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The Fourier-Mukai transform made easy

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Abstract: We propose a slightly modified definition for the Fourier-Mukai transform (on abelian varieties) that makes it much easier to remember various formulas. As an application, we give short proofs for two important theorems: the characterization of GV-sheaves in terms of vanishing, due to Hacon; and fact that M-regularity implies (continuous) global generation, due to Pareschi and Popa.

Keywords: Fourier-Mukai transform, abelian variety, GV-sheaf, m-regularity.

1. Introduction

1.1. The symmetric Fourier-Mukai transform

The Fourier-Mukai transform, first introduced by Mukai [Muk81], is an equivalence of k-linear triangulated categories

$$\mathbf{R}\Phi_{P_A} \colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}(\mathscr{O}_{A^\vee})$$

between the bounded derived category of coherent sheaves on the abelian variety A, and that on the dual abelian variety $A^\vee = \operatorname{Pic}^0_{A/k}$. It is defined by pulling back to $A \times A^\vee$, tensoring by the Poincaré bundle P_A, and then pushing forward to $A^\vee$.

The purpose of this note is to propose a slightly different definition for the Fourier-Mukai transform that makes it easier to remember various formulas. When X is an equidimensional and nonsingular algebraic variety over k, we denote by

$$\mathbf{R}\Delta_X = \mathbf{R}\mathcal{H}om(-, \omega_{X/k}[\dim X]) \colon D^b_{coh}(\mathscr{O}_X) \to D^b_{coh}(\mathscr{O}_X)^{op}$$

the (contravariant) Grothendieck duality functor. In the case $X = \operatorname{Spec} k$, we shall use the simplified notation $\mathbf{R}\Delta_k$.

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Definition 1.1. The exact functor

$$\mathsf{FM}_A = \mathbf{R}\Phi_{P_A} \circ \mathbf{R}\Delta_A \colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}(\mathscr{O}_{A^\vee})^{op}$$

is called the *symmetric Fourier-Mukai transform.*

The word "symmetric" will be justified by the results below. Note that FM_A is a *contravariant* functor; this turns out to be quite useful in practice, for instance in the study of GV-sheaves (see Definition 1.2).

It is clear from Mukai's result [Muk81, Thm 2.2] that FM_A is also an equivalence of categories. One advantage of the new definition is that it respects the symmetry between the two abelian varieties A and $A^\vee$. For example, one can show that

(1) $$\mathsf{FM}_A(k(0)) = \mathscr{O}_{A^\vee} \quad \text{and} \quad \mathsf{FM}_A(\mathscr{O}_A) = k(0).$$

Here $k(0) = e_* \mathscr{O}_{\operatorname{Spec} k}$ means the structure sheaf of the closed point $0 \in A(k)$; we use the same notation also on $A^\vee$. The following theorem further justifies the name "symmetric Fourier-Mukai transform".

Theorem 1.1. *The composed functors $\mathsf{FM}_{A^\vee} \circ \mathsf{FM}_A$ and $\mathsf{FM}_A \circ \mathsf{FM}_{A^\vee}$ are naturally isomorphic to the identity. In other words,*

$$\mathsf{FM}_A \colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}(\mathscr{O}_{A^\vee})^{op}$$

is an equivalence of categories, with quasi-inverse $\mathsf{FM}_{A^\vee}$.

This looks much simpler than Mukai's version [Muk81, Thm. 2.2].

1.2. Basic properties

The effect of pulling back or pushing forward by a homomorphism between abelian varieties is easy to describe; for the case of isogenies, see [Muk81, (3.4)], and for the general case, [CJ15, Proposition 2.3].

Proposition 1.1. *Let $f \colon A \to B$ be a homomorphism of abelian varieties over k. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_B \circ \mathbf{R} f_* = \mathbf{L}\hat{f}^* \circ \mathsf{FM}_A \quad \text{and} \quad \mathsf{FM}_A \circ \mathbf{L} f^* = \mathbf{R}\hat{f}_* \circ \mathsf{FM}_B,$$

where $\hat{f} \colon B^\vee \to A^\vee$ is the induced homomorphism between the dual abelian varieties.

The symmetric Fourier-Mukai transform exchanges translations and tensoring by the corresponding line bundles. Any closed point $a \in A(k)$ determines a translation morphism $t_a \colon A \to A$; on closed points, it is given by the formula $t_a(x) = a + x$. Compare the following result with [Muk81, (3.1)].

Proposition 1.2. *Let $a \in A(k)$ and $\alpha \in A^\vee(k)$ be closed points. Then one has natural isomorphisms of functors*

$$\mathsf{FM}_A \circ (t_a)_* = (P_a \otimes -) \circ \mathsf{FM}_A \quad \text{and} \quad \mathsf{FM}_A \circ (P_\alpha \otimes -) = (t_\alpha)_* \circ \mathsf{FM}_A,$$

where P_a on A, and P_α on $A^\vee$, are the corresponding line bundles.

Together with (1), this leads to the pleasant formulas

$$\mathsf{FM}_A(k(a)) = P_a \quad \text{and} \quad \mathsf{FM}_A(P_\alpha) = k(\alpha), \tag{2}$$

for any pair of closed points $a \in A(k)$ and $\alpha \in A^\vee(k)$.

1.3. GV-sheaves and M-regular sheaves

The symmetric Fourier-Mukai transform has the following simple effect on the standard t-structure on $D^b_{coh}(\mathscr{O}_A)$.

Proposition 1.3. *If $K \in D^{\geq n}_{coh}(\mathscr{O}_A)$, then $\mathsf{FM}_A(K) \in D^{\leq -n}_{coh}(\mathscr{O}_{A^\vee})$.*

In particular, the symmetric Fourier-Mukai transform of a coherent sheaf is a complex of coherent sheaves concentrated in non-positive degrees. This motivates the following definition, first made by Pareschi and Popa [PP11a, Definition 3.1].

Definition 1.2. A coherent sheaf $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ is called a *GV-sheaf* if

$$\mathsf{FM}_A(\mathscr{F}) = \hat{\mathscr{F}}$$

for a coherent sheaf $\hat{\mathscr{F}} \in \mathrm{Coh}(\mathscr{O}_{A^\vee})$. If $\hat{\mathscr{F}}$ is torsion-free, then $\mathscr{F}$ is called *M-regular*.

Our definition of M-regularity differs from the original one in [PP03, Def. 2.1]; the two definitions are equivalent by [PP11b, Thm. 2.8].

Theorem 1.1 shows that $\mathsf{FM}_{A^\vee}(\hat{\mathscr{F}}) = \mathscr{F}$, and so the coherent sheaf $\hat{\mathscr{F}}$ is again a GV-sheaf on $A^\vee$. The property of being a GV-sheaf is therefore self-dual. Using Proposition 1.1 and Proposition 1.2, one shows that the set of closed points in the support of the coherent sheaf $\hat{\mathscr{F}}$ is equal to

$$(\operatorname{Supp} \hat{\mathscr{F}})(k) = \{\, \alpha \in A^\vee(k) \mid H^0(A, \mathscr{F} \otimes P_\alpha^{-1}) \neq 0 \,\}. \tag{3}$$

The minus sign is consistent with the formulas in (2).

