

Homological duality for covering groups of reductive p -adic groups

DRAGOȘ FRĂȚILĂ AND DIPENDRA PRASAD

To Dick Gross, with admiration

Abstract: In this largely expository paper, we extend properties of the homological duality functor $\mathrm{RHom}_{\mathcal{H}}(-, \mathcal{H})$ where \mathcal{H} is the Hecke algebra of a reductive p -adic group, to the case where it is the Hecke algebra of a finite central extension of a reductive p -adic group. The most important properties are that $\mathrm{RHom}_{\mathcal{H}}(-, \mathcal{H})$ is concentrated in a single degree for irreducible representations and that it gives rise to Schneider–Stuhler duality for Ext groups (a Serre functor like property). Our simple proof is self-contained and bypasses the localization techniques of [SS97, Bez04] improving slightly on [NP20]. Along the way we also study Grothendieck–Serre duality with respect to the Bernstein center and provide a proof of the folklore result that on admissible modules this functor is nothing else but the contragredient duality. We single out a necessary and sufficient condition for when these three dualities agree on finite length modules in a given block. In particular, we show this is the case for all cuspidal blocks as well as, due to a result of Roche [Roc02], on all blocks with trivial stabilizer in the relative Weyl group.

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Received June 1, 2021.

2010 Mathematics Subject Classification: Primary 11F70; secondary 22E55.

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1. Introduction

1.1.

Let G be a reductive p -adic group or a covering group (finite central extension) of such a group. The homological duality for the abelian category $\mathcal{M}(G)$ of smooth representations of G is defined at the level of derived categories as

$$D_h := \mathrm{RHom}_{\mathcal{H}}(-, \mathcal{H}) : \mathcal{D}^b(\mathcal{M}(G)) \rightarrow \mathcal{D}^b(\mathcal{M}(G))^{op}$$

where \mathcal{H} is the Hecke algebra of G .

An important property established by Bernstein in his Harvard notes [Ber92] is that if π is irreducible then $D_h(\pi)$ is concentrated in a single degree (and is irreducible). We will denote this representation by $\mathbb{D}_h(\pi)$.

On the other hand, in [SS97] Schneider and Stuhler also prove this property using localization techniques on the Bruhat–Tits building. Moreover, they show that this functor \mathbb{D}_h enjoys a Serre functor like property, namely that for any irreducible representation π of G and any smooth representation π' of G , there is a natural isomorphism

$$(1.1) \quad \mathrm{Ext}_G^i(\pi, \pi')^* \simeq \mathrm{Ext}_G^{d(\pi)-i}(\pi', \mathbb{D}_h(\pi)^\vee),$$

where $d(\pi) \geq 0$ is an integer which is bounded by the split rank of G that we define later. Actually in [SS97], the isomorphism (1.1) was proved only in the subcategory of representations with a fixed central character. The more general version was proved in [NP20] and [BBK18].

One of the aims of this work is to present a direct proof of (1.1) for covering groups that does not make use of [SS97] or of localization techniques. The strategy is to first show that the full homological duality D_h satisfies a Serre functor like property

$$\mathrm{RHom}_{\mathcal{H}}(\pi, \pi')^* \simeq \mathrm{RHom}_{\mathcal{H}}(\pi', D_h(\pi)^\vee)$$

for any smooth representations π and π' with π finitely generated. This is easy and follows from basic adjunctions naturally extended to idempotented algebras. For convenience of the reader, and for lack of a precise reference, we provide all the details in Section 2.3.

The second step, namely showing that $D_h(\pi)$ is concentrated in a single degree for an irreducible representation, is non-formal and is based on several ingredients: Bernstein decomposition, second adjointness and a special property of the algebra governing a cuspidal block (Frobenius-symmetric algebra over its center). Although the proof is already in [Ber92] we hope that the argument that we present is more streamlined.

Another property of the homological functor D_h is that on finite length cuspidal representations it agrees (up to a shift) with the contragredient functor. A sketch of this result for irreducible representations appears already in [Ber92] but we were not able to supply the details so we decided to include a different proof with full details. To this effect, we were led to study the Grothendieck–Serre duality over the Bernstein center $D_{GS/\mathcal{Z}}$ (see Section 12 for the definition). In particular, we provide a proof of the folklore result that $D_{GS/\mathcal{Z}}$ agrees with the contragredient for admissible representations. Moreover, we identify a necessary and sufficient condition (see (FsG)) for the two functors D_h and $D_{GS/\mathcal{Z}}$ to agree (up to a shift) on a Bernstein block. In order to state the condition, we need to introduce some notation.

If $\mathfrak{s} = [L, \rho]$ is a cuspidal datum, up to conjugation and inertia, then there is an equivalence of the Bernstein block $\mathcal{M}(G)_{\mathfrak{s}}$ with the module category $\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ for some algebra $\mathcal{R}_{\mathfrak{s}}$ with center $\mathcal{Z}_{\mathfrak{s}}$ that can be described explicitly (see Theorem 9.12). The condition alluded to before, which we call Frobenius-symmetric-Gorenstein, since it is reminiscent of both these properties, is

$$D_{GS/\mathcal{Z}_{\mathfrak{s}}}(\mathcal{R}_{\mathfrak{s}}) \simeq \mathcal{R}_{\mathfrak{s}}[d] \text{ as } \mathcal{R}_{\mathfrak{s}}\text{-bimodules}$$

where $d = d(\mathfrak{s})$ is the split rank of the center of L (equals the Krull dimension of $\mathcal{Z}_{\mathfrak{s}}$). We are able to check this condition on a cuspidal block $\mathfrak{s} = (\rho, G)$ because in this case $\mathcal{Z}_{\mathfrak{s}}$ is a Laurent-polynomial algebra and $\mathcal{R}_{\mathfrak{s}}$ is an Azumaya algebra (see Proposition 9.4 and Corollary 9.6). Therefore,

we deduce that the homological duality and the Grothendieck–Serre duality agree with the contragredient duality on all finite length cuspidal representations in a given block. Moreover, if $\mathfrak{s} = [L, \rho]$ is a cuspidal datum, by a result of Roche [Roc02, Theorem 3.1], parabolic induction induces an equivalence of the blocks $\mathcal{M}(L)_{[\rho]} \rightarrow \mathcal{M}(G)_{\mathfrak{s}}$ if and only if the stabilizer of the inertia class $[\rho]$ in the relative Weyl group $N_G(L)/L$ is trivial, i.e., if and only if there is no non-trivial $w \in N_G(L)/L$ such that ${}^w\rho \simeq \rho\chi$ for some unramified character χ of L . We also deduce that for these blocks, D_h and $D_{GS/Z}$ agree (up to a shift).

There is yet another duality¹ on smooth representations, namely the Aubert–Zelevinski duality. Originally, it arose in representation theory of finite groups of Lie type where it was defined on the Grothendieck group of finite dimensional representations. Deligne–Lusztig introduced what is now called the Deligne–Lusztig complex in order to have a definition of the involution at the level of representations. For p -adic groups as well as their covering groups, the involution so constructed is usually called the Aubert–Zelevinski involution. It can be proved ([Aub95, BBK18, Bez04]) that the Aubert–Zelevinski involution defines an exact functor from the abelian category $\mathcal{M}(G)$ to itself, hence it extends trivially to a functor on the derived category $\mathcal{D}^b(\mathcal{M}(G))$:

$$D_{AZ}: \mathcal{D}^b(\mathcal{M}(G)) \rightarrow \mathcal{D}^b(\mathcal{M}(G)).$$

The same involution was considered by Schneider–Stuhler in [SS97] and by Bezrukavnikov in [Bez04, Theorem 4.13] where the following isomorphism of functors was proved using localization techniques

$$(1.2) \quad D_{AZ} \circ D_{GS/Z} \simeq D_h.$$

Actually, in [SS97] this was shown only for finite length modules and in the Grothendieck group whereas in [Bez04] the more general result was established. The isomorphism (1.2) was recently revisited in [BBK18] where a simpler proof was given using the geometry of the wonderful compactification.

At the level of Grothendieck groups, it was shown in [Aub95] that D_{AZ} commutes with the contragredient and the commutation formulae with the parabolic induction and restriction functors were proved. None of these are known for representations, say of finite length, of covering groups, as even for linear groups, all these assertions are proved by relating Aubert–Zelevinski

¹It is indeed involutive but *covariant* and so the name duality is slightly misleading.

involution to the homological duality (see (1.2)) which is not yet available for covering groups.

Results similar to Aubert's were proved in the context of smooth representations of covering groups in the work of Ban–Jantzen [BJ, BJ16] (again for the Grothendieck group of finite length representations). In this work we do not address the question of whether (1.2) holds for covering groups nor do we study the Aubert–Zelevinsky involution in this context. An immediate obstacle to generalizing the approach of [BBK18] is that as covering groups are no longer linear, we do not have at our disposal an obvious wonderful compactification. However, in favor of the isomorphism (1.2) is the fact that $[d] \circ D_h = D_{GS/\mathcal{Z}}$ on cuspidal blocks, a result that we prove in Section 12. By the aforementioned result of Roche, this continues to be the case for blocks $\mathfrak{s} = [L, \rho]$ such that the stabilizer of the inertia class of ρ is trivial in the relative Weyl group $N_G(L)/L$.

One reason for interest in homological duality, and the attendant Serre functor property, is in the context of Ext analogues of branching laws, as discussed in [Pra18]. Recall that usually the branching laws in the context of p -adic group are considered for a representation π_1 of a reductive group G to a representation π_2 of a closed subgroup H , as $\mathrm{Hom}_H(\pi_1, \pi_2)$ (and never as $\mathrm{Hom}_H(\pi_2, \pi_1)$). It has been suggested in [Pra18] that the success of these branching laws in the various cases studied stems from the vanishing of $\mathrm{Ext}_H^i(\pi_1, \pi_2), i > 0$, whereas by the Serre functor property, $\mathrm{Ext}_H^i(\pi_2, \pi_1) = 0, i < d(\pi_2)$. In particular, $\mathrm{Hom}_H(\pi_2, \pi_1)$ is typically zero (so no wonder it is never considered!), and shows up only through the higher extension groups, i.e., $\mathrm{Ext}_H^{d(\pi_2)}(\pi_2, \pi_1)$.

A big chunk of the paper is expository: we give an exposition of some basic representation theoretic results in this context to show that they generalize in a naive way to the case of covering groups culminating with Bernstein's decomposition (following closely the linear setting). We also include a sketch of the proof of second adjointness (following the exposition in [Ren10, VI.9.7] of the linear case which makes use of completion functors in order to streamline the argument). We include a short discussion of Grothendieck–Serre duality limiting ourselves to the context that is sufficient for the applications we have in mind. The reader familiar with the above classical results should directly consult §10, §11, §12.

1.2.

Below follows a more precise description of our work. Let $G = \mathbb{G}(F)$ be a p -adic reductive group and let \tilde{G} be a finite covering group of G . Denote by

\mathcal{H} the Hecke algebra of \tilde{G} and by $\mathcal{M}(\tilde{G})$ the category of smooth complex representations of \tilde{G} . The Levi subgroups and parabolic subgroups of \tilde{G} are, by definition, the pullbacks of those of G . We have parabolic induction and restriction functors (Jacquet modules) defined as in the linear case. A representation is called cuspidal if it is killed by all proper parabolic restriction functors.

We denote by $\mathcal{B}(\tilde{G})$ the equivalence classes (conjugation and inertia) of pairs (\tilde{L}, ρ) of a Levi subgroup \tilde{L} and an irreducible cuspidal representation ρ of \tilde{L} .

The following is the Bernstein decomposition for \tilde{G} (see [Ber92, Ber84, Ren10] for the linear case):

Theorem 1.1. *We have a block decomposition*

$$\mathcal{M}(\tilde{G}) \simeq \prod_{\mathfrak{s} \in \mathcal{B}(\tilde{G})} \mathcal{M}(\tilde{G})_{\mathfrak{s}}.$$

Moreover, each block $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ is equivalent to the category of modules over some \mathbb{C} -algebra $\mathcal{R}_{\mathfrak{s}}$ containing a finitely generated commutative subalgebra over which it is finite.

We will outline the proof in Section 7 which follows closely the linear case as exposed in [Ber92] or [Ren10]. To that end, and to fix the notation, we will collect in Sections 1, 2, 3, 4 all the necessary results from the classical theory (i.e., the linear case) as well as some rudiments from category theory that go into the proof of it. This is meant to convince the reader that the same strategy as in the linear case gives also the Bernstein decomposition for covering groups.

Using the second adjointness theorem of Bernstein, we prove the equivalence of $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ with $\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ in Section 9. For completeness, we also give in Section 8 a skeleton of the proof of the second adjointness which is again meant to convince the reader that the proof from the linear case goes through without changes for covering groups. The argument we present follows [Ren10] rather than [Ber92] in that it uses completion of modules in order to streamline the proof. We take advantage of this opportunity to introduce the completion functors in Section 8.1 and we show that (at least on admissible representations) it allows one to recover the lost properties of the invariants $(-)^N$ functor. See also Corollary 8.14 for a far more advanced version (essentially equivalent to the second adjointness theorem).

For an irreducible representation π , the block $\mathfrak{s} = [\tilde{L}, \rho]$ that corresponds to π is called the cuspidal support of π . We denote by $d(\pi) = d(\mathfrak{s}) = d(\tilde{L})$, the dimension of a maximal split torus in the center of the algebraic group \mathbb{L} .

The category of (smooth) representations of \tilde{G} has finite global dimension (see Section 3.4). Consider the homological duality functor D_h on the bounded derived category of \tilde{G} -modules

$$D_h := \mathrm{RHom}_{\mathcal{H}}(-, \mathcal{H}) : \mathcal{D}^b(\mathcal{M}(\tilde{G})) \rightarrow \mathcal{D}^b(\mathcal{M}(\tilde{G}))^{op}.$$

It is surprisingly easy to show that $(\)^\vee \circ D_h$ satisfies a Serre-functor-like property for the full RHom , namely:

Theorem 1.2 (see Corollary 2.23). *For $\pi, \pi' \in \mathcal{D}^b(\mathcal{M}(\tilde{G}))$ with π finitely generated, we have a natural pairing of complexes of \mathbb{C} -vector spaces*

$$\mathrm{RHom}_{\mathcal{H}}(\pi, \pi') \otimes_{\mathbb{C}}^L \mathrm{RHom}_{\mathcal{H}}(\pi', D_h(\pi)^\vee) \rightarrow \mathrm{RHom}_{\mathcal{H}}(\pi, D_h(\pi)^\vee) \rightarrow \mathbb{C}$$

providing a natural isomorphism

$$\mathrm{RHom}_{\mathcal{H}}(\pi, \pi')^* = \mathrm{RHom}_{\mathcal{H}}(\pi', D_h(\pi)^\vee)$$

where $(-)^*$ is taking the dual vector space degree-wise.

Notice that we do not claim that $\mathrm{RHom}_{\mathcal{H}}(\pi, D_h(\pi)^\vee) \rightarrow \mathbb{C}$ is an isomorphism in general but merely that such a canonical map exists.

For a parabolic \tilde{P} with Levi decomposition $\tilde{P} = \tilde{L}N$, we denote by $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$ (resp., $\mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$) the parabolic induction (resp., restriction) functors. Here are further homological properties that one can prove about D_h :

Theorem 1.3. *The functor $D_h : \mathcal{D}^b(\mathcal{M}(\tilde{G})) \rightarrow (\mathcal{D}^b(\mathcal{M}(\tilde{G})))^{op}$ enjoys the following properties*

1. *If $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\mathrm{fl}}$ is a finite length representation in a fixed Bernstein block, then $D_h(\pi)$ is concentrated in degree $d(\mathfrak{s})$,*
2. *$D_h^2 \simeq \mathrm{Id}$,*
3. *If π is cuspidal of finite length and lives in a single block, then*

$$D_h(\pi) = \pi^\vee[-d(\pi)],$$

4. *$D_h \circ \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}} = \mathbf{i}_{\tilde{L}, \tilde{P}^-}^{\tilde{G}} \circ D_h$ and $D_h \circ \mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}} = \mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}} \circ D_h$,*

²In the context of finite dimensional algebras, this functor is called *Nakayama duality*, and it was observed in [BK89, 3.2 Example 3] that it is a Serre functor.

5. $\mathbb{D}_h := D_h[d(\mathfrak{s})]$ restricted to finite length representations in $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ is an exact involution providing an equivalence

$$\mathbb{D}_h : \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}} \xrightarrow{\sim} (\mathcal{M}(\tilde{G})_{\mathfrak{s}^{\vee}}^{\text{fl}})^{\text{op}},$$

where \mathfrak{s}^{\vee} is the contragredient block of \mathfrak{s} .

Part (1) is proved in Section 10.1. The involution statement is shown in Corollary 10.7. Part (3) is showed in Section 12 after a quick interlude on the Grothendieck–Serre duality. The commutation with parabolic induction and restriction is dealt with in Section 10.2. The last part is a consequence of (1),(2) and (3). See also Corollary 10.7.

Let us say a few words about the Grothendieck–Serre duality which we will briefly review in Sections 2.4 and 12 in the generality that we need. Any representation $\pi \in \mathcal{M}(\tilde{G})$, besides having an action of \tilde{G} , has also an action of the Bernstein center \mathcal{Z} , and by definition these two actions commute. Therefore, any operation that one does to a \mathcal{Z} -module will produce again a \tilde{G} -module. The most trivial operation one can think of is taking the contragredient. This works well, in the sense that it is an involution, on admissible representations. One would like to extend the contragredient in a functorial way to all finitely generated modules such that it stays an involution. Working with a fixed Bernstein block at a time, the question now is sent in the commutative algebra court: we have a nice ring (invariants under a finite group of a Laurent polynomial algebra, in particular Cohen–Macaulay) and finitely generated modules over it. We would like to extend the contragredient from finite length modules to all finitely generated modules in such a way that it stays an involution. One way to do this, coming from algebraic geometry, is to use the Grothendieck–Serre duality. Although all the properties are nice and formal, one must now work in the derived category since the abelian category of modules is not preserved anymore under this duality.

For a commutative regular ring A of Krull dimension d (always containing a field k), the normalized dualizing (or canonical) complex is defined to be $\omega_A^{\circ} := \omega_A[d] \in \mathcal{D}^b(A\text{-mod})$, where $\omega_A = \Lambda^d \Omega_A^1$ is the module of top differentials on A . If $A \rightarrow B$ is a finite map of commutative rings and ω_A° is the normalized dualizing complex of A , then $\omega_B^{\circ} := \text{RHom}_A(B, \omega_A^{\circ})$ is the normalized dualizing complex of B .

In our context, the (component of the) Bernstein center $\mathcal{Z}_{\mathfrak{s}}$ is Cohen–Macaulay hence it is finite free over a regular subalgebra. The above paragraph allows us to see that it has a normalized dualizing complex which moreover lives in a single degree (this is a characterization of Cohen–Macaulay rings).

For a commutative ring A with a normalized dualizing complex ω_A° , one defines the Grothendieck–Serre duality as

$$D_{GS}: \mathcal{D}^b(A\text{-mod}) \rightarrow (\mathcal{D}^b(A\text{-mod}))^{op}$$

$$M \mapsto \mathrm{RHom}_A(M, \omega_A^\circ).$$

This has the nice property of being an involution $D_{GS}^2 \simeq \mathrm{Id}$ when restricted to finitely generated modules. It is not obvious how to compute $D_{GS}(M)$ for a particular module M . In general, it is a complex of modules even if M is only a module.

However, a particular case in which $D_{GS}(M)$ is easy to compute is that of finite length modules M (over an arbitrary ring A containing the field $k!$). Indeed, we have $D_{GS}(M) = M^*$ for any $M \in A\text{-mod}$ of finite length. For convenience of the reader we include (the short) proof of this in Section 12.

Going back to the category of smooth representations of \tilde{G} , we have the Grothendieck–Serre duality with respect to the Bernstein’s center (component by component, see Section 12.)

$$D_{GS/Z}: \mathcal{D}^b(\mathcal{M}(\tilde{G})) \rightarrow (\mathcal{D}^b(\mathcal{M}(\tilde{G})))^{op}$$

$$\pi \mapsto \mathrm{RHom}_Z(\pi, \omega_Z^\circ)^{\mathrm{sm}}$$

where the superscript “sm” means taking smooth vectors. On finitely generated representations it is an involution, i.e., $D_{GS/Z}^2 \simeq \mathrm{Id}$. In order to make sure we extended the contragredient from admissible to all finitely generated, one needs to check that $D_{GS/Z}(\pi) = \pi^\vee$ for all admissible representations π of G . This is a folklore result whose (again easy and rather formal) proof we include for the convenience of the reader and for lack of a better reference (see Corollary 12.2).

