

Singular plane sections and the conics in the Fermat quintic threefold

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Dedicated to Herb Clemens, to whom I am grateful for his mathematical insights, generosity, patience, encouragement and empathy

Abstract: We present explicit equations for the space of conics in the Fermat quintic threefold X , working within the space of plane sections of X with two singular marked points. This space of two-pointed singular plane sections has a birational morphism to the space of bitangent lines to the Fermat quintic threefold, which in its turn is birational to a 625-to-1 cover of \mathbb{P}^4 . We illustrate the use of the resulting equations in identifying special cases of one-dimensional families of conics in X .

Keywords: Quintic threefold, conics, Hilbert scheme.

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1. Introduction

The Fermat quintic threefold X is defined in \mathbb{P}^4 by the polynomial

$$(1.1) \quad F(x_0, x_1, x_2, x_3, x_4) = \sum_{i=0}^{i=4} x_i^5.$$

As such it has a large group of automorphisms isomorphic to $S_5 \times \mathbb{Z}_5^4$, which is at the root of many wonderful properties of this manifold. Some of these properties are shared with, and gain a special significance when looking at the Fermat quintic as a member of the Dwork pencil $X_\psi \hookrightarrow \mathbb{P}^4$, given by defining polynomials

$$F_\psi = \sum_{i=0}^{i=4} x_i^5 - 5\psi \prod_{i=0}^{i=4} x_i.$$

Such is the case of the spaces of rational curves in degrees 1 and 2 lying on these threefolds. It is well known that a generic quintic threefold contains 2875 lines and 609,250 conics. On the other hand, all members of the Dwork pencil contain continuous families of lines and of smooth conics.

The lines in the Dwork pencil have been studied in [AK1], [AK2], [M], ending with the remarkably precise and beautiful presentation in [COGS] and its interpretation in [Z]. We briefly outline the history here. The hyperplane $V(x_0+x_1)$ intersects the Fermat quintic X at a cone of vertex $[1 : -1 : 0 : 0 : 0]$ over the Fermat quintic curve

$$x_2^5 + x_3^5 + x_4^5 = 0.$$

A total of 50 such cones can be obtained by permutations of coordinates along with the action of the group \mathbb{Z}_5^4 by

$$(h, [x_0 : x_1 : x_2 : x_3 : x_4]) \longrightarrow [x_0 : x_1\mu^{h_1} : x_2\mu^{h_2} : x_3\mu^{h_3} : x_4\mu^{h_4}]$$

for $h = (h_i)_i \in \mathbb{Z}_5^4$, where μ is a fifth root of 1. The lines through the vertex lying in these cones contribute to 50 components of the Hilbert scheme $\mathcal{H}_1(X)^{\text{red}}$, which are smooth Fermat curves of genus $g = 6$. In [AK1], Alberto Albano and Sheldon Katz showed that these are precisely all the 1-dimensional components of $\mathcal{H}_1(X)$, and they each come with multiplicity 2.

In addition, there are 375 special points of $\mathcal{H}_1(X)$ with multiplicity 5, each representing an element in the orbit of the line

$$(1.2) \quad [u : v] \longrightarrow [u : -u : v : -v : 0] \quad \text{for} \quad [u : v] \in \mathbb{P}^1$$

under the action of $S_5 \times \mathbb{Z}_5^4$. Thus according with the formula by Herb Clemens and Holger Kley in [CK], the family $\mathcal{H}_1(X)$ contributes to the expected

$$(1.3) \quad 50 \cdot 2 \cdot (2g - 2) + 375 \cdot 5 = 2875$$

number of lines on a generic deformation of X .

Within the Dwork pencil, symmetry under an S_3 subgroup of S_5 led Bert van Geemen to 5,000 lines in each X_ψ . Since this number is larger than the 2875 lines on a generic quintic threefold, this implied the existence of 1-dimensional components of $\mathcal{H}_1(X_\psi)$ (see [AK2]). Indeed, in [M] we showed that in general $\mathcal{H}_1(X_\psi)$ consists of two isomorphic curves C_ψ^\pm of genus 626, together with the 375 isolated points shared with $\mathcal{H}_1(X)$. Of the permutation group S_5 of X_ψ , the even permutations act on each curve C_ψ^\pm separately, while the odd ones swap C_ψ^+ and C_ψ^- . Each member of the family X_ψ also has automorphisms given by the subgroup \mathbb{Z}_5^3 of \mathbb{Z}_5^4 consisting of tuples with zero sum, and an additional \mathbb{Z}_5 acts on the parameter ψ . A new parameter φ can be set to account for quotient by this last \mathbb{Z}_5 and for the two components above.

In [COGS] Philip Candelas, Xenia de la Ossa, Bert van Geemen and Duco van Straten found explicit parametrisations for the curves C_φ^+ and C_φ^- , which allowed them to identify the quotients $C_\varphi^\pm/\mathbb{Z}_5^3$ (up to desingularisations) with the special pencil of plane sextics known as the Wiman pencil. A first member of this family was discovered by Wiman in 1897 ([Wi]), as having an unusual automorphism group isomorphic with S_5 ; the other members of the pencil share the same property and were discovered by Edge in [E].

In [Z], Don Zagier observed that the remarkable coordinate presentation of $C_\varphi^\pm/\mathbb{Z}_5^3$ in [COGS] has a more natural invariant theory based – interpretation based on regarding the ambient space for these curves as $\overline{M}_{0,5}$. The moduli space of rational curves with 5 marked points $\overline{M}_{0,5}$ comes with the natural action of the permutation group S_5 . Invariants under this action were also used in [Z] to recover the degree in \mathbb{P}^4 of the hypersurface covered by all the lines in $(X_\psi)_\psi$ – this invariant was calculated in [M] by intersection of Schubert cycles on the Grassmanian $G(2, 5) = G(\mathbb{P}^1, \mathbb{P}^4)$.

So far in the study of lines, the Dwork pencil has been grounds for some remarkable coincidences and an interplay of methods from deformation to

invariant theory, leading to a surprisingly beautiful and concrete presentation. It is natural to wonder if such features will be discernible for higher degree curves. On the one hand, a nice parametrisation will be harder to find due to the increase in the dimension of the ambient space of degree d rational curves in \mathbb{P}^4 , which is $(5d + 1)$. On the other hand, it is to be expected that the rich symmetry will continue to play a role.

Indeed, the context proposed by Don Zagier in [Z] can be extended to degree d rational curves as follows: over suitably well-balanced open sets, the moduli space of stable maps $\overline{M}_{0,0}(\mathbb{P}^4, d)$ has natural local covers $\overline{M}_{0,0}(\mathbb{P}^4, d, t)$, which depend on a fixed coordinate t on \mathbb{P}^1 and are $(\mathbb{C}^*)^4$ -bundles over open sets in $\overline{M}_{0,5d}$. Concretely, a map

$$\begin{aligned} \gamma : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4 \\ [t : 1] &\longrightarrow \left[c_i \prod_{j=1}^d (t - r_{ij}) \right]_{i \in \{0,1,2,3,4\}}, \end{aligned}$$

has an associated tuple $(\text{Im}(\gamma), (r_{ij})_{i \in \{0,1,2,3,4\}, j \in \{0, \dots, d\}})$ in $\overline{M}_{0,5d}$, and the coefficients $(c_i)_i$ give the $(\mathbb{C}^*)^4$ -fibration. The map γ is determined by $(r_{ij})_{i,j}$ only up to quotient by the action of $(S_d)^{n+1}$, and the curve $\text{Im}(\gamma)$ is determined by γ only after quotient by $\mathbb{P}GL_2$.

Proposition 1.1. *For members of the Dwork pencil, the union*

$$\bigcup_{\psi} \overline{M}_{0,0}(X_{\psi}, d, t)$$

is a 625-1 cover over its image in $\overline{M}_{0,5d}$.

Proof. The argument here was first mentioned for degree 1 maps in [AK2]. Consider the rational map on \mathbb{P}^4

$$f((x_0 : \dots : x_4)) = \frac{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5}{x_0 x_1 x_2 x_3 x_4}$$

The condition that $\text{Im}(\gamma) \subset X_{\psi}$ for some ψ is equivalent to $\gamma^* f$ being constant hence having the same zeroes and poles, with multiplicities. In our case, for distinct roots r_{ij} , the condition reduces to

$$\sum_{i=0}^4 c_i^5 (r - r_{i1})^5 \cdots (r - r_{id})^5 = 0, \quad \forall r \in \{r_{ij}\}_{i \in \{0,1,2,3,4\}, j \in \{1, \dots, d\}}.$$

Thus under suitable conditions on r_{ij} -s, the coefficients $(c_i^5)_i$ can be retrieved by solving a linear system of equations. \square

In principle this could reduce the dimension of the ambient space for our space of degree d curves in X_ψ to $(5d - 3)$. Even when $d = 2$, equations in 7 variables may be less than tractable. Moreover, this is only after taking quotients by $\mathbb{P}GL_2$ and $(S_d)^{n+1}$, and for degree $d > 1$ the two actions do not seem to interact nicely.

Our approach to the space of conics will be different. We take from above the idea of working with curves with marked points, but instead of the marked points being traced by the hyperplanes of coordinates, we will focus on singularities of the associated plane sections (namely, intersection points of the conic with the associated cubic). Working in the spaces of planes with 2 such points allows us to reduce to the study of bitangent lines, and in the case of the Fermat quintic this reduces to a finite cover of \mathbb{P}^4 . We are thus able to characterise the space of conics by four equations. These are rather involved and we have not yet started a systematic study exploring their geometric implications – we show here how they can be applied to identify specific families of conics in the Fermat quintic.

Finally, there are a number of reasons why one would be interested in the space of conics for the Fermat quintic and the Dwork pencil. With Clemens' conjecture in mind, it would be interesting to know how the higher degree curves interact with the loci of curves in degrees 1 and 2. A distant but interesting goal concerns possible connections of these spaces to mirror symmetry of the mirror quintic constructed from the Dwork pencil.

2. Examples of conics in Fermat quintic and the Dwork pencil

The group of permutations S_5 acting on \mathbb{P}^4 leaves each Dwork quintic threefold X_ψ invariant. We have found continuous families of conics in these threefolds by looking at planes kept invariant by certain subgroups of S_5 .

The first two examples exploit S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetries. These families were studied by Songyun Xu in [X].