GV-sheaves have several other surprising properties. Let $\mathscr{F}$ be a coherent sheaf on A. For each $j \geq 0$, one has a reduced closed subscheme $S^j(A, \mathscr{F}) \subseteq A^\vee$; its set of closed points is given by

$$S^j(A, \mathscr{F})(k) = \{\, \alpha \in A^\vee(k) \mid H^j(A, \mathscr{F} \otimes P_\alpha^{-1}) \neq 0 \,\}.$$

If $\mathscr{F}$ is a GV-sheaf, then $S^0(A, \mathscr{F}) = \operatorname{Supp} \hat{\mathscr{F}}$, according to (3). The following result is originally due to Hacon [Hac04, Cor. 3.2].

Proposition 1.4. *If $\mathscr{F}$ is a GV-sheaf on A, then*

$$S^0(A, \mathscr{F}) \supseteq S^1(A, \mathscr{F}) \supseteq \cdots \supseteq S^{\dim A}(A, \mathscr{F}),$$

and moreover, one has $\operatorname{codim} S^j(A, \mathscr{F}) \geq j$ *for all* $j \geq 0$.

The converse of this result is true as well: if $\operatorname{codim} S^j(A, \mathscr{F}) \geq j$ for all $j \geq 0$, then $\mathscr{F}$ is a GV-sheaf. In fact, this was the original definition of GV-sheaves in [PP11a].

Another advantage of our definition is that it preserves positivity. Recall that an ample line bundle L on the abelian variety A gives rise to a homomorphism

$$\phi_L \colon A \to A^\vee, \tag{4}$$

with the property that $t_a^* L = L \otimes P_{\phi_L(a)}$ for every closed point $a \in A(k)$. The following result is a version of [Muk81, Prop. 3.1].

Proposition 1.5. *Let L be an ample line bundle on A. Then $\hat{L}$ is an ample vector bundle of rank* $\dim H^0(A, L)$*, and one has*

$$\phi_L^* \hat{L} = i^* L \otimes H^0(A, L)^*,$$

where $i \colon A \to A$ is the inversion morphism.

Note that $\operatorname{rk} \hat{L} = \dim H^0(A, L)$ whereas $\operatorname{rk} L = \dim H^0(A^\vee, \hat{L})$; at least for ample line bundles, the symmetric Fourier-Mukai transform therefore interchanges "rank" and "dimension of the space of global sections". The generalization to arbitrary GV-sheaves is that the generic rank of $\hat{\mathscr{F}}$ is equal to the Euler characteristic

$$\chi(A, \mathscr{F}) = \sum_{j=0}^{\dim A} (-1)^j \dim H^j(A, \mathscr{F}).$$

This follows from Proposition 1.4; compare also [Muk81, Cor. 2.8].

Finally, we note the following useful property of GV-sheaves, which follows almost immediately from Proposition 1.3. As far as I know, this is a new result.

Proposition 1.6. *Let $f\colon A \to B$ be a homomorphism of abelian varieties, and let $\mathscr{F}$ be a GV-sheaf on A. If $f_*\mathscr{F}$ is again a GV-sheaf on B, then*

$$\mathsf{FM}_B(f_*\mathscr{F}) = \hat{f}^*\hat{\mathscr{F}},$$

where $\hat{f}\colon B^\vee \to A^\vee$ is the induced homomorphism between the dual abelian varieties.

Although one can deduce all the results above from what is in Mukai's paper, we are going to prove everything from scratch. This will hopefully convince the reader that the new definition is also easy to work with in practice.

1.4. Applications

As an application, we present relatively short proofs for two important results about GV-sheaves and M-regular sheaves. The following theorem by Hacon [Hac04, Thm 1.2] relates the GV-property to vanishing theorems for ample line bundles.

Theorem 1.2. *A coherent sheaf $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ is a GV-sheaf if, and only if, for every isogeny $\varphi\colon A' \to A$ and every ample line bundle L' on A', one has*

$$H^i(A', L' \otimes \varphi^*\mathscr{F}) = 0 \quad \text{for all } i > 0.$$

Recall that a coherent sheaf $\mathscr{F}$ on an abelian variety is said to be *continuously globally generated* [PP03, Def 2.10] if for every nonempty Zariski-open subset $U \subseteq A^\vee$, the evaluation morphism

$$\bigoplus_{\alpha \in U(k)} H^0(A, \mathscr{F} \otimes P_\alpha^{-1}) \otimes P_\alpha \to \mathscr{F}$$

is surjective. The first half of the following result is due to Pareschi and Popa [PP03, Prop 2.13], the second half to Debarre [Deb06, Prop 3.1].

Theorem 1.3. *Let $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ be an M-regular coherent sheaf on A.*

(a) *$\mathscr{F}$ is continuously globally generated.*
(b) *There is an isogeny $\varphi\colon A' \to A$ such that $\varphi^*\mathscr{F}$ is globally generated.*

In fact, one can prove more generally that if $\mathscr{F}$ is a torsion-free coherent sheaf on A, then $\mathcal{H}^0 \,\mathsf{FM}_A(\mathscr{F})$ is continuously globally generated, and globally generated after a suitable isogeny (see Theorem 4.1 below). This is again a new result.

2. Notation

Let A be an abelian variety over an algebraically closed field k. We denote by

$$m\colon A \times A \to A, \quad i\colon A \to A, \quad e\colon \operatorname{Spec} k \to A$$

the k-morphisms representing composition, inversion, and the identity of the group scheme A; we write $0 \in A(k)$ for the closed point corresponding to e.

We use the notation $A^\vee = \operatorname{Pic}^0_{A/k}$ for the dual abelian variety. On the product $A \times A^\vee$, we have the Poincaré bundle P_A, normalized by the condition that its pullback by $e \times \mathrm{id}$ is trivial; the Poincaré correspondence consists of the line bundle P_A, together with fixed trivializations of $(e \times \mathrm{id})^* P_A$ and $(\mathrm{id} \times e)^* P_A$.

Note that $\operatorname{Pic}^0_{A^\vee/k}$ is canonically isomorphic to A, and that the Poincaré correspondence on $A^\vee \times A$ is the pullback of the Poincaré correspondence on $A \times A^\vee$, under the automorphism that swaps the two factors of the product.