However, beyond finite length representations, $D_{GS/Z}(\pi)$ seems not to have been considered. As a specific example, we may mention that we do not know what $D_{GS}(M)$ is for $M = \mathrm{ind}_K^G \mathbb{C}$, the space of locally constant compactly supported \mathbb{C} -valued functions on $K \backslash G$ for K a compact open subgroup of G . Here is a more specific case. Suppose G is a split group, and $K = \mathfrak{I}$ an Iwahori subgroup of G , with $\mathcal{H}(\mathfrak{I}) = C_c(\mathfrak{I} \backslash G / \mathfrak{I})$, the Iwahori Hecke algebra of G and Z the center of $\mathcal{H}(\mathfrak{I})$. This particular Bernstein component of G is described as the category of modules over $\mathcal{H}(\mathfrak{I})$ by sending a smooth representation M of G (generated by its \mathfrak{I} -fixed vectors) to $M^\mathfrak{I}$ which is a module for $\mathcal{H}(\mathfrak{I})$. Then for $M = \mathrm{ind}_\mathfrak{I}^G \mathbb{C}$, $M^\mathfrak{I} = \mathcal{H}(\mathfrak{I})$ with the left regular representation of $\mathcal{H}(\mathfrak{I})$ on itself. In this case, $D_{GS}(M)$ is the $\mathcal{H}(\mathfrak{I})$ -module

corresponding to $\text{Hom}_Z(\mathcal{H}(\mathfrak{J}), Z)$, i.e. $\text{Ext}_Z^i(\mathcal{H}(\mathfrak{J}), Z) = 0$ for $i > 0$. This follows first by noting that by [Ch-Ka, Theorem 5.4.1], $\text{ind}_{\mathfrak{J}}^G \mathbb{C}$ is isomorphic to the universal unramified principal series $\text{ind}_B^G(\Pi)$ where $\Pi = \text{ind}_{T^\circ}^T \mathbb{C}$, B is a Borel subgroup of G , $T \subset B$ a maximal torus of G and T° , the maximal compact subgroup of T . In this case, it follows from [Roc09, Theorem 1.9.1.1 and Lemma 1.10.4.1] that $M^{\mathfrak{J}} = \mathcal{H}(\mathfrak{J})$ is a free module over Z , proving that $\text{Ext}_Z^i(\mathcal{H}(\mathfrak{J}), Z) = 0$ for $i > 0$. However, we do not know what $\mathcal{H}(\mathfrak{J})$ -module we get for $\text{Hom}_Z(\mathcal{H}(\mathfrak{J}), Z)$, and therefore what is the G -module $D_{GS}(M)$ for $M = \text{ind}_{\mathfrak{J}}^G \mathbb{C}$.

Homological duality allows us to prove part (3) of Theorem 1.3. Additionally, it permits us to understand precisely under what conditions do we have an isomorphism $[d] \circ D_h \simeq D_{GS/Z}$ on a given block. For precise details, we refer to Corollary 12.7 and condition (FsG).

Returning to homological duality and its Serre functor property, let us state the main theorem proved in this work. For $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}}$, a finite length representation in a Bernstein block, we put $\mathbb{D}_h(\pi) = H^{d(\pi)}(D_h(\pi))$ to be the unique non-zero cohomology of $D_h(\pi)$ (see Theorem 1.3(1)).

Theorem 1.4 (see Theorem 11.1). *Let $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ be of finite length and $\pi' \in \mathcal{M}(\tilde{G})$ be arbitrary. Then the following natural pairing is perfect*

$$\text{Ext}_{\tilde{G}}^i(\pi, \pi') \otimes \text{Ext}_{\tilde{G}}^{d(\pi)-i}(\pi', \mathbb{D}_h(\pi)^\vee) \rightarrow \text{Ext}_{\tilde{G}}^{d(\pi)}(\pi, \mathbb{D}_h(\pi)^\vee) \rightarrow \mathbb{C}.$$

The proof is an easy consequence of Theorem 1.2 and Theorem 1.3. It is an improvement on the results of [NP20] where the theorem was proved only for π irreducible.

Remark 1.5.

1. As in the linear case, it would be desirable to have an identification of D_h with $D_{AZ} \circ (-)^\vee$ on the category of finite length representations of a covering group \tilde{G} , where D_{AZ} is the Aubert–Zelevinsky duality, similar to [BBK18], [SS97, Proposition IV.5.1] (Grothendieck group) or [Bez04, Theorem 4.2]. For the moment, it seems we do not know that $D_h(\pi) \cong D_{AZ}(\pi^\vee)$ even for an irreducible representation π of \tilde{G} . We consider this as an important question which is not at all discussed in this paper. However, one can consult [BJ16] for a discussion of Aubert–Zelevinsky duality for finite central extensions of reductive p -adic groups (in the Grothendieck group of representations).

2. As already mentioned, the duality Theorem 1.2 was first proved in [SS97, Duality Theorem] using a more involved argument. Their strategy was to first show the vanishing of Ext groups from Theorem 1.3 (1) and then proceed to prove the isomorphism from Theorem 1.2 using this vanishing. Both steps hinge on nice projective resolutions constructed from the Bruhat-Tits building. The advantage of the proof that we present is that by keeping the homological duality functor D_h at the derived level, the duality theorem becomes very easy and requires no technology. Once the vanishing of Ext-groups is proved, see Theorem 1.3(1) (for which we follow the argument in [Ber92]), we immediately deduce the required duality theorem at the level of abelian categories.

2. Categorical generalities

2.1. Preliminaries on decomposing categories and centers

Given I a set (possibly infinite) and $\mathcal{C}_i, i \in I$, a family of abelian categories, we can define the product category $\mathcal{C} := \prod_{i \in I} \mathcal{C}_i$ in the 2-category of abelian categories through the usual universal property. Concretely, one constructs it as follows:

- the objects are tuples $(X_i)_{i \in I}$ with $X_i \in \mathcal{C}_i$ for every $i \in I$;
- the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}}((X_i)_{i \in I}, (Y_j)_{j \in I}) := \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i);$$

- the projection functors $\pi_i: \mathcal{C} \rightarrow \mathcal{C}_i$ send a tuple $(X_s)_s$ to X_i .

The universal property of a product is immediately verified.

In addition to the projection functors $\pi_i: \mathcal{C} \rightarrow \mathcal{C}_i$, there are also natural inclusion functors

$$\iota_i: \mathcal{C}_i \rightarrow \mathcal{C}$$

sending an object $X \in \mathcal{C}_i$ to the tuple that has X in position i and 0 everywhere else. By construction, the functor ι_i is fully faithful so we can think of the category \mathcal{C}_i as a full subcategory of the product category \mathcal{C} . We will drop the functor ι_i from the notation most of the time. Two such subcategories \mathcal{C}_i and \mathcal{C}_j for $i \neq j$ are (derived) orthogonal, namely

$$\mathrm{Ext}_{\mathcal{C}}^k(X_i, X_j) = 0, \text{ for all } k \geq 0.$$

Moreover we notice that the object $(X_i)_i \in \mathcal{C}$ is the direct sum of the objects $\iota_i(X_i)$ in \mathcal{C} . Indeed, let us check the universal property for direct sums: for an arbitrary object $Y = (Y_i)_i \in \mathcal{C}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}((X_i)_i, (Y_i)_i) &= \prod_i \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i) \\ &= \prod_i \mathrm{Hom}_{\mathcal{C}}(\iota_i(X_i), (Y_j)_j) \\ &= \mathrm{Hom}_{\mathcal{C}}(\oplus_i \iota_i(X_i), (Y_j)_j). \end{aligned}$$

Actually, the same argument shows that $(X_i)_i$ is also the direct product of the objects $\iota_i(X_i)$, $i \in I$, in \mathcal{C} .

Caution: the above does not mean that in \mathcal{C} we always have $\oplus_i A_i \simeq \prod_i A_i$ for any $A_i \in \mathcal{C}$. See also Example 2.2.

As a summary, we have a category \mathcal{C} with full subcategories \mathcal{C}_i , $i \in I$ that are two by two derived orthogonal and moreover every object of \mathcal{C} is a direct sum of objects from \mathcal{C}_i (the subcategories \mathcal{C}_i split the category \mathcal{C}). Conversely, a category \mathcal{C} with full subcategories \mathcal{C}_i , $i \in I$, with the above properties is a direct product of \mathcal{C}_i provided we assume a small technical condition on the categories that we work with which in practice is always verified.

Proposition 2.1 (cf. [Ber84, §1.9]). *Let \mathcal{C} be an abelian category and let $\mathcal{C}_i \subset \mathcal{C}$, $i \in I$ be full abelian subcategories of \mathcal{C} . Assume that*

1. \mathcal{C} admits direct sums indexed by I ,
2. every object of \mathcal{C} can be written as a direct sum of objects of \mathcal{C}_i ,
3. $\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = 0$ for all $X_i \in \mathcal{C}_i$, $X_j \in \mathcal{C}_j$ and $i \neq j$.
4. If $X_i \in \mathcal{C}_i$, $i \in I$ and $f: Y \rightarrow \oplus_i X_i$ is such that $\mathrm{proj}_i \circ f = 0$ for all $i \in I$ then $f = 0$.³

Then the natural functor $\prod_i \mathcal{C}_i \rightarrow \mathcal{C}$ is an equivalence of categories.

Proof. There is a natural functor $\prod_i \mathcal{C}_i \rightarrow \mathcal{C}$ given by $(X_i)_i \mapsto \oplus_i X_i$ that is well defined by assumption (1). By (2) it is essentially surjective.

In order to show that it is an equivalence, it remains to check that it is fully faithful.

Given this observation and taking into account assumptions (3) and (4), we have

$$\mathrm{Hom}_{\prod_i \mathcal{C}_i}((X_i)_i, (Y_i)_i) = \prod_i \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i)$$

³There are pathological examples where this condition is not satisfied.

$$\begin{aligned}
 &= \prod_i \operatorname{Hom}_{\mathcal{C}}(X_i, \oplus_j Y_j) \\
 &= \operatorname{Hom}_{\mathcal{C}}(\oplus_i X_i, \oplus_j Y_j)
 \end{aligned}$$

which proves the fully-faithfulness. □

The following example involving direct sums and direct products might be clarifying.

Example 2.2. Consider the algebra $A = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} = \operatorname{Func}_{\mathcal{C}}(\mathbb{Z}, \mathbb{C})$, i.e., complex valued functions with compact support defined on \mathbb{Z} with pointwise multiplication. It is an algebra that has no unit but it has enough idempotents. We can consider the category of non-degenerate modules over A , call it $\mathcal{C} := A\text{-mod}^{\text{nd}}$, where an object of \mathcal{C} is an A -module M such that for any $m \in M$ there exists an idempotent $e \in A$ such that $em = m$. By considering the delta functions $\delta_i, i \in \mathbb{Z}$, it is clear that any non-degenerate module M satisfies

$$M = \bigoplus_i M_i, \text{ where } M_i := \delta_i M.$$

Notice also that if $M = M_i$ and $M' = M'_i$ then

$$\operatorname{Hom}_{\mathcal{C}}(M, M') = \operatorname{Hom}_{\operatorname{Vec}_{\mathbb{C}}}(M_i, M'_i).$$

Given two non-degenerate A -modules M and M' we see therefore that

$$\operatorname{Hom}_{\mathcal{C}}(M, M') = \operatorname{Hom}_{\mathcal{C}}(\oplus_i M_i, \oplus_j M'_j) = \prod_i \operatorname{Hom}_{\operatorname{Vec}_{\mathbb{C}}}(M_i, M'_i)$$

because there are no A -module morphisms between M_i and M'_j if $i \neq j$.

For any $i \in \mathbb{Z}$ we can consider the full subcategory of \mathcal{C} consisting of those modules M on which δ_i acts as identity. In other words, modules M such that $M = \delta_i M$. We denote it by \mathcal{C}_i and by what we said above we have $\mathcal{C}_i \simeq \operatorname{Vec}_{\mathbb{C}}$ for all $i \in \mathbb{Z}$. From the above Proposition we can deduce an equivalence of categories

$$\mathcal{C} \simeq \prod_{i \in \mathbb{Z}} \operatorname{Vec}_{\mathbb{C}}.$$

Since all the categories \mathcal{C}_i are equivalent to $\operatorname{Vec}_{\mathbb{C}}$ one could confuse the direct sum $\bigoplus_i^{\mathcal{C}} M_i \in \mathcal{C}$ where each M_i is a vector space, to the direct sum $\bigoplus_i^{\operatorname{Vect}_{\mathbb{C}}} M_i \in \mathcal{C}$ where the superscripts refer to the category where the direct sums/products are taken. The former is an “external” direct sum whereas the latter is an “internal” direct sum (all the objects are seen as belonging to the *same* $\operatorname{Vec}_{\mathbb{C}}$).

Moreover, we actually have $\bigoplus_i^{\mathcal{C}} M_i \simeq \prod_i^{\mathcal{C}} M_i$. One direct way to see this is to remember that $\mathcal{C} = A\text{-mod}^{\text{nd}}$ and that the direct product of non-degenerate modules is defined by

$$\begin{aligned} \prod_i^{A\text{-mod}^{\text{nd}}} M_i &= \{(m_i) \in \prod_i^{\text{Vect}_{\mathcal{C}}} M_i \mid \exists e \text{ idempotent in } A \text{ such that } em = m\} \\ &= \{(m_i) \in \prod_i^{\text{Vect}_{\mathcal{C}}} M_i \mid \text{only finitely many } m_i \text{ are non-zero}\}. \end{aligned}$$

Definition 2.3. Given an abelian category \mathcal{C} , we define its center $\mathcal{Z}(\mathcal{C})$ to be $\text{End}(\text{Id}_{\mathcal{C}})$, the endomorphisms of the identity functor.

Remark 2.4. The center of a category is preserved under categorical equivalence.

Since \mathcal{C} is additive we see that $\mathcal{Z}(\mathcal{C})$ is a commutative ring with unit. Moreover, the category \mathcal{C} becomes naturally $\mathcal{Z}(\mathcal{C})$ -enriched, i.e., the Hom spaces in \mathcal{C} have a natural action of $\mathcal{Z}(\mathcal{C})$ making them $\mathcal{Z}(\mathcal{C})$ -modules and \mathcal{C} into a $\mathcal{Z}(\mathcal{C})$ -linear category.

Example 2.5. If $\mathcal{C} = A\text{-mod}$ for an algebra A with unit, then it is not hard to see that $\mathcal{Z}(\mathcal{C})$ is the center $\mathcal{Z}(A)$ of A .

For $e \in \mathcal{Z}(\mathcal{C})$ an idempotent we denote by $e\mathcal{C}$ its image in \mathcal{C} : it is the full subcategory consisting of those objects on which e acts by identity.

A similar argument as in Proposition 2.1 proves the first part of

Proposition 2.6. *If $\mathcal{C} \simeq \prod_i \mathcal{C}_i$ then $\mathcal{Z}(\mathcal{C}) \simeq \prod_i \mathcal{Z}(\mathcal{C}_i)$. Conversely, if $\mathcal{Z}(\mathcal{C}) \simeq \prod_i \mathcal{Z}_i$ then the identity of each \mathcal{Z}_i provides an idempotent $e_i \in \mathcal{Z}(\mathcal{C})$ and we have $\mathcal{C} \simeq \prod_i e_i\mathcal{C}$.*

Proof. Only the last part is non-trivial and it follows by applying Proposition 2.1. □

The following is a generalization of Example 2.5 for algebras without unit but with enough idempotents. An algebra A is said to have enough idempotents if for any element $a \in A$ there exists an idempotent $e \in A$ such that $ae = ea = a$.

A left A -module M is said to be non-degenerate if for any $m \in M$ there exists an idempotent $e \in A$ such that $em = m$. The category of non-degenerate left A -modules is denoted as $A\text{-mod}^{\text{nd}}$.

If A^o is the opposite algebra to A , then $\overline{A} := \text{Hom}_{A^o}(A, A)$ is the space of right A -invariant maps from A to A . We can write $\overline{A} = \text{Hom}_{A^o}(\varinjlim eA, A) = \varprojlim Ae$ where the limit is taken over all the idempotents of A . Clearly \overline{A} is an \overline{A} -bimodule.

Proposition 2.7 ([Ber84, Lemme 1.5] or [Ren10, I.1.7]). *The center of the category $A\text{-mod}^{\text{hd}}$ is identified with the center of \overline{A} as an A -bimodule (i.e., with the A -bimodule endomorphisms of A). It can be further identified with $\varprojlim Z(eAe)$ where e goes over all idempotents of A .*

In our context, the abelian categories that we will encounter will be equivalent to module categories over a unitary ring and as such their center is just the center of the corresponding ring.

We use a standard tool from category theory to detect when an abelian category is equivalent to a module category.

Recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called *faithful* (resp., *fully-faithful*) if for any objects $A, B \in \mathcal{C}$ it induces an injective (resp., bijective) map on Hom spaces:

$$\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)).$$

One calls F *essentially surjective* if every object in \mathcal{D} is isomorphic to the image through F of an object from \mathcal{C} .

A basic theorem in category theory (see [Mac, IV.4 Theorem 1]) says that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if and only if it is essentially surjective and fully-faithful.

An object P of an abelian category \mathcal{A} is called

1. compact (or finite) if $\text{Hom}_{\mathcal{A}}(P, -)$ commutes with arbitrary coproducts (i.e, direct sums),
2. projective if $\text{Hom}_{\mathcal{A}}(P, -)$ is exact,
3. a generator if $\text{Hom}_{\mathcal{A}}(P, -)$ is faithful, a notion that we just recalled above and which for P a projective object amounts to the assertion that for $X \neq 0$, $\text{Hom}_{\mathcal{A}}(P, X) \neq 0$.
4. progenerator if it is projective and a generator.

Proposition 2.8 (see [Par70, 4.11]). *Let \mathcal{A} be an abelian category admitting coproducts. For a compact progenerator P of \mathcal{A} , the functor*

$$\begin{aligned} \mathcal{A} &\rightarrow \mathbf{mod}\text{-End}_{\mathcal{A}}(P) \\ X &\mapsto \text{Hom}_{\mathcal{A}}(P, X) \end{aligned}$$

is an equivalence of categories between \mathcal{A} and the category of left modules over the ring $\text{End}_{\mathcal{A}}(P)$.

2.2. Serre functors

The notion of a Serre functor for an additive category was introduced in [BK89] in order to capture duality phenomena similar to Serre's duality theorem on smooth projective varieties. It is intimately related to questions of representability of certain functors. Moreover this turns out to be a very rigid notion. In particular, if a Serre functor exists then it is unique (up to natural isomorphism).

Definition 2.9 ([Huy06, Definition 1.28][BK89, Definition 3.1]). Let \mathcal{C} be a k -linear category. A Serre functor \mathbb{S} on \mathcal{C} is an equivalence of categories $\mathbb{S}: \mathcal{C} \rightarrow \mathcal{C}$ together with natural isomorphisms

$$\eta_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B)^* \rightarrow \mathrm{Hom}_{\mathcal{C}}(B, \mathbb{S}(A)), \text{ for all } A, B \in \mathcal{C}.$$

Remark 2.10. It is observed in [Huy06, Lemma 1.30] that if one assumes that the categories $\mathcal{C}_1, \mathcal{C}_2$, with Serre functors $\mathbb{S}_1, \mathbb{S}_2$, have finite dimensional Hom spaces, then any functorial isomorphism between the categories $\mathcal{C}_1, \mathcal{C}_2$ commutes with the Serre functors.

An example of a Serre functor, actually one that motivated the notion, is the classical Serre duality. Namely, let X be a smooth projective variety over k of dimension d and denote by ω_X its canonical sheaf. Then Serre duality stipulates a natural isomorphism

$$\mathrm{Ext}_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{G})^* \simeq \mathrm{Ext}_{\mathcal{O}_X}^{d-i}(\mathcal{G}, \mathcal{F} \otimes \omega_X), \text{ for all } \mathcal{F}, \mathcal{G} \in \mathcal{D}^b(\mathrm{Coh}(X)).$$

In other words, the functor

$$- \otimes \omega_X[d]: \mathcal{D}^b(\mathrm{Coh}(X)) \rightarrow \mathcal{D}^b(\mathrm{Coh}(X))$$

is a Serre functor.

A simpler example comes from finite dimensional algebras over a field k and is known as the Nakayama functor, see [BK89, §3.2, Example 3]. Let A be a finite dimensional algebra over k and suppose it has finite homological dimension. (Although there are no nontrivial finite dimensional commutative algebras of finite homological dimension over a field k , there are many non-commutative ones, for example, there are constructions of such algebras using Quivers.) We consider the derived category $\mathcal{C} := \mathcal{D}_{fd}^b(A\text{-mod})$ of finite dimensional left A -modules. There are two duality functors

$$\mathcal{D}_{fd}^b(A\text{-mod}) \xrightarrow{\delta} \mathcal{D}_{fd}^b(\text{mod-}A) \xrightarrow{(-)^\vee} \mathcal{D}_{fd}^b(A\text{-mod}),$$

where $\delta(M) = \mathrm{RHom}_A(M, A)$ and $M^\vee = \mathrm{RHom}_{\mathcal{D}^b(k)}(M, k)$ for any object $M \in \mathcal{D}^b(A\text{-mod})$. We put $D_{Nak} := (-)^\vee \circ \delta$ and call it the Nakayama functor.

The following proposition is very easy to prove from classical adjunctions and our results in Section 2.3 are essentially a detailed version of it for idempotented algebras (see Theorem 2.19):

Proposition 2.11. *The Nakayama functor*

$$D_{Nak} : \mathcal{D}_{fd}^b(A\text{-mod}) \rightarrow \mathcal{D}_{fd}^b(A\text{-mod})$$

is a Serre functor.

Remark 2.12. Notice that we have to restrict ourselves to finite dimensional modules because of the appearance of the dual vector space which provides an equivalence of categories only for finite dimensional modules.

In order to extend the notion to more general categories, in particular to finitely generated modules over an algebra finite over its center, Bezrukavnikov and Kaledin propose the notion of relative Serre functor (see [BK04, §2.1]). Since we do not prove anything about relative Serre functors we prefer to defer this discussion to a later work.