Example 2.1. S_3 -symmetry: Let $S \subset \mathbb{P}^2 \times \mathbb{P}^1$ denote the 1-parameter family of quintic curves $S_\psi \hookrightarrow \mathbb{P}^2$ given by

$$-5\psi^2 ab^2c^2 - 5\psi a^3bc + a^5 + b^5 + c^5 = 0.$$

Let $P \subset \mathbb{P}^4 \times \mathbb{P}^2$ denote the family of planes $P_{[a:b:c]} \subset \mathbb{P}^4$ given by

$$c(x_0 + x_1 + x_2) = ax_4, \quad cx_3 = bx_4 .$$

For each $([a : b : c], \psi) \in S$, the plane $P_{[a:b:c]}$ intersects the quintic X_ψ in the union of a conic and a cubic of equations:

$$\begin{aligned} \text{conic:} \quad & x_0^2 + x_1^2 + x_2^2 + x_0x_1 + x_0x_2 + x_1x_2 - \psi x_3x_4 = 0 \\ \text{cubic:} \quad & (x_0 + x_1)(x_0 + x_2)(x_1 + x_2) + \psi(x_0 + x_1 + x_2)x_3x_4 = 0. \end{aligned}$$

Indeed, if we denote by e_1, e_2, e_3 the elementary symmetric polynomials in x_0, x_1, x_2 and set

$$G_\psi := x_3^5 + x_4^5 + e_1^5 - 5\psi x_3x_4e_1^3 - 5\psi^2x_3^2x_4^2e_1$$

we get the following equation which allows us to factor F_ψ under the conditions set up above:

$$F_\psi - G_\psi = 5(e_1^2 - e_2 - \psi x_3x_4)(e_3 - e_1e_2 - \psi e_1x_3x_4).$$

We note that on the Fermat quintic, the cubic degenerates to a union of lines, while the conic is non-degenerate. The intersection points of the conic with the lines form the orbit under S_3 of the point $[a : -a : ia : ib : ic]$.

The following example was also found independently by Damiano Testa and Roger Heath-Brown by experimenting over finite fields. The plane sections here decompose into a union of one line and two conics. Johannes Walcher in [Wa] extended this to families of conics on the Dwork pencil by considering the case when one \mathbb{Z}_2 subgroup swaps the two conics in each plane.

Example 2.2. $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry for the Fermat quintic:

For $i = 1, 2$, let u_i denote the elementary symmetric polynomials in x_0, x_1 , let v_i denote the elementary symmetric polynomials in x_3, x_2 . Then

$$F = 5u_1 \left(u_2 - \frac{u_1^2}{2} \right)^2 + 5v_1 \left(v_2 - \frac{v_1^2}{2} \right)^2 + x_4^5 - \frac{u_1^5}{4} - \frac{v_1^5}{4}.$$

It follows that, for the curve in \mathbb{P}^2 given by the equation

$$a^{10} + b^{10} - 4b^5c^5 = 0,$$

the following are equations of conics in X :

$$(2.1) \quad a^2(x_0 + x_1) = b^2(x_2 + x_3), \quad bx_4 = c(x_0 + x_1) \quad \text{and}$$

$$(2.2) \quad b(x_0^2 + x_1^2) \pm ia(x_2^2 + x_3^2) = 0.$$

Under a change of variables $-4bc = d^2$ the parametrising curve can be brought to the nice form $a^{10} + b^{10} + d^{10} = 0$. The plane sections given by equations (2.1) also contain the special locus line identified in equation (1.2).

In [Wa] Johannes Walcher also extends this method to look for conics invariant under \mathbb{Z}_2 generated by $(12)(34) \in S_5$, in planes of equations

$$a_1(x_1 + x_2) + a_2(x_3 + x_4) + x_5 = 0, \quad (x_1 - x_2) + a_3(x_3 - x_4) = 0$$

for suitable a_i -s. He found a family consisting entirely of reducible conics obtained by intersecting the van Geemen families of lines, and evidence of other solutions at the limit of computational capabilities.

3. Singular plane sections in the Fermat quintic

We will now begin our systematic study of the space of conics in the Fermat quintic. We start by setting up the ambient space of singular plane sections.

For the rest of the paper, X will denote the Fermat quintic threefold. We use the notation $G(k, n) = G(\mathbb{P}^{k-1}, \mathbb{P}^{n-1})$ for Grassmannians.

For a vector bundle E we will denote by $\mathbb{P}(E)$ the projective bundle of lines in E .

In line with the examples from the previous chapter, it is convenient to describe a plane in \mathbb{P}^4 by the coefficients of two defining linear equations. To do so we consider the canonical isomorphism $\mathbb{C}^5 \cong (\mathbb{C}^5)^\vee$ for a fixed choice of basis on \mathbb{C}^5 , which induces the isomorphism $G(3, 5) \cong G(2, 5)$ sending $\Lambda \subset \mathbb{C}^5$ to $(\mathbb{C}^5/\Lambda)^\vee \subset (\mathbb{C}^5)^\vee$. Hence a plane Λ given by independent equations

$$\sum_{i=0}^4 a_i x_i = 0 \quad \text{and} \quad \sum_{i=0}^4 b_i x_i = 0 \quad \text{in} \quad \mathbb{P}^4$$

will be described by

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix} \in G(2, 5)$$

with Plücker coordinates $[a_i b_j - a_j b_i]_{0 \leq i < j \leq 4}$ in $\mathbb{P}(\wedge^2(\mathbb{C}^5)^\vee) \cong \mathbb{P}(\wedge^3 \mathbb{C}^5) = \mathbb{P}^9$.

For each $\Lambda \in G(3, 5)$, let X_Λ denote the quintic plane curve section $X_\Lambda = X \cap \Lambda$.

Notation 1. We denote by $SP_k(X)$ the subset of $G(3, 5)$ made of planes for which X_Λ has at least k distinct singular points.

$SP_k(X)$ comes with an incidence variety $U_k \subset G(3, 5) \times X^k$ given by

$$U_k(X) = \{(\Lambda, P_1, \dots, P_k) \in SP_k(X) \times X^k; \quad P_i \in X_\Lambda^s, P_i \neq P_j \quad \forall i \neq j\}.$$

The projection map $q_k : U_k(X) \rightarrow SP_k(X)$ forgets the marked points, and there is a natural group action of the permutation group S_k on $U_k(X)$.

Proposition 3.1. *For $k = 1$, the incidence variety above satisfies $U_1(X) \cong \mathbb{P}(\Omega_X^1)$ where Ω_X^1 is the cotangent sheaf of X . As such, $U_1(X)$ can be thought of as a divisor in $\mathbb{P}(E)$ for $E := \mathcal{O}_X(-1)^{\oplus 5}/\mathcal{O}_X(-5)$.*

Consider the rational map given in Plücker coordinates by

$$\begin{aligned} X \times \mathbb{P}^4 &\dashrightarrow G(2, 5) \\ (P, y) &\dashrightarrow [p_i^4 y_j - p_j^4 y_i]_{0 \leq i < j \leq 4} \end{aligned}$$

where $P = [p_0 : p_1 : p_2 : p_3 : p_4]$ and $y = [y_0 : y_1 : y_2 : y_3 : y_4]$.

Let $q' : \mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \dashrightarrow G(2, 5)$ be the restriction of this map to

$$\mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) = \left\{ (P, y) \in X \times \mathbb{P}^4; \quad \sum_{i=0}^4 p_i y_i = 0 \right\}.$$

Then the map $q_1 : U_1(X) \rightarrow SP_1(X) \subset G(3, 5) \cong G(2, 5)$ can be described as the vertical right hand side arrow in the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) & \xrightarrow{q'} & G(2, 5) \\ \downarrow p' & & \uparrow \\ \mathbb{P}(\Omega_X^1) & \xrightarrow{\cong} & U_1(X) \end{array}$$

The vertical arrow p' is the dominant rational map induced by the surjective homomorphism $\Omega_{\mathbb{P}^4}^1|_X \rightarrow \Omega_X^1$. Both p' and q' have singular locus $X' = \{(P, P^4) \in X \times \mathbb{P}^4\} \cong X$ where $P^4 := [p_0^4 : p_1^4 : p_2^4 : p_3^4 : p_4^4]$.

Proof. Indeed, a point P is singular for a plane section X_Λ if and only if $\Lambda \subset \overline{\mathcal{T}}_{X,P}$ for the projective tangent space $\overline{\mathcal{T}}_{X,P}$ of X at P . Hence such planes Λ are parametrised by $\mathbb{P}(\Omega_X^1)$.

The cotangent sequence for X and the restriction of the Euler sequence for \mathbb{P}^4 give the commutative diagram of vector bundles on X with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}_X(-5) & \longrightarrow & \Omega_{\mathbb{P}^4}^1|_X & \longrightarrow & \Omega_X^1 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_X(-5) & \xrightarrow{\nabla_F} & \mathcal{O}_X(-1)^{\oplus 5} & \longrightarrow & E \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{O}_X & \xlongequal{\quad} & \mathcal{O}_X \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where $\nabla_F(P) = (p_0^4, p_1^4, p_2^4, p_3^4, p_4^4)$ and $E := \mathcal{O}_X(-1)^{\oplus 5}/\mathcal{O}_X(-5)$. Hence $\mathbb{P}(\Omega_X^1)$ is the divisor in $\mathbb{P}(E)$ defined as the zero locus of the global section $\sigma \in H^0(\mathcal{O}_{\mathbb{P}(E)}(1))$ given by

$$\mathcal{O}_{\mathbb{P}(E)} \longrightarrow p^*E \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(1).$$

where $p : \mathbb{P}(E) \rightarrow X$ is the projection map. Similarly, based on the middle vertical sequence we can describe $\mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X)$ as the divisor in $\mathbb{P}(\mathcal{O}_X(-1)^{\oplus 5}) = X \times \mathbb{P}^4$ given concretely as the zero locus of $\sum_{i=0}^4 p_i y_i$.

The horizontal surjections in the commutative diagram above induce a dominant rational map $X \times \mathbb{P}^4 \dashrightarrow \mathbb{P}(E)$ and its restriction $\mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \dashrightarrow \mathbb{P}(\Omega_X^1)$, with singular locus $\mathbb{P}(\mathcal{O}_X(-5)) = X'$. For a pair $(P, y) \in (X \times \mathbb{P}^4) \setminus X'$, the vectors $\nabla_F(P)$ and $(y_0, y_1, y_2, y_3, y_4)$ provide the coefficients for two independent linear equations defining a plane Λ in \mathbb{P}^4 . Thus the rational map $X \times \mathbb{P}^4 \dashrightarrow G(2, 5)$ given in Proposition 3.1 is well defined on $(X \times \mathbb{P}^4) \setminus X'$ and constant on the fibres of $(X \times \mathbb{P}^4) \setminus X' \rightarrow \mathbb{P}(E)$, hence descends to a morphism $\mathbb{P}(E) \rightarrow G(2, 5)$. By restriction, the rational map $q' : \mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \dashrightarrow G(2, 5)$ is well defined on $\mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \setminus X'$ and constant on the fibres of $\mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \setminus X' \rightarrow \mathbb{P}(\Omega_X^1)$, descending to a morphism $\mathbb{P}(\Omega_X^1) \rightarrow G(2, 5)$. Composition with the isomorphisms $U_1(X) \cong \mathbb{P}(\Omega_X^1)$ and $G(2, 5) \cong G(3, 5)$ yields exactly $q_1 : U_1(X) \rightarrow SP_1(X) \subset G(3, 5)$. Indeed, with the notations above we have $\Lambda \subset \overline{\mathcal{T}}_{X,P}$ whenever $(P, y) \in \mathbb{P}(\Omega_{\mathbb{P}^4}^1|_X) \setminus X'$. \square

Notation 2. We denote by $\Delta_{2,2}(X)$ the variety of bitangent lines with marked tangency point:

$$\Delta_{2,2}(X) = \{(l, P, Q) \in G(2, 5) \times X \times X; \quad P \neq Q, \quad l \cdot X \geq 2P + 2Q\}$$

which parametrises lines tangent to X at two (distinct) marked points (compare with [CR]).

