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Moishezon morphisms

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Abstract: We try to understand which morphisms of complex analytic spaces come from algebraic geometry. We start with a series of conjectures, and then give some partial solutions.

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A proper, irreducible, reduced analytic space X is *Moishezon* if it is bimeromorphic to a projective variety X^{p}, and a proper morphism of analytic spaces $g : X \to S$ is *Moishezon* if it is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$; see (7) and (10–11) for details.

The aim of this note is to discuss a series of questions about Moishezon morphisms, and give partial solutions to some of them.

We start with a list of conjectures in Section 1. Sections 2–3 are mostly review; new results are in Sections 4–6.

1. Open questions

The theory of Moishezon spaces can be viewed as a special chapter of the theory of algebraic spaces (and later stacks). However, a deformation of a Moishezon space need not be Moishezon, thus we get a theory that is not algebraic. The question we consider is the following.

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- Which morphisms of complex analytic spaces come from algebraic geometry, up to bimeromorphism?

The main open problem may be Conjecture 1 and its special case Conjecture 2. I have very little evidence supporting them and several experts thought that they are likely wrong.

Conjectures 3–4 about the deformation theory of Moishezon spaces go back at least to Hironaka's unpublished thesis.

Conjecture 5 can be viewed as a geometric form of deformation invariance of plurigenera [RT20, Thm.1.2], see also (34).

Conjecture 1. *A proper morphism of analytic spaces $g : X \to S$ is locally Moishezon (11) iff every irreducible component of every fiber is Moishezon.*

Comments 1.1. We check in (16) that the fibers of a Moishezon morphism are Moishezon. If g is smooth, a positive answer is in [RT21, Thm.1.4] and [RT20, Thm.1.2]; see (22).

If $\pi_1(S)$ is finite and S is either Stein or quasi-projective, then maybe g is also globally Moishezon. Easy examples (13) show that finiteness of $\pi_1(S)$ is necessary for the global variant.

Note that we do not assume that g is flat or that X, S are smooth. It is, however, quite likely that the above generality does not matter. The semistable reduction theorem [AK00] suggests that there is a projective, bimeromorphic morphism $S' \to S$ such that the main component of $X \times_S S'$ is bimeromorphic to a morphism $g' : X' \to S'$ such that g' is flat, toroidal and S' is smooth.

A positive answer for $g' : X' \to S'$ does not automatically imply a positive answer for $g : X \to S$, but the method may generalize.

It is reasonable to start with the case when g is flat with mildly singular fibers. The following may well be the key special case, where $\mathbb{D}$ denotes the complex disc.

Conjecture 2. *Let X be a smooth analytic space and $g : X \to \mathbb{D}$ a proper morphism. Assume that the fibers X_s are Moishezon for $s \neq 0$, and X_0 is a simple normal crossing divisor whose irreducible components are Moishezon. Then g is Moishezon.*

The Clemens-Schmid sequence should be a key ingredient here; see [Cle69, Cle77, Cle82] or the survey paper [GS75].

For an arbitrary smooth, proper morphism, the set of Moishezon fibers need not be closed (25), but the following could be true.

Conjecture 3. *Let $g: X \to \mathbb{D}$ be a flat, proper morphism. Assume that X_0 is irreducible with rational singularities and X_s is Moishezon for $s \neq 0$. Then X_0 is Moishezon.*

Comments 3.1. There are 3 preprints [Pop09, Bar17, Pop19] claiming a positive answer if X_0 is smooth. These lie outside my expertise, but my understanding is that not everyone is able to follow the arguments in them. Several cases are proved in [Bar15].

The analogous question for surfaces with cusp singularities has a negative answer; see (14). Cusps are the simplest non-rational surface singularities. This suggested that either log terminal or rational singularities may be the right class here.

Conjecture 4. *Let X be a smooth analytic space and $g: X \to \mathbb{D}$ a proper morphism. Assume that one of the irreducible components of X_0 is of general type. Then g is Moishezon and all other fibers are of general type (over a possibly smaller disc).*

Comments 4.1. Smooth, projective K3 and elliptic surfaces have deformations that are not even Moishezon, so general type may be the best one can hope for. We can harmlessly assume that X_0 is a reduced, simple normal crossing divisor.

If X_0 is irreducible and smooth, this is posed in [Sun80, p.201]; which in turn builds on problems and conjectures in [Iit71, Moi71, Nak75, Uen75].

If X_0 is irreducible, projective and has canonical singularities, a positive answer is given in [Kol22]. Note, however, that smooth Moishezon spaces can have unexpected deformations; see [Cam91, LP92].

Let $g: X \to S$ be a Moishezon morphism. By definition, it is bimeromorphic to a projective morphism $g^{\mathrm{p}}: X^{\mathrm{p}} \to S$. Thus the fibers X_s and X^{p}_s are bimeromorphic to each other for general $s \in S$, but may be quite different for special points $s \in S$. The following conjecture suggests that, over 1-dimensional bases, one can arrange X_s and X^{p}_s to be bimeromorphic to each other for every $s \in S$.

Conjecture 5. *Let $g: X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has canonical (resp. log terminal) singularities. Then g is fiberwise birational (26) to a flat, projective morphism $g^{\mathrm{p}}: X^{\mathrm{p}} \to \mathbb{D}$ such that*

(1) X^{p}_0 has canonical (resp. log terminal) singularities,
(2) X^{p}_s has terminal singularities for $s \neq 0$, and
(3) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

Comments 5.4. This is where singularities inevitably enter the picture. Even if g is a smooth family of projective surfaces, X^{p} may need to be singular; see for example [Kol22, Exmp.4].

If g is smooth and the fibers are of general type, then [RT20, Thm.1.2] implies that the canonical models of the fibers give the optimal choice for $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$. (Here all fibers can have canonical singularities.)

We give a positive answer to the log terminal case, provided X_0 is not uniruled, see (28). The canonical case is discussed in (32).

Remark 6. My aim is to understand how much of the theory of Moishezon spaces fits into algebraic geometry, and especially minimal model theory. The paper [RT20] gave the impetus to try to organize this into a systematic series of questions.

Campana pointed out that several of these questions have analogs for compact spaces of Fujiki's class C, and have a positive answer if we assume that the total space is of class C; see [Cam81].

A very different direction studies the place of the Moishezon property in the theory of compact complex manifolds. Solutions of Conjectures 1–2 are more likely to come from this approach. See [Pop11] for a survey.

2. Moishezon spaces

We give a quick review of the theory of Moishezon spaces.

Definition 7. A proper, irreducible, reduced analytic space X is *Moishezon* if it is bimeromorphic to a projective variety X^{p}. That is, there is a closed, analytic subspace $\Gamma \subset X \times X^{\mathrm{p}}$ such that the coordinate projections $\Gamma \to X$ and $\Gamma \to X^{\mathrm{p}}$ are isomorphisms on Zariski open dense sets.

By Chow's theorem, any 2 such X^{p} are birational to each other, so X acquires a unique algebraic structure.

A proper analytic space X is Moishezon iff the irreducible components of $\operatorname{red} X$ are Moishezon[1]. Thus X is Moishezon iff every irreducible component of its normalization is Moishezon.

8 (Basic theorems)**.** Let X be a proper Moishezon space.