2.3. An abstract duality theorem

The purpose of this section is to prove an abstract duality theorem reminiscent of the Serre functor property of the Nakayama functor $\mathrm{RHom}_A(-, A)^*$ (see [BK89] and Section 2.2). This is achieved in Theorem 2.19 and Corollary 2.25. Although the results presented in this section are well-known and we do not claim any originality, for lack of a precise reference, we provide all the details. We do not strive for maximal generality, so sometimes we make hypotheses which might not be necessary but which hold in our applications to p -adic groups.

Let k be a field and let A be an idempotented k -algebra, i.e., for every $a \in A$ there exists an idempotent $e \in A$ such that $ae = ea = a$. Clearly any unitary algebra is idempotented. We suppose moreover that A has a countable filtered set of idempotents. A left A -module M is said to be *non-degenerate* if $AM = M$, equivalently, if for any $m \in M$, there exists an idempotent $e \in A$ such that $em = m$.

Remark 2.13. For a non-unitary ring A , a free A -module is not necessarily projective. The basic projective left A -modules are Ae with e an idempotent. Any projective finitely generated non-degenerate A -module is a direct summand of $\bigoplus_i Ae_i$ where $\{e_i\}$ is a finite collection of idempotents. This follows

as in the unitary case using the fact that the module is non-degenerate and finitely generated.

We denote by $A\text{-mod}$, the category of all left A -modules, and by $A\text{-mod}^{\text{nd}}$ the full subcategory of non-degenerate left A -modules. We use a similar notation for right A -modules.

Consider the functors

$$(2.1) \quad \begin{aligned} \text{Hom}_A(-, -) &: (A\text{-mod}^{\text{nd}})^{\text{op}} \times A\text{-mod}^{\text{nd}} \rightarrow \mathbb{Z}\text{-mod} \\ - \otimes_A - &: \text{mod}^{\text{nd}}\text{-}A \times A\text{-mod} \rightarrow \mathbb{Z}\text{-mod} \end{aligned}$$

The category of non-degenerate left A -modules, $A\text{-mod}^{\text{nd}}$, has enough projective objects⁴ and so we can derive the functors (2.1) on the first argument since non-degenerate projective modules are still acyclic:

$$(2.2) \quad \begin{aligned} \text{RHom}_A(-, -) &: (\mathcal{D}^b(A\text{-mod}^{\text{nd}}))^{\text{op}} \times \mathcal{D}^b(\text{mod}^{\text{nd}}\text{-}A) \rightarrow \mathcal{D}(\mathbb{Z}\text{-mod}) \\ - \otimes_A^L - &: \mathcal{D}^b(\text{mod}^{\text{nd}}\text{-}A) \times \mathcal{D}^b(A\text{-mod}) \rightarrow \mathcal{D}(\mathbb{Z}\text{-mod}) \end{aligned}$$

Remark 2.14. The functor $\text{Hom}_A(-, A): (A\text{-mod}^{\text{nd}})^{\text{op}} \rightarrow \text{mod}\text{-}A$ does not land inside the subcategory of non-degenerate modules. This can already be seen for A itself: $\text{Hom}_A(A, A) \simeq \varprojlim_e eA$ where the inverse limit is taken over the poset of idempotents e in A (can take a filtered subset). However, if M is a *finitely generated* non-degenerate module then $\text{Hom}_A(M, A)$ is non-degenerate.

For two A -modules $M, N \in A\text{-mod}^{\text{nd}}$, there is a canonical morphism

$$(2.3) \quad \text{can}_{N,M}: \text{Hom}_A(M, A) \otimes_A N \rightarrow \text{Hom}_A(M, N)$$

that extends to the derived category $\mathcal{D}^b(A\text{-mod}^{\text{nd}})$

$$(2.4) \quad \text{can}_{M,M}: \text{RHom}_A(M, A) \otimes_A^L N \rightarrow \text{RHom}_A(M, N).$$

Definition 2.15. A non-degenerate module M over A is said to be *perfect* if it has a finite resolution by finitely generated non-degenerate projective A -modules. An object of $\mathcal{D}^b(A\text{-mod}^{\text{nd}})$ is said to be perfect if it is isomorphic to a finite complex of finitely generated non-degenerate projective A -modules.

The next lemma tells us that $\text{can}_{N,M}$ is an isomorphism when M is perfect:

⁴We do not know if the category of all left modules over A has enough projective objects, but this plays no role for us.

Lemma 2.16. *If $M, N \in \mathcal{D}^b(A\text{-mod}^{\text{nd}})$ and M is perfect, the canonical morphism (2.4)*

$$\text{RHom}_A(M, A) \otimes_A^L N \rightarrow \text{RHom}_A(M, N)$$

is an isomorphism.

Proof. In this proof, all modules are non-degenerate. We split the proof into several steps.

First, notice that for $M = Ae$ with e an idempotent we have canonical identifications $\text{RHom}_A(Ae, N) = \text{Hom}_A(Ae, N) = eN$ and in particular $\text{Hom}_A(Ae, A) = eA$ is a projective right A -module and so one can compute the derived tensor product with it. Hence in this situation we have

$$\text{RHom}_A(Ae, A) \otimes_A^L N = eA \otimes_A N = eN = \text{Hom}_A(Ae, N)$$

and we are done.

Second, notice that the morphism $\text{RHom}_A(M, A) \otimes_A^L N \rightarrow \text{RHom}_A(M, N)$ in the statement of the lemma is compatible with finite direct sums (in both arguments). Since a finitely generated projective module is a direct summand of $\bigoplus_{i=1}^n Ae_i$ for some idempotents e_i we deduce the validity of the lemma for M a finitely generated projective module.

Next, let M be a perfect complex, quasi-isomorphic to P_\bullet where each P_r is a finitely generated non-degenerate projective module.

For $N = N_\bullet \in \mathcal{D}^b(A\text{-mod}^{\text{nd}})$, $\text{RHom}_A(M, N)$ is computed by the totalization of a double complex with entries $\text{Hom}_A(P_r, N_s)$ which by what we said above is isomorphic to $\text{Hom}_A(P_r, A) \otimes_A N_s$ through can_{P_r, N_s} . The latter are the entries of the double complex $\text{Hom}_A(P_\bullet, A) \otimes_A N_\bullet$ whose totalization computes $\text{RHom}_A(M, A) \otimes_A^L N$. The naturality of (2.3) ensures that these isomorphisms commute with all the differentials in the double complexes and therefore the total complexes are isomorphic. In other words, the natural map $\text{can}_{N, M}: \text{RHom}_A(M, A) \otimes_A^L N \rightarrow \text{RHom}_A(M, N)$ is an isomorphism. \square

Definition 2.17. For an A -module P , we denote by P^{nd} its non-degenerate submodule, i.e., the subspace of P consisting of elements that are fixed by some idempotent in A .

It is clear that a morphism from a non-degenerate A -module to an arbitrary A -module lands inside its non-degenerate part. Said otherwise, we have

Proposition 2.18. *The right adjoint of the natural inclusion $A\text{-mod}^{\text{nd}} \rightarrow A\text{-mod}$ is the functor $(-)^{\text{nd}}$ of taking the non-degenerate part of a module. Moreover $(-)^{\text{nd}}$ is exact, so the adjunction extends trivially to derived categories.*

Contragredient. Any non-degenerate left or right A -module M has a natural structure of k -vector space. We can therefore construct the contra-
 gradient module $M^\vee := \text{Hom}_k(M, k)^{\text{nd}}$ as the non-degenerate k -linear maps from M to k . As $M \rightarrow M^\vee$ is an exact functor from $A\text{-mod}^{\text{nd}}$ to $\text{mod}^{\text{nd}}\text{-}A$, it extends naturally to the bounded derived categories

$$(-)^\vee : \mathcal{D}^b(A\text{-mod}^{\text{nd}}) \rightarrow \mathcal{D}^b(\text{mod}^{\text{nd}}\text{-}A)$$

by applying it degree-wise.

For a right A -module M , a left A -module N , both non-degenerate and a k -vector space V , we have a natural adjunction morphism

$$\tau_{M,N,V} : \text{Hom}_k(M \otimes_A N, V) \rightarrow \text{Hom}_A(N, \text{Hom}_k(M, V)^{\text{nd}})$$

that can be checked (as in the classical situation) to be an isomorphism. This isomorphism extends to bounded derived categories and we get the usual derived tensor-hom adjunction

$$(2.5) \quad \tau_{M,N,V} : \text{RHom}_k(M \otimes_A^L N, V) \xrightarrow{\sim} \text{RHom}_A(N, \text{RHom}_k(M, V)^{\text{nd}})$$

For a complex of vector spaces $V \in \mathcal{D}^b(k)$, we denote by V^* , the complex of dual vector spaces.

Denote by $\mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}})$, the full subcategory of $\mathcal{D}^b(A\text{-mod}^{\text{nd}})$ consisting of perfect complexes.

Theorem 2.19. *The following functor*

$$\begin{aligned} D_{\text{Nak}} : \mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}}) &\longrightarrow \mathcal{D}^b(A\text{-mod}^{\text{nd}}) \\ M &\mapsto \text{RHom}_A(M, A)^\vee, \end{aligned}$$

satisfies the Serre functor duality, i.e., for any perfect object M and any object $N \in \mathcal{D}^b(A\text{-mod}^{\text{nd}})$, we have a natural isomorphism of k -vector spaces:

$$\text{Hom}_{\mathcal{D}^b(A)}(M, N)^* \simeq \text{Hom}_{\mathcal{D}^b(A)}(N, D_{\text{Nak}}(M)).$$

Remark 2.20. A Serre functor (see Section 2.2) is, moreover, required to be an equivalence of categories. In our situation, in order for D_{Nak} to be an equivalence, we must restrict to finite dimensional modules (because of duality over

k). In the context of representations of p -adic groups this means restricting to admissible modules. Replacing the contragredient by Grothendieck–Serre duality over the Bernstein center extends D_{Nak} from admissible modules to finitely generated modules and indeed gives a Serre functor (relative to the center) for the whole category of finitely generated representations.⁵ This statement already appears in [BBK18].

Proof. Apply Lemma 2.16 and the tensor-hom adjunction (2.5) to get natural quasi-isomorphisms

$$\begin{aligned} \text{RHom}_k(\text{RHom}_A(M, N), k) &\simeq \text{RHom}_k(\text{RHom}_A(M, A) \overset{L}{\otimes}_A N, k) \\ &\simeq \text{RHom}_A(N, \text{RHom}_k(\text{RHom}_A(M, A), k)^{\text{nd}}) \\ &\simeq \text{RHom}_A(N, \text{RHom}_A(M, A)^\vee) \\ &\simeq \text{RHom}_A(N, D_{\text{Nak}}(M)). \end{aligned}$$

Taking H^0 gives the desired result. □

Putting $N = M, D_{\text{Nak}}(M)$ above, we get the following:

Corollary 2.21. *Suppose M is a non-degenerate irreducible left A -module which belongs to $\mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}})$ and for which Schur’s lemma holds. Then,*

$$\text{Hom}_{\mathcal{D}^b(A)}(D_{\text{Nak}}(M), D_{\text{Nak}}(M))^* \simeq k.$$

Proof. It follows from the following chain of natural isomorphisms

$$\begin{aligned} k &\simeq \text{Hom}_{\mathcal{D}^b(A)}(M, M)^* \simeq \text{Hom}_{\mathcal{D}^b(A)}(M, D_{\text{Nak}}(M)) \\ &\simeq \text{Hom}_{\mathcal{D}^b(A)}(D_{\text{Nak}}(M), D_{\text{Nak}}(M))^*. \end{aligned} \quad \square$$

Remark 2.22. The above corollary shows that for M irreducible, $D_{\text{Nak}}(M)$ is indecomposable. So if $D_{\text{Nak}}(M)$ is concentrated in degree 0, then it is isomorphic to M . In particular, if M is irreducible and projective, then $D_{\text{Nak}}(M) \simeq M$. If we apply it to a finite dimensional semisimple algebra A , then we get the simple fact that D_{Nak} leaves stable every irreducible module. See Corollary 12.4 for a statement for p -adic groups. In general however, it looks like $D_{\text{Nak}}(M)$ might be an interesting, non-trivial, involution.

Taking $N = D_{\text{Nak}}(M)$ in the above theorem we get a canonical map

$$(2.6) \quad \text{can}_M : \text{Hom}_{\mathcal{D}^b(A)}(M, D_{\text{Nak}}(M)) \rightarrow k$$

⁵We are indebted to Roman Bezrukavnikov for explaining this to us.

and moreover, unwinding the definitions (see the proof of Theorem 2.19), we obtain

Corollary 2.23. *For $M, N \in \mathcal{D}^b(A\text{-mod}^{\text{nd}})$ with M perfect, the natural pairing*

$$\text{Hom}_{\mathcal{D}^b(A)}(M, N) \times \text{Hom}_{\mathcal{D}^b(A)}(N, D_{\text{Nak}}(M)) \rightarrow \text{Hom}_{\mathcal{D}^b(A)}(M, D_{\text{Nak}}(M)) \xrightarrow{\text{can}_M} k$$

is perfect giving rise to the isomorphism

$$\text{Hom}_{\mathcal{D}^b(A)}(M, N)^* \simeq \text{Hom}_{\mathcal{D}^b(A)}(N, D_{\text{Nak}}(M))$$

from Theorem 2.19.

Define the *homological duality* functor as

$$(2.7) \quad \begin{aligned} D_h: \mathcal{D}^+(A\text{-mod}) &\longrightarrow \mathcal{D}^-(\text{mod-}A)^{op} \\ M &\mapsto \text{RHom}_A(M, A) \end{aligned}$$

The same formula sends a right module into a left module. By abuse of notation we will still write D_h for this functor.

Notice that without further hypotheses, the target category of D_h is only bounded from below.

Proposition 2.24. *The functor D_h preserves perfect complexes and is an involution on $\mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}})$.*

Proof. Notice that if P is a finitely generated projective left A -module, then $\text{Hom}_A(P, A)$ is a finitely generated projective right A -module. It follows that D_h preserves perfect complexes.

Let us now prove that $D_h^2 \simeq \text{Id}$ on perfect complexes. There is a natural transformation $\text{ev}: \text{Id} \rightarrow D_h^2$ coming from the evaluation morphism

$$\text{ev}_P: P \rightarrow \text{Hom}_A(\text{Hom}_A(P, A), A).$$

Note that ev_P is compatible with finite direct sums, i.e., $\text{ev}_{\oplus P_i} = \oplus_i \text{ev}_{P_i}$. Hence in order to show that ev_P is an isomorphism for every finitely generated projective, it is enough to show it for $P = Ae$, where e is an idempotent. But then $\text{Hom}_A(P, A) = eA$ and $\text{Hom}_A(eA, A) = Ae = P$ and so clearly ev_P is an isomorphism.

Since ev is a natural transformation (functorial) it behaves well on complexes and hence we deduce that ev_{P^\bullet} is also an isomorphism for every perfect complex $P^\bullet \in \mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}})$. □

We denote by $\mathcal{D}_{fg}^b(A\text{-mod}^{\text{nd}})$, the bounded derived category of finitely generated, non-degenerate, left A -modules. (Similarly for right A -modules.)

Corollary 2.25. *Suppose A has both left and right finite homological dimension. Then the homological duality D_h gives a functor*

$$D_h: \mathcal{D}_{fg}^b(A\text{-mod}^{\text{nd}}) \rightarrow \mathcal{D}_{fg}^b(\text{mod}^{\text{nd}}\text{-}A)^{op}$$

whose square is isomorphic to the identity, i.e., $D_h^2 \simeq \text{Id}$.

Proof. Since A has finite left and right homological dimension, we have

$$\mathcal{D}_{fg}^b(A\text{-mod}^{\text{nd}}) = \mathcal{D}_{\text{perf}}^b(A\text{-mod}^{\text{nd}})$$

and similarly for right modules. Now the conclusion follows from Proposition 2.24. \square

2.4. Grothendieck–Serre duality and homological duality

We provide some general discussion of Grothendieck–Serre and homological duality, and then we apply it in Section 12 to the context of smooth representation of p -adic groups.

The tools that we will use are the dualizing (or canonical) complex $\omega_A^\circ \in \mathcal{D}^b(A\text{-mod})$ for a commutative ring as well as the exceptional pull-back functor $f^!: D^+(A\text{-mod}) \rightarrow D^+(B\text{-mod})$ for any finite map of algebras $f: A \rightarrow B$. We will provide some of the details without striving for optimal generality. For a thorough discussion of these matters, one could consult [Har66, Ch III] or [Sta21, 0A7A]. Check [Sta21, 0AU3] for a summary.

For an algebra R , we denote by $R\text{-mod}$ the abelian category of left modules over R . We put $\mathcal{D}^+(R\text{-mod})$ for the bounded below derived category of $R\text{-mod}$ and $\mathcal{D}^b(R\text{-mod})$ for the *bounded* derived category. For right modules we use the notation $\text{mod}\text{-}R$. The full subcategory of $R\text{-mod}$ consisting of finite length modules is denoted by $R\text{-mod}^{\text{fl}}$ and similarly for right modules.

To simplify the discussion of dualizing complexes and upper-shriek functoriality, we start with the following definition:

Definition 2.26. For the polynomial ring $A = k[X_1, \dots, X_d]$, we put $\omega_A^\circ = A[d]$ as an object in $\mathcal{D}^b(A\text{-mod})$ and call it the normalized dualizing complex.

Remark 2.27. Notice that in general one speaks of dualizing complexes over a ring A (or more generally over a scheme X), which, however, are not unique in general. In order to make it unique, one normalizes it in such a way to make

it compatible with exceptional pullback functors $f^!$, i.e., such that $f^!\omega_Y^\circ = \omega_X^\circ$ for any map of schemes $f: X \rightarrow Y$. Since a discussion about $f^!$ for a general map of algebras $f: A \rightarrow B$ would take us too far afield, we prefer to start by Definition 2.26 and only discuss $f^!$ for finite maps.

As a trivial example, notice that for a field k , we have $\omega_k^\circ = k[0]$.

If $f: A \rightarrow B$ is a map of (commutative) algebras, then the natural restriction functor $B\text{-mod} \rightarrow A\text{-mod}$ is exact and extends in an obvious way to derived categories. We will denote this functor by $f_*: \mathcal{D}^b(B\text{-mod}) \rightarrow \mathcal{D}^b(A\text{-mod})$. Notice that it can also be described as $f_* = {}_A B_B \otimes_B -$ where we have denoted by ${}_A B_B$, the A - B -bimodule B .

Lemma 2.28. *Let $f: A \rightarrow B$ be a finite map of commutative algebras. Then the restriction functor $f_*: \mathcal{D}^+(B\text{-mod}) \rightarrow \mathcal{D}^+(A\text{-mod})$ is left adjoint to*

$$f^! := \text{RHom}_A(B, -): \mathcal{D}^+(A\text{-mod}) \rightarrow \mathcal{D}^+(B\text{-mod}).$$

Proof. Since $f_* = {}_A B_B \otimes -$ the tensor-Hom adjunction gives for a B -module M and an A -module N the following natural isomorphism:

$$\text{RHom}_A(B \otimes_B M, N) = \text{RHom}_B(M, \text{RHom}_A(B, N)).$$

In other words, $f^!$ is right adjoint to f_* . □

Remark 2.29. The previous proof does not need the map f to be finite. If the map f is not finite, the right adjoint $\text{RHom}_A(B, -)$ is not denoted by $f^!$. In general, $f^!$ is right adjoint to a functor $f_!$ (restriction with compact support) that we will not define (see for example [Har66, Appendix] or [Sta21, 0G4Z]). See [Sta21, 0A9Y] for a construction of $f^!$ in general. If f is a finite map, then $f_* = f_!$, so the right adjoint of f_* deserves the name $f^!$.

Definition 2.30. Suppose we have defined for a commutative k -algebra A , a (normalized) dualizing complex $\omega_A^\circ \in \mathcal{D}^b(A\text{-mod})$. Then for any finite map $f: A \rightarrow B$ of commutative algebras, we put $\omega_B^\circ := f^!(\omega_A^\circ)$ and call it the (normalized) dualizing complex of B .

Remark 2.31. (1) This definition of ω_B° seems to depend on A and on the map f . It turns out that this is not the case, see [Sta21, 0A7A] and more generally [Sta21, 0BZI] for details.

(2) By construction, the object ω_A° has finite injective dimension for the polynomial algebra A because A is smooth. If $A \rightarrow B$ is any finite map, also the dualizing complex ω_B° will have finite injective dimension. This can be seen using adjunction and the fact that B has a finite projective resolution as an

A -module since A is smooth.

(3) Here is a special case of the above considerations. If R is a local Cohen-Macaulay ring with canonical module ω_R , and A is an algebra over R which is Cohen-Macaulay, and is a finitely generated R module, then if $c = \dim R - \dim A$,

$$\omega_A = \text{Ext}_R^c[A, \omega_R],$$

is a canonical module for A , and the other Ext groups are zero.

Example 2.32. For any ideal $I \leq A$, the map $A \rightarrow A/I$ is finite, hence $\omega_{A/I}^\circ = \text{RHom}_A(A/I, \omega_A^\circ)$. In particular, for \mathfrak{m} a maximal ideal of A with residue field k , we recover $\text{RHom}_A(k, \omega_A^\circ) = \omega_k^\circ = k[0]$.

Example 2.33. If A is a commutative finite dimensional k -algebra, then $\omega_A^\circ = A^*[0]$. Indeed, using the finite map $k \rightarrow A$, and the fact that $\omega_k^\circ = k[0]$, we deduce immediately the equality $\omega_A^\circ = \text{RHom}_k(A, k) = A^*[0]$.