For a closed point $\alpha \in A^\vee(k)$, we denote by P_α the corresponding line bundle on A; to be precise, P_α is the restriction of P_A to the closed subscheme $A \times \{\alpha\}$. Similarly, P_a means the line bundle on $A^\vee$ corresponding to a closed point $a \in A(k)$. We also denote by

$$P_{(a,\alpha)} = P_A\big|_{(a,\alpha)}$$

the fiber of the Poincaré bundle at the closed point $(a, \alpha) \in (A \times A^\vee)(k)$. Using the canonical isomorphism $(i \times i)^* P_A = P_A$, we obtain

$$P_{(a,\alpha)} = P_a\big|_\alpha = P_\alpha\big|_a = P_{(-a,-\alpha)} = P_{-\alpha}\big|_{-a} = P_{-a}\big|_{-\alpha}. \tag{5}$$

All these isomorphisms are uniquely determined by the Poincaré correspondence.

3. Proofs

We use the notation $D^b_{coh}(\mathscr{O}_A)$ for the derived category of cohomologically bounded and coherent complexes of sheaves of $\mathscr{O}_A$-modules; this category is equivalent to the (much smaller) bounded derived category of $\mathrm{Coh}(\mathscr{O}_A)$. In Mukai's original definition, the Fourier-Mukai transform is the exact functor

$$\mathbf{R}\Phi_{P_A}\colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}(\mathscr{O}_{A^\vee}),$$

given by the (slightly abbreviated) formula $\mathbf{R}\Phi_{P_A} = \mathbf{R}(p_2)_*\big(P_A \otimes p_1^*\big)$. Here p_1 and p_2 are the projections from $A \times A^\vee$ to the two factors:

$$\begin{tikzcd} A \times A^\vee \arrow[r, "p_2"] \arrow[d, "p_1"] & A^\vee \\ A \end{tikzcd}$$

3.1. Equivalence of categories

Let us start by checking the two examples in (1). The first one is very easy: Grothendieck duality, applied to the morphism $e\colon \operatorname{Spec} k \to A$, gives

$$\mathbf{R}\Delta_A\big(k(0)\big) = e_*\mathbf{R}\Delta_{\operatorname{Spec} k}\big(\mathscr{O}_{\operatorname{Spec} k}\big) = e_*\mathscr{O}_{\operatorname{Spec} k} = k(0),$$

and so the symmetric Fourier-Mukai transform of $k(0)$ is

$$\mathsf{FM}_A\big(k(0)\big) = \mathbf{R}\Phi_{P_A}\big(k(0)\big) = \mathscr{O}_A.$$

The second isomorphism comes from the fixed trivialization of $(e \times \mathrm{id})^* P_A$.

The other example in (1) needs more work. To help us describe

$$\mathsf{FM}_A(\mathscr{O}_A) = \mathbf{R}\Phi_{P_A}\Big(\omega_A[\dim A]\Big) = \mathbf{R}(p_2)_*\Big(P_A \otimes p_1^*\omega_A[\dim A]\Big), \tag{6}$$

we first recall the following result about the cohomology of line bundles on abelian varieties [Mum70, §8(vii)].

Lemma 3.1. *If $\alpha \in A^\vee(k)$ is nontrivial, then $H^n(A, P_\alpha) = 0$ for all $n \in \mathbb{N}$.*

Proof. The proof is by induction on $n \geq 0$. The case $n = 0$ is easy: if s is a nontrivial global section of P_α, then i^*s is a nontrivial global section of $i^*P_\alpha = P_{-\alpha}$, and so $s \otimes i^*s$ is a nontrivial global section of $P_\alpha \otimes P_{-\alpha} = \mathscr{O}_A$; this clearly implies $\alpha = 0$. To deal with the general case, we factor the identity morphism $\mathrm{id}\colon A \to A$ into the following two morphisms:

$$A \xrightarrow{\mathrm{id} \times e} A \times A \xrightarrow{m} A$$

On the level of cohomology, we obtain a factorization of the identity as

$$H^n(A, P_\alpha) \to H^n(A \times A, m^* P_\alpha) \to H^n(A, P_\alpha).$$

Because $m^* P_\alpha = p_1^* P_\alpha \otimes p_2^* P_\alpha$, the Künneth formula gives

$$H^n(A \times A, m^* P_\alpha) = \bigoplus_{i+j=n} H^i(A, P_\alpha) \otimes H^j(A, P_\alpha),$$

which is zero by induction. The conclusion is that $H^n(A, P_\alpha) = 0$. $\square$

The remainder of the argument is taken from [Mum70, §13], with a few minor tweaks. Consider the cohomology sheaves

$$\mathscr{F}_j = R^j (p_2)_* \Big(P_A \otimes p_1^* \omega_A [\dim A] \Big) = R^{\dim A + j} (p_2)_* (P_A \otimes p_1^* \omega_A)$$

of the complex in (6); they are coherent sheaves on the dual abelian variety $A^\vee$. Since the line bundle $P_A \otimes p_1^* \omega_A$ is flat over $A^\vee$, we can apply the base change theorem and Lemma 3.1 to conclude that each $\mathscr{F}_j$ is supported on the closed point $0 \in A^\vee(k)$; being coherent, $\mathscr{F}_j$ therefore has finite length. For dimension reasons, we obviously have $\mathscr{F}_j = 0$ for $j > 0$.

The next task is to show that $\mathscr{F}_j = 0$ also for $j < 0$. This can be done by using Serre duality. Consider the Leray spectral sequence

$$E_2^{i,j} = H^i(A^\vee, \mathscr{F}_j) \Longrightarrow H^{\dim A + i + j}(A \times A^\vee, P_A \otimes p_1^* \omega_A).$$

We have $E_2^{i,j} = 0$ for $i > 0$ (because $\operatorname{Supp} \mathscr{F}_j$ is zero-dimensional); the spectral sequence therefore degenerates at E_2 and gives us isomorphisms

$$H^{\dim A + j}(A \times A^\vee, P_A \otimes p_1^* \omega_A) = H^0(A^\vee, \mathscr{F}_j).$$

In particular, this group vanishes for $j > 0$. By Serre duality, the k-vector space on the left is dual to

$$H^{\dim A - j}(A \times A^\vee, P_A^{-1} \otimes p_2^* \omega_{A^\vee})$$

and this vanishes for $j < 0$, for the same reason as before. But then $H^0(A^\vee, \mathscr{F}_j) = 0$, and because the sheaf $\mathscr{F}_j$ has finite length, we see that $\mathscr{F}_j = 0$ for every $j < 0$.