(1) There is a projective variety X' and a bimeromorphic morphism $X' \to X$ (Chow lemma).
(2) For every $x \in X$ there is a pointed quasi projective scheme (x', X') and an étale morphism $(x', X') \to (x, X)$.

[1] This is not standard terminology.

(3) If X is normal then there is a proper variety Y and a finite group G acting on Y such that $X \cong Y/G$. (Note that usually Y can not be chosen projective.)
(4) If $Z \to X$ is finite, then Z is Moishezon.
(5) If $X \to Y$ is surjective, then Y is Moishezon.
(6) Assume that X is smooth. Then the usual Hodge decomposition $H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^p(X, \Omega_X^q)$ holds.
(7) $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ are algebraic spaces whose irreducible components are proper (but the connected components may have infinitely many irreducible components). The connected components of the space of divisors $\mathrm{Chow}_{n-1}(X)$ are proper.
(8) If X has rational singularities then it is projective iff it is Kähler.

Hints of proofs. Note that (1) is not obvious [Moi66]. It also follows from the more general results of [Hir75], and one can easily modify the arguments in [Sta15, Tag 088U]; the key step is probably [Sta15, Tag 0815].

(2) is quite hard; see [Art70].

For (3), cover X with finitely many $X'_i \to X$ as in (2). Then normalize X in the Galois closure of the field extensions $\mathbb{C}(X'_i)/\mathbb{C}(X)$. One can then use this to get $Z \to X$ as a quotient of a finite morphism $Z' \to X'$ to obtain (4).

For (5), using (4) and (1) we may assume that $X \to Y$ is generically finite, say of degree d, and X is projective. Then $y \mapsto [g^{-1}(y)] \in S^d X$ gives a bimeromorphic embedding of Y into the dth symmetric power of X.

By direct computation, the existence of a Hodge decomposition is invariant under smooth blow ups, thus we get (6). A better argument is in [Uen83, Prop.1.3].

For (7) see [Art69, Bar75, Cam80, Fuj82, Kol96] and the more complete treatment [BM20].

The smooth case of (8) is proved in [Moi66], the singular one in [Nam02].

Remark. The complements of closed analytic subsets form the open subsets of the Zariski topology. Note, however, that 2 open subsets can be biholomorphic to each other even if they are not birational. This is the main reason why one usually does not define 'Moishezon' for non-proper analytic spaces.

3. Moishezon morphisms

Definition 9 (Projective morphisms)**.** A proper morphism of analytic spaces $g : X \to S$ is *projective* if X can be embedded into $\mathbf{P}_S := \mathbb{P}^N \times S \to S$ for some N. Note that some authors allow $\mathbf{P}_S \to S$ to be any (locally trivial)

$\mathbb{P}^N$-bundle. The 2 versions are equivalent if S is Stein or quasi-projective (the cases we are mostly interested in) but not in general.

Definition 10 (Moishezon morphisms)**.** [Moi74, Fuj82] Assume now that S is reduced. A proper morphism of analytic spaces $g : X \to S$ is *Moishezon* iff the following equivalent conditions hold.

(1) $g : X \to S$ is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$. That is, there is a closed subspace $Y \subset X \times_S X^{\mathrm{p}}$ such that the coordinate projections $Y \to X$ and $Y \to X^{\mathrm{p}}$ are bimeromorphic.
(2) There is a projective morphism of algebraic varieties $G : \mathbf{X} \to \mathbf{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathbf{S}$ such that X is bimeromorphic to $\mathbf{X} \times_{\mathbf{S}} S$.

Here (2) $\Rightarrow$ (1) is clear. To see the converse, note that $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$ is flat over a dense, Zariski open subset $S^\circ \subset S$, thus we get a meromorphic map $\phi : S \dashrightarrow \mathrm{Hilb}(\mathbb{P}^N)$. The pull-back of the universal family over $\mathrm{Hilb}(\mathbb{P}^N)$ is then bimeromorphic to X.

Comment. This is the right notion if S is Stein or quasi projective, but, as with projectivity, there are different versions in general.

Assume that X is normal and the maps

$$X \overset{\phi}{\dashrightarrow} X^{\mathrm{p}} \overset{\iota}{\hookrightarrow} \mathbf{P}_S \tag{10.3}$$

show that $X \to S$ is Moishezon. Then $\iota \circ \phi : X \dashrightarrow \mathbf{P}_S$ is defined outside a codimension 2 closed subset, and $(\iota \circ \phi)^* \mathcal{O}_{\mathbf{P}_S}(1)$ extends to a rank 1 reflexive sheaf L on X. This L 'certifies' that X is Moishezon. This gives another equivalent characterization (in case X is normal, and S is Stein or quasi projective.)

(4) There is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \mathrm{Proj}_S(g_* L)$ is bimeromorphic onto the closure of its image.

We call such a sheaf L *very big* (over S)[2]. L is *big* (over S) if $L^{[m]}$ is very big for some $m > 0$, where $L^{[m]}$ denotes the reflexive hull of the mth tensor power.

Note that L is big (resp. very big) on $X \to S$ iff it is big (resp. very big) on $X^\circ \to S^\circ$ on some dense, Zariski open $S^\circ \subset S$.

Warning. By contrast it can happen that $X^\circ \to S^\circ$ is Moishezon but $X \to S$ is not, since the L° that certifies Moishezonness need not extend to X; see (14).

[2] *Very* big is not standard terminology, but it matches *very* ample.

Definition 11 (Locally Moishezon morphisms)**.** [Moi74] A proper morphism of analytic spaces $g : X \to S$ is *locally Moishezon* if S is covered by (Euclidean) open sets $S_i \subset S$ such that each $g^{-1}(S_i) \to S_i$ is Moishezon.

Comment. This follows standard usage of 'locally' in algebraic geometry and it works best for the purposes of Conjecture 1. However, it is *not* equivalent to the definition in [Fuj82].

Example 12. Let $g : X \to S$ be a proper morphism of analytic spaces, S Moishezon. Then g is Moishezon iff X is Moishezon.

Example 13. Let Z be a normal, projective variety with discrete automorphism group. Let $g : X \to S$ be a fiber bundle with fiber Z over a connected base S. Then g is Moishezon $\Leftrightarrow$ g is projective $\Leftrightarrow$ the monodromy is finite.

There are rational and K3 surfaces with infinite, discrete automorphism group. These lead to fiber bundles over the punctured disc $\mathbb{D}^\circ$ that are locally Moishezon but not globally Moishezon.

Example 14. [Loo81] studies flat, proper morphisms $g : X \to \mathbb{D}$ where X_0 is an Inoue surface (which is not Moishezon) with a cusp (which is log canonical), yet X_s is a smooth rational surface for $s \neq 0$.

Next we look at fibers of Moishezon morphisms.

Lemma 15. *Let $g : X \to S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then* $\operatorname{Ex}(g) \to S$ *is Moishezon.*

Proof. We prove the special case when the smooth locus of S is dense in $g(\operatorname{Ex}(g))$. This is a harmless assumption if S is Stein (or quasi-projective), since we can compose g with a finite $S \to \mathbb{C}^{\dim S}$ (or with a quasi-finite $S \to \mathbb{P}^{\dim S}$). A more heavy handed approach, which works in general, is to use a resolution $S' \to S$ and replace X with the normalization of the main component of $X \times_S S'$.