As ω_A has finite injective dimension (see Remark 2.31(2)), the boundedness of the derived categories in the following definition is justified:

Definition 2.34. For a commutative k -algebra A with normalized dualizing complex ω_A° , we define the Grothendieck–Serre duality functor as

$$D_{GS} := \text{RHom}_A(-, \omega_A^\circ): \mathcal{D}^b(A\text{-mod}) \rightarrow (\mathcal{D}^b(A\text{-mod}))^{op}.$$

Remark 2.35. The canonical evaluation morphism $\text{Id} \rightarrow D_{GS}^2$ is an isomorphism on finitely generated modules. This follows by the definition of a dualizing complex, see [Sta21, 0A7C].

Example 2.36. For A a finite dimensional commutative k -algebra we have that $D_{GS} = (-)^*$. To see this, use $\omega_A^\circ = A^*[0]$, discussed in Example 2.33, and the tensor-Hom adjunction:

$$\text{RHom}_A(M, \omega_A^\circ) = \text{RHom}_A(M, \text{Hom}_k[A, k]) = \text{RHom}_k(M, k).$$

Lemma 2.37. For any commutative algebra A with normalized dualizing complex ω_A° , the functor D_{GS} restricts to a duality

$$D_{GS}: A\text{-mod}^{\text{fl}} \rightarrow (A\text{-mod}^{\text{fl}})^{op}$$

naturally identified with the contragredient duality $(-)^* = \text{Hom}_k(-, k)$.

Proof. Let M be a finite length A -module. Then there exists a finite codimension ideal $I \leq A$ such that the A -module structure on M factors through the (finite) map of algebras $A \rightarrow A/I$. Using the right adjoint of restriction (see Lemma 2.28) together with the definition of the normalized dualizing complex, we get natural isomorphisms of A -modules:

$$\begin{aligned} D_{GS}(M) &= \mathrm{RHom}_A(M, \omega_A^\circ) \\ &= \mathrm{RHom}_{A/I}(M, \mathrm{RHom}_A(A/I, \omega_A^\circ)) \\ &= \mathrm{RHom}_{A/I}(M, \omega_{A/I}^\circ) \\ &= \mathrm{RHom}_{A/I}(M, (A/I)^*) \\ &= M^* \end{aligned}$$

where in the last two equalities we have used Example 2.36. □

Remark 2.38. Part of the content of the above lemma is that $D_{GS}(M)$ is concentrated in a single degree if M is a finite length A -module. This is not a priori obvious from the definition of D_{GS} and it is a feature of dualizing complexes. The fact that it lives in degree 0 has to do with the normalization that we chose.

Remark 2.39. In this remark, we recall a part of the duality theory for Cohen-Macaulay local rings R of dimension d with residue field k , and with a canonical module ω_R ($\omega_R = R$ if R is Gorenstein) which we recall has the following two properties:

1. ω_R has finite injective dimension.
2. $\dim_k \mathrm{Ext}_R^d(k, \omega_R) = 1$.

Then if M is a Cohen-Macaulay R -module of dimension t , we have:

1. $\mathrm{Ext}_R^i(M, \omega_R) = 0$ for $i \neq d - t$.
2. $\mathrm{Ext}_R^{d-t}(M, \omega_R)$ is Cohen-Macaulay of dimension t .
3. $\mathrm{Ext}_R^{d-t}(\mathrm{Ext}_R^{d-t}(M, \omega_R), \omega_R) \cong M$.

We would like to apply all this to non-commutative algebras. From now on, we consider R a (possibly) non-commutative k -algebra together with an algebra map $A \rightarrow Z(R)$ from a commutative k -algebra A to the center of R . Suppose moreover that R is finite as an A -module and that A has a (normalized) dualizing complex ω_A° . The category of left R -modules becomes linear over A and therefore one can consider the following functor

$$D_{GS/A}: \mathcal{D}^b(R\text{-mod}) \rightarrow (\mathcal{D}^b(\text{mod-}R))^{op}$$

$$M \mapsto \mathrm{RHom}_A(M, \omega_A^\circ)$$

from the (bounded) derived category of left R -modules to the derived category of right R -modules. We will abuse notation and denote by the same symbol the analogous functor from right modules to left modules.

Definition 2.40. A *duality* on a k -linear category \mathcal{C} is a k -linear functor $D: \mathcal{C} \rightarrow \mathcal{C}^{op}$ such that $D^2 \simeq \mathrm{Id}$.

Notice that for abelian and triangulated categories, a duality is necessarily exact.

Corollary 2.41. *In the above setting, the functor $D_{GS/A}$ restricts to a duality*

$$D_{GS/A}: R\text{-mod}^{\mathrm{fl}} \rightarrow (\mathbf{mod}^{\mathrm{fl}}\text{-}R)^{op}$$

that can moreover be identified with the contragredient $(-)^*$.

Proof. Observe that since R is a finite A -module, any finite length R -module restricts to a finite length A -module. We can therefore apply Lemma 2.37 to deduce that $D_{GS/A}(M) \simeq M^*$ as A -modules for any M a finite length left (resp., right) R -module. The naturality implies the isomorphism $D_{GS/A}(M) \simeq M^*$ holds also as right (resp., left) R -modules. \square

Remark 2.42. Notice that the proof above still applies to R -modules that are of finite length as A -modules even if we do not assume that R is finite over A .

Recall that for a k -algebra R , the homological duality was defined in Section 2.3 as the functor

$$D_h := \mathrm{RHom}_R(-, R): \mathcal{D}^\pm(R\text{-mod}) \rightarrow \mathcal{D}^\mp(\mathbf{mod}\text{-}R)^{op},$$

where it was shown that it is a duality on perfect complexes. If moreover R has finite global dimension, then we get a duality on the bounded derived category of finitely generated R -modules (see Proposition 2.24):

$$D_h: \mathcal{D}_{fg}^b(R\text{-mod}) \rightarrow \mathcal{D}_{fg}^b(\mathbf{mod}\text{-}R)^{op}.$$

In general, there is no reason for this functor to preserve any abelian subcategories or to have good properties. However, in the case of representations of p -adic groups it does (see Theorem 10.2) thanks to second-adjointness and the

following technical condition that is satisfied on the cuspidal blocks (keeping the assumptions on R and A)

$$\begin{aligned}
 & \text{there is an integer } d \text{ such that} \\
 (\text{FsG}) \quad & D_{GS/A}(R) \simeq R[d] \\
 & \text{as } R\text{-bimodules}
 \end{aligned}$$

Remark 2.43. The condition (FsG) is satisfied for all cuspidal blocks (see Corollary 9.6) but it is not satisfied for all Bernstein blocks for a p -adic group, for example it does not hold for the Iwahori block for $SL_2(F)$ as we discuss in Example Example 12.9 below.

Remark 2.44. The name (FsG) is inspired by the notions of *Frobenius symmetric* and *Gorenstein* algebras. Recall that a finite dimensional algebra A over a field k is called *Frobenius* if there exists a nondegenerate bilinear form $B : A \times A \rightarrow k$ such that $B(ab, c) = B(a, bc)$ for all a, b, c in A , and is called *Frobenius symmetric* if furthermore $B(a, b) = B(b, a)$ for all a, b in A . One can see that a finite dimensional k algebra A is Frobenius if and only if there exists a codimension one subspace of A not containing any nonzero right-ideal of A . Before proceeding further, recall that if M is a right- A module (resp., left- A module), then $\text{Hom}_k(M, k)$ is naturally a left- A module (resp., right- A module). Now note that given a bilinear form $B : A \times A \rightarrow k$, the associated map $A \rightarrow \text{Hom}_k(A, k)$ is equivariant for the right- A module structures if and only if $B(ab, c) = B(a, bc)$ for all a, b, c in A , whereas the map $A \rightarrow \text{Hom}_k(A, k)$ is equivariant for the left- A module structures if and only if $B(ba, c) = B(a, cb)$ for all a, b, c in A . Thus a finite dimensional associative algebra A over k is Frobenius if and only if $\text{Hom}_k(A, k) \cong A$ as right A -modules, whereas $\text{Hom}_k(A, k) \cong A$ is a bi- A -module isomorphism if and only if the bilinear form $B : A \times A \rightarrow k$ is both Frobenius and symmetric. If A is commutative and local then A is Frobenius if and only if A is Gorenstein. Among non-commutative algebras, the best known Frobenius symmetric algebra is the group algebra $k[G]$ of a finite group G for k a field of characteristic zero or nonzero.

Proposition 2.45. *Keeping the same assumptions, suppose moreover that R satisfies condition (FsG) above. Then the (shifted by d) homological duality functor $[d] \circ D_h$ is isomorphic to $D_{GS/A}$. In particular, it restricts to a duality on finite length R -modules*

$$[d] \circ D_h : R\text{-mod}^{\text{fl}} \rightarrow (\text{mod}^{\text{fl}}\text{-}R)^{op}$$

that is moreover isomorphic to the contragredient.

Proof. Using condition (FsG), tensor-hom adjunction and Corollary 2.41 we get natural isomorphisms of functors:

$$\begin{aligned}
 [d] \circ D_h &= [d] \circ \mathrm{RHom}_R(-, R) \\
 &= \mathrm{RHom}_R(-, \mathrm{RHom}_A(R, \omega_A^\circ)) \\
 &= \mathrm{RHom}_A(-, \omega_A^\circ) \\
 &= D_{GS/A} \\
 &= (-)^* \quad \text{for finite length modules.} \quad \square
 \end{aligned}$$

Remark 2.46. If in Condition (FsG), we replace R -bimodule by left R -module, then we still get that $[d] \circ D_h$ sends $R\text{-mod}^{\mathrm{fl}}$ to $\mathbf{mod}^{\mathrm{fl}}\text{-}R$ but we can not identify it with the contragredient.

The above proof gives us a bit more:

Corollary 2.47. *Keeping the same assumptions, we have that $[d] \circ D_h \simeq D_{GS/A}$ on all R -modules if and only if condition (FsG) is satisfied.*

Proof. By the definition of $D_{GS/A}$ and adjunction we have

$$\begin{aligned}
 D_{GS/A} &= \mathrm{RHom}_A(-, \omega_A^\circ) \\
 &= \mathrm{RHom}_R(-, \mathrm{RHom}_A(R, \omega_A^\circ))
 \end{aligned}$$

and, by Yoneda’s lemma, this last functor is isomorphic to

$$[d] \circ D_h = \mathrm{RHom}_R(-, R[d])$$

if and only if condition (FsG) is satisfied. □

Let us consider a particular case that will be of importance to us. Suppose that R is a finite projective A -algebra where $A = k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$. We have that $\omega_A^\circ = A[d]$ and then condition (FsG) becomes

$$(2.8) \quad \mathrm{Hom}_A(R, A) \simeq R \text{ as } R\text{-bimodules.}$$

Corollary 2.48. *Keeping the above notation and assuming condition (2.8), we have the following isomorphisms of functors when restricted to finite-length R -modules*

$$\mathrm{RHom}_R(-, R)[d] \simeq \mathrm{RHom}_A(-, A)[d] \simeq \mathrm{Hom}_k(-, k).$$

3. Conventions on groups and representations

3.1. Central extensions

Let $G = \mathbb{G}(F)$ be the locally compact group of F -rational points of a reductive linear algebraic group \mathbb{G} over a non-archimedean local field F . Let \tilde{G} be a finite central extension of G with kernel a finite abelian group μ :

$$1 \rightarrow \mu \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

that is moreover a topological covering. In this situation it is proved, for example in [BJ, Lemma 2.2], that \tilde{G} admits a basis of open sets around the identity element of \tilde{G} consisting of compact open subgroups lifted from G (i.e., for which the restriction of the map $\tilde{G} \rightarrow G$ is an isomorphism). So \tilde{G} is a totally disconnected group or an l -group in the terminology of [Ber92].

3.2. Representations

In this section, we recall some notions around the representation theory of locally compact totally disconnected groups G . All representations will be on complex vector spaces. One can consult [BZ76, Ber92, Ren10] for details. We will work with the category $\mathcal{M}(G)$ of smooth complex representations of such a group G .

Let $H \leq G$ be a closed subgroup of G . Then restricting a representation from G to H we obtain an exact functor between the categories of smooth representations

$$\mathrm{Res}_H^G: \mathcal{M}(G) \rightarrow \mathcal{M}(H).$$

The restriction functor has a right adjoint given by induction

$$\mathrm{Ind}_H^G: \mathcal{M}(H) \rightarrow \mathcal{M}(G)$$

which is defined by,

$$\mathrm{Ind}_H^G(V) := \{f: G \rightarrow V \mid f(hg) = hf(g), \text{ for all } h \in H \text{ and } g \in G\}^{\mathrm{sm}}.$$

The pair of functors $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ has the adjointness property (for V a representation of G , and W of H)

$$\mathrm{Hom}_H[\mathrm{Res}_H^G V, W] = \mathrm{Hom}_G[V, \mathrm{Ind}_H^G(W)],$$

which goes under the name of Frobenius reciprocity for which we refer to [Ber92] or [Ren10, III.2.5].

The induction functor admits a subfunctor $\text{ind}_H^G \subset \text{Ind}_H^G$ called *compact induction* consisting of functions with compact support modulo H :

$$\text{ind}_H^G(V) := \{f \in \text{Ind}_H^G(V) \mid H \backslash \text{supp}(f) \text{ is compact}\}.$$

In case G/H is compact we clearly have $\text{ind}_H^G = \text{Ind}_H^G$.

Definition 3.1. If V is a smooth representation of G then the contragredient representation of V is defined to be the smooth part of the linear dual $V^\vee := (V^*)^{\text{sm}}$.

One can prove (see [Ren10, III.2.7]) that induction and compact induction are related to each other through the contragredient. More precisely, for V a smooth representation of H we have

$$\text{Ind}_H^G(V^\vee) = \text{ind}_H^G(V \otimes \delta_{H \backslash G})^\vee,$$

where $\delta_{H \backslash G}$ is the modular character of G divided by the one of H .

Suppose now that $H \leq G$ is open. Since H is the complement of the union of all non-trivial left cosets Hg , $g \in G$, it is also a closed subgroup.

Lemma 3.2 (see [Ren10, III.2.6.5]). *If $H \leq G$ is an open subgroup then the restriction functor Res_H^G is right adjoint to the compact-induction functor ind_H^G . In particular, ind_H^G sends projective objects to projective objects.*

3.3. Projectives and injectives

We continue with G being a locally compact, totally disconnected group. Let $\mathcal{M}(G)$ be the abelian category of smooth representations of G . Let $\mathcal{H}(G)$ be the Hecke algebra of locally constant compactly supported functions on G endowed with convolution (one needs to choose a left invariant Haar measure on G). The algebra $\mathcal{H}(G)$ is an algebra without unit but with a rich supply of idempotents⁶ because G has a basis of neighborhoods of identity consisting of open compact subgroups. A representation V of $\mathcal{H}(G)$ is said to be *non-degenerate* if it has the property that $\mathcal{H}(G)V = V$. There is the well-known equivalence of the category of smooth representations of G and the category of non-degenerate representations of $\mathcal{H}(G)$:

$$\mathcal{M}(G) \simeq \mathcal{H}(G)\text{-mod}^{\text{nd}}.$$

⁶It is what is called an *idempotent algebra*: for every finite set of element a_i , there is an idempotent e such that $ea_i = a_i e = a_i$ for each a_i .

Remark 3.3. Suppose that G admits a countable basis of neighborhoods of identity consisting of compact open subgroups. Then the Hecke algebra $\mathcal{H}(G)$ is a projective object in $\mathcal{M}(G)$. Indeed, one writes $\mathcal{H}(G) = \bigcup_{i=1}^\infty \mathcal{H}(G)e_i$ where $e_i = e_{K_i}$ are idempotents corresponding to a countable basis of compact open subgroups of G . Since $\text{Hom}_G(\mathcal{H}(G)e_i, V) = V^{K_i}$, each of the $\mathcal{H}(G)e_i$ is a projective object in $\mathcal{M}(G)$, and then one notices that $\text{Hom}_G(\mathcal{H}(G), -) = \lim_i \text{Hom}_G(\mathcal{H}(G)e_i, -)$ and this latter inverse limit is exact because the transition functions are surjective (and hence the Mittag-Leffler condition is automatically satisfied).

The abelian category $\mathcal{M}(G)$ has a good supply of injective and projective objects, for example for any open compact subgroup K of G , $\text{ind}_K^G(\mathbb{C})$ is a projective object (see Lemma 3.2), and its smooth dual $\text{Ind}_K^G(\mathbb{C})$ is an injective object.

We use $\text{Ext}_G^i(V, V')$ to denote Ext groups in $\mathcal{M}(G)$.

3.4. Homological dimension

Here $G = \mathbb{G}(F)$ and \tilde{G} is a central extension of G as defined in Section 3.1.

It is shown in [Ber92, Theorem 29] that the category of smooth representations of G has finite homological dimension. The argument uses the building of G to give a resolution of the trivial module by projective modules which are sums of representations of G of the form $\text{ind}_K^G(\mathbb{C})$ (where K are compact open subgroups of G):

$$0 \rightarrow P_d \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{C} \rightarrow 0,$$

where d is the split rank of G . Tensoring this resolution with any G -module V , and observing that

$$\text{ind}_K^G(\mathbb{C}) \otimes V = \text{ind}_K^G(V|_K),$$

we find a projective resolution of length $\leq d$ for any G -module V . This argument with the resolution of the trivial module \mathbb{C} for G works just as well for covering groups $\tilde{G} \rightarrow G$ since $\text{ind}_K^G(\mathbb{C})$ treated as a \tilde{G} -module is $\text{ind}_{\tilde{K}}^{\tilde{G}}(\mathbb{C})$, and $\text{ind}_K^G(\mathbb{C}) \otimes V = \text{ind}_{\tilde{K}}^{\tilde{G}}(V|_{\tilde{K}})$ is a projective \tilde{G} -module. Therefore, $\text{Ext}_{\tilde{G}}^i(V, V') = 0$ for any representations V, V' of \tilde{G} , if $i > d$.

Once we have the Bernstein decomposition for covering groups one can prove exactly as for linear group that the category of smooth representations $\mathcal{M}(\tilde{G})$ is noetherian (Theorem 7.15). Since it is also of finite homological dimension, any finitely generated module admits a finite resolution by finitely

generated projective modules. (If d is the projective dimension of a finitely generated representation V , resolve V by finitely generated projective modules $P_{d-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow V \rightarrow 0$, and then observe as in the proof of Hilbert Syzygy that the kernel of the map $P_{d-1} \rightarrow P_{d-2}$ must be projective.) This will be useful when we apply the abstract results from Section 2.3 to representation theory in Section 10 and Section 11 as all finitely generated modules will be perfect.

4. Splitting the category of representations

4.1. Compact representations

In this section, G denotes an arbitrary locally compact td-group which is countable at infinity. The most important result we need is that compact representations split the category of smooth representations of G .

Definition 4.1. A smooth representation (π, V) of G is said to be compact if all its matrix coefficients have compact support.

Remark 4.2. The existence of a compact irreducible representation implies that G has compact center.

The first important theorem about compact representations is that any compact representation which appears as a subquotient in any smooth representation appears as a direct summand by [BZ76, Theorem 2.44]. The following theorem is an easy consequence of it:

Theorem 4.3. *Let ρ be a compact irreducible representation of G . Then matrix coefficients of ρ provide us with an injective map of $G \times G$ -modules*

$$\rho \boxtimes \rho^\vee \subset \mathcal{H}(G),$$

and this is the only subquotient of $\mathcal{H}(G)$ isomorphic to $\rho \boxtimes \rho^\vee$.

Moreover, if ρ' is another irreducible compact representation of G , non-isomorphic to ρ , then the $G \times G$ representation $\rho \boxtimes \rho'^\vee$ does not appear as a subquotient of $\mathcal{H}(G)$.

Proof. There is a natural map from $\rho \boxtimes \rho^\vee$ to functions on G given by

$$v \otimes \ell \in \rho \boxtimes \rho^\vee \mapsto F_{v \otimes \ell}(g) = \ell(gv),$$

which if ρ is a compact smooth representation of G lands inside compactly supported smooth functions on G , i.e., in $\mathcal{H}(G)$. If ρ is furthermore irreducible, the resulting map $\rho \boxtimes \rho^\vee \subset \mathcal{H}(G)$ is injective.

As any compact representation which appears as a subquotient in any smooth representation appears as a direct summand by [BZ76, Theorem 2.44], the rest of the theorem is a direct consequence of the Frobenius reciprocity:

$$\mathrm{Hom}_{G \times G}[\mathrm{ind}_{\Delta G}^{G \times G}(\mathbb{C}), \rho \boxtimes \rho'^{\vee}] = \mathrm{Hom}_{\Delta G}[\rho \boxtimes \rho'^{\vee}, \mathbb{C}]. \quad \square$$

We denote by $\mathcal{M}(G)_c$ the full subcategory of $\mathcal{M}(G)$ consisting of those smooth representations of G all whose irreducible subquotients are compact representations. It is clearly a subcategory closed under subquotients, direct sums and extensions.

We denote by $\mathcal{M}(G)_{nc}$ the full subcategory of $\mathcal{M}(G)$ formed of representations that have no compact irreducible subquotient.

For \mathcal{S} , a collection of irreducible representations of G , we denote by $\mathcal{M}(G)_{[\mathcal{S}]}$ the subcategory of $\mathcal{M}(G)$ formed of representations such that all their irreducible subquotients are isomorphic to an object in \mathcal{S} . Denote by $\mathcal{M}(G)_{[\mathrm{out} \mathcal{S}]}$ the subcategory of representations such that none of their irreducible subquotients are isomorphic to an object in \mathcal{S} .