For every space Y , we define Δ_Y as the diagonal $\Delta_Y := \text{Im}(Y \hookrightarrow Y \times Y)$.

In the following discussion we will identify \mathbb{Z}_4^4 with the quotient of the group homomorphism $i : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4^5$ given by $i(k_0) = (k_0, k_0, k_0, k_0, k_0)$, and as such we will write elements of \mathbb{Z}_4^4 as equivalence classes $[k] := [k_i]_{0 \leq i \leq 4}$ with $k_i \in \mathbb{Z}_4$.

Consider the action of \mathbb{Z}_4^4 on $\mathbb{P}^4 \times \mathbb{P}^4$

$$(3.1) \quad \sigma : \mathbb{Z}_4^4 \times (\mathbb{P}^4 \times \mathbb{P}^4) \longrightarrow \mathbb{P}^4 \times \mathbb{P}^4$$

$$(3.2) \quad ([k], ([p_i]_i, [q_i]_i)) \longrightarrow ([p_i]_i, [\sqrt{-1}^{k_i} q_i]_i),$$

and let $\sigma_{[k]} = \sigma([k], _)$ for each $[k] = [k_i]_{0 \leq i \leq 4}$ as above.

Let $X_{[k]} := \sigma_{[k]}(\Delta_{\mathbb{P}^4}) \cap (X \times X)$ and $X_\sigma := \mathbb{Z}_4^4 \cdot (\Delta_{\mathbb{P}^4}) \cap (X \times X) = \bigcup_{[k] \in \mathbb{Z}_4^4} X_{[k]}$.

Theorem 3.2. *We have a commutative diagram*

$$\begin{CD} U_2(X) @>\cong>> (\mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)) \setminus \Delta_{\mathbb{P}(\Omega_X^1)} @<< \hookrightarrow \mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1) \\ @V f^\circ VV @VV \downarrow V @VV p V \\ \Delta_{2,2}(X) @<< \hookrightarrow (X \times X) \setminus \Delta_X @<< \hookrightarrow X \times X \end{CD}$$

such that the vertical map f° extends to a surjective morphism $f : \overline{U}_2(X) \rightarrow \overline{\Delta}_{2,2}(X)$, where $\overline{\Delta}_{2,2}(X)$ is the closure of $\Delta_{2,2}(X)$ in $X \times X$ and $\overline{U}_2(X)$ is the closure of $U_2(X)$ in $\mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)$.

For every $[k] \in \mathbb{Z}_4^4$ non-trivial, $X_{[k]}$ is a 2-dimensional subvariety of $\overline{\Delta}_{2,2}(X)$ and f has \mathbb{P}^1 -fibres over all points of $X_{[k]} \setminus X_{[0]}$. Outside $X_\sigma = \bigcup_{[k] \in \mathbb{Z}_4^4} X_{[k]}$, f is an isomorphism.

Proof. The two projections $U_2(X) \rightarrow U_1(X)$ together with the isomorphism in Proposition 3.1 give an open embedding $U_2(X) \rightarrow \mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)$, whose image is exactly $(\mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)) \setminus \Delta_{\mathbb{P}(\Omega_X^1)}$.

Given a plane $\Lambda \in G(3, 5)$, two distinct points P and Q are singular in the plane section X_Λ iff $PQ \subset \Lambda \subseteq \overline{\mathcal{T}}_{X,P} \cap \overline{\mathcal{T}}_{X,Q}$. Hence $(PQ, P, Q) \in \Delta_{2,2}(X)$. This defines f° .

For generic $(PQ, P, Q) \in \Delta_{2,2}(X)$ we have a unique plane Λ as above, namely $\Lambda = \overline{\mathcal{T}}_{X,P} \cap \overline{\mathcal{T}}_{X,Q}$. The special points $(P, Q) \in X \times X$ where $\overline{\mathcal{T}}_{X,P} = \overline{\mathcal{T}}_{X,Q}$ are those for which

$$[\nabla_F(P)] := [p_0^4 : p_1^4 : p_2^4 : p_3^4 : p_4^4] = [q_0^4 : q_1^4 : q_2^4 : q_3^4 : q_4^4] =: [\nabla_F(Q)],$$

namely those pairs in $X \times X$ which are also in $\mathbb{Z}_4^4 \cdot (\Delta_{\mathbb{P}^4})$. In other words, the restriction of f° on the preimage of $\Delta_{2,2}(X) \setminus X_\sigma$ gives a bijective map to $\Delta_{2,2}(X) \setminus X_\sigma$.

$\Delta_{2,2}(X)$ is a closed subset of $(X \times X) \setminus \Delta_X$ given in $(\mathbb{P}^4 \times \mathbb{P}^4) \setminus \Delta_{\mathbb{P}^4}$ by equations

$$(3.3) \quad \sum_{i=0}^4 q_i^h p_i^{5-h} = 0 \text{ for } h \in \{0, 1, 4, 5\}$$

for distinct $P = [p_0 : p_1 : p_2 : p_3 : p_4]$ and $Q = [q_0 : q_1 : q_2 : q_3 : q_4] \in X$, hence its components will have dimension at least 4.

In the case when $Q = [k] \cdot P$ for some $[k] = [\sqrt{-1}^{k_i}]_i \in \mathbb{Z}_4^4$ non-trivial, the equations (3.3) for $h = 0, 4$ and for $h = 1, 5$ coincide, hence $X_{[k]} \hookrightarrow \overline{\Delta}_{2,2}(X)$ and $X_{[k]}$ is isomorphic to the surface $V(\sum_{i=0}^4 p_i^5, \sum_{i=0}^4 p_i^5 \sqrt{-1}^{k_i})$ in X , while $X_{[0]} \cong X$. Thus all components of X_σ have dimension smaller than 4 and hence f° is a dominant map.

With the identifications above, f° is just the restriction to

$$(\mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)) \setminus \Delta_{\mathbb{P}(\Omega_X^1)}$$

of the projection map $p : \mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1) \rightarrow X \times X$ which remembers the two marked points coming from the two copies of $\mathbb{P}(\Omega_X^1) \cong U_1(X)$. As such, f° extends to $f : \overline{U}_2(X) \rightarrow \overline{\Delta}_{2,2}(X)$, which is just the restriction of p on $(\mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1)) \setminus \Delta_{\mathbb{P}(\Omega_X^1)}$. Since $\overline{U}_2(X)$ and $\overline{\Delta}_{2,2}(X)$ are compact and f° is dominant, we get that f is a surjective morphism. In fact, we have seen that f is invertible outside $X_\sigma = \bigcup_{[k] \in \mathbb{Z}_4^4} X_{[k]}$.

For $(P, Q) \in X_{[k]}$ with $[k] \in \mathbb{Z}_4^4$ non-trivial, the fibre $f^{-1}((P, Q))$ is the pencil \mathbb{P}^1 parametrising planes Λ with $PQ \subset \Lambda \subset \overline{\mathcal{T}}_{X,P} = \overline{\mathcal{T}}_{X,Q}$. \square

The surjective map $q_2 : U_2(X) \rightarrow SP_2(X)$ can be extended to $q_2 : \overline{U}_2(X) \rightarrow \overline{SP}_2(X)$ by restricting the projection $q : \mathbb{P}(\Omega_X^1) \times_{G(3,5)} \mathbb{P}(\Omega_X^1) \rightarrow$

$G(3, 5)$. The same map can also be described as the composition

$$\overline{U}_2(X) \rightarrow X \times X \dashrightarrow G(3, 5),$$

where the right side map is the restriction of the rational map $g : \mathbb{P}^4 \times \mathbb{P}^4 \dashrightarrow G(3, 5)$ which via the isomorphism $G(3, 5) \cong G(2, 5)$ is defined in Plücker coordinates by

$$([p_0 : p_1 : p_2 : p_3 : p_4], [q_0 : q_1 : q_2 : q_3 : q_4]) \xrightarrow{g} [p_i^4 q_j^4 - p_j^4 q_i^4]_{0 \leq i < j \leq 4}.$$

We note that this map can be resolved by blowing up along the locus $\mathbb{Z}_4 \cdot \Delta_{\mathbb{P}^4}$ discussed above.

Coordinates for $\overline{\Delta}_{2,2}(X)$

Notation 3. To set up our new coordinates, consider the rational map $\phi : \overline{\Delta}_{2,2}(X) \dashrightarrow \mathbb{P}^4$ obtained as a composition $\phi = u \circ \pi$ between the projection $\pi : \overline{\Delta}_{2,2}(X) \rightarrow \mathbb{P}^4 \times \mathbb{P}^4$ and the rational map

$$u : \mathbb{P}^4 \times \mathbb{P}^4 \dashrightarrow \mathbb{P}^4, \quad u(P, Q) = U := [u_0 : u_1 : u_2 : u_3 : u_4]$$

where $u_i = q_i/p_i$ for all i .

Consider the involution $\tau : \overline{\Delta}_{2,2}(X) \rightarrow \overline{\Delta}_{2,2}(X)$ which swaps P and Q . There is a corresponding map $\iota : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ such that the following diagram is commutative

$$\begin{array}{ccc} \overline{\Delta}_{2,2}(X) & \xrightarrow{\tau} & \overline{\Delta}_{2,2}(X) \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{P}^4 & \xrightarrow{\iota} & \mathbb{P}^4 \end{array}$$

The map $\iota : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ is the Cremona transformation given on an open set by $u_i \rightarrow 1/u_i$.

We will now prepare some notations for more concrete coordinate calculations:

Notation 4. The discriminant $\delta := \prod_{j>k}(u_j - u_k)$ and elementary symmetric polynomials

$$e_0 = 1, \quad e_1 = \sum_{j=0}^4 u_j, \quad e_2 = \sum_{4 \geq j > k \geq 0} u_j u_k, \quad \dots, \quad e_5 = \prod_{j=0}^4 u_j$$

will be useful in our computations. We will also work with the S_4 -symmetric polynomials

$$e_0(i) = 1, \quad e_1(i) = \sum_{j \in \{0, \dots, 4\} \setminus \{i\}} u_j, \quad e_2(i) = \sum_{j, k \in \{0, \dots, 4\} \setminus \{i\}; j > k} u_j u_k,$$

etc. and the partial discriminants

$$d_i := \prod_{j, k \in \{0, \dots, 4\} \setminus \{i\}; j > k} (u_j - u_k).$$

Let $\varphi : X \rightarrow H \cong \mathbb{P}^3$ be the quotient map for the action of $(\mathbb{Z}_5)^4$ on X ,

$$\varphi([x_0 : x_1 : x_2 : x_3 : x_4]) = [x_0^5 : x_1^5 : x_2^5 : x_3^5 : x_4^5]$$

which maps X to the hyperplane $H = V(\sum_{i=0}^4 y_i)$ in $\text{Proj} \mathbb{C}[y_0, y_1, y_2, y_3, y_4]$. We note that φ is ramified exactly along the coordinate hyperplanes $H_i = V(x_i)$.