Let E_0 be a g-exceptional divisor. Set $(g_0 : X_0 \to S_0) := (g : X \to S)$ and $Z_0 := g_0(E_0)$.

If $g_i : X_i \to S_i$ and $E_i \subset X_i$ are already defined, we set $Z_i := g_i(E_i)$. Let S_{i+1} be the normalization of the blow-up $B_{Z_i}S_i$, and $g_{i+1} : X_{i+1} \to S_{i+1}$ the normalization of the graph of $X_i \to S_i \dashrightarrow S_{i+1}$. Let $E_{i+1} \subset X_{i+1}$ denote the bimeromorphic transform of E_i. (Note that $X_{i+1} \to X_i$ is an isomorphism over an open subset of E_i.)

Let $a(E_i, S_i)$ denote the vanishing order of the Jacobian of g_i along E_i. By an elementary computation we get that

$$a(E_{i+1}, S_{i+1}) \leq a(E_i, S_i) + 1 - \operatorname{codim}(Z_i \subset S_i).$$

Thus eventually we reach the situation when $\operatorname{codim}(Z_i \subset S_i) = 1$, hence $E_i \to Z_i$ is generically finite.

Note that each $Z_{i+1} \to Z_i$ is projective, thus $E_i \to Z_0$ is Moishiezon by (8.4), and so is $E_0 \to S$. □

The following is the easy direction of Conjecture 1.

Corollary 16. *The fibers of a proper, Moishezon morphism are Moishezon.*

Proof. Let $g : X \to S$ be a proper, Moishezon morphism. It is bimeromorphic to a projective morphism $X^{\mathrm{p}} \to S$. We may assume X^{p} to be normal. Let Y be the normalization of the closure of the graph of $X \dashrightarrow X^{\mathrm{p}}$.

Fix now $s \in S$. Let $Z_s \subset X_s$ be an irreducible component and $W_s \subset Y_s$ an irreducible component that dominates Z_s. By (8.5) it is enough to show that W_s is Moishezon.

If $\pi : Y \to X^{\mathrm{p}}$ is generically an isomorphism along W_s, then W_s is bimeromorphic to an irreducible component of X^{p}_s, hence Moishezon. Otherwise $W_s \subset \operatorname{Ex}(\pi)$. Now $\operatorname{Ex}(\pi) \to X^{\mathrm{p}}$ is Moishezon by (15) and $\dim \operatorname{Ex}(\pi) < \dim Y = \dim X$. So W_s is contained in a fiber of $\operatorname{Ex}(\pi) \to S$, hence Moishezon by induction on the dimension. □

Remark 17. More generally, if $g : X \to S$ is proper and Moishezon and $T \to S$ is a morphism of analytic spaces then $X \times_S T \to T$ is also proper and Moishezon.

The rest of this section is a study of the set of Moishezon fibers for arbitrary proper morphisms of analytic spaces. It is mostly a summary of some of the results of [RT20], with occasional changes.

Definition 18. Let $g : X \to S$ be a proper morphism of normal analytic spaces and L a line bundle on X. Set

(1) $\mathrm{VB}_S(L) := \{s \in S : L_s \text{ is very big on } X_s\} \subset S$,
(2) $\mathrm{GT}_S(X) := \{s \in S : X_s \text{ is of general type}\} \subset S$,
(3) $\mathrm{MO}_S(X) := \{s \in S : X_s \text{ is Moishezon}\} \subset S$,
(4) $\mathrm{PR}_S(X) := \{s \in S : X_s \text{ is projective}\} \subset S$.

Lemma 19. *Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces and L a line bundle on X. Then $\mathrm{VB}_S(L) \subset S$ is*

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. By passing to an open subset of S, we may assume that g is flat, g_*L is locally free and commutes with restriction to fibers. We get a meromorphic map $\phi : X \dashrightarrow \mathbb{P}_S(g_*L)$. There is thus a smooth, bimeromorphic model $\pi : X' \to X$ such that $\phi \circ \pi : X' \to \mathbb{P}_S(g_*L)$ is a morphism.

After replacing X by X' and again passing to an open subset of S, we may assume that g is flat, g_*L is locally free, commutes with restriction to fibers, and $\phi : X \to \mathbb{P}_S(g_*L)$ is a morphism. Let $Y \subset \mathbb{P}_S(g_*L)$ denote its image and $W \subset X$ the Zariski closed set of points where $\pi : X \to Y$ is not smooth. Set $Y^\circ := Y \setminus \phi(W)$ and $X^\circ := X \setminus \phi^{-1}(\phi(W))$. The restriction $\phi^\circ : X^\circ \to Y^\circ$ is then smooth and proper.

We assume that $\phi^{-1}(y)$ is a single point for a dense set in Y, hence for a dense set in Y°. Since ϕ° is smooth and proper, it is then an isomorphism. Thus ϕ is bimeromorphic on every irreducible fiber that has a nonempty intersection with X°. □

Corollary 20. *Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then* $\mathrm{GT}_S(X) \subset S$ *is*

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. Using resolution of singularities, we may assume that X is smooth. By passing to an open subset of S, we may also assume that S and g are smooth. By [HM06] there is an m (depending only on $\dim X_s$) such that $|mK_{X_s}|$ is very big whenever X_s is of general type. Thus (19) applies to $L = mK_X$. □

The following is essentially proved in [RT21, Thm.1.4] and [RT20, Thm. 1.2].

Theorem 21. *Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Then* $\mathrm{MO}_S(X) \subset S$ *is*

(1) either contained in a countable union $\cup_i Z_i$, where $Z_i \subsetneq S$ are Zariski closed,
(2) or $\mathrm{MO}_S(X)$ *contains a dense, open subset of S.*

Furthermore, if $R^2 g_*\mathcal{O}_X$ is torsion free then (2) *can be replaced by*

(3) $\mathrm{MO}_S(X) = S$ *and g is locally Moishezon.*

Remark 21.4. A positive answer to Conjecture 3 for smooth morphisms would imply that in fact $\mathrm{MO}_S(X) = S$ always holds in case (21.2); see (22).

Proof. Assume first that $R^2 g_* \mathcal{O}_X$ is torsion free.

As in [RT20, 3.15], the push-forward of the exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^\times \to 1$$

gives

$$R^1 g_* \mathcal{O}_X^\times \to R^2 g_* \mathbb{Z}_X \xrightarrow{e_2} R^2 g_* \mathcal{O}_X.$$

We may pass to the universal cover of S and assume that $R^2 g_* \mathbb{Z}_X$ is a trivial $H^2(X_s, \mathbb{Z})$-bundle.

Let $\{\ell_i\}$ be those global sections of $R^2 g_* \mathbb{Z}_X$ such that $e_2(\ell_i) \in H^0(S, R^2 g_* \mathcal{O}_X)$ is identically 0, and $\{\ell'_j\}$ the other global sections. The ℓ_i then lift back to global sections of $R^1 g_* \mathcal{O}_X^\times$, hence to line bundles L_i on X.