The following theorem summarizes the main properties of compact representations, cf. [BZ76, Theorem 2.44]:

Theorem 4.4. *For \mathcal{S} a finite collection of compact irreducible representations of G , we have:*

1. *The category $\mathcal{M}(G)_{[\mathcal{S}]}$ is semisimple: all the objects in $\mathcal{M}(G)_{[\mathcal{S}]}$ are isomorphic to a direct sum of objects in \mathcal{S} .*
2. *There is an equivalence of categories*

$$(4.1) \quad \mathcal{M}(G) \simeq \mathcal{M}(G)_{[\mathcal{S}]} \times \mathcal{M}(G)_{[\mathrm{out} \mathcal{S}]}.$$

Moreover, the category $\mathcal{M}(G)_c$ admits a decomposition

$$(4.2) \quad \mathcal{M}(G)_c = \prod_{\tau} \mathcal{M}(G)_{[\tau]}$$

where the product runs over all isomorphism classes of compact irreducible representations of G . In particular, $\mathcal{M}(G)_c$ is a semisimple category.

It is natural to ask if we can always decompose a representation into a direct sum of compact and non-compact. This is not completely automatic from Theorem 4.4, and one needs a further finiteness condition:

Consider the following condition (see [Ren10, IV.1.7 (KF)]) called “compact finite”:

(KF) for any compact open subgroup $K \leq G$ there is only a finite number of isomorphism classes of compact irreducible representations of G having a non-zero K -fixed vector.

Theorem 4.5. *If the group G satisfies the above condition (KF), then we have a decomposition of categories*

$$(4.3) \quad \mathcal{M}(G) = \mathcal{M}(G)_c \times \mathcal{M}(G)_{nc}.$$

Remark 4.6. Notice that in light of the theorem Theorem 4.5, every compact representation is projective-injective in $\mathcal{M}(G)$. This is a remarkably strong homological property that will be useful in the sequel.

In our situation, the condition (KF) is satisfied thanks to the uniform admissibility theorem; see Theorem 6.4.

4.2. Compact modulo center

We have seen that compact representations behave as nicely as one could hope but for many interesting groups there are no such representations. It turns out that the issue comes from the non-compactness of the center. In this subsection we present an analogue of the decomposition of categories Theorem 4.4 and Theorem 4.5.

Definition 4.7. A representation of G is called *compact modulo center* if its matrix coefficients have compact support modulo $Z(G)$.

Denote by G° the subgroup of $G = \mathbb{G}(F)$, for \mathbb{G} a reductive group over F , defined by:

$$G^\circ = \cap \{\ker|\chi| : G \rightarrow \mathbb{R}^+\},$$

where $\chi : G \rightarrow F^\times$ are the algebraic characters of G defined over F . The subgroup G° of G has the property that $G/G^\circ \simeq \mathbb{Z}^d$ for some $d \geq 0$. Further, $G^\circ Z(G)(F)$ is a normal subgroup of finite index in $G(F)$. The subgroup $G^\circ \subset G(F)$ contains all compact subgroups of $G(F)$ and, in fact, G° is generated by all compact subgroups of G . Since we could not find a convenient reference, we provide a proof for which we thank Sandeep Varma.

Lemma 4.8. *The subgroup G° is generated by all compact subgroups of G .*

Proof. Let $G^d = [G, G]$ be the derived subgroup of G , and $j : G^s \rightarrow G^d$, the simply connected cover of G^d . The group $G(F)$ operates by conjugation on G^d , and this action lifts to G^s , thus $G(F)$ acts on the set of Iwahori subgroups of $G^s(F)$ which are permuted transitively by $G^s(F)$. Let \mathfrak{J} be a fixed Iwahori subgroup of $G^s(F)$. We know that the only elements of $G^s(F)$ that normalize \mathfrak{J} belong to \mathfrak{J} , and the stabilizer of \mathfrak{J} in G° , call it $N(\mathfrak{J})$, is a compact subgroup of $G(F)$ (this uses the assertion contained in the Lemma for tori which is equivalent to the assertion that anisotropic tori are the same as compact tori). Hence $N(\mathfrak{J})$ surjects onto $G^\circ/j(G^s(F))$. It follows that if we can prove the assertion of the Lemma for the simply connected group G^s , then the Lemma follows for the group G .

The assertion of the Lemma for the simply connected group G^s , assuming that it is isotropic (else $G^s(F)$ is compact and there is nothing to prove), follows from the Bruhat decomposition $G^s(F) = \mathfrak{J}W^{\text{aff}}\mathfrak{J}$ where W^{aff} is a Coxeter group which comes equipped with a set of generators $\{s \in S\}$ of order 2 where each s treated as an element of $G^s(F)$ is compact (since $\mathfrak{J} \cup \mathfrak{J}s\mathfrak{J}$ is a compact parahoric subgroup of $G^s(F)$). \square

We put $\mathcal{X}(G) := \text{Hom}_{\text{gr}}(G/G^\circ, \mathbb{C}^\times)$ and call it the group of *unramified characters* of G .

Remark 4.9. A representation π of G is compact modulo center if and only if its restriction $\pi|_{G^\circ}$ is compact.

The following well-known proposition, although simple, is the main ingredient allowing one to pass from G° to G (since $G^\circ \cdot Z(G)$ is of finite index in G , the proof of this proposition is the same as the Clifford theory in finite groups):

Proposition 4.10 ([BZ76, Lemmas 3.26 and 3.29]).

1. Let (V, π) be an irreducible representation of G . The irreducible representations of G° appearing in $\text{Res}_{G^\circ}^G(\pi)$ are all conjugate under G . Moreover, the representation $\text{Res}_{G^\circ}^G(\pi)$ is semisimple and of finite length.
2. Let (V_1, π_1) and (V_2, π_2) be two irreducible representations of G . Then the following are equivalent.

(a) $\text{Res}_{G^\circ}^G(\pi_1) \simeq \text{Res}_{G^\circ}^G(\pi_2)$,

(b) there exists an unramified character $\chi \in \mathcal{X}(G)$ such that

$$\pi_1 \otimes \chi \simeq \pi_2,$$

(c) $\text{Hom}_{G^\circ}(\text{Res}_{G^\circ}^G(\pi_1), \text{Res}_{G^\circ}^G(\pi_2)) \neq 0$.

We denote by $\mathcal{M}(G)_{\text{cs}}$, the full subcategory of $\mathcal{M}(G)$ formed of those representations all of whose subquotients are compact modulo center. The set of irreducible objects of $\mathcal{M}(G)_{\text{cs}}$ is denoted by $\text{Irr}(G)_{\text{cs}}$.

Definition 4.11. We say that two irreducible representations $\rho_1, \rho_2 \in \text{Irr}(G)_{\text{cs}}$ are in the same *inertia class* if there exists a character $\chi \in \mathcal{X}(G)$ such that $\rho_1 \simeq \chi\rho_2$. We write the corresponding equivalence relation as $\rho_1 \sim \rho_2$, and denote the inertia class containing $\rho \in \text{Irr}(G)_{\text{cs}}$ by the square bracket $[\rho]$. We denote the set of inertia classes in $\text{Irr}(G)_{\text{cs}}$ by $[\text{Irr}(G)_{\text{cs}}]$.

Given $\pi \in \text{Irr}(G)_{\text{cs}}$, we denote by $\mathcal{M}(G)_{[\pi]}$, the full subcategory of $\mathcal{M}(G)$ consisting of those smooth representations of G whose restriction to G° have all the irreducible subquotients only among those of $\pi|_{G^\circ}$ (a finite set of compact representations of G°).

Remark 4.12. Proposition 4.10 says that two irreducible representations of G are in the same inertia class if and only if their restriction to G° are isomorphic. Therefore the objects of the category $\mathcal{M}(G)_{[\pi]}$ are those smooth representations of G all of whose irreducible subquotients are in the same inertia class as π .

Using the above proposition and Theorem 4.4, one immediately deduces:

Theorem 4.13 ([Ber84]). *For an irreducible compact modulo center representation π of G , we have a decomposition of the category $\mathcal{M}(G)$ as*

$$\mathcal{M}(G) \simeq \mathcal{M}(G)_{[\pi]} \times \mathcal{M}(G)_{[\text{out } \pi]}.$$

Moreover, the category $\mathcal{M}(G)_{\text{cs}}$ decomposes as

$$\mathcal{M}(G)_{\text{cs}} = \prod_{[\pi] \in [\text{Irr}(G)_{\text{cs}}]} \mathcal{M}(G)_{[\pi]}.$$

The full subcategory of $\mathcal{M}(G)$ formed of those representations that have no irreducible subquotient that is compact modulo center is denoted by $\mathcal{M}(G)_{\text{ind}}$. Using Proposition 4.10 and Theorem 4.5 one deduces easily:

Theorem 4.14 ([Ber84]). *If the group G° satisfies condition (KF) then we have a decomposition of categories*

$$\mathcal{M}(G) = \mathcal{M}(G)_{\text{cs}} \times \mathcal{M}(G)_{\text{ind}}.$$

4.3. The simplest Bernstein component

The goal of this subsection is to formulate and prove an analogue of Theorem 4.3

Fix $\pi \in \mathcal{M}(G)$ an irreducible, compact modulo center representation of G . Then $\pi \boxtimes \pi^\vee \in \mathcal{M}(G \times G)$ is irreducible and compact modulo center. We consider the Hecke algebra $\mathcal{H}(G) = \text{ind}_{\Delta G}^{G \times G} \mathbb{C}$ as a $G \times G$ -module and we try to understand the part of $\mathcal{H}(G)$, denoted by $\mathcal{H}(G)[\pi \boxtimes \pi^\vee]$, that lives in $\mathcal{M}(G \times G)_{[\pi \boxtimes \pi^\vee]}$ (see Theorem 4.13).

Let us fix $\pi_0 \subset \pi|_{G^\circ}$, an irreducible representation of G° . Put $G_1 = \{g \in G \mid \pi_0 \simeq {}^g \pi_0\} \leq G$. It is a normal subgroup of G containing $G^\circ Z(G)$ which is of finite index in G , hence G_1 is of finite index in G . Denote by Σ the group G/G_1 and its order by f .

Put $H := \Delta(G)(G^\circ \times G^\circ) \leq G \times G$. The following is the main result of this subsection:

Proposition 4.15. *For $\pi \in \mathcal{M}(G)$ an irreducible, compact modulo center representation of G , we have an isomorphism of $G \times G$ representations*

$$(4.4) \quad \mathcal{H}(G)[\pi \boxtimes \pi^\vee] \simeq \text{ind}_{\Delta(G_1)(G^\circ \times G^\circ)}^{G \times G} (\pi_0 \boxtimes \pi_0^\vee).$$

Further,

$$(4.5) \quad (\pi \boxtimes \pi^\vee) \otimes \text{ind}_H^{G \times G} \mathbb{C} \simeq \text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ}) \otimes \mathcal{H}(G)[\pi \boxtimes \pi^\vee] \\ \simeq e^2 f \mathcal{H}(G)[\pi \boxtimes \pi^\vee],$$

where $\text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ})$ is a finite dimensional representation of G/G° , treated as a representation of $G \times G$ trivial on $H = \Delta(G)(G^\circ \times G^\circ) \leq G \times G$.

Proof. Put $\mathcal{S} = \{\pi' \mid \pi' \subset \pi \boxtimes \pi^\vee|_{G^\circ \times G^\circ} \text{ irreducible}\}$. The set \mathcal{S} is a finite set of compact irreducible representations of $G^\circ \times G^\circ$ and given Theorem 4.4 we can write

$$\mathcal{H}(G)|_{G^\circ \times G^\circ} = \mathcal{H}(G)_{[\mathcal{S}]} \oplus \mathcal{H}(G)_{[\text{out } \mathcal{S}]}$$

as $G^\circ \times G^\circ$ -representations. Moreover, since \mathcal{S} is stable under conjugation by $G \times G$, the above decomposition holds as $G \times G$ -modules. Let us write $\mathcal{H}(G)[\pi \boxtimes \pi^\vee]$ for the $G \times G$ representation $\mathcal{H}(G)_{[\mathcal{S}]}$.

For $\sigma \in \Sigma$, let $\pi_\sigma := \sigma \pi_0 \subset \pi$, an irreducible sub G° -representation of $\pi|_{G^\circ}$. Notice that π_σ is isomorphic to the conjugated representation ${}^\sigma \pi_0$ where the action on π_0 is conjugated by σ . The irreducible summands of $\pi|_{G^\circ}$ are

isomorphic to π_σ for some $\sigma \in \Sigma$ and $\pi_\sigma \not\simeq \pi_{\sigma'}$ if $\sigma \neq \sigma'$. Let's write the restriction of π to G° as (for an integer $e \geq 1$):

$$\pi|_{G^\circ} = \bigoplus_{\sigma \in \Sigma} e\pi_\sigma.$$

For an irreducible representation π of G , the set of unramified characters $\chi : G/G^\circ \rightarrow \mathbb{C}^\times$ with $\pi \otimes \chi \simeq \pi$ is a finite abelian group, call it $X(\pi)$. Then, the space $\text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ})$ comes equipped with an action of the abelian group G/G° , diagonalizing which gives a canonical basis (up to scalars) which are nothing but the intertwining operators $\pi \otimes \chi \simeq \pi$, thus one sees that $\dim \text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ}) = e^2 f$ is the number of the intertwining operators $\pi \otimes \chi \simeq \pi$, and $\text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ})$ comes equipped with a basis e_χ such that $g \cdot e_\chi = \chi(g)e_\chi$.

The G° -representation π_0 does not extend to a representation of G_1 but the $(G^\circ \times G^\circ)$ -representation $\pi_0 \boxtimes \pi_0^\vee$ does extend to a representation of $\Delta(G_1)(G^\circ \times G^\circ)$: this is most easily seen by noticing that π_0 extends to a projective representation of G_1 , and therefore $\pi_0 \boxtimes \pi_0^\vee$ is canonically a representation of $\Delta(G_1)$, hence of $\Delta(G_1)(G^\circ \times G^\circ)$. The representations $\pi_\sigma \boxtimes \pi_{\sigma'}^\vee$ are also representations of $\Delta(G_1)(G^\circ \times G^\circ)$ as they can be written as $(\sigma, \sigma')\pi_0 \boxtimes \pi_0^\vee$.

In what follows, let τ be the irreducible representation of $H = \Delta(G)(G^\circ \times G^\circ)$,

$$\tau = \bigoplus_{\sigma \in \Sigma} \pi_\sigma \boxtimes \pi_\sigma^\vee = \text{ind}_{\Delta(G_1)(G^\circ \times G^\circ)}^H (\pi_0 \boxtimes \pi_0^\vee).$$

By induction in stages: $G \times G \supset \Delta(G)(G^\circ \times G^\circ) \supset \Delta(G)$, we can write:

$$\mathcal{H}(G) = \text{ind}_{\Delta(G)}^{G \times G} \mathbb{C} = \text{ind}_H^{G \times G} \mathcal{H}(G^\circ),$$

as a representation of $G \times G$, where we recall that $H = \Delta(G)(G^\circ \times G^\circ) \leq G \times G$ is a normal subgroup of $G \times G$ with a natural action on $\mathcal{H}(G^\circ)$ (where $\Delta(G)$ acts on G° by the conjugation action, and $G^\circ \times G^\circ$ acts on G° by left and right translations). Since the set \mathcal{S} is stable under conjugation by $G \times G$, we deduce the following

$$\mathcal{H}(G)[\pi \boxtimes \pi^\vee] = \text{ind}_H^{G \times G} (\mathcal{H}(G^\circ)_{[\mathcal{S}]})$$

Theorem 4.3 tells us that in $\mathcal{M}(G^\circ \times G^\circ)$, we have a natural isomorphism

$$\mathcal{H}(G^\circ)_{[\mathcal{S}]} \simeq \bigoplus_{\sigma \in \Sigma} \pi_\sigma \boxtimes \pi_\sigma^\vee \simeq \text{ind}_{\Delta(G_1)(G^\circ \times G^\circ)}^{\Delta(G)(G^\circ \times G^\circ)} (\pi_0 \boxtimes \pi_0^\vee),$$

which is moreover an isomorphism of H -modules, proving the isomorphism (4.4).

All the irreducible subrepresentations of $\pi \boxtimes \pi^\vee|_H$ are of the form $(\sigma, 1)\tau \simeq (\sigma, 1)_\tau$, hence

$$(4.6) \quad \pi \boxtimes \pi^\vee|_H = \bigoplus_{\sigma \in \Sigma} e^2(\sigma, 1)\tau.$$

Put these together and use the Mackey’s relation $V \otimes \text{ind}_K^G W = \text{ind}_K^G(V|_K \otimes W)$ to get

$$\begin{aligned} (\pi \boxtimes \pi^\vee) \otimes \text{ind}_H^{G \times G}(\mathbb{C}) &\simeq \text{ind}_H^{G \times G}(\pi \boxtimes \pi^\vee|_H) \\ &\simeq e^2 f \text{ind}_H^{G \times G}(\tau) \\ &\simeq e^2 f \text{ind}_{\Delta(G_1)(G^\circ \times G^\circ)}^{G \times G}(\pi_0 \boxtimes \pi_0^\vee) \\ &\simeq \text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ}) \otimes \text{ind}_{\Delta(G_1)(G^\circ \times G^\circ)}^{G \times G}(\pi_0 \boxtimes \pi_0^\vee) \\ &\simeq \text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ}) \otimes \mathcal{H}(G)[\pi \boxtimes \pi^\vee], \end{aligned}$$

where in the 2nd isomorphism above, we have used (4.6) together with the fact that H is a normal subgroup of $G \times G$, and $\text{ind}_H^{G \times G}(\tau) = \text{ind}_H^{G \times G}(\tau^{(\sigma, 1)})$ for all $\sigma \in \Sigma$; the last isomorphism is the isomorphism (4.4), and the isomorphism previous to that follows since $\text{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ})$ consists of characters of G/G° , treated as a character of $G \times G$ trivial on $\Delta(G)(G^\circ \times G^\circ)$. Thus we have proved the isomorphism (4.5). \square

Remark 4.16. The isomorphism (4.5) of the above proposition can be interpreted as saying that for the cuspidal representation $\pi \boxtimes \pi^\vee$ of $G \times G$, the corresponding Bernstein component in $\mathcal{H}(G)$ “contains” all representations $(\pi\alpha) \boxtimes (\pi\alpha)^\vee$ where α runs over all the unramified characters of G/G° . Since $\pi \otimes \chi \simeq \pi$ for $e^2 f$ many characters, this number shows up as multiplicity in the right hand side of (4.5). This means that for each irreducible cuspidal representation π of G , $\pi \boxtimes \pi^\vee$ “appears” in $\mathcal{H}(G)$ with multiplicity 1, where the precise meaning of “appears” is to be understood in the derived sense, see Proposition 4.23.

Remark 4.17. Suppose the group G is either a reductive p -adic group, or a finite cover of it. Then for a standard parabolic $P = LN$ inside G , and any cuspidal pair $[L, \rho]$ defining a Bernstein block \mathfrak{s} (see Section 7), similar to Proposition 4.15, it is natural to propose the following isomorphism in $\mathcal{M}(G \times G)$

$$\text{ind}_{P \times P^-}^{G \times G}(\rho \boxtimes \rho^\vee \otimes \text{ind}_{\Delta(L)(L^\circ \times L^\circ)}^{L \times L}(\mathbb{C})) \simeq e^2 f w \mathcal{H}(G)[\mathfrak{s} \times \mathfrak{s}^\vee],$$

where e, f are defined as before for the cuspidal representation ρ of L , and w is the order of $N_G(L, [\rho])/L$ where $N_G(L, [\rho])$ is the normalizer of L preserving ρ up to an unramified character. This will be a kind of Plancherel decomposition in the smooth category; see [Hei04] for some results in this direction.

The following corollary is a consequence of the proof of the above proposition.

Corollary 4.18. *The component of $\mathcal{H}(G)$ in $\mathcal{M}(G)_{[\pi_1 \boxtimes \pi_2^\vee]}$ is zero unless $\pi_2 \simeq \pi_1 \chi$ for some unramified character χ of G .*

Remark 4.19. In the language of Bernstein decomposition reviewed in Section 7, this is equivalent to saying that the Bernstein component of $\mathcal{H}(G)$ corresponding to $\pi_1 \boxtimes \pi_2^\vee$ is zero unless $\pi_2 \simeq \pi_1 \chi$ for some unramified character χ of G .

We note the following corollary that will be useful to us in Section 9 when studying contragredients. It also appears in [AS00, Lemma B.5]

Corollary 4.20. *For any irreducible representation $\pi \in \mathcal{M}(G)$, compact modulo center, we have an isomorphism of G -modules*

$$\mathrm{Hom}_G(\mathrm{ind}_{G^\circ}^G(\pi|_{G^\circ}), \mathcal{H}(G)) \simeq \mathrm{ind}_{G^\circ}^G(\pi^\vee|_{G^\circ}),$$

where in the left hand side, one considers $\mathrm{Hom}_G(-, \mathcal{H}(G))$ for the left G -module structure on $\mathcal{H}(G)$, and as $\mathcal{H}(G)$ is both left and right G -module, $\mathrm{Hom}_G(-, \mathcal{H}(G))$ continues to be a right G -module.