Theorem 3.3. *The map $\phi = u \circ \pi : \overline{\Delta}_{2,2}(X) \dashrightarrow \mathbb{P}^4$ is generically 625 : 1, unramified over its regular points. There is a commutative diagram*

$$\begin{array}{ccc} \overline{\Delta}_{2,2}(X) & \xrightarrow{\pi_1} & X \\ \phi \downarrow & & \downarrow \varphi \\ \mathbb{P}^4 & \xrightarrow{m} & H \end{array}$$

where for $U = [u_0 : u_1 : u_2 : u_3 : u_4]$, we have $m(U) = [M_i(U)]_{i \in \{0, \dots, 4\}}$ given by

$$M_i(U) := (-1)^i d_i n_i, \quad \text{where} \quad n_i := e_2(i)^2 - e_1(i) e_3(i).$$

Along the horizontal rows have $\pi_1 \circ \tau = \pi_2$ and $m \circ \iota(U) = [M_i(U) u_i^5]_{i \in \{0, \dots, 4\}}$.

The base locus of m is made of 25 planes $V(u_i - u_j, u_k - u_l)$ for $i \neq j, k \neq l$ and $\{i, j\} \neq \{k, l\}$ (of which 10 correspond to the case $|\{i, j\} \cap \{k, l\}| = 1$ and 15 to the case $\{i, j\} \cap \{k, l\} = \emptyset$), 10 irreducible quartic surfaces $V(u_i - u_j, n_i)$ for $i > j$, and a degree 6 irreducible component $V(e_2, e_3)$.

Proof. Let $V \cong (\mathbb{C}^*)^4$ and W denote the main open sets in \mathbb{P}^4 and $G(3, 5)$ respectively, obtained by removing the coordinate hyperplanes (for the Plücker coordinates in the case of $G(3, 5)$).

In projective coordinates $P = [p_0 : p_1 : p_2 : p_3 : p_4]$ and $Q = [q_0 : q_1 : q_2 : q_3 : q_4]$ and

$$PQ = \{tP + sQ = [tp_i + sq_i]_{i \in \{0,1,2,3,4\}}; \quad [t : s] \in \mathbb{P}^1\},$$

the condition $(PQ, P, Q) \in \Delta_{2,2}(X)$ translates into the four equations (3.3). With the new coordinates introduced in Notation 3, equations (3.3) are written as

$$(3.4) \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ u_0 & u_1 & u_2 & u_3 & u_4 \\ u_0^4 & u_1^4 & u_2^4 & u_3^4 & u_4^4 \\ u_0^5 & u_1^5 & u_2^5 & u_3^5 & u_4^5 \end{pmatrix} \begin{pmatrix} p_0^5 \\ p_1^5 \\ p_2^5 \\ p_3^5 \\ p_4^5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which admits the solution $p_i^5 = M_i(U)$, for $M_i(U)$ defined as in the statement of the Theorem. This defines the map m . The base locus of m is $V(\{M_i(U)\}_{i \in \{0, \dots, 4\}})$, which consists of 25 planes $V(u_i - u_j, u_k - u_l)$ for $i < j \leq k < l$, 10 irreducible quartic surfaces $V(u_i - u_j, n_i)$ for $i > j$, and a degree 6 irreducible component $V(e_2, e_3)$.

Indeed, $(u_i - u_j)$ is a factor of three sections $M_k(U) \in H^0(\mathcal{O}_{\mathbb{P}^4}(10))$, and also of $M_i(U) - M_j(U)$. This explains the first two types of components.

The last component is in fact $V(\{n_i\}_{i \in \{0, \dots, 4\}}) = V(e_2, e_3)$. Indeed, on the one hand

$$e_2 = e_2(i) + u_i e_1(i) \quad \text{and} \quad e_3 = u_i e_2(i) + e_3(i)$$

so $V(e_2, e_3) \subseteq V(n_i)$. In addition, within the ideal $(n_i)_{i \in \{0, \dots, 4\}}$ we have the identities

$$n_i(u_j - u_k) + n_j(u_k - u_i) + n_k(u_i - u_j) = (u_j - u_k)(u_k - u_i)(u_i - u_j)e_2$$

for all triples (i, j, k) while the following symmetric sums satisfy

$$\sum_{i=0}^4 n_i = 3e_2^2 - 4e_1e_3, \quad \sum_{i=0}^4 u_i n_i = e_2e_3, \quad \sum_{i=0}^4 u_i^2 n_i = 3e_3^2 - 4e_4e_2,$$

hence $V(e_2, e_3) = V(\{n_i\}_{i \in \{0, \dots, 4\}})$.

The primary decomposition of the ideal $(\{M_i(U)\}_{i \in \{0, \dots, 4\}})$, checked with the help of Macaulay2, shows that all the components of the base locus of m are accounted for by the list stated in the Theorem. There are no embedded

components, hence the map m can be resolved by successive blow-ups of \mathbb{P}^4 along the loci made of these components and their intersections, starting with the smallest dimension one. Blow-ups of $V(\{M_i(U)\}_{i \in \{0, \dots, 4\}})$ are part of the construction of $\overline{\mathcal{M}}_{0,7}$ as discussed below. The other components of the singular locus above need to be considered separately.

We note that for the involution ι

$$e_k(\iota(U))e_5 = e_{5-k}(U), \quad d_i(\iota(U))e_5^3 = d_i(U)u_i^3, \quad n_i(\iota(U))e_5^2 = n_i(U)u_i^2$$

hence $M_i(\iota(U))e_5^5 = M_i(U)u_i^5$ which is indeed the descent of the relation $\pi_1 \circ \tau = \pi_2$. \square

Resolving the maps ϕ , τ and m

Let $H_i = V(x_i)$ denote the coordinate hyperplanes in \mathbb{P}^4 . To resolve both ι and φ simultaneously, it is enough to blow-up \mathbb{P}^4 along $\bigcap_{i \in J} H_i$ for all $J \subset \{0, 1, 2, 3, 4\}$ with $\#J > 1$, done successively from smallest to largest strata. The resulting space is a Losev-Manin moduli space of rational curves with marked points. To partially resolve m , we can consider $\overline{\mathcal{M}}_{0,7}$, obtained by successive blow-ups of \mathbb{P}^4 along $\bigcap_{\alpha \in J} H_\alpha$ for $J \subset \{0, 1, 2, 3, 4\} \cup \{(i, j); 0 \leq i < j \leq 4\}$, where $H_{(i,j)} = V(x_i - x_j)$. Let $\tilde{\Delta}_{2,2}(X)$ denote the strict transform of $\overline{\Delta}_{2,2}(X)$ in $\overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{M}}_{0,7}$. We then obtain a lift of the earlier diagram to

$$\begin{array}{ccc} \tilde{\Delta}_{2,2}(X) & \xrightarrow{\tilde{\tau}} & \tilde{\Delta}_{2,2}(X) \\ \tilde{\phi} \downarrow & & \tilde{\phi} \downarrow \\ \overline{\mathcal{M}}_{0,7} & \xrightarrow{\tilde{\iota}} & \overline{\mathcal{M}}_{0,7}. \end{array}$$

This also partially resolves the map m , though there remains the locus at the intersection of n_i -s which would need a separate blow-up.

4. Parametrising conics with high contact at singular points of plane sections of the Fermat quintic

Consider the singular plane sections of the Fermat quintic parametrised by $\overline{SP}_2(X)$, and their incidence variety $\overline{U}_2(X)$. We fix an element $(\Lambda, P, Q) \in U_2(X)$. For the next steps we assume $(PQ, P, Q) \in \Delta_{2,2}(X) \setminus X_\sigma$, though in the last sections we will discuss how the methods developed here can be adapted to points in X_σ .

We would like to find conditions for a conic $C \hookrightarrow \Lambda$ to be a component of X_Λ . We will again approach this problem gradually.

We write the fixed points as $P = [p_0 : p_1 : p_2 : p_3 : p_4]$, $Q = [q_0 : q_1 : q_2 : q_3 : q_4]$ and $\phi(PQ, P, Q) = U = [u_0 : u_1 : u_2 : u_3 : u_4]$. (Even though it is convenient to use homogeneous coordinate notations, we fix affine representatives for the above). In the previous section we have seen how, under the condition $(PQ, P, Q) \in \Delta_{2,2}(X)$, the points P and Q can be recovered (though only up to the action of the group \mathbb{Z}_5^4) from the data $U = [u_i]_i$, via the formulae $[p_i^5]_{i \in \{0, \dots, 4\}} := m(U) = [M_i(U)]_{i \in \{0, \dots, 4\}}$ and $q_i^5 := M_i(U)u_i^5$ given in Theorem 3.3. Furthermore, if the section X_Λ contains a conic, then in general we expect to find other singularities $S \in X_\Lambda^s$, so that $(\Lambda, P, S) \in U_2(X)$. Then similarly for $S = [s_i]_i$ we could consider new coordinates $v_i = s_i/p_i$, such that up to the action of the group \mathbb{Z}_5^4 , both P and S can be written in terms of $V = [v_i]_i$. The search for such S motivates us to perform a change of coordinates as follows:

Notation 5. Consider the map

$$\nu_P : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4, \quad \nu_P([v_i]_{i \in \{0,1,2,3,4\}}) = [p_i v_i]_{i \in \{0,1,2,3,4\}}.$$

Note that here the values $\{p_i\}_i$ are fixed. For the next steps we will assume that $P \in V = \mathbb{P}^4 \setminus \bigcup_i H_i$ so that ν_P is an isomorphism. We will work on the domain of ν_P , namely on $\mathbb{P}^4 = \text{Proj}\mathbb{C}[v_0, v_1, v_2, v_3, v_4]$, and describe X , Λ as well as P , Q via the change of coordinates $\nu_P^{-1}([x_i]_i) = [x_i/p_i]_i$. The coordinates x_i are pulled-back as $\nu_P^*(x_i) = p_i v_i$, and P is identified with $\nu_P^{-1}(P) = [1] := [1 : 1 : 1 : 1 : 1]$, and Q with $\nu_P^{-1}(Q) = [U] = [u_0 : u_1 : u_2 : u_3 : u_4]$.