If there is an L_i such that $\mathrm{VB}_S(L_i)$ contains a dense open subset of S, then $X \to S$ is Moishezon by (19) and we are done. Otherwise, we claim that

$$\text{(21.5)} \qquad \mathrm{MO}_S(X) \subset \cup_i \mathrm{VB}_S(L_i) \bigcup \cup_j \bigl(e_2(\ell'_j) = 0\bigr).$$

To see this assume that $s \notin \cup_j \bigl(e_2(\ell'_j) = 0\bigr)$. Then every line bundle on X_s is numerically equivalent to some $L_i|_{X_s}$. Since being big is preserved by numerical equivalence, we see that X_s has a big line bundle $\Leftrightarrow L_i|_{X_s}$ is big for some $i \Leftrightarrow L_i|_{X_s}$ is very big for some i. This completes the case when $R^2 g_* \mathcal{O}_X$ is torsion free.

In general, the torsion subsheaf of $R^2 g_* \mathcal{O}_X$ is supported on a Zariski closed, proper subset, hence (21.2) gives that if (21.1) does not hold then $\mathrm{MO}_S(X)$ contains a Zariski dense open subset of S. $\square$

Corollary 22. *Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces whose fibers are Moishezon. Then g is locally Moishezon.*

Proof. If X_s is Moishezon, then Hodge theory (8.6) tells us that $H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s})$ is surjective for every i. Thus $R^2 g_* \mathcal{O}_X$ is locally free by (24), hence (21.3) applies. $\square$

There are many complex manifolds for which Hodge decomposition holds; these are called *cohomologically Kähler* manifolds or $\partial\bar{\partial}$-manifolds. We also get the following variant.

Corollary 23. *Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Assume that $\mathrm{MO}_S(X)$ contains a dense, open subset of S and all fibers are cohomologically Kähler. Then g is locally Moishezon.* $\square$

We have used the following result of [DJ74]; see also [Nak87, 3.13] and [Kol20, 2.64].

Theorem 24. *Let $g : X \to S$ be a smooth, proper morphism of analytic spaces. Assume that $H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s})$ is surjective for every i for some $s \in S$. Then $R^i g_* \mathcal{O}_X$ is locally free in a neighborhood of s for every i.* $\square$

Example 25 (25.1)**.** Let $X \to D^{20}$ be a universal family of K3 surfaces. A smooth, compact surface is Moishezon iff it is projective. The projective fibers of $X \to D^{20}$ correspond to a countable union of hypersurfaces $H_{2g} \subset D^{20}$.

(25.2) Let $E \subset \mathbb{P}^2$ be a smooth cubic. Fix $m \geq 10$ and let $X \to D$ be the universal family of surfaces obtained by blowing up m distinct points $p_i \in E$, and then contracting the birational transform of E. (So D is open in E^m.) If such a surface is projective then there are positive n_i such that $\sum_i n_i[p_i] \sim nH|_E$ where H is the line class on $\mathbb{P}^2$ and $n = \frac{1}{3}\sum_i n_i$.

Here $X \to D$ is Moishezon and the projective fibers correspond to a countable union of hypersurfaces $H_i \subset D$. All fibers have log canonical singularities.

4. 1-parameter families

Definition 26. Let $g_i : X^i \to S$ be a proper morphisms. A bimeromorphic map $\phi : X^1 \dashrightarrow X^2$ is *fiberwise bimeromorphic* if ϕ induces a bimeromorphic map $\phi_s : X^1_s \dashrightarrow X^2_s$ for every $s \in S$.

If X^1, X^2 are fiberwise bimeromorphic to each other then X^1_s, X^2_s are bimeromorphic to each other for every $s \in S$, but the latter is only a sufficient condition in general.

We study whether a flat, proper, Moishezon morphism $g : X \to \mathbb{D}$ is fiberwise bimeromorphic to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$. The next examples suggest that the answer is

- negative if g is very singular,
- positive if g is mildly singular, and
- even if g is smooth, g^{p} usually can not be chosen smooth.

Example 27. Let $g : X \to \mathbb{D}$ be a smooth, projective morphism. Assume that $\operatorname{Pic}(X) \cong \mathbb{Z}$ but $\operatorname{rank} \operatorname{Pic}(X_0) \geq 2$.

Let $Z \subset X_0$ be a smooth, ample divisor whose class is not in the image of $\operatorname{Pic}(X) \to \operatorname{Pic}(X_0)$. Blow up Z to get $g' : X' \to \mathbb{D}$. Here $X'_0 \cong X_0$ has normal bundle $\mathcal{O}_{X_0}(-Z)$, hence it is contractible. We get a non-projective, Moishezon morphism $h : Y \to \mathbb{D}$.

Conjecture 27.1. In most cases, $h : Y \to \mathbb{D}$ is not fiberwise birational to a flat, projective morphism.

The next result is a positive answer to the log terminal case of (5), provided X_0 is not uniruled. See (32) for a discussion of the canonical case.

Theorem 28. *Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that*

(1) X_0 has log terminal singularities and
(2) X_0 is not uniruled.

Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ (possibly over a smaller disc) such that

(3) X_0^{p} has log terminal singularities,
(4) X_s^{p} is not uniruled and has terminal singularities for $s \neq 0$, and
(5) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

Remark 28.6. Conjecturally we can also achieve that $K_{X^{\mathrm{p}}}$ is relatively nef. The main obstacle is that (algebraic) minimal models are currently known to exist only in the general type case.

29 (Proof of (28))**.** The basic plan is similar to the proof of properness of the KSB moduli space; see [KSB88, Sec.5] or [Kol20, Sec.2.5].

We take a resolution of singularities $Y \to X$ such that $Y \to \mathbb{D}$ is projective, and then take a relative minimal model of $Y \to \mathbb{D}$. We hope that it gives what we want. There are, however, several obstacles. Next we discuss these, and their solutions, but for all technical details we refer to later sections.

(29.1) We need to control the singularities of X. First (39) reduces us to the case when K_X is $\mathbb{Q}$-Cartier. We assume this from now on. Then (30) implies that the pair (X, X_0) is plt.

(29.2) After a base change $z \mapsto z^r$ we get $g^r : X^r \to \mathbb{D}$. For suitable r, there is a semi-stable, projective resolution $h : Y \to \mathbb{D}$; we may also choose it to be equivariant for the action of the cyclic group $G \cong \mathbb{Z}_r$. All subsequent steps will be G-equivariant. We denote by X_0^Y the birational transform of X_0 and by E_i the other irreducible components of Y_0.

(29.3) We claim that Y_s is not uniruled for $s \neq 0$. Indeed, for smooth families being uniruled is a deformation invariant property, and by Matsusaka's theorem [Kol96, IV.1.7], we would get that X_0^Y is uniruled. Thus K_{Y_s} is pseudo-effective by [BDPP13].

(29.4) The required relative minimal model theorem is known only when the general fibers are of log general type. To achieve this, let H be an ample,

G-equivariant divisor such that Y_0+H is snc. For $\epsilon>0$ we get a pair $(Y,\epsilon H)$ whose general fibers $(Y_s,\epsilon H_s)$ are of log general type since K_{Y_s} is pseudo-effective. For such algebraic families, relative minimal models are known to exist [BCHM10]. We also know that $(Y,Y_0+\epsilon H)$ is dlt for $0<\epsilon\ll 1$.

