = 0$, which is exactly the pencil of planes containing L by our previous argument. $\square$

Remark 31. Both cases from the proof of the proposition actually occur (for the second case, it suffices that $g(x, y, z)$ above has order 4).

Note that Propositions 29 and 30 provide the missing ingredients for the proof of the Main Claim 16 in 4.3. Thereby the proof of the first statement of Theorem 1 is now complete.

7. 14 singularities are nodes

The aim of this section is to prove the following result which covers the second part of Theorem 1:

Theorem 32. *Let $X \subset \mathbb{P}^3$ be a normal quartic with 14 singular points. Then all singularities are nodes.*

Proof. The minimal resolution S of X is a K3 surface by Proposition 14, and the singular points are nodes or biplanar double points ($u = 0$) by Corollary 15.

Assume that we have a singular point P which is not of type A_1. Just like in the proof of Proposition 29, any genus one fibration $S \to \mathbb{P}^1$ induced by two singular points admits 12 disjoint smooth rational curves in the fibres. By Proposition 22 this is the maximum possible for an elliptic fibration.

If the fibration is not induced by P and another singular point, P lies in exactly one fibre of X and the fundamental cycle is supported on the corresponding fibre of S. Hence Corollary 25 implies that the fibration is quasi-elliptic.

Any singular point $Q \neq P$ thus admits at least 6 quasi-elliptic fibrations induced by a pair of singular points Q, Q' which are nodes or biplanar double points. Hence we infer from Proposition 30 and the proof of Proposition 19 that $\deg(X^\vee) \geq 6$ and $b \leq 2$. More precisely, by Remark 3, P can only have type A_2 or A_3, and in the former case there may be a second singular point P' of type A_2.

In fact, we can say more about the configuration of singularities relative to Q. Namely Q is collinear with at least 5 pairs of singularities (possibly including P), for else it would induces at least 8 quasi-elliptic fibrations, and $\deg(X^\vee) \geq 8$ would give a contradiction using (1).

We pick one such pair not involving P, say $Q, Q', Q'' \in L \subset X$, and consider the induced quasi-elliptic fibration

$$\pi : S \to \mathbb{P}^1.$$

The fibres are the cubics C residual to L in the respective plane H containing L. Except possibly for the cubic containing L as a component, these cubics are all reduced, since they meet L in the three points Q, Q', Q''. Recall moreover that the exceptional (-2)-curve resolving a node not on L also appears always with multiplicity 1, hence the only fibres of π which may not be reduced are those containing exceptional curves lying above the singular points of type A_n with $n \geq 2$ and the one containing L. Since $b \leq 2$, this makes for at most 3 fibres.

Since there are 5 pairs of singular points collinear with Q, there has to be a pair of nodes left which lie on a reduced fibre (since no plane can contain more than 6 singular points (cf. 4.2), so all pairs of points $\neq (Q', Q'')$ collinear with Q lie on different fibres). In particular, this reduced fibre has at least 4 components. However, by [CD89, Prop. 5.5.10] the possible fibre types of a quasi-elliptic fibration are a priori

$$\mathrm{II}, \quad \mathrm{III}, \quad \mathrm{I}_{2n}^* \ (n \geq 0), \quad \mathrm{III}^*, \quad \mathrm{II}^*. \tag{13}$$

Of these, only fibres of type $\mathrm{II}, \mathrm{III}$ are reduced, with one or two components. This gives the required contradiction. Hence all singularities of X are nodes. $\square$

8. Proof of Theorem 1: the non-supersingular case

To complete the proof of Theorem 1, it remains to analyse the non-supersingular case.

Proposition 33. *Let $X \subset \mathbb{P}^3$ be a normal quartic such that a minimal resolution is not a supersingular K3 surface. Then X contains at most 13 singular points.*

Proof. By the general part of Theorem 1, we only have to rule out: X contains 14 singularities. Assuming this, all singular points are nodes by Theorem 32, and the minimal resolution S is a K3 surface (non-supersingular by assumption). We continue to study the fibrations $\pi_{i,j}$ induced by pairs of nodes (P_i, P_j). The proof of Theorem 32 shows that the fibres contain 12 disjoint (-2)-curves, so by Proposition 26 there are two additive fibres; by Corollary 24, the possible types are

$$\mathrm{I}_{2n}^*(n \geq 0), \quad \mathrm{I}_1^*, \quad \mathrm{IV}^*, \quad \text{and} \quad \mathrm{III}^*. \tag{14}$$