Assume $u_i \neq u_j$ for $i \neq j$. Let $F(x) = \sum_{i=0}^4 x_i^5 \in H^0(\mathcal{O}_{\mathbb{P}^4}(5))$. We identify the Fermat quintic $X = V(F)$ with $\nu_P^{-1}(X) = V(\nu_P^*(F))$ where in the new coordinates

$$(4.1) \quad \nu_P^*(F)(v) = \sum_{i=0}^4 p_i^5 v_i^5 = \sum_{i=0}^4 M_i(U) v_i^5.$$

Similarly, for the plane Λ , defined in the proof of Theorem 3.2 by

$$\Lambda = \overline{\mathcal{T}}_{X,P} \cap \overline{\mathcal{T}}_{X,Q} = V\left(\sum_{i=0}^4 p_i^4 x_i, \sum_{i=0}^4 q_i^4 x_i\right),$$

we will now identify Λ with $\nu_P^{-1}(\Lambda)$ given by equations

$$(4.2) \quad \nu_P^* \left(\sum_{i=0}^4 p_i^4 x_i \right) = \sum_{i=0}^4 M_i(U) v_i = 0 \text{ and}$$

$$(4.3) \quad \nu_P^* \left(\sum_{i=0}^4 q_i^4 x_i \right) = \sum_{i=0}^4 M_i(U) u_i^4 v_i = 0,$$

Hence the new coordinates allow us to write all important equations based on the fixed data $[U] = [u_i]_i$, in which terms we already know two fixed singular points $P, Q \in X_\Lambda$.

We also note that the involution τ swapping P and Q generates the following changes:

$$(4.4) \quad U = [u_i]_i \rightarrow \iota(U) = \left[\frac{1}{u_i} \right]_i,$$

$$(4.5) \quad M_i(U) \rightarrow M_i(\iota(U)) = \frac{M_i(U) u_i^5}{e_5^5},$$

$$(4.6) \quad v_i \rightarrow \frac{v_i}{u_i}$$

for $i \in \{0, 1, 2, 3, 4\}$.

We will now start looking for conditions that insure

$$C \cdot X_\Lambda \geq 3P + 3Q \quad \text{in } \Lambda.$$

Notation 6. We now consider the conics in \mathbb{P}^4 given by

$$(4.7) \quad \gamma_b : \mathbb{P}^1 \rightarrow \mathbb{P}^4, \quad \gamma_b([t : s]) = [t^2 + b_i t s + u_i b_5^2 s^2]_i,$$

so that $\gamma_b([1 : 0]) = P$ and $\gamma_b([0 : 1]) = Q$, and depending on the parameter $b = (b_0, b_1, b_2, b_3, b_4, b_5) \in \mathbb{C}^6$. We will denote $C_{[b]} = \text{Im}(\gamma_b)$.

The parameter b was chosen so that under the action of the remaining automorphism group $\mathbb{C}^* = \text{Spec} \mathbb{C}[\lambda, \frac{1}{\lambda}] \subset \mathbb{P}GL_2$,

$$\gamma_b([\lambda t : s]) = \left[t^2 + \frac{b_i}{\lambda} t s + u_i \left(\frac{b_5}{\lambda} \right)^2 s^2 \right]_i = \gamma_{b/\lambda}([t : s]),$$

so that we have a \mathbb{P}^5 -family of conics $C_{[b]}$ with $[b] \in \mathbb{P}^5$. Let $o = (0, 0, 0, 0, 0, 1)$.

Proposition 4.1. *Consider the $(\mathbb{P}^5 \setminus \{[o]\})$ -family of conics passing through P and Q with the parametrisation (4.7) above. Let $B := [b_0 : b_1 : b_2 : b_3 : b_4]$. Then the condition*

$$C_b \subset \Lambda \text{ and } C_b \cdot X_\Lambda \geq 3P + 3Q \text{ in } \Lambda$$

is equivalent to $B \in \Lambda \cap \Gamma_U \cap \Gamma_{\iota(U)}$, where Γ_U and $\Gamma_{\iota(U)}$ are the quadric surfaces in \mathbb{P}^4 given by equations

$$(4.8) \quad \sum_{i=0}^4 M_i(U)v_i^2 = 0 \quad \text{for} \quad \Gamma_U \quad \text{and}$$

$$(4.9) \quad \sum_{i=0}^4 M_i(U)u_i^3v_i^2 = 0 \quad \text{for} \quad \Gamma_{\iota(U)}.$$

Proof. We have chosen the conic parametrisation in (4.7) so that B represents its tangent direction at $0 = [0 : 1]$ and $\infty = [1 : 0]$. The incidence conditions at P and Q above are equivalent to B being in the tangent cones to X_Λ at P and Q . The equations (4.8) and (4.9) above define the two tangent cones. In coordinates, we can write

$$f_b(t) = \nu_P^*(F)(\gamma_b([t : 1])) = \sum_{i=0}^4 M_i(U)(t^2 + b_i t s + u_i b_5^2 s^2)^5.$$

Then the conditions for $C_b \subset \Lambda$ and $C_b \cdot X_\Lambda \geq 3P + 3Q$ in Λ can be calculated as

$$\left. \frac{\partial^k f_b}{\partial t^k} \right|_{t=0} = 0 \quad \text{for } k \in \{0, 1, 2, 8, 9, 10\}.$$

Conditions for $(10 - k)$ are equivalent to $\left. \frac{\partial^k f_b}{\partial t^k} \right|_{t=\infty} = 0$ and can be obtained from conditions for k by extending the transformation ι :

$$\begin{aligned} U = [u_i]_i &\rightarrow \iota(U) = \left[\frac{1}{u_i} \right]_i, \\ M_i(U) &\rightarrow M_i(\iota(U)) = \frac{M_i(U)u_i^5}{e_5^5}, \\ b_i &\rightarrow \frac{b_i}{u_i} \text{ for } i \in \{0, \dots, 4\}. \end{aligned}$$

We obtain the following set of conditions for $k \in \{0, 1, 2\}$ at $t = 0$ and $t = \infty$:

$$\begin{aligned}
 k = 0: \quad & \sum_{i=0}^4 M_i(U) = 0 & \text{and} & \quad \sum_{i=0}^4 M_i(U) u_i^5 = 0, \\
 k = 1: \quad & \sum_{i=0}^4 M_i(U) b_i = 0 & \text{and} & \quad \sum_{i=0}^4 M_i(U) u_i^4 b_i = 0, \\
 k = 2: \quad & \sum_{i=0}^4 M_i(U) b_i^2 = 0 & \text{and} & \quad \sum_{i=0}^4 M_i(U) u_i^3 b_i^2 = 0.
 \end{aligned}$$

The identities for $k = 0$ correspond to $P, Q \in X$ and impose no conditions on the family of conics. The identities for $k = 1$ are equivalent to $B := [b_0 : b_1 : b_2 : b_3 : b_4] \in \Lambda$ and the cases $k = 2$ to $B \in \Gamma_U \cap \Gamma_{\iota(U)}$, the intersection of the two tangent cones. \square

Next we will choose a third point $R \in \Lambda \setminus \{PQ\}$ and thus set a system of coordinates $[x : y : z]$ for any point $V = [xP + yQ + zR] \in \Lambda$.

Lemma 4.2. *In general, the conic $\Gamma_U \cap \Lambda$ is a union of two lines intersecting at P while $\Gamma_{\iota(U)} \cap \Lambda$ is a union of two lines intersecting at Q .*

Proof. Indeed, for every $b \in \mathbb{P}^5$ whose projection on $V(b_5) \cong \mathbb{P}^4$ is $B \in \Gamma_U \cap \Lambda$, the indexed conic C_b must satisfy

$$C_b \cdot X_\Lambda \geq 3P + 2Q \quad \text{in } \Lambda.$$

Since X_Λ is singular at both P and Q , in general we expect that the tangent cone to X_Λ at P is a union of lines l_P^1 and l_P^2 , and similarly for the tangent cone at Q . Then each conic C_b must pass through Q and be tangent to one of l_P^1 or l_P^2 . For each l_P^i , there is a \mathbb{P}^2 -family of such C_b , with free parameter b_5 and B varying along l_P^i . Similarly for $\Gamma_{\iota(U)} \cap \Lambda$.

We now write the equations of the conics $\Gamma_U \cap \Lambda$ and $\Gamma_{\iota(U)} \cap \Lambda$ in coordinates $[x : y : z]$ on Λ . Let $V = [xP + yQ + zR] = [v_i]_i$, with $v_i = x + yu_i + zl_i$. The quadratic polynomial defining $\Gamma_U \cap \Lambda$ is

$$\begin{aligned}
 (4.10) \quad & Q_U(V) = \sum_{i=0}^4 M_i(U) (x + yu_i + zl_i)^2 \quad \text{hence} \\
 & = y^2 \left(\sum_{i=0}^4 M_i(U) u_i^2 \right) + 2yz \left(\sum_{i=0}^4 M_i(U) u_i l_i \right) + z^2 \left(\sum_{i=0}^4 M_i(U) l_i^2 \right)
 \end{aligned}$$

since the missing terms came with coefficients

$$\sum_{i=0}^4 M_i(U) = \sum_{i=0}^4 M_i(U)u_i = \sum_{i=0}^4 M_i(U)l_i = 0.$$

Similarly for $\Gamma_{\iota(U)} \cap \Lambda$ we have

$$\begin{aligned} (4.11) \quad Q_{\iota(U)}(V) &= \sum_{i=0}^4 M_i(U)u_i^3(x + yu_i + zl_i)^2 \\ &= x^2 \left(\sum_{i=0}^4 M_i(U)u_i^3 \right) + 2xz \left(\sum_{i=0}^4 M_i(U)u_i^3 l_i \right) + z^2 \left(\sum_{i=0}^4 M_i(U)u_i^3 l_i^2 \right), \end{aligned}$$

since the missing terms have coefficients

$$\sum_{i=0}^4 M_i(U)u_i^5 = \sum_{i=0}^4 M_i(U)u_i^4 = \sum_{i=0}^4 M_i(U)u_i^4 l_i = 0$$

Thus each of $\Gamma_U \cap \Lambda$ and $\Gamma_{\iota(U)} \cap \Lambda$ is a union of two lines in Λ , passing through P and Q respectively. □

Lemma 4.3. *Consider quadratic polynomials associated to the scaled discriminants of (4.10) and (4.11), respectively:*

$$\begin{aligned} \Delta(V) &= \left(\sum_{i=0}^4 M_i(U)u_i v_i \right)^2 - \left(\sum_{i=0}^4 M_i(U)u_i^2 \right) \left(\sum_{i=0}^4 M_i(U)v_i^2 \right), \text{ and} \\ \Delta'(V) &= \left(\sum_{i=0}^4 M_i(U)u_i^3 v_i \right)^2 - \left(\sum_{i=0}^4 M_i(U)u_i^3 \right) \left(\sum_{i=0}^4 M_i(U)u_i^3 v_i^2 \right) \end{aligned}$$

Δ and Δ' are exchanged under the existing involution (up to a factor of e_5^{10}), and

$$\Delta(xP + yQ + zR) = \Delta(R)z^2, \quad \Delta'(xP + yQ + zR) = \Delta'(R)z^2,$$

for all $R \in \Lambda$.