(29.5) Although our family is not algebraic, [KNX18] treats the relative MMP for semi-stable, projective morphisms to a disc. The precise results are recalled in (36). Thus we get a relative minimal model

$$\phi:(Y,\epsilon H)\dashrightarrow (Y^{\mathrm{m}},\epsilon H^{\mathrm{m}}),$$

and $(Y^{\mathrm{m}},Y_0^{\mathrm{m}}+\epsilon H^{\mathrm{m}})$ is dlt. Here H^{m} is $\mathbb{Q}$-Cartier for general choice of ϵ by [Ale15, Lem.1.5.1], thus $(Y^{\mathrm{m}},Y_0^{\mathrm{m}})$ is also dlt.

Remark. We have a choice here whether to take the minimal or the canonical model. The minimal model has milder singularities, but it is not unique. Conjecturally, the canonical model Y^{c} is independent of $0<\epsilon\ll 1$, but this is known only in dimensions ≤ 3.

(29.6) We claim that ϕ contracts all the E_i. Since (X^r,X_0) is plt, all the E_i have discrepancy >-1. Thus the E_i are contained in the restricted, relative base locus of K_Y+Y_0 by (31.2). For ϵ small enough, the E_i are also contained in the restricted, relative base locus of $K_Y+Y_0+\epsilon H$ by (31.1). Thus any MMP contracts the E_i. On the other hand, X_0^Y can not be contracted, so $X\dashrightarrow Y^{\mathrm{m}}$ is fiberwise birational.

(29.7) Note that h is smooth away from Y_0, thus $(Y_s,\epsilon H_s)$ is terminal for $s\neq 0$ and $0\le\epsilon\ll 1$. Since H_s is ample, we do not contract it, so $(Y_s^{\mathrm{m}},\epsilon H_s^{\mathrm{m}})$ is still terminal. Hence so is Y_s^{m}, giving (4).

(29.8) As we noted, $(Y^{\mathrm{m}},Y_0^{\mathrm{m}})$ is dlt, hence plt since Y_0^{m} is irreducible. Thus Y_0^{m} is log terminal by the easy direction of (30). □

The following results were also used in the proof of (28).

Proposition 30 (Inversion of adjunction I)**.** *Let X be a normal, complex analytic space, $X_0\subset X$ a Cartier divisor and Δ an effective $\mathbb{R}$-divisor such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then $(X,X_0+\Delta)$ is plt in a neighborhood of X_0 iff $(X_0,\Delta|_{X_0})$ is klt.*

Proof. The proof given in [Kol92, Sec.17] or [KM98, Sec.5.4] applies with minor changes, using the complex analytic vanishing theorems proved in [Tak85] and [Nak87]. □

31 (Divisorial restricted base locus)**.** The basic theory is in [ELM+09, 1.12–21] and an extension to the non-projective case is outlined in [FKL16, Sec.5].

Let $g : X \to S$ be a proper, Moishezon morphism, X normal. The (relative) *base locus* of a Weil divisor F is

$$B(F) := \operatorname{Supp} \operatorname{coker}\bigl[g^* g_* \mathcal{O}_X(F) \to \mathcal{O}_X(F)\bigr].$$

Its divisorial part is denoted by $B^{\mathrm{div}}(F)$; we think of it as a Weil divisor.

Let D be an $\mathbb{R}$-divisor on X. Its *stable divisorial base locus* is the $\mathbb{R}$-divisor

$$\mathbf{B}^{\mathrm{div}}(D) := \lim_{m\to\infty} \tfrac{1}{m} B^{\mathrm{div}}(\lfloor mD \rfloor),$$

and its *restricted divisorial base locus* is

$$\mathbf{B}_{-}^{\mathrm{div}}(D) := \sup_{A} \mathbf{B}^{\mathrm{div}}(D + A),$$

where A runs through all big $\mathbb{R}$-divisors on X that satisfy $\mathbf{B}^{\mathrm{div}}(A) = \emptyset$. This could be an infinite linear combination of prime divisors.

An important observation of [FKL16] is that all the projective theorems on the *divisorial* restricted base locus carry over to proper schemes and Moishezon varieties. We need 2 properties:

(31.1) Let $X \to S$ be a proper, Moishezon morphism, D an $\mathbb{R}$-divisor on X, and A a big $\mathbb{R}$-divisor on X such that $\mathbf{B}^{\mathrm{div}}(A) = \emptyset$. Then, for every prime divisor $F \subset X$,

$$\operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D) = \lim_{\epsilon\to 0} \operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D + \epsilon A).$$

(31.2) Let $X_i \to S$ be proper, Moishezon morphisms, $h : X_1 \to X_2$ a proper, bimeromorhic morphism, D_2 a pseudo-effective, $\mathbb{R}$-Cartier divisor on X_2, and E an effective, h-exceptional divisor. Then

$$\mathbf{B}_{-}^{\mathrm{div}}(E + h^* D_2) \geq E.$$

32 (Canonical case of Conjecture 5)**.** An argument similar to (29) should prove the canonical case, but there are 3 difficulties.

The reduction to the case when K_X is $\mathbb{Q}$-Cartier again follows from (39). Then we need to show that the pair (X, X_0) is canonical. This is proved (though not stated) in [Nak04, 5.2]. This is also a special case of the general inversion of adjunction; a quite roundabout proof for Moishezon morphisms is given in (40).

In (29) next we run the MMP for $K_Y + \epsilon H$, which is the same as MMP for $K_Y + Y_0 + \epsilon H$ since Y_0 is numerically relatively trivial.

In the canonical case we would need to run the MMP for $K_Y + X_0^Y + \eta\sum E_i + \epsilon H$, where we choose ϵ, η small, positive. The arguments of [KNX18] do not cover this case, but I expect that a method similar to [KNX18] would prove this. The forthcoming [LM21] should also cover this case.

If X_0 is of general type, then the canonical model of (Y, X_0^Y) gives what we want.

In general, arguing as in (36) we should get a minimal model $g^{\mathrm{m}} : (Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}}) \to \mathbb{D}$. Here $\eta\sum E_i^{\mathrm{m}}$ is omitted since the E_i get contracted. General theory tells us that

$$\operatorname{discrep}(Y_0^{\mathrm{m}}) \geq \operatorname{discrep}(Y_0^{\mathrm{m}}, \operatorname{Diff}_{Y_0^{\mathrm{m}}} \epsilon H^{\mathrm{m}}) \geq -\epsilon.$$

We can choose ϵ arbitrarily small, but Y_0^{m} may depend on ϵ, so we can not just take a limit as $\epsilon \to 0$. This is a problem that appears even if we start with a projective, algebraic family.

At this point we could appeal to one of the ACC conjectures (33) which says that, for ϵ small enough, we must have $\operatorname{discrep}(Y_0^{\mathrm{m}}) \geq 0$. That is, Y_0^{m} is canonical.

The necessary result is known in dimensions ≤ 3, but it is likely to be quite difficult in general. So an alternate approach to our situation would be better.

33 (A gap conjecture)**.** The following is a special case of [Sho96, Conj.4.2]

(1) For every $n \geq 1$ there is an $\epsilon(n) > 0$ such that if X is an n-dimensional variety and $\operatorname{discrep} X > -\epsilon(n)$, then in fact $\operatorname{discrep} X \geq 0$ (that is, X has canonical singularities).

In dimension 2 this can be read off from the classifiation of log terminal singularities (these have $\operatorname{discrep} X > -1$). We get the optimal value $\epsilon(2) = \frac{1}{3}$ and equality holds for $\mathbb{C}^2/\frac{1}{3}(1,1)$.