We distinguish three cases:

8.1 If there are 3 collinear nodes, then they give sections of the induced fibration, and the 11 exceptional curves above the other nodes embed into the negative definite root lattices which are the orthogonal complements of the sections. On the multiplicative fibres, this imposes no general restrictions, but additive fibres can, by inspection of the singular fibres, as described in the proof of Proposition 22, only support one disjoint smooth rational curve less. This is because the sections necessarily intersect only the simple fibre components. Therefore the number of disjoint rational curves not meeting one of the three sections is at most $N_v - 1$, where we recall that

$$N_v \leq \frac{1}{2}\lfloor e(F_v) + \delta_v \rfloor. \tag{15}$$

Hence, with two additive fibres, there can only be 10 disjoint (-2)-curves supported on the orthogonal complement of the sections, contradiction.

8.2 Thus we may assume that there are no three collinear nodes (i.e., any two nodes are companions). Note that this implies that any 3 nodes lie on a unique plane. If some connecting line is contained in X, then the line and the two nodes give sections of the fibration, with 12 disjoint (-2)-curves in the fibres. Hence the argument from 8.1 applies to establish a contradiction.

8.3 We can therefore assume that no line $\overline{P_i P_j}$ is contained in X. We continue by restricting the possible additive fibre types. They arise from the quartic curve $X \cap H$ by blowing up the nodes in the plane H: two of them give bisections of the fibration while the others give (-2)-fibre components, which cannot be such that their multiplicity in the fibre is ≥ 3.

Only the additive fibre type I_0^* can be realized of the possible types listed in (14) (as a double conic with 6 nodes; see part III). The reason is based on the fact that this is the only one with only one component with multiplicity at least 2, while the others have several components appearing with multiplicity at least 2, indeed at least 3 components except for the case of I_1^*.

Indeed, the blow ups of nodes appear with multiplicity 1, hence the plane section $X \cap H$ must be non reduced. In particular there are at most 2 components appearing with multiplicity at least 2.

To exclude the case of I_1^*, we need to exclude that $X \cap H$ consists of two double lines. In this case there are at most 4 nodes in H, since the intersection point of the two double lines cannot be a node, and there are no 3 collinear nodes by assumption, hence the number of irreducible components of the fibre is at most 4, a contradiction.

Therefore each fibration $\pi_{i,j}$ admits two such fibres. This turns out to be too restrictive: in fact, we have seen that for each pair $\mathcal{P}$ of nodes, there are exactly two planes π containing the pair, each containing six nodes.

Consider then the number of pairs as above $(\mathcal{P}, \pi), \mathcal{P} \subset \pi$. The number is therefore $(13) \cdot 14$. But since each such plane π contains exactly 15 such pairs $\mathcal{P}$, we have obtained a contradiction. $\square$

8.4. Proof of Theorem 1 Since the triple point case was covered in [Cat21b], we only have to deal with double point singularities. The general statement that a normal quartic in characteristic 2 contains at most 14 singularities was proved in 4.1 (using Propositions 29 and 30). That 14 singular points necessarily form nodes was proved in Theorem 32. An irreducible component was exhibited in Theorem 11. Finally, the result that there are fewer than 14 singularities if the resolution is not a supersingular K3 surface, was proved in Proposition 33. $\square$

Acknowledgement

Thanks to the anonymous referees for their comments which helped us improve the paper. We would also like to thank Stephen Coughlan for an interesting conversation.

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Fabrizio Catanese
Lehrstuhl Mathematik VIII
Mathematisches Institut der Universität Bayreuth
NW II
Universitätsstr. 30
95447 Bayreuth
Germany
and
Korea Institute for Advanced Study
Hoegiro 87
Seoul, 133–722
Republic of Korea
E-mail: Fabrizio.Catanese@uni-bayreuth.de

Matthias Schütt
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
Germany
and
Riemann Center for Geometry and Physics
Leibniz Universität Hannover
Appelstrasse 2
30167 Hannover
Germany
E-mail: schuett@math.uni-hannover.de