So far, we have shown that the natural morphism

$$\mathbf{R}(p_2)_* \Big(P_A \otimes p_1^* \omega_A [\dim A] \Big) \to \mathscr{F}_0$$

is an isomorphism. To conclude the proof of (1), we need to argue that $\mathscr{F}_0 = k(0)$. Another application of the base change theorem yields

$$e^*\mathscr{F}_0 = H^{\dim A}(A, \omega_A) = k.$$

By Nakayama's lemma, it follows that $\mathscr{F}_0 = \mathscr{O}_Z$ for a (maybe nonreduced) closed subscheme $Z \subseteq A^\vee$ supported on the closed point $0 \in A^\vee(k)$. We now use the universal property of the Poincaré correspondence to show that Z is reduced, and hence that $\mathscr{F}_0 = k(0)$. Let $\mathscr{F} \in \mathrm{Coh}(A^\vee)$ be an arbitrary coherent sheaf. Grothendieck duality, applied to the second projection $p_2 \colon A \times A^\vee \to A^\vee$, gives us an isomorphism

$$\mathrm{Hom}(\mathscr{O}_Z, \mathscr{F}) = \mathrm{Hom}(\mathscr{F}_0, \mathscr{F}) = \mathrm{Hom}(P_A, p_2^*\mathscr{F}),$$

functorial in $\mathscr{F}$. In particular, the identity endomorphism of $\mathscr{O}_Z$ corresponds to a nontrivial morphism $P_A \to p_2^*\mathscr{O}_Z$, hence to a nontrivial morphism

$$P_A\big|_{A\times Z} \to \mathscr{O}_{A\times Z}.$$

Because the restriction of P_A to the closed subscheme $A \times \{0\}$ is trivial, Nakayama's lemma tells us that this morphism is an isomorphism. In other words, the restriction of P_A to the subscheme $A \times Z$ is trivial; by the universal property of the Poincaré correspondence, the subscheme in question therefore has to be contained in $A \times \{0\}$, which means exactly that Z is reduced.

Having shown that $\mathsf{FM}_A(\mathscr{O}_A) = k(0)$, we can now explain why the symmetric Fourier-Mukai transform is an equivalence of categories.

Proof of Theorem 1.1. Since we can interchange the role of A and $A^\vee$, we only need to prove that the functor

$$\mathsf{FM}_{A^\vee} \circ \mathsf{FM}_A = \mathbf{R}\Phi_{P_{A^\vee}} \circ \mathbf{R}\Delta_{A^\vee} \circ \mathbf{R}\Phi_{P_A} \circ \mathbf{R}\Delta_A$$

is naturally isomorphic to the identity. To do this, we first consider

$$\mathbf{R}\Delta_{A^\vee} \circ \mathbf{R}\Phi_{P_A} \circ \mathbf{R}\Delta_A \colon D^b_{coh}(\mathscr{O}_A) \to D^b_{coh}(\mathscr{O}_{A^\vee}).$$

A brief computation using Grothendieck duality shows that this functor is an integral transform, whose kernel is the complex

$$P_A^{-1} \otimes p_2^*\omega_{A^\vee}[\dim A]$$

on the product $A \times A^\vee$. After composing with $\mathbf{R}\Phi_{P_{A^\vee}}$ and swapping the order of the factors, we obtain another integral transform, whose kernel is now the complex

$$\mathbf{R}(p_{12})_*\Big(p_3^*\omega_{A^\vee}[\dim A] \otimes p_{13}^* P_A^{-1} \otimes p_{23}^* P_A\Big) \tag{7}$$

on $A \times A$. Theorem 1.1 will be proved once we show that (7) is isomorphic to the structure sheaf of the diagonal in $A \times A$.

Let $s\colon A \times A \to A$ be defined as $s = m \circ (\mathrm{id} \times i)$; the formula on closed points is $s(x, y) = x - y$. Considering $p_{13}^* P_A^{-1} \otimes p_{23}^* P_A$ as a correspondence between $A \times A$ and $A^\vee$, and using the universal property of the Poincaré correspondence, we obtain

$$p_{13}^* P_A^{-1} \otimes p_{23}^* P_A = (s \times \mathrm{id})^* P_A.$$

We can now apply flat base change in the commutative diagram

$$\begin{tikzcd}
A \times A \times A^\vee \arrow[r, "p_{12}"] \arrow[d, "s \times \mathrm{id}"] & A \times A \arrow[d, "s"] \\
A \times A^\vee \arrow[r, "p_1"] & A
\end{tikzcd}$$

and conclude that the complex in (7) is isomorphic to

$$s^*\,\mathbf{R}(p_1)_*\Big(P_A \otimes p_2^*\omega_{A^\vee}[\dim A]\Big) = s^*\,\mathsf{FM}_{A^\vee}(\mathscr{O}_{A^\vee}) = s^* k(0) = \Delta_* \mathscr{O}_A,$$

where $\Delta\colon A \to A \times A$ is the diagonal embedding. Here we used the fact that the symmetric Fourier-Mukai transform of the structure sheaf $\mathscr{O}_{A^\vee}$ is the structure sheaf $k(0)$ of the closed point $0 \in A(k)$, by (1). $\square$

3.2. Functoriality

In this section, we investigate how the symmetric Fourier-Mukai transform interacts with pushing forward and pulling back by homomorphisms between abelian varieties. Let $f\colon A \to B$ be a homomorphism, and denote by $\hat{f}\colon B^\vee \to A^\vee$ the induced homomorphism between the dual abelian varieties.

Proof of Proposition 1.1. It will be enough to show that

$$\mathsf{FM}_B \circ \mathbf{R} f_* = \mathbf{L}\hat{f}^* \circ \mathsf{FM}_A;$$

the second identity in the theorem follows from this with the help of Theorem 1.1. Using the definition of FM_B and Grothendieck duality, we get

$$\mathsf{FM}_B \circ \mathbf{R}f_* = \mathbf{R}\Phi_{P_A} \circ \mathbf{R}\Delta_B \circ \mathbf{R}f_* = \mathbf{R}\Phi_{P_B} \circ \mathbf{R}f_* \circ \mathbf{R}\Delta_A.$$

This reduces the problem to proving that

$$\mathbf{R}\Phi_{P_B} \circ \mathbf{R}f_* = \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_A}. \tag{8}$$

We make use of the following commutative diagram:

$$\begin{tikzcd}
A^\vee & B^\vee \arrow[l, "\hat{f}"'] & & \\
A \times A^\vee \arrow[u, "p_2"] & A \times B^\vee \arrow[l, "\mathrm{id} \times \hat{f}"'] \arrow[u, "p_2"] \arrow[r, "f \times \mathrm{id}"] \arrow[d, "p_1"] & B \times B^\vee \arrow[r, "p_2"] \arrow[d, "p_1"] & B^\vee \\
 & A \arrow[r, "f"] & B &
\end{tikzcd}$$

The universal property of the Poincaré correspondence gives $(f \times \mathrm{id})^* P_B = (\mathrm{id} \times \hat{f})^* P_A$. Using the projection formula and flat base change, we have

$$\begin{aligned}
\mathbf{R}\Phi_{P_B} \circ \mathbf{R}f_* &= \mathbf{R}(p_2)_*\big(P_B \otimes p_1^* \mathbf{R}f_*\big) = \mathbf{R}(p_2)_*\big(P_B \otimes \mathbf{R}(f \times \mathrm{id})_* p_1^*\big) \\
&= \mathbf{R}(p_2)_*\big((f \times \mathrm{id})^* P_B \otimes p_1^*\big) = \mathbf{R}(p_2)_*\big((\mathrm{id} \times \hat{f})^* P_A \otimes p_1^*\big) \\
&= \mathbf{R}(p_2)_* \mathbf{L}(\mathrm{id} \times \hat{f})^* \big(P_A \otimes p_1^*\big) = \mathbf{L}\hat{f}^* \, \mathbf{R}(p_2)_*\big(P_A \otimes p_1^*\big) \\
&= \mathbf{L}\hat{f}^* \circ \mathbf{R}\Phi_{P_A}.
\end{aligned}$$

This calculation establishes Proposition 1.1. □

Next, we investigate the effect of translations by closed points.