Proof. By the Mackey theory, since $(G^\circ \times G)H = G \times G$, it follows that the restriction of $\mathrm{ind}_H^{G \times G}(\mathbb{C})$ to $G^\circ \times G$ is $\mathbb{C} \otimes \mathrm{ind}_{G^\circ}^G \mathbb{C}$. Hence by the isomorphism (4.5), and Frobenius reciprocity:

$$\begin{aligned} e^2 f \mathrm{Hom}_G(\mathrm{ind}_{G^\circ}^G(\pi|_{G^\circ}), \mathcal{H}(G)) &\simeq e^2 f \mathrm{Hom}_G(\mathrm{ind}_{G^\circ}^G(\pi|_{G^\circ}), \mathcal{H}(G)[\pi \boxtimes \pi^\vee]) \\ &\simeq \mathrm{Hom}_G(\mathrm{ind}_{G^\circ}^G(\pi|_{G^\circ}), \pi \otimes \pi^\vee \otimes \mathrm{ind}_H^{G \times G}(\mathbb{C})) \\ &\simeq \mathrm{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ} \otimes (\pi^\vee \otimes \mathrm{ind}_{G^\circ}^G \mathbb{C})) \\ &\simeq \mathrm{Hom}_{G^\circ}(\pi|_{G^\circ}, \pi|_{G^\circ}) \otimes (\pi^\vee \otimes \mathrm{ind}_{G^\circ}^G \mathbb{C}) \\ &\simeq e^2 f \mathrm{ind}_{G^\circ}^G(\pi^\vee|_{G^\circ}), \end{aligned}$$

where in the 4th isomorphism we have used the following lemma.

Lemma 4.21. *Let H_1 (resp., H_2) be a totally disconnected group, and π_1 (resp., π_2) any smooth representation of H_1 (resp., H_2). If π_1 has finite length,*

then treating the representation $\pi_1 \otimes \pi_2$ of $H_1 \times H_2$ as a representation of $H_1 \subset H_1 \times H_2$, we have,

$$\mathrm{Hom}_{H_1}(\pi_1, \pi_1 \otimes \pi_2) \simeq \mathrm{Hom}_{H_1}(\pi_1, \pi_1) \otimes \pi_2$$

as representations of H_2 . □

The following lemma will be used in the next proposition.

Lemma 4.22. *Let G be a totally disconnected group, G° a normal subgroup of G such that G/G° is a discrete group. Let M, N be two smooth representations of G . Assume that M is an irreducible representation of G which restricted to G° is a finite sum of irreducible representations which are both injective and projective as a smooth representation of G° . Then,*

$$\mathrm{Ext}_G^i(M, N) = H^i(G/G^\circ, \mathrm{Hom}_{G^\circ}(M, N)).$$

Proof. Consider a projective resolution of M as a G -module

$$(4.7) \quad \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Since M restricted to G° is both an injective and a projective representation of G° , assume that P_i restricted to G° is a direct sum of copies of the irreducible components of M restricted to G° .

The exact sequence (4.7) gives rise to an exact sequence of G/G° -modules (the exactness follows since we are assuming as discussed in the last paragraph that P_i are projective modules restricted to G°):

$$(4.8) \quad 0 \rightarrow \mathrm{Hom}_{G^\circ}(M, N) \rightarrow \mathrm{Hom}_{G^\circ}(P_0, N) \rightarrow \mathrm{Hom}_{G^\circ}(P_1, N) \rightarrow \cdots.$$

To prove the lemma it suffices to prove that for P a projective module for G° , and N arbitrary smooth module for G , $\mathrm{Hom}_{G^\circ}(P, N)$ is an injective module for G/G° (as then the exact sequence (4.8) will be an injective resolution of $\mathrm{Hom}_{G^\circ}(M, N)$ as G/G° -modules). Injectivity of $\mathrm{Hom}_{G^\circ}(P, N)$ as a G/G° -module is a simple consequence of the adjunction (for Q any module for G/G°):

$$\mathrm{Hom}_G(Q, \mathrm{Hom}_{G^\circ}(P, N)) = \mathrm{Hom}_G(Q \otimes P, N) = \mathrm{Hom}_G(P, \mathrm{Hom}(Q, N)). \quad \square$$

The next result, based on Proposition 4.15, seems worth including as its proof is fairly elementary. A generalization of this proposition to finite length representations plus functoriality is proved in Corollary 12.4. Recall that $G/G^\circ \simeq \mathbb{Z}^d$ for some integer $d \geq 0$.

Proposition 4.23. *Let π be an irreducible, compact modulo center representation of G . Then $\mathrm{RHom}_G^\bullet(\pi, \mathcal{H}(G))$, lives only in degree $d \geq 0$ (where d is the integer for which $G/G^\circ \simeq \mathbb{Z}^d$), where it is isomorphic to π^\vee .*

Proof. By Proposition 4.15, the proof of this proposition boils down to proving that (for $H = \Delta(G)(G^\circ \times G^\circ)$), $\mathrm{RHom}_G^\bullet(\pi, \pi \boxtimes \pi^\vee \otimes \mathrm{ind}_H^{G \times G}(\mathbb{C}))$ lives only in degree d and

$$\mathrm{Ext}_G^d(\pi, \pi \boxtimes \pi^\vee \otimes \mathrm{ind}_H^{G \times G}(\mathbb{C})) = (e^2 f)\pi^\vee.$$

We apply Lemma 4.22 to $M = \pi$ (treated as a module for $G \times G$ on which $1 \times G$ acts trivially), and $N = \pi \boxtimes \pi^\vee \otimes \mathrm{ind}_H^{G \times G}(\mathbb{C})$. Note that $\mathrm{ind}_H^{G \times G}(\mathbb{C})$ restricted to $G^\circ \times G$ is $\mathrm{ind}_{G^\circ \times G^\circ}^{G^\circ \times G}(\mathbb{C})$. Thus, N restricted to $G^\circ \times G \subset G \times G$ is $\pi|_{G^\circ} \otimes \pi^\vee \otimes \mathrm{ind}_{G^\circ}^G(\mathbb{C})$. Thus

$$\mathrm{Hom}_{G^\circ}[\pi, \pi \otimes \pi^\vee \otimes \mathrm{ind}_H^{G \times G}(\mathbb{C})] = \mathrm{Hom}_{G^\circ}[\pi, \pi] \otimes \pi^\vee \otimes \mathrm{ind}_{G^\circ}^G(\mathbb{C}),$$

as a module for $(G/G^\circ) \times G$ where the actions of G/G° and $1 \times G$ coincide on $\mathrm{ind}_{G^\circ}^G(\mathbb{C})$. Therefore

$$H^i(G/G^\circ, \mathrm{Hom}_{G^\circ}[M, N]) = H^i(G/G^\circ, \mathrm{Hom}_{G^\circ}[\pi, \pi] \otimes \mathrm{ind}_{G^\circ}^G(\mathbb{C})) \otimes \pi^\vee.$$

Now we know that $\mathrm{Hom}_{G^\circ}[\pi, \pi]$ is a representation of G/G° of dimension $e^2 f$, therefore,

$$\mathrm{Hom}_{G^\circ}[\pi, \pi] \otimes \mathrm{ind}_{G^\circ}^G(\mathbb{C}) \simeq e^2 f \mathrm{ind}_{G^\circ}^G(\mathbb{C}),$$

both as a module for $G \times 1$ and $1 \times G$.

Hence we are reduced to understanding $H^i(\mathbb{Z}^d, \mathbb{C}[\mathbb{Z}^d])$ which is well-known to be zero for $i < d$, and $H^d(\mathbb{Z}^d, \mathbb{C}[\mathbb{Z}^d]) = \mathrm{Ext}_{\mathbb{Z}^d}^d(\mathbb{C}, \mathbb{C}[\mathbb{Z}^d]) \simeq \mathbb{C}$, completing the proof of the Proposition. \square

Remark 4.24. Later, in Corollary 12.4, we will prove that Proposition 4.23 remains true for any finite length cuspidal representation of a p -adic group or of a finite cover of it. We take this occasion to mention that the Bernstein block even of a cuspidal representation is not totally trivial: although it is true that a finite length indecomposable cuspidal representation of \tilde{G} is the successive extension of a fixed irreducible cuspidal representation ρ of \tilde{G} , it is not true that such a finite length indecomposable cuspidal representation of \tilde{G} is of the form $\rho \otimes \lambda$ where ρ is an irreducible cuspidal representation of \tilde{G} , and λ is a finite dimensional indecomposable representation of $\tilde{G}/\tilde{G}^\circ$ (which is the case when $\rho|_{\tilde{G}^\circ}$ is irreducible, in which case, indeed this Bernstein block is rather simple).

5. Basic notions of representation theory

We place ourselves in the setting $G = \mathbb{G}(F)$ for \mathbb{G} a reductive group over a non-archimedean local field F and \tilde{G} a covering group of G (see Section 3.1). All the basic notions for linear groups have an analogue for the covering group $\tilde{G} \rightarrow G$.

5.1. Parabolics, Levi and the Weyl group

A parabolic subgroup \tilde{P} for \tilde{G} is simply the preimage of a parabolic subgroup P of G . Similarly, for Levi subgroups and tori⁷ in \tilde{G} . A Levi decomposition $P = LN$ for $P \leq G$ lifts to a Levi decomposition $\tilde{P} = \tilde{L}N$ where $N \leq \tilde{G}$ is the unique lift of N to \tilde{G} that is normalized by \tilde{L} (in characteristic zero such a lift is obvious, in general see [MW94, Appendix Lemma]).

For P a parabolic subgroup of G with Levi decomposition LN we denote by P^- the opposite parabolic with Levi decomposition LN^- . Similarly, for their preimages in \tilde{G} .

The Weyl group of \tilde{G} is defined simply to be the Weyl group of G .

5.2. Parabolic induction and restriction

The notion of parabolic restriction (Jacquet modules), and parabolic induction, make sense for \tilde{G} , and they have the same adjointness properties as in the linear case.

Let $\tilde{P} = \tilde{L}N$ be a parabolic with a Levi decomposition. The (normalized) functors of parabolic induction and parabolic restriction are defined as follows (for details see [Ren10, VI.1] whose convention for the modulus character is opposite that of [BZ76] and [BZ77], which explains interchange of the factors $\delta_{\tilde{P}}^{\pm 1/2}$ of [BZ76] and [BZ77] with their inverses here):

$$\begin{aligned}
 \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}: \mathcal{M}(\tilde{L}) &\rightarrow \mathcal{M}(\tilde{G}) \\
 \pi &\mapsto \text{ind}_{\tilde{P}}^{\tilde{G}}(\pi \otimes \delta_{\tilde{P}}^{-1/2}) \\
 \mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}}: \mathcal{M}(\tilde{G}) &\rightarrow \mathcal{M}(\tilde{L}) \\
 \tau &\mapsto (\text{Res}_{\tilde{P}}^{\tilde{G}}(\tau) \otimes \delta_{\tilde{P}}^{1/2})_N
 \end{aligned}$$

⁷Notice that \tilde{T} is not necessarily abelian but this is immaterial to us.

where $(-)_N$ is the functor of coinvariants under N and $\delta_{\tilde{P}}: \tilde{P} \rightarrow \mathbb{R}_+^\times$ is the modular character of \tilde{P} . Both functors are exact.

Frobenius reciprocity states that $\mathbf{i}_{L, \tilde{P}}^{\tilde{G}}$ is right adjoint to $\mathbf{r}_{L, \tilde{P}}^{\tilde{G}}$. As a consequence, parabolic induction preserves limits and parabolic restriction preserves colimits.

It is easy to establish that parabolic induction preserves admissibility (see [Ren10, III.2.3]): this is because the double-coset $\tilde{K} \backslash \tilde{G} / \tilde{P}$ is finite for any open compact $\tilde{K} \leq \tilde{G}$. The analogous result for parabolic restriction is not so immediate (see Corollary 6.11).

It is also rather straightforward to show that parabolic restriction preserves finite type representations. However, the analogous result for parabolic induction is not so trivial and the proof is based on Bernstein’s decomposition Theorem 7.14.

We define cuspidal representations of \tilde{G} just as in the linear case: those smooth representations of \tilde{G} whose all nontrivial parabolic restrictions vanish (i.e., all nontrivial Jacquet modules vanish).

By similar arguments as in the linear case (see [Ren10, VI.5.1]), the geometric lemma holds for covering groups \tilde{G} too and allows one to calculate the Jordan–Hölder series of the Jacquet modules of a parabolically induced representation of \tilde{G} . Then one can conclude as in the linear case that every irreducible representation of \tilde{G} is a subquotient of a parabolically induced representation of \tilde{G} induced from a cuspidal representation of a Levi subgroup of \tilde{G} .

5.3. Iwahori decomposition

Definition 5.1. A compact open subgroup K of G (or of \tilde{G}) is said to have an Iwahori factorization with respect to a parabolic $P = LN$ with opposite parabolic $P^- = LN^-$ if the natural map given by multiplication:

$$m: K^+ \times K^0 \times K^- \rightarrow K,$$

is a bijection, where

$$\begin{aligned} K^+ &= K \cap N, \\ K^0 &= K \cap L, \\ K^- &= K \cap N^-. \end{aligned}$$

For central extensions \tilde{G} , we replace $P = LN$ with $\tilde{P} = \tilde{L}N$ and $\tilde{P}^- = \tilde{L}N^-$ and everything makes sense to define Iwahori factorization in \tilde{G} .

5.4. Cartan decomposition

The Cartan decomposition for the group G plays an important role in the proof of the uniform admissibility theorem. The analogous theorem for \tilde{G} is proved by taking the inverse image from G . Let us recall some notions before we state the theorem.

Recall from Section 4.2 the subgroup $G^\circ \leq G$. Let \tilde{G}° be the subgroup of \tilde{G} which is the preimage of $G^\circ \leq G$. We put $\Lambda(\tilde{G}) := \tilde{G}/\tilde{G}^\circ = G/G^\circ$; it is a finite rank free \mathbb{Z} -module.

Definition 5.2. The set of unramified characters of \tilde{G} is defined to be

$$\mathcal{X}(\tilde{G}) := \text{Hom}(\tilde{G}/\tilde{G}^\circ, \mathbb{C}^\times) = \text{Hom}(G/G^\circ, \mathbb{C}^\times).$$

It is clear that the image of $Z(\tilde{G})$ has finite index in $\tilde{G}/\tilde{G}^\circ$ and hence the rank of $Z(\tilde{G})$ is the same as the rank of $Z(G)$. In other words, we have $\mathcal{X}(\tilde{G}) \simeq (\mathbb{C}^\times)^{d_G}$ where d_G is the split rank of $Z(G)$. Notice that by definition, the unramified characters of \tilde{G} are the same as those of G (identified through the map $\tilde{G} \rightarrow G$).

Construction. For \mathbb{L} , a reductive algebraic group over F , we denote by $A_{\mathbb{L}}$, the maximal split torus contained in the center of \mathbb{L} , and denote $d(\mathbb{L}) := \dim(A_{\mathbb{L}})$, and call it the split rank of the center of \mathbb{L} . We extend this definition to any finite central extension \tilde{L} of $L = \mathbb{L}(F)$, defining $d(\tilde{L}) = d(\mathbb{L}) = \dim(A_{\mathbb{L}})$.

Put $A := A_{\mathbb{L}}(F)$ and notice that the image of $A/A^\circ \rightarrow L/L^\circ$ has finite index. Moreover, the surjection $A \rightarrow A/A^\circ$ admits a section that is a group homomorphism (use a uniformizer in F) and so we get an injection $A/A^\circ \hookrightarrow L$ whose image in L/L° has finite index.

If \tilde{L} is a finite central extension of L , then the natural map $Z(\tilde{L}) \rightarrow Z(L)$ induces a map of lattices $\Lambda(Z(\tilde{L})) \rightarrow \Lambda(Z(L))$ whose image has finite index. We can therefore find a lift $\Lambda(Z(\tilde{L})) \rightarrow \tilde{L}$ that is a group homomorphism and moreover its image in $\Lambda(\tilde{L}) = \Lambda(L)$ has finite index. Summarizing we have a diagram

$$\begin{array}{ccccc} \tilde{L} & \longrightarrow & \Lambda(\tilde{L}) & \xrightarrow{=} & \Lambda(L) \\ \uparrow & & \uparrow & & \uparrow \\ Z(\tilde{L}) & \longrightarrow & \Lambda(Z(\tilde{L})) & \hookrightarrow & \Lambda(Z(L)) \end{array}$$

and from the discussion above we have a section $\Lambda(Z(\tilde{L})) \rightarrow Z(\tilde{L})$ which is a group homomorphism. Denote its image by C_A ; it is a central subgroup of \tilde{L} .

Fix also finitely many elements $F_A := \{f_1, \dots, f_r \in \tilde{L}\}$ that lift some system of representatives for $\Lambda(\tilde{L})/\Lambda(Z(\tilde{L}))$.

All in all, we have that $C_A F_A$ provides a set-theoretic lift of $\Lambda(\tilde{L})$ to \tilde{L} that is moreover a group homomorphism when restricted to $\Lambda(Z(\tilde{L}))$.

Apply the above discussion to a parabolic subgroup $\tilde{P} = \tilde{L}N$ and recall the notion of dominant cocharacters of L with respect to P . Let L^+ denote the preimage of dominant cocharacters of L through the map $L \rightarrow \text{Hom}_{\text{gr}}(L, \mathbb{G}_m)^*$ given by $m \mapsto (\chi \mapsto \chi(m))$. We let \tilde{L}^+ denote its preimage in \tilde{L} under $\tilde{L} \rightarrow L$. We can choose the above F_A to lie in \tilde{L}^+ and similarly we can consider $C_A^+ \subset C_A$ the subset of dominant elements in C_A .

Let P_0 be a fixed minimal parabolic subgroup of G with Levi decomposition $P_0 = L_0 N_0$ and denote by $\tilde{P}_0 = \tilde{L}_0 N_0$ the corresponding subgroups in \tilde{G} .

Let $A = A_0$ be the split component of P_0 and $C_A F_A \subset \tilde{L}_0$ as above.

Theorem 5.3 (Cartan decomposition, see [BT]). *There exists a maximal compact subgroup K_0 of G such that for its inverse image \tilde{K}_0 inside \tilde{G} , we have:*

1. $\tilde{G} = \tilde{P}_0 \tilde{K}_0 = \tilde{K}_0 \tilde{P}_0$
2. $\tilde{G} = \bigsqcup_{af \in C_A^+ F_A} \tilde{K}_0 a f \tilde{K}_0$.

Put $\mathcal{H}_0 = \mathcal{H}(\tilde{K}_0, \tilde{K})$ to be the Hecke algebra of \tilde{K} -biinvariant functions on \tilde{K}_0 where $\tilde{K} \leq \tilde{K}_0$ is any compact open subgroup of \tilde{K}_0 . It is a finite dimensional algebra over \mathbb{C} .

A rather easy consequence of this theorem, which is essential in the proof of the uniform admissibility theorem (Theorem 6.4 below), is the following decomposition of the Hecke algebra:

Theorem 5.4. *With the notation as above, let $K \leq K_0 \leq G$ be compact open subgroups of G , and $\tilde{K} \leq \tilde{K}_0 \leq \tilde{G}$ be the compact open subgroup of \tilde{G} obtained by taking their inverse images in \tilde{G} . Then the Hecke algebra $\mathcal{H}(\tilde{G}, \tilde{K})$ decomposes as*

$$\mathcal{H}(\tilde{G}, \tilde{K}) = \mathcal{H}_0 D \mathcal{C} \mathcal{H}_0$$

where D is a vector subspace spanned by functions indexed by F_A and \mathcal{C} is a subalgebra isomorphic to the group algebra of $C_A \simeq \Lambda(Z(\tilde{L}_0))$.

6. Basic theorems

As before, \tilde{G} is a covering group (see Section 3.1) of a reductive p -adic group G , where $G = \mathbb{G}(F)$ for \mathbb{G} a reductive group over F , a non-archimedean local

field. All the basic theorems concerning representations of reductive p -adic groups hold also for \tilde{G} with the same proofs. We will give precise references to [Ren10] where the analogous results for G are proved.

The groups G and \tilde{G} do not have any compact representations in general because their centers may be non-compact. However, we can ask for the next best thing (see Section 4.2):

Definition 6.1. An irreducible smooth representation $(V, \pi) \in \mathcal{M}(\tilde{G})$ is called compact modulo center if all its matrix coefficients have compact support in $\tilde{G}/Z(\tilde{G})$.

The classical theorem of Harish-Chandra still holds with the same proof:

Theorem 6.2 (Harish-Chandra, see [Ren10, VI.2.1] for linear groups). *An irreducible smooth representation (π, V) of \tilde{G} is cuspidal if and only if it is compact modulo center.*

Using the easy fact that parabolic induction preserves admissibility, the fact that all compact representations are admissible and the above theorem we can prove the admissibility of irreducible representations (for a proof in the linear case see for example [Ren10, VI.2.2] or [Ber92, Theorem 12]):

Theorem 6.3. *Any irreducible representation of \tilde{G} is admissible.*

Not only irreducible representations are admissible but in fact they are so in a uniform way. Below is the precise formulation of the Uniform Admissibility Theorem due to Bernstein whose proof is based on the decomposition of the bi-invariant Hecke algebra given in Theorem 5.4 and some tricky linear algebra:

Theorem 6.4 (uniform admissibility, [Ber74]). *Let $\tilde{K} \leq \tilde{G}$ be an open compact subgroup. There exists a constant $c = c(\tilde{K})$ such that for all irreducible representations (π, V) of \tilde{G} we have $\dim(V^{\tilde{K}}) \leq c$.*

One can also look at [Ren10, VI.2.3] for a proof.