Proof. Exchange under involution (up to scalar) can be checked directly. Evaluating the expressions above at $V = P = [1 : 1 : 1 : 1 : 1]$, and $V = Q = [u_i]_i$ directly gives $\Delta(Q) = 0$ and $\Delta'(P) = 0$. On the other hand, as $p_i^5 = M_i(U)$

form the solution in (3.4),

$$\sum_{i=0}^4 M_i(U) = \sum_{i=0}^4 M_i(U)u_i = \sum_{i=0}^4 M_i(U)u_i^4 = \sum_{i=0}^4 M_i(U)u_i^5 = 0$$

hence $\Delta(P) = 0$ and $\Delta'(Q) = 0$. For general $v_i = x + yu_i + zl_i$, we get

$$\begin{aligned} \Delta(V) &= -\sum_{i>j}^4 M_i(U)M_j(U)(x(u_i - u_j) + z(u_il_j - u_jl_i))^2 \\ &= -2xz \sum_{i>j}^4 M_i(U)M_j(U)(u_i - u_j)(u_il_j - u_jl_i) + z^2\Delta(R), \quad \text{and} \\ &\sum_{i>j}^4 M_i(U)M_j(U)(u_i - u_j)(u_il_j - u_jl_i) \\ &= \sum_{i,j}^4 M_i(U)M_j(U)u_iu_jl_i - \sum_{i,j}^4 M_i(U)M_j(U)u_i^2l_j \\ &= \left(\sum_i^4 M_i(U)u_il_i\right) \left(\sum_j^4 M_j(U)u_j\right) - \left(\sum_{i,j}^4 M_i(U)u_i^2\right) \left(\sum_j^4 M_j(U)l_j\right) \\ &= 0 \end{aligned}$$

which shows that $\Delta(V) = \Delta(R)z^2$. Similarly for Δ' . \square

The above Lemma indicates that any choice of point R in $\Lambda \setminus PQ$ is equally suited for factoring the quadratic polynomials (4.10) and (4.11).

Notation 7. Let $S_{mn}(U)$ be the symmetric polynomials given by

$$(4.12) \quad \sum_{i=0}^4 M_i(U)u_i^m l_i^n = \delta(U)S_{mn}(U).$$

These are well defined as the sums $T_{mn}(U) := \sum_{i=0}^4 M_i(U)u_i^m l_i^n$ are symmetric polynomials in $\mathbb{C}[u_i]_i$ which are multiples of the discriminant $\delta(U)$. Indeed, $T_{mn}(U)|_{u_j=u_k} = 0$ because in the case $u_j = u_k$ we have $M_i(U) = (-1)^i d_i n_i = 0$ for all $i \neq j, k$ while $M_j(U) = -M_k(U)$ and hence $l_j = l_k$ by equations (4.2) applied to $R = [l_i]_i \in \Lambda$ (in the case $M_j(U) = -M_k(U) \neq 0$).

Recall that Δ and Δ' are swapped under the involution (up to a factor of e_5^{10}). Up to a factor of e_5^5 , the involution τ described in (4.4) swaps $S_{m,n}(U)$ with $S_{(5-m-n),n}(\iota(U))$.

Below are the relevant values of $S_{m,0}$ in terms of elementary symmetric polynomials:

$$(4.13) \quad S_{00} = 0, \quad S_{10} = 0, \quad S_{20} = e_2, \quad S_{30} = e_3, \quad S_{40} = 0, \quad S_{50} = 0.$$

Indeed, the null terms are due to equation (3.4) with solutions $p_i^5 = M_i(U)$, where $(-1)^i M_i(U)$ are the 4×4 minors of the associated matrix. By the same token,

$$T_{20} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ u_0 & u_1 & u_2 & u_3 & u_4 \\ u_0^2 & u_1^2 & u_2^2 & u_3^2 & u_4^2 \\ u_0^4 & u_1^4 & u_2^4 & u_3^4 & u_4^4 \\ u_0^5 & u_1^5 & u_2^5 & u_3^5 & u_4^5 \end{vmatrix} \quad \text{and} \quad T_{30} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ u_0 & u_1 & u_2 & u_3 & u_4 \\ u_0^3 & u_1^3 & u_2^3 & u_3^3 & u_4^3 \\ u_0^4 & u_1^4 & u_2^4 & u_3^4 & u_4^4 \\ u_0^5 & u_1^5 & u_2^5 & u_3^5 & u_4^5 \end{vmatrix}$$

are 5×5 minors of a 6×6 Vandermonde matrix $V = (u_i^j)_{0 \leq i, j \leq 5}$. We note that T_{23} and $-T_{30}$ are entries in the adjoint V_{adj} of the Vandermonde matrix, which can be calculated based on the matrix multiplication formula $V_{\text{adj}}V = \prod_{0 \leq i < j \leq 5} (u_j - u_i) \mathbf{I}_6$. Hence T_{23} and $-T_{30}$ are the coefficients of u^3 and u^2 in the polynomial

$$p(u) = \delta(U) \prod_{i=0}^4 (u - u_i) = \delta(U) (u^5 - e_1 u^4 + e_2 u^3 - e_3 u^2 + e_4 u - e_5).$$

Consider the extension of algebras $\mathbb{C}[u_i]_i \hookrightarrow \mathbb{C}[u_i]_i[\sqrt{\Delta(R)}, \sqrt{\Delta'(R)}]$. From now on, we will work in the associated 4-to-1 cover of \mathbb{P}^4 .

Corollary 4.4. *Consider the \mathbb{P}^5 -family of conics passing through P and Q given by parametric equations (4.7). Let $B := [b_0 : b_1 : b_2 : b_3 : b_4]$. Then within this family there are in general four \mathbb{P}^1 -families of conics which satisfy*

$$C_b \subset \Lambda \text{ and } C_b \cdot X_\Lambda \geq 3P + 3Q \text{ in } \Lambda$$

with $b = [B : b_5]$ where $B = [\alpha_\pm + \beta_\pm u_i + l_i]_{i \in \{0,1,2,3,4\}}$ and $[\alpha_\pm : 1]$ are the solutions of the quadratic equation (4.10) and $[\beta_\pm : 1]$ are the solutions of the quadratic equation (4.11) for a suitable choice of point $R = [l_i]_i \in \Lambda$. Namely, if $S_{20} \neq 0$ and $S_{30} \neq 0$ then

$$S_{20}\beta_\pm = -S_{11} \pm \sqrt{\Delta(R)}, \quad S_{30}\alpha_\pm = -S_{31} \pm \sqrt{\Delta'(R)},$$

$$\Delta(R) = S_{11}^2 - S_{20}S_{02} \text{ and } \Delta'(R) = S_{31}^2 - S_{30}S_{32}.$$

Thus we get 4 families of conics exactly when $S_{20} \neq 0$, $S_{30} \neq 0$ and $\Delta(R) \neq 0$, $\Delta'(R) \neq 0$.

5. Equations for the conic and cubic curves in suitable singular plane sections of the Fermat quintic

Assume that for a chosen plane Λ with marked points P and Q , the section X_Λ decomposes into a conic C and a cubic C' . In this section we will calculate the equations of C and C' in Λ , in terms of the coordinates $[x : y : z]$ on Λ determined by the coordinate simplex PQR .

The conic C will be a member in one of the pencils described in Corollary 4.4. Let this pencil be parametrised by $b = [B : b_5]$ where $B = \alpha P + \beta Q + R = (\alpha + \beta u_i + l_i)_{i \in \{0,1,2,3,4\}}$ for a choice of roots α and β for the quadratic equations in (4.10) and (4.11) respectively. For simplicity we will denote $\lambda = b_5^2$.

In the chosen coordinates on Λ for which $P = [1 : 0 : 0]$, $Q = [0 : 1 : 0]$ and $R = [0 : 0 : 1]$, the parametric equations (4.7) translate to

$$\begin{aligned} \gamma_b : \mathbb{P}^1 &\rightarrow \mathbb{P}^2, & \gamma_b([t : s]) &= [t^2 P + ts B + \lambda s^2 Q] = \\ & & &= [t^2 + \alpha ts : \beta ts + \lambda s^2 : ts]. \end{aligned}$$

After the change of coordinates on $\Lambda \cong \mathbb{P}^2$

$$[x : y : z] \rightarrow \left[x - \alpha z : z : \frac{1}{\lambda}(y - \beta z) \right]$$

the parametrisation of C becomes the Veronese embedding, hence C has defining polynomial

$$(5.1) \quad f(x, y, z) = (x - \alpha z)(y - \beta z) - \lambda z^2 = (\alpha\beta - \lambda)z^2 - (\beta x + \alpha y)z + xy.$$

Thinking of this as a polynomial in z , we denote the coefficients

$$f_2 = d := \alpha\beta - \lambda, \quad f_1 = -(\beta x + \alpha y), \quad f_0 = xy.$$

Equations for the paired cubic

Assume $fg = F_\Lambda$ where F_Λ is the polynomial of the quintic section X_Λ . We will write the cubic polynomial $g(x, y, z) = \sum_{k=0}^3 g_k z^k$, with $g_k(x, y)$ homogeneous polynomial of degree $(3 - k)$. We also write the equation of the quintic

section X_Λ , noting the common factor $\delta(U)$ which comes from (4.12) above.

$$\begin{aligned} \delta(U)F_\Lambda(x, y, z) &= \sum_{i=0}^4 M_i(U)(x + u_i y + l_i z)^5 \\ &= \delta(U) \sum_{j+l+k=5} \frac{5!}{j!l!k!} S_{jk}(U)x^l y^j z^k \end{aligned}$$

and based on the initial values of $S_{jk}(U)$ as listed in (4.13) above.

$$\begin{aligned} F_\Lambda(x, y, z) &= 10x^2y^2(S_{20}x + S_{30}y) + (20x^3yS_{11} + \\ &+ 30x^2y^2S_{21} + 20xy^3S_{31})z + \\ &+ \sum_{k=2}^4 \sum_{j+l=5-k} \frac{5!}{j!l!k!} S_{jk}x^l y^j z^k \end{aligned}$$

Notation 8. We denote

$$k(x, y) := S_{20}x + S_{30}y = e_2x + e_3y \text{ and } l(x, y) := \beta x + \alpha y.$$

By identifying coefficients $\sum_{i+j=k} g_i f_j = F_k$ for $k \in \{0, 1, 2, 3\}$, we determine the coefficients $g_k(x, y)$ of the cubic equation recursively:

$$(5.2) \quad g_0(x, y) = 10xy(S_{20}x + S_{30}y) = 10xyk(x, y);$$

$$(5.3) \quad g_1(x, y) = (20x^2S_{11} + 30xyS_{21} + 20y^2S_{31}) + 10k(x, y)l(x, y);$$

$$(5.4) \quad xyg_2 = F_2 + g_1l - g_0d;$$

$$(5.5) \quad xyg_3 = F_3 + g_2l - g_1d.$$

Differentiating equation (5.4) with respect to x and y , and since $g_2 = xg_{2x} + yg_{2y}$ we get

$$(5.6) \quad g_2 = \frac{1}{2}(F_{2xy} + \alpha g_{1x} + \beta g_{1y} + lg_{1xy} - dg_{0xy});$$