Proof of Proposition 1.2. Once again, it suffices to prove that

$$\mathsf{FM}_A \circ (t_a)_* = (P_a \otimes -) \circ \mathsf{FM}_A$$

because the other identity follows from this with the help of Theorem 1.1. Using Grothendieck duality, we get a natural isomorphism of functors

$$\mathsf{FM}_A \circ (t_a)_* = \mathbf{R}\Phi_{P_A} \circ \mathbf{R}\Delta_A \circ (t_a)_* = \mathbf{R}\Phi_{P_A} \circ (t_a)_* \circ \mathbf{R}\Delta_A,$$

and so the problem is reduced to showing that

$$\mathbf{R}\Phi_{P_A} \circ (t_a)_* = (P_a \otimes -) \circ \mathbf{R}\Phi_{P_A}.$$

We use the following commutative diagram:

$$\begin{tikzcd} A \times A^\vee \arrow[r, "t_a \times \mathrm{id}"] \arrow[d, "p_1"] & A \times A^\vee \arrow[r, "p_2"] \arrow[d, "p_1"] & A^\vee \\ A \arrow[r, "t_a"] & A & \end{tikzcd}$$

Since $(t_a \times \mathrm{id})^* P_A = p_2^* P_a \otimes P_A$, we have

$$\begin{aligned} \mathbf{R}\Phi_{P_A} \circ (t_a)_* &= \mathbf{R}(p_2)_*\Big(P_A \otimes p_1^*(t_a)_*\Big) = \mathbf{R}(p_2)_*\Big(P_A \otimes (t_a \times \mathrm{id})_* p_1^*\Big) \\ &= \mathbf{R}(p_2)_*\Big((t_a \times \mathrm{id})^* P_A \otimes p_1^*\Big) = \mathbf{R}(p_2)_*\Big(p_2^* P_a \otimes P_A \otimes p_1^*\Big) \\ &= P_a \otimes \mathbf{R}(p_2)_*(P_A \otimes p_1^*) = P_a \otimes \mathbf{R}\Phi_{P_A}, \end{aligned}$$

which is exactly what we needed. □

3.3. GV-sheaves

In this section, we prove the assertions about GV-sheaves in the introduction.

Proof of Proposition 1.3. The functor FM_A is exact and contravariant, and so it suffices to show that $\mathsf{FM}_A(\mathscr{F}) \in D^{\leq 0}_{coh}(\mathscr{O}_{A^\vee})$ for every coherent sheaf $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$. Observe that the j-th cohomology sheaf of the complex

$$\mathbf{R}\Delta_A(\mathscr{F}) = \mathbf{R}\mathcal{H}om(\mathscr{F}, \omega_A[\dim A])$$

is supported in codimension $\geq j + \dim A$; equivalently, the dimension of its support is $\leq -j$. In the spectral sequence

$$E_2^{i,j} = R^i(p_2)_*\Big(P \otimes p_1^* R^j\Delta_A(\mathscr{F})\Big) \Longrightarrow \mathcal{H}^{i+j}\,\mathsf{FM}_A(\mathscr{F}),$$

we therefore have $E_2^{i,j} = 0$ whenever $i > -j$, for dimension reasons. We conclude that $\mathcal{H}^n\,\mathsf{FM}_A(\mathscr{F}) = 0$ whenever $n > 0$, which is what we needed to show. □

Suppose now that $\mathscr{F}$ is a GV-sheaf; as in Definition 1.2, we let $\hat{\mathscr{F}} = \mathsf{FM}_A(\mathscr{F})$. Let us compute the restriction of the coherent sheaf $\hat{\mathscr{F}}$ to a closed point $\alpha \in A^\vee(k)$. We can write this restriction in the form

$$\mathbf{L}e^*(t_{-\alpha})_*\hat{\mathscr{F}},$$

with $t_{-\alpha}\colon A^\vee \to A^\vee$ denoting translation by $-\alpha$. We can then combine the identities in Proposition 1.1 and Proposition 1.2 to obtain an isomorphism

$$\mathbf{L}e^*(t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}) = \mathbf{L}e^* \mathsf{FM}_A\big(P_{-\alpha} \otimes \mathscr{F}\big) = \mathbf{R}\Delta_k \mathbf{R}p_*\big(P_{-\alpha} \otimes \mathscr{F}\big), \tag{9}$$

where $p\colon A \to \operatorname{Spec} k$ is the structure morphism. In particular

$$\hat{\mathscr{F}}\big|_\alpha = \operatorname{Hom}_k\Big(H^0(A, \mathscr{F} \otimes P_{-\alpha}), k\Big), \tag{10}$$

and so $\alpha \in A^\vee(k)$ belongs to the support of $\hat{\mathscr{F}}$ if and only if $H^0(A, \mathscr{F} \otimes P_{-\alpha}) \neq 0$, proving (3) from the introduction.

This seems like a good place to prove Proposition 1.4 about the codimension of the loci $S^j(A, \mathscr{F})$.

Proof of Proposition 1.4. Let $\alpha \in A^\vee(k)$ be a closed point. It will be convenient to use the symbol $i_\alpha = t_\alpha \circ e\colon \operatorname{Spec} k \hookrightarrow A^\vee$ for the resulting closed embedding. After taking cohomology in (9), we obtain

$$L^{-j} i_\alpha^* \hat{\mathscr{F}} = \operatorname{Hom}_k\Big(H^j(A, \mathscr{F} \otimes P_{-\alpha}), k\Big).$$

This implies of course that

$$S^j(A, \mathscr{F})(k) = \left\{\, \alpha \in A^\vee(k) \;\middle|\; L^{-j} i_\alpha^* \hat{\mathscr{F}} \neq 0 \,\right\}.$$

All the assertions in the proposition are now general facts about coherent sheaves on nonsingular varieties. For the convenience of the reader, we recall the necessary result (and its simple proof) in the following paragraph. ☐

Let X be a nonsingular algebraic variety over k; for a closed point $x \in X(k)$, we denote by $i_x\colon \operatorname{Spec} k \hookrightarrow X$ the resulting closed embedding. For any coherent sheaf $\mathscr{F}$ on X, there are closed algebraic subsets $S^{-j}(\mathscr{F}) \subseteq X$, indexed by $j \in \mathbb{N}$; on closed points, they are given by the formula

$$S^{-j}(\mathscr{F})(k) = \left\{\, x \in X \;\middle|\; L^{-j} i_x^* \mathscr{F} \neq 0 \,\right\}.$$

Here is the result that we used to prove Proposition 1.4.