The 3-dimensional case is much harder; see [Jia20]. The optimal value is $\epsilon(3) = \frac{1}{13}$ and an extremal case is the cyclic quotient singularity $\mathbb{C}^3/\frac{1}{13}(3,4,5)$.

Special cases (in all dimensions) are proved in [Nak16].

Remark 34. The deformation invariance of plurigenera for smooth, proper morphisms with Moishezon fibers is proved in [RT20, Thm.1.2].

The canonical case of Conjecture 5 would show that the projective case implies the Moishezon case. However, the hard part in [RT20] is to show that g is Moishezon, so using Conjecture 5 would only yield a longer proof.

5. Approximating Moishezon morphisms

We discuss 2 ways of approximating a projective (resp. Moishezon) morphism $g : X \to \mathbb{D}$ by morphisms between projective (resp. Moishezon) varieties. This allows us to prove some results for Moishezon morphisms $g : X \to \mathbb{D}$.

35 (Algebraic approximation of projective morphisms)**.** Let $g : Y \to \mathbb{D}$ be a projective morphism with relatively ample line bundle L. For later purposes we also specify a finite set of relative Cartier divisors $E^i \subset Y$.

Then $(Y_0, L_0 := L|_{Y_0}, E^i_0 := E^i|_{Y_0})$ is a projective, polarized scheme marked with effective Cartier divisors. (For now Y_0 can be even nonreduced.)

(Y_0, L_0, E^i_0) has a universal deformation space $G_S : (\mathbf{Y}_S, \mathbf{L}_S, \mathbf{E}^i_S) \to S$, where $S, \mathbf{Y}_S$ are quasi-projective schemes, G_S is flat and projective, $\mathbf{L}_S$ is G_S-ample and the $\mathbf{E}^i_S$ are relative Cartier divisors.

The original family gives a holomorphic $\phi_S : \mathbb{D} \to S$. Next we resolve the singularities of the image, and then replace S first by the Zariski closure of $\phi_S(\mathbb{D})$ and then by its resolution. Denote the latter by B. We obtain the following data.

(1) A smooth $\mathbb{C}$-variety B,
(2) a flat, projective morphism $G : (\mathbf{Y}, \mathbf{L}, \mathbf{E}^i) \to B$, where $\mathbf{L}$ is G-ample, the $\mathbf{E}^i$ are relative Cartier divisors, and
(3) a holomorphic map $\phi : \mathbb{D} \to B(\mathbb{C})$,

such that,

(4) $\big((Y, L, E^i) \to \mathbb{D}\big) \cong \big(\phi^*(\mathbf{Y}, \mathbf{L}, \mathbf{E}^i) \to \mathbb{D}\big)$ and
(5) $\phi(\mathbb{D})$ is smooth and Zariski dense in B.

We call $G : (\mathbf{Y}, \mathbf{L}, \mathbf{E}^i) \to B$ an *algebraic envelope* of $g : (Y, L, E^i) \to \mathbb{D}$.

Note that we have no control over the dimension of B. However, if Y is smooth, then so is $\mathbf{Y}$.

Since B is smooth, the holomorphic curve $\phi(\mathbb{D}) \subset B$ can be approximated by algebraic curves to any order. Thus, for any fixed $m > 0$ we get

(6) a smooth, pointed algebraic curve (c, C),
(7) a flat, projective morphism $(Y_C, L_C, E^i_C) \to C$, and
(8) an isomorphism $(Y, L, E^i)_m \cong (Y_C, L_C, E^i_C)_m$,

where the subscript m denotes the mth order infinitesimal neighborhood of the central fibers.

If $m = 0$ then we only get that the central fibers Y_0 and $(Y_C)_0$ are isomorphic. The case $m = 1$ carries much more information: the smoothness of the total space along the central fiber and the normal bundles of the irreducible components of the central fiber are also preserved.

As in [KNX18], algebraic envelopes can be used to show that MMP works for projective morphism over Riemann surfaces.

Proposition 36. *Let $g:(Y,\Delta)\to\mathbb{D}$ be projective, with irreducible, normal general fibers. Assume in addition that $(Y,Y_0+\Delta)$ is dlt and $K_{Y_s}+\Delta_s$ is klt and big for almost every $s\in\mathbb{D}$. Then*

(1) there is a relative minimal model $g^{\mathrm{m}}:(Y^{\mathrm{m}},\Delta^{\mathrm{m}})\to\mathbb{D}$, and
(2) the relative canonical model $g^{\mathrm{c}}:(Y^{\mathrm{c}},\Delta^{\mathrm{c}})\to\mathbb{D}$ exists.

Assume in addition that there is an irreducible divisor $Y_0^*\subset Y_0$ such that $Y_0\setminus Y_0^*$ is contained in the stable, relative base locus of $K_Y+\Delta$. Then

(3) $Y_0^*\dashrightarrow Y_0^{\mathrm{m}}\to Y_0^{\mathrm{c}}$ are birational,
(4) $(Y^{\mathrm{m}},Y_0^{\mathrm{m}}+\Delta^{\mathrm{m}})$ and $(Y^{\mathrm{c}},Y_0^{\mathrm{c}}+\Delta^{\mathrm{c}})$ are plt, and
(5) $(Y_0^{\mathrm{m}},\mathrm{Diff}_{Y_0^{\mathrm{m}}}\Delta^{\mathrm{m}})$ and $(Y_0^{\mathrm{c}},\mathrm{Diff}_{Y_0^{\mathrm{c}}}\Delta^{\mathrm{c}})$ are klt.
(6) If Δ is $\mathbb{R}$-Cartier then $(Y^{\mathrm{m}},Y_0^{\mathrm{m}})$ is plt and Y_0^{m} is log terminal.
(7) If the coefficients in Δ are sufficiently general, then $(Y^{\mathrm{c}},Y_0^{\mathrm{c}})$ is plt and Y_0^{c} is log terminal.

Proof. Claims (1–2) are basically proved in [KNX18]. Unfortunately, the main result [KNX18, Thm.2] is formulated to apply to the Calabi-Yau case. However, [KNX18, Props.8–14] contain a complete proof, though not a clear statement.

The rest of the proof uses only the conclusions of (1–2).

Any MMP $Y\dashrightarrow Y^{\mathrm{m}}$ contracts the stable base locus of $K_Y+\Delta$, thus Y_0^{m} is irreducible and $Y_0^*\dashrightarrow Y_0^{\mathrm{m}}$ is thus bimeromorphic. Also $(Y^{\mathrm{m}},Y_0^{\mathrm{m}}+\Delta^{\mathrm{c}})$ is dlt, hence plt since Y_0^{m} is irreducible. Since $Y^{\mathrm{m}}\to Y^{\mathrm{c}}$ does not contract Y_0^{m}, we see that $(Y^{\mathrm{c}},Y_0^{\mathrm{c}}+\Delta^{\mathrm{c}})$ is also plt. This is (4), and (5) follows by the easy direction of adjunction [Kol13, 4.8].