Here $F_2(x, y) = 10(S_{02}x^3 + 3S_{12}x^2y + 3S_{22}xy^2 + S_{32}y^3)$ hence $F_{2xy}(x, y) = 60(S_{12}x + S_{22}y)$. Also $g_{0xy}(x, y) = 20k(x, y)$ and $g_{1xy} = 30S_{21} + 10k(\alpha, \beta)$, while g_{1y}, g_{1x} , are of degree 1 hence evaluating at $P = (1, 0)$ and $Q = (0, 1)$ we may rewrite

$$\begin{aligned} g_{1x} &= xg_{1xx} + yg_{1xy} = 2xg_1(P) + yg_{1xy}, \\ g_{1y} &= xg_{1xy} + yg_{1yy} = xg_{1xy} + 2yg_1(Q). \end{aligned}$$

Similarly for the constant coefficient g_3 we get

$$g_3 = F_{3xy} + \alpha g_{2x} + \beta g_{2y} - dg_{1xy}.$$

Here $F_3(x, y) = 10(S_{03}x^2 + 2S_{13}xy + S_{23}y^2)$ hence $F_{3xy}(x, y) = 20S_{13}$. Also since $\deg(g_2) = 1$,

$$g_{2x} = g_2(P) \quad \text{and} \quad g_{2y} = g_2(Q).$$

The $[x : y]$ coordinates of the intersection points of C and C' are given by the zeroes of the resultant $R(f, g, z)$ which is a degree 6 homogeneous polynomial in x, y , written in terms of $(f_i)_i$ and $(g_i)_i$. We know that $R(f, g, z)$ is a multiple of xy since $P, Q \in C \cap C'$.

6. Equations for the the parameter space of conics in the Fermat quintic

We continue with the set-up where we parametrise plane sections together with two marked singular points. Previously we wrote the equations of the conic and cubic components of a plane section X_Λ , which were obtained from identifying coefficients of z^k in the equation of X_Λ , for $k \in \{0, 1, 2, 3\}$. The equations for $k \in \{3, 4, 5\}$ will yield 5 necessary and sufficient conditions for the existence of such conics in X : The equation (5.5) requires that $xy|(F_3 + g_2l - g_1d)$ (the equivalent conditions from equations (5.2)–(5.4) are automatically satisfied due to the context set-up in the previous section). Equivalently, the polynomial $F_3 + g_2l - g_1d$ is zero when evaluated at $P = (1, 0)$ and $Q = (0, 1)$. Similarly for $k \in \{4, 5\}$. We obtain

Theorem 6.1. *Let Λ be a plane in \mathbb{P}^4 spanned by the points $P = [1 : 1 : 1 : 1 : 1]$, $Q = [u_0 : u_1 : u_2 : u_3 : u_4]$ and $R = [l_i]_i$. Consider the sums S_{mn} defined by equation (4.12), and the extension $\mathbb{C}[u_0, u_1, u_2, u_3, u_4][\alpha, \beta]$ with α, β satisfying*

$$S_{30}\alpha^2 + 2S_{31}\alpha + S_{32} \quad \text{and} \quad S_{20}\beta^2 + 2S_{11}\beta + S_{02}.$$

Then the plane spanned by P, Q, R cuts the Fermat quintic along a conic and a cubic (possibly degenerate) passing through P and Q iff the following equations are satisfied for some parameter d :

$$(6.1) \quad 10S_{03} + \beta g_2(P) - g_1(P)d = 0 \quad \text{and}$$

$$(6.2) \quad 10S_{23} + \alpha g_2(Q) - g_1(Q)d = 0$$

$$(6.3) \quad 5S_{04} + \beta g_3 - g_2(P)d = 0$$

$$(6.4) \quad 5S_{14} + \alpha g_3 - g_2(Q)d = 0 \quad \text{and}$$

$$(6.5) \quad S_{05} - g_3d = 0.$$

Equations (6.1) and (6.2) above are linear in d ; the remaining equations are quadratic in d , since both g_2 and g_3 are linear in d . In detail, the coefficients are

$$(6.6) \quad g_1(P) = 20S_{11} + 10S_{20}\beta, \quad g_1(Q) = 20S_{31} + 10S_{30}\alpha,$$

$$(6.7) \quad g_2(P) = 30S_{12} + \alpha g_1(P) + \beta g_{1xy} - 10S_{20}d$$

$$(6.8) \quad g_2(Q) = 30S_{22} + \beta g_1(Q) + \alpha g_{1xy} - 10S_{30}d,$$

$$(6.9) \quad g_3 = 20S_{13} + \alpha g_2(P) + \beta g_2(Q) - dg_{1xy},$$

where as seen previously $g_{1xy} = 30S_{21} + 10k(\alpha, \beta) = 30S_{21} + 10S_{20}\alpha + 10S_{30}\beta$.

Equation (6.5) is invariant under the involution τ , while the same involution swaps (6.1) and (6.2) as well as (6.3) and (6.4). Hence for symmetry reasons it seems optimal to substitute d from (6.1) into (6.3) and d from (6.2) into (6.4) thus obtaining two equations in $\mathbb{C}[u_i]_i[\alpha, \beta]$ which are pairs under τ . We can use a combination of (6.1) and (6.2) to eliminate d from (6.5). Finally, (6.1) and (6.2) together produce a fourth equation in $\mathbb{C}[u_i]_i[\alpha, \beta]$.

Choices of coordinate simplex for the plane Λ

The equations for the conics, cubics in X and their parameter spaces found in the previous section all depend on the choice of coordinates the plane Λ . The coordinate simplex we worked with has two fixed vertices P and Q , while the third vertex R can be chosen to be any point on $\Lambda \setminus PQ$. This choice will determine the format of the entries S_{mn} in the equations above, the format of α, β and the parameter d . At the same time, the degree 4 extension $\mathbb{C}[u_i]_i[\alpha, \beta]$ of $\mathbb{C}[u_i]_i$ does not depend on the choice of the point R due to Corollary 4.4.

It will be interesting to determine exactly how the system of equations found in the previous section varies with the choice of R , and use this to gain insights into the geometry of the space of conics determined by these equations. Within the constraints of this article we were not able to delve deeper into this topic but we hope to do this elsewhere.

In this section we will only briefly discuss and compare two particular choices for the point $R \in \Lambda$, their advantages and drawbacks. Recall that the plane Λ was given by the equations (4.2). Recall that $\delta(U)S_{mn} = \sum_{i=0}^4 M_i(U)u_i^m l_i^n$.

Option 1. Let $R = [l_i]_i$ be given by the equations

$$l_i = S_{80}u_i^5 - S_{90}u_i^4$$

Indeed, R satisfies the equations (6.10) as we have

$$\begin{aligned} \sum_{i=0}^4 M_i(U)u_i^4 l_i &= S_{80} \sum_{i=0}^4 M_i(U)u_i^9 - S_{90} \sum_{i=0}^4 M_i(U)u_i^8 \\ &= S_{80}S_{90} - S_{90}S_{80} = 0, \\ \sum_{i=0}^4 M_i(U)l_i &= S_{80} \sum_{i=0}^4 M_i(U)u_i^5 - S_{90} \sum_{i=0}^4 M_i(U)u_i^4 \\ &= S_{80}S_{50} - S_{90}S_{40} = 0 \end{aligned}$$

as $S_{50} = S_{40} = 0$ from the construction of $M_i(U)$ in the proof of Proposition 3.3.

The terms S_{m0} have initial values:

$$S_{00} = 0, \quad S_{10} = 0, \quad S_{20} = e_2, \quad S_{30} = e_3, \quad S_{40} = 0, \quad S_{50} = 0.$$

All further values can be found recursively from the Newton-type identity

$$S_{m,0} = e_1 S_{m-1,0} - e_2 S_{m-2,0} + e_3 S_{m-3,0} - e_4 S_{m-4,0} + e_5 S_{m-5,0},$$

which follows from the identity $\sum_{i=0}^4 M_i(U)q(u_i)u_i^{m-5} = 0$ for

$$q(u) := \prod_{i=0}^4 (u - u_i) = u^5 - e_1 u^4 + e_2 u^3 - e_3 u^2 + e_4 u - e_5.$$

In turn, the general terms S_{mn} are determined by the recurrence

$$S_{mn} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} S_{80}^k S_{90}^{n-k} S_{m+4n+k,0}.$$

Indeed, this follows from applying the binomial formula to the term $l_i^n = (S_{80}u_i^5 - S_{90}u_i^4)^n$ in the definition of S_{mn} .

We thus have an algorithmic procedure of finding all the coefficients entering the equations of the space of conics. However, the complexity of the terms S_{mn} as polynomials of e_i increases fast with m, n . For the time being we have been unable to extract geometrically meaningful information from this procedure, but it's possible that more patterns will emerge from the interplay of the algebra and geometry here.

Option 2. Our second choice R' is defined geometrically as the point of intersection of the planes $\Lambda = \Lambda(U)$ and Λ' in \mathbb{P}^4 , where Λ' is given by equations

$$(6.10) \quad \sum_{i=0}^4 M_i(U)u_i^2 v_i = 0 \quad \text{and} \quad \sum_{i=0}^4 M_i(U)u_i^3 v_i = 0,$$

With the notations from the previous sections, up to a \mathbb{C}^* -factor we have $R' = [l'_i]_i$ with

$$(6.11) \quad l'_i = e_2(i) \prod_{j \neq i} n_j,$$

where n_j are as defined in Theorem 3.3.

Below are the first few values of $S'_{m,n}$ in terms of elementary symmetric polynomials:

$$\begin{aligned} S'_{00} = 0, \quad S'_{10} = 0, \quad S'_{20} = e_2, \quad S'_{30} = e_3, \quad S'_{40} = 0, \quad S'_{50} = 0, \\ S'_{01} = 0, \quad S'_{11} = 0, \quad S'_{21} = \prod_i n_i, \quad S'_{31} = 0, \quad S'_{41} = 0. \end{aligned}$$

Comparing with the coefficients in formulae (6.6)–(6.9), we see that simplifies the system (6.1)–(6.5) and in particular eliminates the parameter d . All the remaining terms are products of $\prod_i n_i$ due to our choice in formula (6.11). Then by computer assisted calculations we verified

$$\begin{aligned} S'_{02} &= -e_2 e_3^2 S_{80} \prod_i n_i, \\ S'_{12} e_2 + S'_{02} e_3 &= -e_2 e_3^2 (e_3^2 e_2^2 e_1 + e_5 e_3^3 + e_4^2 e_3 e_2 e_1 - e_3^2 e_2) \prod_i n_i. \end{aligned}$$

The next term S'_{22} is the transform of S'_{12} under the involution τ and S'_{32} is the transform of S'_{02} under the same involution (up to a factor of e_5^k).