Lemma 3.2. *Let X be a nonsingular algebraic variety over k, and let $\mathscr{F}$ be a coherent sheaf on X.*

(a) One has $S^{-(j+1)}(\mathscr{F}) \subseteq S^{-j}(\mathscr{F})$.
(b) Every irreducible component of $S^{-j}(\mathscr{F})$ has codimension $\geq j$.

Proof. The basic fact is that every finitely generated module over a local ring R has an (essentially unique) minimal free resolution, and that when R is regular, the length of this resolution is at most $\dim R$.

Let us prove the first assertion. It amounts to saying that $L^{-j} i_x^* \mathscr{F} = 0$ implies $L^{-(j+1)} i_x^* \mathscr{F} = 0$. The stalk $\mathscr{F}_x$ is a finitely generated module over the local ring $\mathscr{O}_{X,x}$, and in its minimal free resolution, the rank of the free $\mathscr{O}_{X,x}$-module in degree $-j$ is equal to $\dim L^{-j} i_X^* \mathscr{F}$. If $L^{-j} i_X^* \mathscr{F} = 0$, then the minimal free resolution has length at most j, and this trivially implies that $L^{-(j+1)} i_x^* \mathscr{F} = 0$.

Now let us prove the second assertion. Fix an irreducible component Z of $S^{-j}(\mathscr{F})$. After localizing at the generic point of Z, we obtain a finitely generated module $\mathscr{F}_Z$ over the local ring $\mathscr{O}_{X,Z}$. By Serre's theorem, $\mathscr{O}_{X,Z}$ is a regular local ring of dimension $c = \operatorname{codim}(Z, X)$, and so the minimal free resolution of $\mathscr{F}_Z$ has length at most c. Since $\mathscr{F}$ is coherent, there is a Zariski-open subset $U \subseteq X$, containing the generic point of Z, such that $\mathscr{F}\big|_U$ has a free resolution of length at most c. This implies that $L^{-j} i_x^* \mathscr{F} = 0$ for every $j > c$ and every $x \in U(k)$, and since $Z \subseteq S^{-j}(\mathscr{F})$, the conclusion is that $c \geq j$. ☐

We finish our discussion of the symmetric Fourier-Mukai transform and its properties by proving Proposition 1.6 from the introduction.

Proof of Proposition 1.6. In $D^b_{coh}(\mathscr{O}_B)$, we have a distinguished triangle

$$f_* \mathscr{F} \to \mathbf{R} f_* \mathscr{F} \to \tau_{\geq 1} \mathbf{R} f_* \mathscr{F} \to \cdots,$$

where $\tau_{\geq 1} \colon D^b_{coh}(\mathscr{O}_B) \to D^{\geq 1}_{coh}(\mathscr{O}_B)$ is one of the truncation functors for the standard t-structure. Applying FM_B, which is exact and contravariant, we obtain another distinguished triangle

$$\mathsf{FM}_B(\tau_{\geq 1} \mathbf{R} f_* \mathscr{F}) \to \mathsf{FM}_B(\mathbf{R} f_* \mathscr{F}) \to \mathsf{FM}_B(f_* \mathscr{F}) \to \cdots$$

in $D^b_{coh}(\mathscr{O}_{B^\vee})$. The complex on the left belongs to $D^{\leq -1}_{coh}(\mathscr{O}_{B^\vee})$, due to Proposition 1.3; also $\mathsf{FM}_B(\mathbf{R} f_* \mathscr{F}) = \mathbf{L} \hat{f}^* \, \mathsf{FM}_A(\mathscr{F}) = \mathbf{L} \hat{f}^* \hat{\mathscr{F}}$ by Proposition 1.1. Going from the distinguished triangle to the long exact sequence in cohomology leads to

$$\hat{f}^* \hat{\mathscr{F}} = \mathcal{H}^0 \, \mathsf{FM}_B(f_* \mathscr{F}) = \mathsf{FM}_B(f_* \mathscr{F}),$$

due to the fact that $f_* \mathscr{F}$ is also a GV-sheaf. ☐

4. Applications

Now let us turn to the applications to GV-sheaves and M-regular sheaves. We start by proving two simple lemmas. Recall the notation

$$K_1 \boxtimes K_2 = p_1^* K_1 \otimes p_2^* K_2 \in D^b_{coh}(\mathscr{O}_{A\times B})$$

for objects $K_1 \in D^b_{coh}(\mathscr{O}_A)$ and $K_2 \in D^b_{coh}(\mathscr{O}_B)$.

Lemma 4.1. *We have a natural isomorphism of bifunctors*

$$\mathsf{FM}_{A\times B} \circ \boxtimes = \mathsf{FM}_A \boxtimes \mathsf{FM}_B$$

from $D^b_{coh}(\mathscr{O}_A) \times D^b_{coh}(\mathscr{O}_B)$ *to* $D^b_{coh}(\mathscr{O}_{A^\vee}) \times D^b_{coh}(\mathscr{O}_{B^\vee})$.

Proof. This follows from the isomorphism $P_{A\times B} = p_{13}^* P_A \otimes p_{24}^* P_B$ on the abelian variety $A \times B \times A^\vee \times B^\vee$, by the usual base change arguments. □

The following lemma shows the effect of tensoring by an ample line bundle.

Lemma 4.2. *Let L be an ample line bundle on A, and $K \in D^b_{coh}(\mathscr{O}_A)$. Then the k-vector space* $\operatorname{Hom}_k\big(H^i(A, L\otimes K), k\big)$ *is isomorphic to a direct summand of*

$$H^{-i}\big(A^\vee, L \otimes \phi_L^* \mathsf{FM}_A(K)\big) \otimes H^0(A, L)^*,$$

where $\phi_L\colon A \to A^\vee$ is the homomorphism in (4).

Proof. If we apply Proposition 1.1 to the morphism $p\colon A \to \operatorname{Spec} k$, we obtain

$$\mathbf{R}\Delta_{\operatorname{Spec} k}\mathbf{R}p_*(L\otimes K) = \mathbf{L}e^* \mathsf{FM}_A(L\otimes K).$$

Now $L\otimes K = \mathbf{L}\Delta^*(L\boxtimes K)$, and the diagonal homomorphism $\Delta\colon A \to A\times A$ is dual to the multiplication morphism $m\colon A^\vee \times A^\vee \to A^\vee$. Another application of Proposition 1.1, followed by Lemma 4.1, therefore gives

$$\mathsf{FM}_A(L\otimes K) = \mathsf{FM}_A\, \mathbf{L}\Delta^*(L\boxtimes K) = \mathbf{R}m_*\big(\hat{L} \boxtimes \mathsf{FM}_A(K)\big),$$

due to the fact that L is a GV-sheaf. In the rest of the proof, we use the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\phi_L"] \arrow[dr, bend right, "p"'] & A^\vee \arrow[r, "(i,\mathrm{id})"] \arrow[d, "p"] & A^\vee \times A^\vee \arrow[d, "m"] \\
& \operatorname{Spec} k \arrow[r, "e"] & A^\vee
\end{tikzcd}$$