If Δ is $\mathbb{Q}$-Cartier then so is Δ^{m}, hence (6) follows from [KM98, 2.27]. A similar argument works for (7) using [Ale15, Lem.1.5.1]. □

37 (Algebraic approximation of Moishezon morphisms)**.** Let $f:X\to\mathbb{D}$ be a proper, Moishezon morphism and $h:Y\to X$ a proper morphism such that $g:=f\circ h:Y\to\mathbb{D}$ is projective with relatively ample line bundle L. Also choose relative Cartier divisors E^i on Y. Assume also that X_0 is seminormal (though this is probably ultimately not necessary).

We can apply (35.1–5) to get an algebraic envelope $G:(\mathbf{Y},\mathbf{L},\mathbf{E}^i)\to B$. By [Art70], after an étale base change, we may assume that $h_0:Y_0\to X_0$ extends to $H:\mathbf{Y}\to\mathbf{X}$ where $F:\mathbf{X}\to B$ is an algebraic space.

Comment 37.1. General extension theory, as in [Art70, MR71], tells us only that we have

$$H_0 : \mathbf{Y}_0 = Y_0 \xrightarrow{h_0} X_0 \xrightarrow{\tau} \mathbf{X}_0,$$

where τ is a finite homeomorphism. Then we use that the functor of simultaneous seminormalizations is formally representable. The projective case is discussed in [Kol11]; see [Kol20, 9.61] for algebraic spaces. By our assumption, the identity is a simultaneous seminormalization over the completion of $\phi(\mathbb{D})$, which is Zariski dense. Thus $\mathbf{X} \to B$ has seminormal fibers, hence $X_0 \cong \mathbf{X}_0$, as claimed.

Assume next that fibers of f over $\mathbb{D}^\circ$ satisfy a property $\mathcal{P}$ that is Zariski open in families (for example smooth, normal or reduced). Then general fibers of F also satisfy $\mathcal{P}$. As before, $\phi(\mathbb{D}) \subset B$ can be approximated by algebraic curves to any order. Thus, for any fixed $m > 0$ we get

(2) a smooth, pointed algebraic curve (c, C),
(3) morphisms $h_c : (Y_C, L_C, E^i_C) \to X_C \to C$, where
  (a) $g_C : (Y_C, L_C) \to C$ is flat, projective,
  (b) $f_C : X_C \to C$ is a flat algebraic space,
  (c) general fibers of f_C satisfy $\mathcal{P}$, and
(4) an isomorphism $\big((Y, L, E^i) \to X\big)_m \cong \big((Y_C, L_C, E^i_C) \to X_C\big)_m$,

where the subscript m denotes the mth order infinitesimal neighborhood of the central fibers.

Corollary 38. *Let $f : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism, X normal. Assume that it has a resolution $h : Y \to X$ where $g := f \circ h : Y \to \mathbb{D}$ is projective and Y_0 a reduced, snc divisor. Then X has a canonical modification $\pi : X^{\mathrm{c}} \to X$. (That is, X^{c} has canonical singularities and $K_{X^{\mathrm{c}}}$ is π-ample.)*

Proof. Let $H : \mathbf{Y} \to \mathbf{X}$ and $F : \mathbf{X} \to B$ be an algebraic envelope as in (37).

Note that canonical modifications are unique and commute with étale morphisms. They exist for quasi-projective varieties over $\mathbb{C}$ by [BCHM10], hence every algebraic space of finite type over $\mathbb{C}$ has a canonical modification.

Let $\Pi : \mathbf{X}^{\mathrm{c}} \to \mathbf{X}$ denote the canonical modification of $\mathbf{X}$. Since $\mathbf{Y} \to B$ is locally stable, so is $\mathbf{X}^{\mathrm{c}} \to B$; cf. [Kol20, Sec.4.8].

By pull-back we get a locally stable morphism $\pi : X^{\mathrm{c}} \to X \to \mathbb{D}$ whose general fibers are canonical. Since $(X^{\mathrm{c}}, X^{\mathrm{c}}_0)$ is lc and X^{c}_0 is a Cartier divisor, we see that X^{c} has canonical singularities. □

The following extends [KSB88, Sec.3] to Moishezon morphisms, see also [Kol20, Sec.5.5].

Corollary 39. *Let $f : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 is log terminal. Then X has a canonical modification $\pi : X^{\mathrm{c}} \to X$, X_0^{c} is log terminal and π is fiberwise birational.*

Proof. After a ramified base change $\tilde{\mathbb{D}} \to \mathbb{D}$ with group $G := \mathbb{Z}/r$, we can apply (38) to $\tilde{X} \to \tilde{\mathbb{D}}$ to get $\tilde{\pi} : \tilde{X}^{\mathrm{c}} \to \tilde{X}$.

As in [Kol20, 5.32] we get that $\tilde{X}_0^{\mathrm{c}}$ is log terminal and $\tilde{X}_0^{\mathrm{c}} \to \tilde{X}_0$ is birational. Set $X^{\mathrm{c}} := \tilde{X}^{\mathrm{c}}/G$.

The base change group acts trivially on the central fiber $\tilde{X}_0^{\mathrm{c}}$, hence $X_0^{\mathrm{c}} \cong \tilde{X}_0^{\mathrm{c}}$ is also log terminal. Finally the pair $\bigl(\tilde{X}^{\mathrm{c}}, \tilde{X}_0^{\mathrm{c}}\bigr)$ is log canonical, hence so is $\bigl(X^{\mathrm{c}}, X_0^{\mathrm{c}}\bigr)$ by [KM98, 5.20]. Thus X^{c} is canonical. $\square$

6. Inversion of adjunction

The proof of the general inversion of adjunction theorem given in [Kol13, 4.9] relies on the MMP. For our purposes we would need the relative MMP for projective morphisms over an analytic base. This is currently not known. However, [Nak87] discusses the first steps, [Kol22] some special cases, and ongoing work of Lyu-Murayama [LM21] settles the general case.[3]

As a stopgap measure, we go around this problem for Moishezon morphisms using approximations. While ultimately this will not be necessary, the key Lemma 42 is of independent interest.

Theorem 40. *Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism and Δ an effective $\mathbb{Q}$-divisor on X. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then*

$$\operatorname{discrep}(X, X_0 + \Delta_0) = \operatorname{totaldiscrep}(X_0, \Delta_0),$$

where on the left we use only those exceptional divisors whose centers on X have nonempty intersection with X_0.

Note that here the $\leq$ part is easy [Kol13, 4.8]. The known proofs of the $\geq$ part use the MMP, and the cases settled in [KNX18] do not seem enough.

We start the proof of (40) with a discussion on snc divisors and then with a general result which says that discrepancies can be computed from the 1st order neighborhood of the exceptional set.

[3]Added in proof: See also Osamu Fujino, *Minimal model program for projective morphisms between complex analytic spaces* (2022)

41 (Simple normal crossing divisors)**.** It would be convenient to recognize simple normal crossing divisors (abbreviated as *snc*) from an infinitesimal neighborhood of the special fiber. At first sight, this seems impossible. Consider for example the family

$$g : (\mathbb{C}^3, D := (xy = z^{m+1})) \to \mathbb{C}_z,$$

where D is not an snc divisor. The mth order infinitesimal neighborhood of the special fiber is defined by $z^{m+1} = 0$, hence isomorphic to the mth order neighborhood of the snc family

$$g : (\mathbb{C}^3, B := (xy = 0)) \to \mathbb{C}_z.$$

There is also the added problem that snc in the Euclidean topology is not the same as snc in the Zariski topology. (For example, $(y^2 = x^2 + x^3$ is snc in the Euclidean topology but not in the Zariski topology.)