For establishing the Bernstein decomposition, and prior to that, verifying the condition (KF) which leads to Theorem 4.5, one makes use of the following

Corollary 6.5 (see [Ren10, VI.2.4] for the linear case). *Given an open compact subgroup $\tilde{K} \leq \tilde{G}$ there is a compact modulo center subset $\Omega \subset \tilde{G}$ such that for any irreducible cuspidal representation (V, π) of \tilde{G} and any vector $v \in V^{\tilde{K}}$ the function*

$$g \mapsto e_{\tilde{K}} \cdot \pi(g)(v)$$

has support contained in Ω .

The following theorem, which goes under the name of «*the geometric lemma*», is extensively used. It is due to Bernstein and Zelevinsky [BZ77, 2.12].

Theorem 6.6 (see [BZ77, 2.12]). *Let $\tilde{P} = \tilde{M}N$ and $Q = \tilde{L}U$ be two parabolic subgroups of \tilde{G} with Levi decompositions. Let ρ be an irreducible cuspidal representation of \tilde{M} and put $\tau := \mathbf{r}_{\tilde{L}, \tilde{Q}}^{\tilde{G}} \mathbf{i}_{\tilde{M}, \tilde{P}}^{\tilde{G}}(\rho)$. Then we have*

1. *If \tilde{L} has no Levi subgroup conjugate to \tilde{M} , then $\tau = 0$.*
2. *If \tilde{M} is not conjugate to \tilde{L} , then τ has no cuspidal subquotient.*
3. *If \tilde{M} and \tilde{L} are standard and conjugate, then τ has a filtration with subquotients isomorphic to ${}^w\rho$ for $w \in W(\tilde{L}, \tilde{M})/W_{\tilde{L}}$. In particular, τ is cuspidal.*

In the above statement $W_{\tilde{L}}$ is the Weyl group of \tilde{L} and $W(\tilde{L}, \tilde{M})$ is the subset of the Weyl group of \tilde{G} conjugating \tilde{L} into \tilde{M} .

The next result, proved using the above theorem, shows that the Jordan–Hölder factors of a parabolically induced representation of an irreducible cuspidal are independent of the chosen parabolic and depend only on the conjugacy class of the cuspidal datum.

Theorem 6.7 (see [BZ77, 2.9]). *Let $\tilde{P} = \tilde{L}N$ and $\tilde{P}' = \tilde{L}'N'$ be two parabolic subgroups in \tilde{G} with Levi decompositions. Let $\rho \in \text{Irr}_{\text{cs}}(\tilde{L})$ and $\rho' \in \text{Irr}_{\text{cs}}(\tilde{L}')_{\text{cs}}$ and $\pi = \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(\rho)$ and $\pi' = \mathbf{i}_{\tilde{L}', \tilde{P}'}^{\tilde{G}}(\rho')$ be the induced representations. Then π has finite length and the following are equivalent.*

1. *The cuspidal data (\tilde{L}, ρ) and (\tilde{L}', ρ') are conjugate,*
2. *The Jordan–Hölder series of π and π' are equivalent,*
3. *The Jordan–Hölder series of π and π' have a common element.*

In particular, if $\tilde{L} = \tilde{L}'$ and $\rho = \rho'$, this shows that π and π' have the same Jordan–Hölder series, independent of the chosen parabolic.

We can now deduce

Corollary 6.8. *Parabolic induction preserves finite length representations.*

Proof. If the parabolic induction involves a representation which is irreducible cuspidal then the corollary follows from the above theorem. In general, any irreducible representation is a subrepresentation of a parabolically induced representation of an irreducible cuspidal representation. We conclude the above corollary by the transitivity of parabolic induction. \square

In order to show the analogous result for parabolic restriction, we need to use the following theorem due to Howe whose proof for covering groups is the same as in the linear case:

Theorem 6.9 (see [BZ77, Thm 4.1] or [Ren10, VI.6.3]). *A smooth representation of \tilde{G} is of finite length if and only if it is admissible and finitely generated.*

A crucial tool that goes into the proof of Howe's theorem is Jacquet's lemma. Later, in Theorem 8.13, we will state a generalized version of it since it is necessary for the second adjointness. Until then, let us give the classical statement

Theorem 6.10 (Jacquet's lemma). *Let V be an admissible representation of \tilde{G} , $\tilde{K} \leq \tilde{G}$ an open compact subgroup admitting an Iwahori decomposition with respect to $\tilde{P} = \tilde{L}N$. Then the following canonical map is surjective*

$$V^{\tilde{K}} \rightarrow (V_N)^{\tilde{K} \cap \tilde{L}}.$$

Immediately from Jacquet's lemma we obtain

Corollary 6.11. *Parabolic restriction preserves admissibility.*

Use Howe's theorem, the above corollary and the easy fact that parabolic restriction preserves finite type to deduce

Corollary 6.12. *Parabolic restriction preserves finite length representations.*

7. The Bernstein decomposition

We give an exposition of a small part of a theory due to Bernstein which allows one to decompose the category $\mathcal{M}(\tilde{G})$ of smooth complex representations of \tilde{G} as a direct product of certain indecomposable full subcategories, now called the Bernstein components of $\mathcal{M}(\tilde{G})$. The results are due to [Ber84] where it is also stated that they hold for finite central extensions of reductive p -adic groups. Indeed, no essential modifications are needed to adapt the proof from [Ber84] to the case of finite central extensions. However, in the following we follow mostly the exposition of the linear case from [Ren10, VI.7] giving precise references in appropriate places.

The idea is that whereas \tilde{G} does not have compact representations, the group \tilde{G}° does, and moreover, from Harish-Chandra's Theorem 6.2, all irreducible cuspidal representations of \tilde{G} restrict to compact representations of \tilde{G}° . So using the results of Section 4.2, we can decompose $\mathcal{M}(\tilde{G})$ into a cuspidal part and an induced part. Using induction we express the induced part in a similar way.

7.1. Cuspidals split

In this section, we sketch the decomposition of $\mathcal{M}(\tilde{G})$ into a product of cuspidal and induced.

We define $\mathcal{M}(\tilde{G})_{\text{cs}}$ (resp., $\mathcal{M}(\tilde{G})_{\text{ind}}$) to be the full subcategory of $\mathcal{M}(\tilde{G})$ formed of representations all of whose irreducible subquotients are cuspidal (resp., that have no cuspidal irreducible subquotients). The set $\text{Irr}_{\text{cs}}(\tilde{G})$ denotes the set of irreducible cuspidal representations of \tilde{G} .

Remark 7.1. The category $\mathcal{M}(\tilde{G})_{\text{cs}}$ (resp., $\mathcal{M}(\tilde{G})_{\text{ind}}$) is the pullback of the subcategory of compact representations (resp., non-compact representations) from $\mathcal{M}(\tilde{G}^\circ)$ through the functor $\text{Res}_{\tilde{G}^\circ}^{\tilde{G}}$.

Let $(V, \pi) \in \text{Irr}_{\text{cs}}(\tilde{G})$ and denote by $[\pi]$ its inertia class, i.e., its orbit under $\mathcal{X}(\tilde{G})$. From Proposition 4.10 we have that the restriction of π to \tilde{G}° depends only on the inertia class and we know moreover that this restriction has finite length and is semisimple.

Definition 7.2. We define $\mathcal{M}(\tilde{G})_{[\pi]}$ to be the full subcategory of $\mathcal{M}(\tilde{G})$ formed of those representations all of whose subquotients belong to $[\pi]$. Similarly, define $\mathcal{M}(\tilde{G})_{[\text{out } \pi]}$.

Remark 7.3. Here is a different way of thinking about $\mathcal{M}(\tilde{G})_{[\pi]}$. If $\tau_1, \tau_2, \dots, \tau_l$ are the irreducible summands of π restricted to \tilde{G}° , then the category $\mathcal{M}(\tilde{G})_{[\pi]}$ consists precisely in the representations of \tilde{G} whose restriction to \tilde{G}° are direct sums of $\tau_i, i = 1, \dots, l$, i.e., those representations of \tilde{G} whose restriction to \tilde{G}° belongs to $\mathcal{M}(\tilde{G}^\circ)_{[\mathcal{A}]}$, where $\mathcal{A} = \{\tau_i \mid i = 1, \dots, l\}$. (See Section 4.1.)

Harish-Chandra's Theorem 6.2 together with the results from Section 4.2 give

Theorem 7.4. *For an irreducible cuspidal representation π of \tilde{G} , we have a decomposition of the category $\mathcal{M}(\tilde{G})$ as*

$$\mathcal{M}(\tilde{G}) \simeq \mathcal{M}(\tilde{G})_{[\pi]} \times \mathcal{M}(\tilde{G})_{[\text{out } \pi]}.$$

One uses the uniform admissibility Theorem 6.4 to check the condition (KF) in Section 4.1 and then applies Theorem 4.13 and Theorem 4.14 to get the following decomposition

Theorem 7.5 (see [Ren10, VI.3.5]). *The subcategories $\mathcal{M}(\tilde{G})_{\text{cs}}$ and $\mathcal{M}(\tilde{G})_{\text{ind}}$ split the category $\mathcal{M}(\tilde{G})$:*

$$(7.1) \quad \mathcal{M}(\tilde{G}) = \mathcal{M}(\tilde{G})_{\text{cs}} \times \mathcal{M}(\tilde{G})_{\text{ind}}$$

and moreover the cuspidal part decomposes as

$$(7.2) \quad \mathcal{M}(\tilde{G})_{\text{cs}} = \prod_{[\pi] \in [\text{Irr}(\tilde{G})_{\text{cs}}]} \mathcal{M}(\tilde{G})_{[\pi]}.$$

Since the center of a product of categories is the product of their centers (see Proposition 2.6), we have

Corollary 7.6.

$$\mathcal{Z}(\mathcal{M}(\tilde{G})) = \prod_{[\pi] \in [\text{Irr}_{\text{cs}}(\tilde{G})]} \mathcal{Z}_{[\pi]} \times \mathcal{Z}(\mathcal{M}(\tilde{G})_{\text{ind}}).$$

It turns out that for an irreducible cuspidal representation π the center $\mathcal{Z}_{[\pi]}$ of $\mathcal{M}(\tilde{G})_{[\pi]}$ is not very hard to determine. Recall the notation $\Lambda(\tilde{G}) = \tilde{G}/\tilde{G}^\circ$ and $\mathcal{X}(\tilde{G}) = \text{Hom}_{\text{gr}}(\Lambda(\tilde{G}), \mathbb{C}^\times)$ and notice that the group algebra $\mathbb{C}[\Lambda(\tilde{G})]$ is the algebra of regular functions on the algebraic variety (a torus) $\mathcal{X}(\tilde{G})$.

We first need a lemma whose easy proof follows by considerations of the central character.

Lemma 7.7 ([Ren10, V.2.7]). *Given $\pi \in \text{Irr}_{\text{cs}}(\tilde{G})$, its stabilizer in $\mathcal{X}(\tilde{G})$,*

$$\mathcal{G}_\pi := \text{Stab}_{\mathcal{X}(\tilde{G})}(\pi) = \left\{ \chi \in \mathcal{X}(\tilde{G}) \mid \pi \otimes \chi \cong \pi \right\},$$

is finite.

Theorem 7.8 (see [Ren10, VI.10] or [Ber84, 1.12–1.14] for a slick proof). *Given $\pi \in \text{Irr}_{\text{cs}}(\tilde{G})$, we have a canonical isomorphism*

$$\mathcal{Z}(\mathcal{M}(\tilde{G})_{[\pi]}) \simeq \mathbb{C}[\Lambda(\tilde{G})]^{\mathcal{G}_\pi} = \mathcal{O}(\mathcal{X}(\tilde{G})/\mathcal{G}_\pi).$$

In particular, $\mathcal{Z}(\mathcal{M}(\tilde{G})_{[\pi]})$ is isomorphic to a ring of Laurent polynomials and hence is smooth of Krull dimension equal to the rank of $\Lambda(\tilde{G})$.

Actually, one can do a bit better and find an equivalence of categories $\mathcal{M}(\tilde{G})_{[\pi]} \simeq \mathbf{mod}\text{-}\mathcal{R}_{[\pi]}$ with $\mathcal{R}_{[\pi]} := \text{End}_{\tilde{G}}(\Pi_{[\pi]})$ and $\Pi_{[\pi]} := \text{ind}_{\tilde{G}^\circ}^{\tilde{G}}(\text{Res}_{\tilde{G}^\circ}^{\tilde{G}}(\pi))$. This will be discussed in Section 9.

7.2. Induced representations

In this section, we look at the category $\mathcal{M}(\tilde{G})_{\text{ind}}$ and decompose it into blocks. The main input is the geometric lemma (see [Ren10, VI.5.1] for a proof that works also for \tilde{G}).

Definition 7.9.

1. A cuspidal datum is a couple (\tilde{L}, ρ) where \tilde{L} is a Levi subgroup of \tilde{G} and $\rho \in \text{Irr}(\tilde{L})_{\text{cs}}$ is an irreducible cuspidal representation of \tilde{L} .
2. We say that two cuspidal data $(\tilde{L}, \rho), (\tilde{M}, \tau)$ are conjugate (or associate) if there exists $g \in \tilde{G}$ such that

$$\tilde{L} = {}^g \tilde{M} \text{ and } \rho = {}^g \tau,$$

3. We say that two cuspidal data $(\tilde{L}, \rho), (\tilde{M}, \tau)$ define the same inertial support if there exists $g \in \tilde{G}$ and $\chi \in \mathcal{X}(\tilde{L})$ such that

$$\tilde{L} = {}^g \tilde{M} \text{ and } \rho = {}^g \tau \chi.$$

We denote by $\Omega(\tilde{G})$ the set of cuspidal data up to conjugation and by $\mathcal{B}(\tilde{G})$ the cuspidal data up to conjugation and inertia.

Theorem 6.7 guarantees that the following notion is well defined.

Definition 7.10. Let $\pi \in \mathcal{M}(\tilde{G})$ be an irreducible representation. We define the cuspidal support of π to be a cuspidal datum $(\tilde{L}, \rho) \in \Omega(\tilde{G})$ (well-defined up to conjugation in \tilde{G}) such that π appears in the Jordan–Hölder series of the parabolic induction $i_{L, \tilde{P}}^{\tilde{G}}(\rho)$ for some parabolic subgroup \tilde{P} with Levi decomposition $\tilde{P} = \tilde{L}N$.

Remark 7.11. In order to decompose $\mathcal{M}(\tilde{G})_{\text{ind}}$ we would like to pull back, for each cuspidal datum up to conjugation $(\tilde{L}, \rho) \in \Omega(\tilde{G})$, the decomposition from (7.1) for \tilde{L} to \tilde{G} for each irreducible cuspidal representation of \tilde{L} . However, we need to take into account also the inertia because $\mathcal{M}(\tilde{L})_{[\rho]}$ is an indecomposable subcategory.

Recall the set $\mathcal{B}(\tilde{G})$ of cuspidal data up to conjugation and inertia. For (\tilde{L}, ρ) a cuspidal datum we denote by $[\tilde{L}, \rho]_{\tilde{G}}$ its class in $\mathcal{B}(\tilde{G})$. If no confusion can arise, we drop the subscript \tilde{G} .

Given $[\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$, Theorem 6.7 allows us to define unambiguously the subcategory of representations of \tilde{G} all whose irreducible subquotients have cuspidal support in $[\tilde{L}, \rho]$:

$$\mathcal{M}(\tilde{G})_{[\tilde{L}, \rho]} = \left\{ \pi \in \mathcal{M}(\tilde{G}) \mid \begin{array}{l} \text{all irreducible subquotients of } \pi \text{ as a} \\ \tilde{G}\text{-module have cuspidal support in } [\tilde{L}, \rho] \end{array} \right\}.$$

The next lemma gives another characterization of the category $\mathcal{M}(\tilde{G})_{[\tilde{L}, \rho]}$.

Lemma 7.12. *Let $[\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$ and $\tilde{P} = \tilde{L}N$ be a parabolic subgroup with Levi \tilde{L} . Then the subcategory $\mathcal{M}(\tilde{G})_{[\tilde{L}, \rho]}$ is the smallest full subcategory of $\mathcal{M}(\tilde{G})$ closed under subquotients and containing $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(\mathcal{M}(\tilde{L})_{[\rho]})$.*

Proof. It follows easily from Theorem 6.6 and Theorem 6.7. □

Use Frobenius reciprocity, Theorem 6.6 and exactness of parabolic induction and restriction to deduce the following proposition.

Proposition 7.13. *If $[\tilde{L}, \rho], [\tilde{L}', \rho']$ are distinct elements of $\mathcal{B}(\tilde{G})$, then the subcategories $\mathcal{M}(\tilde{G})_{[\tilde{L}, \rho]}$ and $\mathcal{M}(\tilde{G})_{[\tilde{L}', \rho']}$ are (derived) orthogonal, i.e., there are no non-zero Ext groups between them.*

The above proposition together with Proposition 2.1 imply that we have a fully faithful functor

$$\prod_{\mathfrak{s} \in \mathcal{B}(\tilde{G})} \mathcal{M}(\tilde{G})_{\mathfrak{s}} \hookrightarrow \mathcal{M}(\tilde{G}).$$

Proving that this functor is essentially surjective⁸ gives the Bernstein decomposition for $\mathcal{M}(\tilde{G})$:

Theorem 7.14 (see [Ren10, VI.7.2]). *The category of smooth representations of \tilde{G} decomposes into blocks indexed by $\mathcal{B}(\tilde{G})$:*

$$(7.3) \quad \mathcal{M}(\tilde{G}) = \prod_{\mathfrak{s} \in \mathcal{B}(\tilde{G})} \mathcal{M}(\tilde{G})_{\mathfrak{s}}.$$

Proof. Using Theorem 7.5 what is left to show is that every representation $\pi \in \mathcal{M}(\tilde{G})_{\text{ind}}$ can be written as a direct sum of representations $\pi_{\mathfrak{s}} \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$:

$$(7.4) \quad \pi \simeq \bigoplus_{\mathfrak{s} \in \mathcal{B}(\tilde{G})} \pi_{\mathfrak{s}} \quad \text{for some } \pi_{\mathfrak{s}} \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}.$$

First, one observes that (7.4) holds for an induced representation of a cuspidal representation from a Levi. This is tautological using Lemma 7.12.

Next, one notices that if $\pi' \subset \pi$ is a subrepresentation and π satisfies (7.4) then the same is true of π' . Namely, consider $\pi'_{\mathfrak{s}} = \pi_{\mathfrak{s}} \cap \pi'$, then $\pi' = \bigoplus_{\mathfrak{s} \in \mathcal{B}(\tilde{G})} \pi'_{\mathfrak{s}}$ as follows easily only using the fact that no nonzero subquotients of $\pi_{\mathfrak{s}}$ for different \mathfrak{s} are isomorphic (analogous assertion in group theory is called Goursat’s Lemma).

⁸A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called essentially surjective if for any $X \in \mathcal{D}$ there exists $A \in \mathcal{C}$ such that $F(A) \simeq X$.

Finally, one needs to show that every representation $\pi \in \mathcal{M}(\tilde{G})_{\text{ind}}$ can be embedded into a sum of induced representations. This is done using a pair of adjoint functors that are direct sums of all parabolic induction functors (resp., parabolic restriction) indexed by all the standard parabolic subgroups. For details, see [Ren10, VI.7.2 second Lemme]. \square

As a consequence one can show that $\mathcal{M}(\tilde{G})$ is a noetherian category and this allows one to show that parabolic induction preserves finite type representations. Below we sketch the strategy.

Recall that in an abelian category \mathcal{A} an object X is said to be

- noetherian if all increasing sequences of subobjects of X are stationary.
- of finite type if whenever $X = \cup_i X_i$ with $X_i \subset X$, there exist finitely many indices i_1, \dots, i_n such that $X = \cup_{k=1}^n X_{i_k}$, where we recall that in an abelian category, the union of subobjects is defined by $\cup_i X_i := \text{Im}(\oplus_i X_i \rightarrow X)$.

An abelian category is called noetherian if all its finite type objects are noetherian. The archetypal example is the category of modules over a *unitary* noetherian ring.

Thanks to Bernstein’s decomposition theorem and the fact that each cuspidal block is equivalent to modules over a noetherian algebra (see Proposition 9.3) one can show:

Theorem 7.15 (see [BZ76, §4.19], [Ber84, Rem 3.12], [Ren10, VI.7.5]). *The category of smooth representation $\mathcal{M}(\tilde{G})$ is noetherian.*

Using the above theorem and the simple fact that parabolic restriction preserves finite type (see the discussion in Section 5.2) one shows:

Theorem 7.16 (see [Ren10, VI.7.5]). *Parabolic induction preserves finite type representations.*

This theorem is used in Lemma 9.10 to show that the generator $\Pi_{\mathfrak{s}}$ of a non-cuspidal block is of finite type.

8. The second adjointness theorem

Recall that the Frobenius reciprocity theorem asserts that for \tilde{P} a parabolic subgroup of \tilde{G} with Levi subgroup \tilde{L} , the functor $\mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$ is left adjoint to $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$. It is natural to look for an adjoint in the other direction.

Although showing that $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$ admits a right adjoint is not difficult (see below), identifying this adjoint precisely is quite hard and is the content of

Bernstein’s second adjointness theorem: $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$ is left adjoint to $\mathbf{r}_{\tilde{L}, \tilde{P}^-}^{\tilde{G}}$, where \tilde{P}^- is the opposite parabolic of \tilde{P} .