Base change for the smooth morphism m yields an isomorphism

$$\mathbf{R}\Delta_k \mathbf{R}p_*(L \otimes K) = \mathbf{R}p_*\big(i^*\hat{L} \otimes \mathsf{FM}_A(K)\big).$$

Since ϕ_L is an isogeny, $\mathbf{R}p_*\big(i^*\hat{L} \otimes \mathsf{FM}_A(K)\big)$ is isomorphic to a direct summand of

$$\mathbf{R}(p \circ \phi_L)_* \phi_L^*\big(i^*\hat{L} \otimes \mathsf{FM}_A(K)\big).$$

Using Proposition 1.5, we can rewrite this in the form

$$\mathbf{R}p_*\big(\phi_L^* i^*\hat{L} \otimes \phi_L^* \mathsf{FM}_A(K)\big) = \mathbf{R}p_*\big(L \otimes \phi_L^* \mathsf{FM}_A(K)\big) \otimes H^0(A, L)^*.$$

We now get the desired result by taking cohomology. □

We can now prove Hacon's criterion for GV-sheaves in terms of vanishing.

Proof of Theorem 1.2. Suppose that $\mathscr{F}$ is a GV-sheaf on A. The pullback of $\mathscr{F}$ by an isogeny $\varphi\colon A' \to A$ is again a GV-sheaf on A'; this follows from the identities in Proposition 1.1 because

$$\mathsf{FM}_{A'}(\varphi^*\mathscr{F}) = \hat{\varphi}_* \mathsf{FM}_A(\mathscr{F}).$$

Here we used the fact that isogenies are flat and finite. We can therefore assume without loss of generality that $\varphi = \mathrm{id}_A$. Now let L be any ample line bundle on A. To prove the first half of the theorem, we need to show that

$$H^i(A, L \otimes \mathscr{F}) = 0 \quad \text{for } i > 0.$$

Lemma 4.2 reduces the problem to showing that

$$H^{-i}\big(A^\vee, L \otimes \phi_L^* \mathsf{FM}_A(\mathscr{F})\big) = 0 \quad \text{for } i > 0.$$

But this is obvious because $\mathsf{FM}_A(\mathscr{F})$ is a sheaf (and ϕ_L is flat).

Next, we prove the converse. Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ has the property stated in the theorem. We need to explain why $\mathcal{H}^i \mathsf{FM}_A(\mathscr{F}) = 0$ for every $i \neq 0$; in fact, by Proposition 1.3, it would be enough to consider only $i < 0$. After choosing a sufficiently ample line bundle L' on $A^\vee$, the problem reduces to showing that

$$H^i\big(A^\vee, L' \otimes \mathsf{FM}_A(\mathscr{F})\big) = 0 \quad \text{for } i \neq 0.$$

Another application of Lemma 4.2, but with the roles of A and $A^\vee$ interchanged, shows that it is enough to prove the vanishing

$$H^{-i}(A^\vee, L' \otimes \phi_{L'}^* \mathscr{F}) = 0 \quad \text{for } i \neq 0,$$

where $\phi_{L'}\colon A^\vee \to A$ is the isogeny determined by L'. This is obvious for $i > 0$, and for $i < 0$, it is exactly the condition in the statement of the theorem. ☐

In the remainder of the paper, we relate M-regularity to (continuous) global generation. The underlying idea is extremely simple. Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ is a GV-sheaf, and set $\hat{\mathscr{F}} = \mathsf{FM}_A(\mathscr{F}) \in \mathrm{Coh}(\mathscr{O}_{A^\vee})$. Let $S \subseteq A^\vee(k)$ be any set of closed points. A small calculation shows that the evaluation morphism

$$\bigoplus_{\alpha \in S} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F} \tag{11}$$

corresponds, under the (contravariant!) functor FM_A, to the morphism

$$\hat{\mathscr{F}} \to \prod_{\alpha \in S} \hat{\mathscr{F}} \otimes k(\alpha); \tag{12}$$

here $k(\alpha) = (t_\alpha \circ e)_* \mathscr{O}_{\mathrm{Spec}\, k}$. Now suppose that $\mathscr{F}$ is M-regular, and that the set $S \subseteq A^\vee(k)$ is dense in the Zariski topology. Since $\hat{\mathscr{F}}$ is torsion-free, the morphism in (12) must be injective, and so we should expect the evaluation morphism in (11) to be surjective. The only problem is that infinite sums are not coherent (and infinite products are not even quasi-coherent), which means that our results about FM_A do not apply. This can be dealt with by making a pointwise argument.

We now turn the above idea into a rigorous proof.

Proposition 4.1. *Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ is M-regular. Let $S \subseteq A^\vee(k)$ be any subset that is dense in the Zariski topology. Then there is a finite subset $F \subseteq S$ such that the evaluation morphism*

$$\bigoplus_{\alpha \in F} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F}$$

is surjective.

Proof. Let $a \in A(k)$ be any closed point. Consider the evaluation morphism

$$\mathrm{ev}_\alpha\colon H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F}.$$

To show that ev_α is surjective in a neighborhood of a closed point $a \in A(k)$, it is enough (by Nakayama's lemma) to show that the induced morphism

$$H^0(A, \mathscr{F} \otimes P_{-\alpha}) \to (\mathscr{F} \otimes P_{-\alpha})\big|_a \tag{13}$$

is surjective. By Proposition 1.2, we have

$$\mathsf{FM}_A(\mathscr{F} \otimes P_{-\alpha}) = (t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}) = (t_{-\alpha})_* \hat{\mathscr{F}}.$$

Combining this with the formula in (10), we get

$$\begin{aligned}(\mathscr{F} \otimes P_{-\alpha})\big|_a &= \mathrm{Hom}_k\bigl(H^0(A^\vee, (t_{-\alpha})_* \hat{\mathscr{F}} \otimes P_{-a}), k\bigr)\\ &= \mathrm{Hom}_k\bigl(H^0(A^\vee, \hat{\mathscr{F}} \otimes P_{-a}), k\bigr),\end{aligned}$$

using the translation invariance of P_{-a}. It follows that the morphism in (13) is dual to the morphism

$$H^0(A^\vee, \hat{\mathscr{F}} \otimes P_{-a}) \to (\hat{\mathscr{F}} \otimes P_{-a})\big|_\alpha. \tag{14}$$

The conclusion is that ev_α is surjective in a neighborhood of a closed point $a \in A(k)$ if and only if the morphism in (14) is injective.