The full pattern of identities among the terms S_{mn} and their significance for the solutions of the equations in the previous section is yet to emerge.

7. Plane sections in the exceptional locus of $\overline{U}_2(X) \rightarrow \overline{\Delta}_{2,2}(X)$

In this section we will illustrate the application of the equations obtained in Theorem 6.1 to the study of conics found for plane sections in some of the components of the exceptional locus of $\overline{U}_2(X) \rightarrow \overline{\Delta}_{2,2}(X)$.

While the equations in Theorem 6.1 were set up on $\Delta_{2,2}(X) \setminus X_\sigma$, we note that these equations can be extended to work for points outside the regular sets of maps ϕ and m . Namely, for a triple $(\Lambda, P, Q) \in \overline{U}_2(X)$, we can unpack the change of coordinates ν_P^{-1} introduced in Section 4, so that the equations (6.6)–(6.9) can be translated following the rules

$$(7.1) \quad M_i(U) = p_i^5, \quad q_i = p_i u_i, \quad h_i = p_i l_i \quad \text{and hence}$$

$$(7.2) \quad S_{mn} = \sum_{i=0}^4 p_i^{5-n-m} q_i^m h_i^n.$$

Here h_i are the coordinates of the point R introduced in Section 4. Rather than reversing the benefits of Theorem 3.3, this coordinate choice can be interpreted as working over a blow-up of \mathbb{P}^4 as discussed in Section 3 (an intermediate space between \mathbb{P}^4 and $\overline{M}_{0,7}$ like the Losev-Manin spaces would be sufficient).

Consider the case when (Λ, P, Q) is in the exceptional locus of the map $f : \overline{U}_2(X) \rightarrow \overline{\Delta}_{2,2}(X)$. Recall that this is the locus in the preimage of the set $X_{[k]} = \sigma_{[k]}(\Delta_{\mathbb{P}^4}) \cap (X \times X)$ for the action defined by formula (3.1), namely $P = [p_i]_i$ and $Q = [p_i \sqrt{-1}^{k_i}]_i$ with $[k_i]_i \in \mathbb{Z}_4^4$.

Recall that in this case, $\overline{\mathcal{T}}_{X,P} = \overline{\mathcal{T}}_{X,Q}$ and hence Λ varies in the pencil given by $PQ \subset \Lambda \subset \overline{\mathcal{T}}_{X,P}$. As well, P and Q are both singular points of the quintic surface $X \cap \overline{\mathcal{T}}_{X,P} = V(\sum_{i=0}^4 x_i^5, \sum_{i=0}^4 x_i p_i^4)$. The tangent cones $C_P(X)$ and $C_Q(X)$ to the surface $X \cap \overline{\mathcal{T}}_{X,P}$ at P and Q respectively, are quadratic surfaces defined in $\overline{\mathcal{T}}_{X,P}$ by the polynomials

$$\sum_{i=0}^4 x_i^2 p_i^3 \quad \text{and} \quad \sum_{i=0}^4 x_i^2 q_i^3 = \sum_{i=0}^4 x_i^2 p_i^3 \sqrt{-1}^{3k_i}$$

respectively. These correspond to the quadratic equations in Proposition 4.7.

To parametrise Λ as above, we pick two distinct points $S, T \in \overline{\mathcal{T}}_{X,P} \setminus PQ$ so that together P, Q, S, T span $\overline{\mathcal{T}}_{X,P}$, and we will let $R = uS + vT = [h_i]_i$ for $[u : v] \in \mathbb{P}^1$. We can then define Λ as the projective plane spanned by P, Q, R . The tangent cones satisfy

$$X_\Lambda \cap C_P(X) = C_P(X_\Lambda) \quad \text{and} \quad X_\Lambda \cap C_Q(X) = C_Q(X_\Lambda)$$

and Corollary 4.4 applies, with with the notations set in (7.1). Then the program from Sections 5 and 6 follows accordingly.

We will illustrate how this works in the case of $k = [0 : 0 : 0 : 2 : 2] \in \mathbb{Z}_4^4$. We denote by $\Sigma_{[k]}$ the image of the first projection of $X_{[k]}$ on X , namely

$\Sigma_{[k]} = V(\sum_{i=0}^4 x_i^5, \sum_{i=0}^4 x_i^5 \sqrt{-1}^{ki})$, a degree 25 surface in \mathbb{P}^4 . For k as chosen above and μ a fifth root of 1, we have

$$\Sigma_{[k]} = \bigcup_{h \in \mathbb{Z}_5} V\left(\sum_{i=0}^2 x_i^5, x_3 + \mu^h x_4\right),$$

a union of 5 of the 50 cones making up the variety covered by lines in X . We will focus on the case when $h = 0$. Of course, the other cones are obtained from X_k by the action of $S_5 \times \mathbb{Z}_5^4$, hence they will behave similarly in respect to conics.

We will set the following (affine) coordinates, with $a^5 + b^5 + c^5 = 0$:

$$\begin{aligned} P &= [a : b : c : 1 : -1], & Q &= [a : b : c : -1 : 1], \\ S &= [0 : -b : 0 : 0 : b^5], & T &= [0 : 0 : -c : c^5 : 0]. \end{aligned}$$

It can be readily verified that these four points span $\overline{T}_{X,P} = V(a^4x_0 + b^4x_1 + c^4x_2 + x_3 + x_4)$ in the case $bc \neq 0$. The other cases can be treated similarly after a permutation of coordinates. For simplicity we will use an affine parameter, writing $R = tT + S$ with $t \in \mathbb{A}^1$. Then since $p_i^2 = q_i^2$, the coefficients $S_{m,n}$ satisfy $S_{m0} = p_0^5 + p_1^5 + p_2^5 + (-1)^m(p_3^5 + p_4^5) = 0$ for all $m \in \{0, 1, 2, 3, 4, 5\}$, and

$$\begin{aligned} S_{01} &= S_{21} = S_{41} = 0, \\ S_{11} &= S_{31} = -2(tc^5 + b^5); \\ S_{02} &= S_{22} = t^2(c^5 + c^{10}) + b^5 - b^{10}; \\ S_{12} &= S_{32} = t^2(c^5 - c^{10}) + b^5 + b^{10}; \\ S_{03} &= S_{23} = -t^3(c^5 - c^{15}) - b^5 + b^{15}; \\ S_{13} &= -t^3(c^5 + c^{15}) - b^5 - b^{15}; \\ S_{14} &= t^4(c^5 - c^{20}) + b^5 + b^{20}; \\ S_{04} &= t^4(c^5 + c^{20}) + b^5 - b^{20}; \\ S_{05} &= -t^5(c^5 - c^{25}) - b^5 + b^{25}. \end{aligned}$$

In particular, the forms in (4.10) and (4.11) equal y (respectively x) times linear factor and admit common zeros along the line $PQ = V(z)$ (with the notations from (4.10) and (4.11)). Conics tangent to this line at both P and Q are those unions of lines $PQ \cup l \subset X$. Assuming $S_{11} \neq 0$, the remaining case is when $\alpha = -S_{12}/(2S_{11})$ and $\beta = -S_{02}/(2S_{11})$.

After simplifying and reverting to the coefficient $\lambda = \alpha\beta - d$, the equations from Theorem 6.1 become

$$\begin{aligned} 10S_{03} + 30\beta S_{12} &= -20S_{11}\lambda; \\ 10S_{23} + 30\alpha S_{22} &= -20S_{31}\lambda; \\ 5S_{04} + 20\beta S_{13} + 20\beta^2 S_{02} &= -20S_{32}\lambda; \\ 5S_{14} + 20\alpha S_{13} + 20\alpha^2 S_{12} &= -20S_{02}\lambda; \\ S_{05} - 5S_{04}\alpha - 5S_{14}\beta - 60\alpha\beta S_{13} - 40\alpha^2\beta S_{32} - 40\alpha\beta^2 S_{02} &= -20S_{13}\lambda. \end{aligned}$$

We will denote the five quantities above as q_1, q_2, \dots, q_5 . Based on the formulae for α, β and S_{mn} above, the equations for q_1 and q_2 are in fact identical. Eliminating λ and denominators from the remaining equations and substituting $b^5 + c^5 = -a^5$ we obtain the following polynomial equations:

$$\begin{aligned} f_3 &:= q_3 S_{11}^2 - q_1 S_{12} S_{11} = -20t^2(t-1)^2 b^5 c^5 a^5 = 0; \\ f_4 &:= q_4 S_{11}^2 - q_1 S_{02} S_{11} = -20t^2(t-1)^2 b^5 c^5 a^5 = 0; \\ f_5 &:= q_5 S_{11}^3 - q_1 S_{13} S_{11}^2 = 0, \end{aligned}$$

The solution set is found based on evaluating

$$\begin{aligned} f_5|_{t=0} &= 2b^{20}(b^{20} - 1), & f_5|_{t=1} &= 2a^4(a^{20} - 1), \\ f_5|_{c=0} &= 2b^{20}(b^{20} - 1), & f_5|_{b+c=0} &= 2(t-1)^8 b^{20}(b^{20} - 1) \end{aligned}$$

and similarly for $b = 0$ and $b + \mu^h c = 0$, where μ is a fifth root of unity.

The solutions split into 3 symmetric cases: $t = 0$ and $b^{20}(b^{20} - 1) = 0$, or $t = \infty$ and $c^{20}(c^{20} - 1) = 0$ or $t = 1$ and $a^4(a^{20} - 1) = 0$. In the last case let $a^4 = \mu$. Then the points P, Q and R satisfy the equations

$$c(x_0 + \mu^{-1}x_3 + \mu^{-1}x_4) = ax_2 \quad \text{and} \quad cx_1 = bx_2,$$

Thus the solutions Λ are in the orbit of the family in Example 2.1 under the automorphism group $S_5 \times \mathbb{Z}_5^4$.

It remains to look at the case when $S_{11} = 0$, namely $t = -b^5/c^5$. Then either our conic is the double line PQ or $S_{12} = S_{02} = 0$ hence $c^5 = -b^5$ and $t = \pm 1$. The solutions in this case are contained in the orbit of the family in Example 2.2. Perhaps not surprisingly, we obtain the following

Corollary 7.1. *All the planes containing both a line and a smooth conic in X are in the orbits of Examples 2.1 and 2.2 under the natural action of the automorphism group $S_5 \times \mathbb{Z}_5^4$.*

This is not exhaustive study of the consequences of Theorem 6.1. A full systematic study of the solution set of the resulting equations, as well as of the exceptional loci involved in our constructions will require a more extensive discussion which is yet to be finalised. In the general case, it is likely that more insights will be needed to find patterns and reduce the complexity of the equations. Optimistically, we would hope that these will be able to be brought to a point where the degree 2 equivalent of equation (1.3) can be verified.

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