We can, however, solve both problems by a simple bookkeeping convention.

Let M be a complex manifold and $\{E_i : i \in I\}$ (reduced) divisors on M. We say that $(M, E_i : i \in I)$ is a *marked snc pair* if for every $p \in M$ there are

(1) local analytic coordinates $z_1, \dots, z_n$, and
(2) an injection $\sigma : \{1, \dots, r\} \hookrightarrow I$ for some $0 \le r \le m$,

such that

(3) $E_{\sigma(i)} = (z_i = 0)$ near p, and
(4) the other E_j do not contain p.

With this definition we have the following.

Claim 41.5. Let $E_1, \dots, E_r, E_{r+1}, \dots, E_m$ and $E'_{r+1}, \dots, E'_m$ be reduced divisors on a complex manifold M. Assume that

(a) $(M, E_1 + \dots + E_m)$ is a marked snc pair, and
(b) E_j and E'_j have the same restriction on $E_1 \cup \dots \cup E_r$ for all $j > r$.

Then $(M, E_1 + \dots + E_r + E'_{r+1} + \dots + E'_m)$ is also a marked snc pair in a neighborhood of $E_1 \cup \dots \cup E_r$. □

Lemma 42. *Let $(X, \Delta = \sum d_j D_j)$ be a normal, algebraic or analytic pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $B_X \subset X$ be a Cartier divisor. Let*

$$p : (Y, \textstyle\sum_i B_i + \sum_j \bar{D}_j + \sum_\ell E_\ell) \to (X, B_X + \operatorname{Supp} \Delta)$$

be a log resolution, where $B := \sum_i B_i = \operatorname{red} p^{-1}(B_X)$, $\bar{D}_j$ is the birational transform of D_j and the E_ℓ are the other p-exceptional divisors.

Then the discrepancies $a(*, X, \Delta)$ of all p-exceptional divisors (whose centers have nonempty intersection with B_X) can be computed from

(1) $B_{(2)} := \operatorname{Spec} \mathcal{O}_Y/\mathcal{O}_Y(-2B)$, *and*
(2) *the divisors* $\bar{D}_j|_B$ *and* $E_\ell|_B$.

Proof. After replacing X by a smaller neighbood of B_X, we may assume that B is a deformation retract of Y. In particular, the centers of all p-exceptional divisors have nonempty intersection with B_X, and numerical equivalence of divisors is determined by their restriction to B.

The discrepancies b_i and e_ℓ are uniquely determined by the conditions

$$K_Y + \textstyle\sum_i' b_i B_i + \sum_j d_j \bar{D}_j + \sum_\ell e_\ell E_\ell \equiv_p p^*(K_X + \Delta), \tag{42.3}$$

where in $\sum_i'$ we sum over the p-exceptional divisors in B. Restricting to B and using adjunction we get

$$\textstyle\sum_i' b_i B_i|_B + \sum_\ell e_\ell E_\ell|_B \equiv_p B|_B - K_B - \sum_j d_j \bar{D}_j|_B + (p|_B)^*(K_X + \Delta). \tag{42.4}$$

Note that $B_{(2)}$ determines the $B_i|_B$ and hence $B|_B$. Thus the right hand side is known and the b_i, e_ℓ are the unique solution to (42.4). □

Corollary 43. *Using the notation of (42), assume that there is another pair with a log resolution*

$$p' : (Y', \textstyle\sum_i B_i' + \sum_j \bar{D}_j' + \sum_\ell E_\ell' + F') \to (X', B_{X'}' + \operatorname{Supp} \Delta')$$

such that there is an isomorphism

$$\phi : (B_{(2)}' \hookleftarrow B' \xrightarrow{p'} B_{X'}') \cong (B_{(2)} \hookleftarrow B \xrightarrow{p} B_X),$$

that sends $\bar{D}_j'|_{B'}$ to $\bar{D}_j|_B$ and $E_\ell'|_{B'}$ to $E_\ell|_B$ for every j, ℓ. Then

(1) *corresponding divisors have the same discrepancies, and*
(2) *divisors in F' have discrepancy 0.*

Proof. Note that (42.4) gives us that

$$\textstyle\sum_i' b_i B_i'|_{B'} + \sum_\ell e_\ell E_\ell'|_{B'} + 0 \cdot F' \equiv_{p'} B'|_{B'} - K_{B'} - \sum_j \bar{D}_j'|_{B'} + (p'|_{B'})^*(K_{X'} + \Delta').$$

Since this equation has a unique solution, b_i, e_ℓ give the discrepancies over X'. □

The following example illustrates the role of the divisor F' in (43).

Example 44. Let $X = (x^2 - y^2 + z^2 = t^4) \subset \mathbb{C}^4$, $B = (t = 0)$ and Y the small resolution obtained by blowing up $(x - y = z - t^2 = 0)$. (Here $\Delta = 0$ and E s empty.) Next set $X' = (x^2 - y^2 + z^2 = 0) \subset \mathbb{C}^4$, $B' = (t = 0)$ and Y' the resolution obtained by blowing up $(x = y = z = 0)$.

The 1st order neigborhoods are isomorphic, but on Y' we have an exceptional divisor F'. Note that if we replace t^4 by t^{2m+2}, we have isomorphisms of mth order infinitesimal neighborhoods as well.

Thus we can not tell whether a singularity is terminal or canonical by looking at mth order infinitesimal neighborhoods for some fixed m.

45 (Proof of (40))**.** Write $\Delta = \sum d_j D_j$.

Let $h : Y \to (X, \sum D_j)$ be a log resolution such that $Y \to \mathbb{D}$ is projective. Let $\bar{D}_j \subset Y$ denote the birational transform of D_j, and let $E_i \subset Y$ be the exceptional divisors that dominate $\mathbb{D}$.

By (37.2–4) there are a smooth, pointed algebraic curve (c, C), a flat, proper morphism of algebraic spaces $X^{\mathrm{a}} \to C$ and a projective resolution $h^{\mathrm{a}} : Y^{\mathrm{a}} \to X^{\mathrm{a}}$ such that

$$(45.1) \qquad \left(h^{\mathrm{a}} : (Y^{\mathrm{a}}, \textstyle\sum \bar{D}^{\mathrm{a}}_j + \sum E^{\mathrm{a}}_j) \to X^{\mathrm{a}}\right)_1 \cong \left(h : (Y, \textstyle\sum D_j + \sum E_i) \to X\right)_1.$$

Note that $h^{\mathrm{a}}(E^{\mathrm{a}}_j) \cap X^{\mathrm{a}}_0 = h(E_j) \cap X_0$, thus the E^{a}_j are h^{a}-exceptional. (As in (44), there may be other h^{a}-exceptional divisors.)

By (41), $(Y^{\mathrm{a}}, \sum \bar{D}^{\mathrm{a}}_j + \sum E^{\mathrm{a}}_j)$ is also an snc pair, we are thus in the situation of (43). Since inversion of adjunction holds for the algebraic pair $(X^{\mathrm{a}}, X^{\mathrm{a}}_0 + \Delta^{\mathrm{a}})$, it also holds for $(X, X_0 + \Delta)$.

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