Now $H^0(A^\vee, \hat{\mathscr{F}} \otimes P_{-a})$ is a finite-dimensional k-vector space. Since $\hat{\mathscr{F}}$ is a torsion-free coherent sheaf, and $S \subseteq A^\vee(k)$ is dense in the Zariski topology, we can therefore find a *finite* subset $S(a) \subseteq S$ such that the morphism

$$H^0(A^\vee, \hat{\mathscr{F}} \otimes P_{-a}) \to \prod_{\alpha \in S(a)} (\hat{\mathscr{F}} \otimes P_{-a})\big|_\alpha$$

is injective. It follows that the dual morphism

$$\bigoplus_{\alpha \in S(a)} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \to (\mathscr{F} \otimes P_{-\alpha})\big|_a$$

is surjective, and hence that the evaluation morphism

$$\bigoplus_{\alpha \in S(a)} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F}$$

is surjective on a Zariski-open subset containing the given closed point $a \in A(k)$. Finitely many of these open subsets cover A; consequently, there is a finite subset $F \subseteq S$ with the property that

$$\bigoplus_{\alpha \in F} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F}$$

is surjective. This completes the proof. $\square$

We now show that M-regularity implies (continuous) global generation.

Proof of Theorem 1.3. The first assertion follows immediately from Proposition 4.1. For the second one, let $S \subseteq A^\vee(k)$ be the set of all torsion points; this is certainly dense in the Zariski topology. By Proposition 4.1, there is a finite subset $F \subseteq S$ such that

$$\bigoplus_{\alpha \in F} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes P_\alpha \to \mathscr{F}$$

is surjective. Let $\varphi\colon A' \to A$ be an isogeny with the property that $\hat{\varphi}(\alpha) = 0$ for every $\alpha \in F$; for example, multiplication by the greatest common divisor of the orders of the points in F will do. The induced morphism

$$\bigoplus_{\alpha \in F} H^0(A, \mathscr{F} \otimes P_{-\alpha}) \otimes \mathscr{O}_{A'} \to \varphi^* \mathscr{F}$$

is then still surjective, and so $\varphi^*\mathscr{F}$ is globally generated. □

In fact, the argument above leads to the following stronger result.

Theorem 4.1. *Suppose that $\mathscr{F} \in \mathrm{Coh}(\mathscr{O}_A)$ is torsion-free. Let $S \subseteq A(k)$ be any subset that is dense in the Zariski topology. Then there is a finite subset $F \subseteq S$ such that the evaluation morphism*

$$\bigoplus_{a \in F} H^0(A^\vee, \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}) \otimes P_a \to \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})$$

is surjective.

Proof. Let $a \in A(k)$ and $\alpha \in A^\vee(k)$ be any pair of closed points. We have

$$\begin{aligned} \mathbf{R}\Delta_k \mathbf{L}e^*(t_{-a})_*(\mathscr{F} \otimes P_{-\alpha}) &= \mathbf{R}p_*\big((t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \\ &= \mathbf{R}p_*\big(\mathsf{FM}_A(\mathscr{F}) \otimes t_{-\alpha}^* P_{-a}\big) \end{aligned}$$

using Proposition 1.1, Proposition 1.2, and the projection formula (for $t_{-\alpha}$). Our fixed trivializations for $(e \times \mathrm{id})^* P_A$ and $(\mathrm{id} \times e)^* P_A$ determine an isomorphism

$$(\mathrm{id} \times t_{-\alpha})^* P_A = p_1^* P_{-\alpha} \otimes P_A,$$

which we can use to rewrite the identity above in the form

$$\mathbf{R}\Delta_k \mathbf{L}e^*(t_{-a})_*(\mathscr{F} \otimes P_{-\alpha}) = \mathbf{R}p_*\big(\mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \otimes P_{(-a,-\alpha)}.$$

Here $P_{(-a,-\alpha)}$ is the fiber of the Poincaré bundle at $(-a,-\alpha) \in (A \times A^{\vee})(k)$. Taking cohomology in degree zero, we see that

$$(\mathscr{F} \otimes P_{-\alpha})\big|_a \quad \text{is dual to} \quad H^0\big(A^{\vee}, \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \otimes P_a\big|_{\alpha},$$

using the identity in (5) to replace $P_{(-a,-\alpha)}$. Similarly, we have

$$\mathbf{R}\Delta_k \mathbf{R}p_*(P_{-\alpha} \otimes \mathscr{F}) = \mathbf{L}e^*(t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}),$$

and since $\mathsf{FM}_A(\mathscr{F}) \in D^{\leq 0}_{coh}(\mathscr{O}_{A^{\vee}})$ by Proposition 1.3, it follows that

$$H^0(A, \mathscr{F} \otimes P_{-\alpha}) \quad \text{is dual to} \quad L^0 e^*(t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}) = \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})\big|_{\alpha}.$$

The conclusion is that that the evaluation morphism

$$H^0(A, \mathscr{F} \otimes P_{-\alpha}) \to (\mathscr{F} \otimes P_{-\alpha})\big|_a$$

on A is dual to the evaluation morphism

$$H^0\big(A^{\vee}, \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \otimes P_a\big|_{\alpha} \to L^0 e^*(t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}).$$

on $A^{\vee}$. Note that the second morphism fits into a commutative diagram

$$\begin{tikzcd}
H^0\big(A^{\vee}, \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \otimes P_a\big|_{\alpha} \arrow[r] \arrow[d] & L^0 e^*(t_{-\alpha})_* \mathsf{FM}_A(\mathscr{F}) \arrow[d, equal] \\
H^0\big(A^{\vee}, \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F}) \otimes P_{-a}\big) \otimes P_a\big|_{\alpha} \arrow[r] & \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})\big|_{\alpha}.
\end{tikzcd}$$

Now we can argue as before to prove the assertion.

Fix a closed point $\alpha \in A^{\vee}(k)$. Since $\mathscr{F}$ is torsion-free, and $S \subseteq A(k)$ is dense in the Zariski topology, there is a finite subset $S(\alpha) \subseteq S$ such that the morphism

$$H^0(A, \mathscr{F} \otimes P_{-\alpha}) \to \prod_{a \in S(\alpha)} (\mathscr{F} \otimes P_{-\alpha})\big|_a$$

is injective. After dualizing, it follows that the morphism

$$\bigoplus_{a \in S(\alpha)} H^0\big(A^{\vee}, \mathcal{H}^0 \mathsf{FM}_A \otimes P_{-a}\big) \otimes P_a\big|_{\alpha} \to \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})\big|_{\alpha}$$

is surjective. By Nakayama's lemma, this means that

$$\bigoplus_{a \in S(\alpha)} H^0\big(A^{\vee}, \mathcal{H}^0 \mathsf{FM}_A \otimes P_{-a}\big) \otimes P_a \to \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})$$

is surjective on a Zariski-open subset containing the closed point $\alpha \in A^\vee(k)$. Finitely many of these open subsets cover $A^\vee$; consequently, there is a finite subset $F \subseteq S$ with the property that

$$\bigoplus_{a \in F} H^0(A^\vee, \mathcal{H}^0 \mathsf{FM}_A \otimes P_{-a}) \otimes P_a \to \mathcal{H}^0 \mathsf{FM}_A(\mathscr{F})$$

is surjective. This finishes the proof. □

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