In this section, we will sketch the main steps in the proof following the exposition in [Ren10, VI.9] where all the details are spelled out for linear groups. In loc.cit., the argument is streamlined using the notion of *completion* of a representations of \tilde{G} which we recall below. Other references for the second adjointness theorem for the linear case include [Ber92, Theorem 19], [Bus01]; a geometric proof is given in [BK15].

8.1. Completion

We recall the notion of completion for smooth G -modules where G is a locally compact, totally disconnected group admitting a countable basis of neighborhoods of 1 given by compact open subgroups. For V a smooth G -module its completion \overline{V} is a G -module (not smooth anymore) sitting between V and $(V^\vee)^*$.

This notion will play a simplifying role in the proof of the second adjointness theorem and it also helps clarify the relationship between N -invariants and N -coinvariants.

Notice that for a representation V of G and compact open subgroups $L \subset K$ with corresponding idempotents $e_K, e_L \in \mathcal{H}(G)$, we have a natural map $e_L V \rightarrow e_K V$ sending w to $e_K w$.

Definition 8.1. For V , a smooth representation of G , we define the functor

$$\begin{aligned} \overline{(\)} &: \mathcal{M}(G) \rightarrow \text{Rep}_C(G) \text{ by} \\ \overline{V} &:= \varprojlim (e_K V) = \varprojlim (V^K) \end{aligned}$$

where the inverse limit is taken over all compact open subgroups of G .

A few remarks are in order.

Remark 8.2.

1. Since the set of compact open subgroups of G is invariant under conjugation by G , it follows that \overline{V} is a G -module and that $\overline{V}^K = V^K$. Clearly $V \subset \overline{V}$ and the smooth part of \overline{V} is V .
2. The completion functor is exact because the Mittag-Leffler condition is automatically satisfied.
3. Another convenient way of writing the completion functor is to note that $\text{Hom}_G(\mathcal{H}(G), V)$ acquires a G -module structure since $\mathcal{H}(G)$ is a

$G \times G$ -module, and since $\mathcal{H}(G) = \varinjlim \mathcal{H}(G)e_K$, it follows that

$$\overline{V} = \text{Hom}_G(\mathcal{H}(G), V).$$

Note that the exactness of the functor $V \rightarrow \overline{V}$ is equivalent to $\mathcal{H}(G)$ being a projective G -module.

Proposition 8.3. *For V a smooth representation of G , we have*

$$\overline{V^\vee} = V^*.$$

Proof. Suppose given $B: V \times W \rightarrow \mathbb{C}$ a pairing of smooth G -representations inducing an injective map of representation $V \hookrightarrow W^*$. Since $(W^*)^K = (W^K)^*$ for any smooth representation W of G and any compact open subgroup $K \subset G$, we obtain

$$\overline{V} = \varprojlim (V^K) \subset \varprojlim (W^K)^* = (\varinjlim W^K)^* = W^*.$$

In particular, taking $B: V^\vee \times V \rightarrow \mathbb{C}$ to be the natural pairing and using that $(V^\vee)^K = (V^*)^K$ for all open compact subgroups K , we obtain the claimed equality. \square

Applying it to the standard bilinear form $B: \mathcal{H}(G) \times \mathcal{H}(G) \rightarrow \mathbb{C}$, we find

Corollary 8.4. *The completion $\overline{\mathcal{H}}(G)$ of $\mathcal{H}(G)$ is the space of distributions D on G such that $e_K * D$ is a compactly supported distribution on G for all idempotents e_K .*

In particular, coupled with Proposition 2.7, we deduce:

Corollary 8.5. *The center of $\mathcal{M}(G)$ can be described as $\overline{\mathcal{H}}(G)^G$, i.e., the space of invariant distributions D on G such that $e_K * D$ is a compactly supported distribution on G for all idempotents e_K .*

Remark 8.6. It is known that for smooth representations taking invariants with respect to a unipotent subgroup N is not a well behaved functor (for example, it is not exact) and as such it is almost never used. On the other hand, taking coinvariants under N leads to nice exact functors (Jacquet functors) which are extremely important in the theory. The completion allows us to reconcile invariants and coinvariants for N as shown below.

Proposition 8.7. *The functor $(-)^N \circ \overline{(-)} \circ (-)^\vee: \mathcal{M}(G) \rightarrow \text{Vect}_{\mathbb{C}}$ is exact.*

Proof. Notice that we have a natural isomorphism $(V^*)^N = (V_N)^*$ for any representation V of G . In particular, for a smooth representation V of G ,

using Proposition 8.3 we obtain

$$\overline{V^V}^N = (V^*)^N = (V_N)^*.$$

Taking N -coinvariants is an exact functor because N is an increasing union of compact open subgroups. So the functor $(-)^N \circ \overline{(-)} \circ (-)^V$ identifies with the composition of two exact functors $(-)^* \circ (-)_N$ which concludes the proof. \square

In other words, taking N -invariants on completions of contragredient G -modules is an exact functor. In particular, we deduce

Corollary 8.8. *The functor $(-)^N \circ \overline{(-)}$ is exact on admissible modules.*

8.2. Adjoints

From now on, G is a reductive p -adic group and \tilde{G} is a finite covering group of it (see Section 3.1). We fix $\tilde{P} = \tilde{L}N$, a parabolic subgroup with a Levi decomposition in \tilde{G} , and we denote by \tilde{P}^- , the opposite parabolic subgroup with Levi decomposition $\tilde{L}N^-$. We denote by $\delta_{\tilde{P}}$ the modulus character of \tilde{P} .

The induction functor $\text{ind}_{\tilde{P}}^{\tilde{G}}: \mathcal{M}(\tilde{P}) \rightarrow \mathcal{M}(\tilde{G})$ has a different, more algebraic, expression in terms of the Hecke algebras. First, recall that we have an equivalence of categories

$$\mathcal{M}(\tilde{G}) \simeq \mathcal{H}(\tilde{G})\text{-mod}^{\text{nd}}$$

between smooth representations of \tilde{G} and non-degenerate $\mathcal{H}(\tilde{G})$ -modules and that this holds also for \tilde{P} and \tilde{L} .

The following result gives a different incarnation of the induction from \tilde{P} to \tilde{G} :

Proposition 8.9 ([Ren10, Théorème III.2.6]). *There is a natural isomorphism of functors*

$$\mathcal{H}(\tilde{G}) \otimes_{\mathcal{H}(\tilde{P})} - \simeq \text{ind}_{\tilde{P}}^{\tilde{G}} \circ (- \otimes_{\mathbb{C}} \delta_{\tilde{P}}).$$

Now consider the functor of taking \tilde{G} -smooth vectors (or the non-degenerate submodule for the Hecke algebra)

$$(-)_{\text{nd-}\tilde{G}}: \mathcal{H}(\tilde{G})\text{-mod} \rightarrow \mathcal{H}(\tilde{G})\text{-mod}^{\text{nd}}$$

which to a module associates its non-degenerate submodule. Similarly for $\mathcal{H}(\tilde{P})$ -modules.

Remark 8.10. The functor $(-)\text{nd-}\tilde{G}$ is right adjoint to the inclusion functor

$$\mathcal{H}(\tilde{G})\text{-mod}^{\text{nd}} \hookrightarrow \mathcal{H}(\tilde{G})\text{-mod}.$$

Proposition 8.11. *The functor*

$$\text{ind}_{\tilde{P}}^{\tilde{G}} \circ (- \otimes_{\mathbb{C}} \delta_{\tilde{P}}): \mathcal{M}(\tilde{P}) \rightarrow \mathcal{M}(\tilde{G})$$

is left adjoint to

$$(-)\text{nd-}\tilde{P} \circ \text{Res}_{\tilde{P}}^{\tilde{G}} \circ \overline{(-)}: \mathcal{M}(\tilde{G}) \rightarrow \mathcal{M}(\tilde{P}).$$

Proof. The usual tensor–Hom adjunction has the following analogue here: the functor

$$\mathcal{H}(\tilde{G}) \otimes_{\mathcal{H}(\tilde{P})} -: \mathcal{M}(\tilde{P}) \rightarrow \mathcal{M}(\tilde{G})$$

is left adjoint to

$$(-)\text{nd-}\tilde{P} \circ \text{Hom}_{\mathcal{H}(\tilde{G})}(\mathcal{H}(\tilde{G}), -): \mathcal{M}(\tilde{G}) \rightarrow \mathcal{M}(\tilde{P})$$

where $\mathcal{H}(\tilde{G})$ is viewed as a left $\mathcal{H}(\tilde{G})$ -module and a right $\mathcal{H}(\tilde{P})$ -module.

Therefore the functor $\text{Hom}_{\mathcal{H}(\tilde{G})}(\mathcal{H}(\tilde{G}), -): \mathcal{M}(\tilde{G}) \rightarrow \mathcal{H}(\tilde{P})\text{-mod}$ is identified with $\text{Res}_{\tilde{P}}^{\tilde{G}} \circ \overline{(-)}$.

Now we conclude using the fact that the image of a non-degenerate module is a non-degenerate module. \square

We immediately get some expression of the right adjoint of parabolic induction:

Corollary 8.12. *The functor $\mathbf{i}_{L,\tilde{P}}^{\tilde{G}}: \mathcal{M}(\tilde{L}) \rightarrow \mathcal{M}(\tilde{G})$ is left adjoint to*

$$(-)^N \circ (-)\text{nd-}\tilde{P} \circ \text{Res}_{\tilde{P}}^{\tilde{G}} \circ \overline{(-)}.$$

Proof. Use Remark 8.10 and that the right adjoint of the inflation functor $\text{Res}_{\tilde{P}}^{\tilde{L}}: \mathcal{M}(\tilde{L}) \rightarrow \mathcal{M}(\tilde{P})$ is $(-)^N$. \square

The key in the proof of Bernstein’s second adjointness is the generalized Jacquet Lemma which was proved first by Casselman for admissible representations and later generalized by Bernstein to all smooth representations. See [Ber92, p.65 Jacquet’s Lemma], [Bus01], and [Ren10, VI.9.1]. The same proof works for finite central extensions:

Theorem 8.13. *Let $\tilde{K} \leq \tilde{G}$ be an open compact admitting an Iwahori decomposition with respect to $\tilde{P} = \tilde{L}N$. Then for any representation $V \in \mathcal{M}(\tilde{G})$, the projection*

$$V^{\tilde{K}} \rightarrow (V_N)^{\tilde{K} \cap \tilde{L}}$$

is surjective and has a natural (functorial in V) section, call it $s_{\tilde{K}} : (V_N)^{\tilde{K} \cap \tilde{L}} \rightarrow V^{\tilde{K}}$. Further, for $\tilde{K}' \subset \tilde{K}$ open compact, both admitting Iwahori decompositions with respect to $\tilde{P} = \tilde{L}N$, we have the commutative diagram (see [Ren10, VI.9.6.6]):

$$\begin{array}{ccc} (V_N)^{\tilde{K}' \cap \tilde{L}} & \xrightarrow{s_{\tilde{K}'}} & V^{\tilde{K}'} \\ e_{\tilde{K} \cap \tilde{L}} \downarrow & & \downarrow e_{\tilde{K}} \\ (V_N)^{\tilde{K} \cap \tilde{L}} & \xrightarrow{s_{\tilde{K}}} & V^{\tilde{K}}. \end{array}$$

Recall that for a \tilde{G} -module V , we defined its completion to be

$$\overline{V} := \text{Hom}_{\tilde{G}}(\mathcal{H}(\tilde{G}), V) = \lim_{\tilde{K}} V^{\tilde{K}}$$

where the limit is taken over all the open compact subgroups \tilde{K} of \tilde{G} . The previous theorem is actually equivalent to the following very nice looking statement that is proved for linear groups in [Ren10, VI.9.7]. The same proof works for finite central extensions. (The commutativity of the diagram in Theorem 8.13 allows one to construct a map $\overline{V}_N \rightarrow \overline{V}$ which is then proved to land inside the space of N^- -invariants.)

Corollary 8.14. *If $\tilde{P} = \tilde{L}N$ is a parabolic with opposite $\tilde{P}^- = \tilde{L}N^-$ and V is a smooth representation of \tilde{G} then the natural map*

$$\overline{V}^{N^-} \rightarrow \overline{V}_N$$

is an isomorphism of \tilde{L} -representations.

In representation theory of p -adic groups taking invariants is not a well behaved functor and one only works with co-invariants. In some sense, this corollary recovers the lost properties of invariants provided we are willing to work with completed representations.

Using Corollary 8.14 one can now prove Bernstein’s second adjointness theorem easily (we follow the exposition in [Ren10, VI.9.7] for linear groups). Other references for the linear case are [Ber92, Theorem 19], [Bus01], and [BK15].

Theorem 8.15. *The functor $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}$ is left adjoint to $\mathbf{r}_{\tilde{L}, \tilde{P}^-}^{\tilde{G}}$.*

Proof. We have the following natural isomorphisms:

$$\begin{aligned}
 \mathrm{Hom}_{\tilde{G}}(\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(V), W) &= \mathrm{Hom}_{\tilde{G}}(\mathrm{ind}_{\tilde{P}}^{\tilde{G}} \mathrm{Res}_{\tilde{P}}^{\tilde{L}}(\delta_P^{1/2} \otimes V), W) && \text{by definition} \\
 &= \mathrm{Hom}_{\tilde{G}}(\mathcal{H}(\tilde{G}) \otimes_{\mathcal{H}(\tilde{P})} \mathrm{Res}_{\tilde{P}}^{\tilde{L}}(\delta_P^{-1/2} \otimes V), W) && \\
 &&& \text{by Proposition 8.9} \\
 &= \mathrm{Hom}_{\tilde{P}}(\mathrm{Res}_{\tilde{P}}^{\tilde{L}}(\delta_P^{-1/2} \otimes V), \overline{W}) && \text{by Proposition 8.11} \\
 &= \mathrm{Hom}_{\tilde{L}}(\delta_P^{-1/2} \otimes V, \overline{W}^N) && \text{adjunction } \mathrm{Res}_{\tilde{P}}^{\tilde{L}} \dashv (-)^N \\
 &= \mathrm{Hom}_{\tilde{L}}(V, \delta_P^{1/2} \otimes \overline{W}_{N^-}) && \text{by Corollary 8.14} \\
 &= \mathrm{Hom}_{\tilde{L}}(V, \delta_P^{1/2} \otimes W_{N^-}) && \text{by Remark 8.10} \\
 &= \mathrm{Hom}_{\tilde{L}}(V, \mathbf{r}_{\tilde{L}, \tilde{P}^-}^{\tilde{G}}(W)) && \text{by definition.}
 \end{aligned}$$

□

Since every functor that admits an exact right adjoint preserves projective objects, we deduce the rather non-trivial fact:

Corollary 8.16. *The functor $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}: \mathcal{M}(\tilde{L}) \rightarrow \mathcal{M}(\tilde{G})$ sends projective objects to projective objects.*

9. Blocks as module categories

In this section, we describe each block $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$, $\mathfrak{s} \in \mathcal{B}(\tilde{G})$, as the category of modules over some algebra $\mathcal{R}_{\mathfrak{s}}$ and prove some basic homological properties of this algebra.

The key results of this section are Corollary 9.6 and Proposition 9.7.

9.1. Cuspidals

Fix (\tilde{L}, ρ) a cuspidal data and denote by $\mathfrak{s} \in \mathcal{B}(\tilde{G})$ its equivalence class (conjugation and inertia).

Define the following representation of \tilde{L} :

$$\Pi_{[\rho]} := \mathrm{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\rho|_{\tilde{L}^\circ}) = \rho \otimes \mathrm{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\mathbb{C}),$$

where we recall that \tilde{L}° is the subgroup of \tilde{L} generated by all compact subgroups.

Proposition 9.1. *The representation $\Pi_{[\rho]}$ is a finite type projective generator in $\mathcal{M}(\tilde{L})_{[\rho]}$.*

Proof. First, we show the projectivity. Since $\tilde{L}^\circ \leq \tilde{L}$ is an open subgroup, the functor $\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}$ is also left adjoint to $\text{Res}_{\tilde{L}^\circ}^{\tilde{L}}$ (see Lemma 3.2) which is exact, hence $\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}$ preserves projective objects. It is therefore enough to show that $\rho|_{\tilde{L}^\circ}$ is projective which follows immediately from Harish-Chandra’s Theorem 6.2 and the fact that compact representations are projective by Section 4.

Let us now show that $\Pi_{[\rho]}$ is a generator of $\mathcal{M}(\tilde{L})_{[\rho]}$. Let π be an arbitrary representation in $\mathcal{M}(\tilde{L})_{[\rho]}$. We must show that there’s a non-zero morphism $\Pi_{[\rho]} \rightarrow \pi$. Recall (paragraph before Theorem 7.4) that the category $\mathcal{M}(\tilde{L})_{[\rho]}$ is defined as consisting of those smooth representations of \tilde{L} all whose irreducible subquotients belong to $[\rho]$. Since $\Pi_{[\rho]}$ is a projective representation of \tilde{L} , it is enough to prove that an irreducible representation belonging to $\mathcal{M}(\tilde{L})_{[\rho]}$, thus isomorphic to $\chi\rho$ for some unramified character $\chi \in \mathcal{X}(\tilde{L})$, appears as a quotient of $\Pi_{[\rho]}$. By the adjunction Lemma 3.2, we have

$$\text{Hom}_{\tilde{L}}(\Pi_{[\rho]}, \chi\rho) = \text{Hom}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ}, \chi\rho|_{\tilde{L}^\circ}),$$

and the latter is non-zero as it contains the identity.

In order to show that $\Pi_{[\rho]}$ is of finite type, one notices that $\tilde{L}^\circ \backslash \tilde{L}$ is a discrete group and leads to a basis of $\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\rho|_{\tilde{L}^\circ})$ indexed by these cosets and a basis of ρ . The action of \tilde{L} permutes this basis and hence, by choosing a finite generating set of $\rho|_{\tilde{L}^\circ}$, we obtain a finite generating set of $\Pi_{[\rho]}$ as an \tilde{L} -representation, proving that $\Pi_{[\rho]}$ is of finite type. \square

We put $\mathcal{R}_{[\rho]} := \text{End}_{\tilde{L}}(\Pi_{[\rho]})$. Generalities from category theory (see Proposition 2.8), give us the following corollary:

Corollary 9.2. *We have an equivalence of categories*

$$\begin{aligned} \mathcal{M}(\tilde{L})_{[\rho]} &\longrightarrow \mathbf{mod}\text{-}\mathcal{R}_{[\rho]} \\ \pi &\mapsto \text{Hom}_{\tilde{L}}(\Pi_{[\rho]}, \pi). \end{aligned}$$

Below we give a description of the algebra $\mathcal{R}_{[\rho]}$ by generators and relations.

Recall that $\mathcal{G}_\rho \subset \text{Hom}_{\text{gr}}(\Lambda, \mathbb{C}^\times) = \mathcal{X}(\tilde{L})$ denotes the stabilizer of ρ , where $\Lambda = \tilde{L}/\tilde{L}^\circ$, i.e., $\mathcal{G}_\rho = \{\chi \in \text{Hom}_{\text{gr}}(\Lambda, \mathbb{C}^\times) \mid \rho \otimes \chi \simeq \rho\}$. The group \mathcal{G}_ρ naturally acts on the complex torus $\mathcal{X}(\tilde{L})$ by translations, and hence also on the algebra of regular functions on $\mathcal{O}(\mathcal{X}(\tilde{L})) = \mathbb{C}[\Lambda] := \mathcal{A}$ which we denote by $\chi(f) = {}^x f$

for $\chi \in \mathcal{G}_\rho$. By Schur's lemma, the isomorphism $\rho \otimes \chi \simeq \rho$ defines a unique map – up to scalars – from the space underlying ρ to itself, hence provides us with a 2-cocycle $c \in H^2(\mathcal{G}_\rho, \mathbb{C})$, which in turn defines a twisted group algebra $\mathbb{C}[\mathcal{G}_\rho, c]$, the algebra generated by the intertwining operators from ρ to itself through the isomorphisms $\rho \otimes \chi \simeq \rho$.

Proposition 9.3 ([Ber92, Proposition 28]). *The algebra $\mathcal{R}_{[\rho]}$ has the following presentation:*

1. As a vector space $\mathcal{R}_{[\rho]} = \mathcal{A} \otimes \mathbb{C}[\mathcal{G}_\rho, c]$,
2. \mathcal{A} and $\mathbb{C}[\mathcal{G}_\rho, c]$ are subalgebras,
3. $fb_\chi = b_\chi^x f$ for all $f \in \mathcal{A}$ and $\chi \in \mathcal{G}_\rho$, with $b_\chi = 1 \otimes \chi \in \mathcal{A} \otimes \mathbb{C}[\mathcal{G}_\rho, c] = \mathcal{R}_{[\rho]}$.

Proof. The proof is fairly elementary but we will give a sketch.

First, notice that by adjunction, we have a natural isomorphism of vector spaces

$$(9.1) \quad \mathcal{R}_{[\rho]} = \text{End}_{\tilde{L}}(\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\rho|_{\tilde{L}^\circ})) \simeq \text{End}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ}) \otimes \mathcal{A}.$$

Since $\text{End}_{\tilde{L}}(\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\mathbb{C})) = \mathcal{A}$, and since $\text{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\rho|_{\tilde{L}^\circ}) \simeq \rho \otimes \text{ind}_{\tilde{L}^\circ}^{\tilde{L}}(\mathbb{C})$, this exhibits \mathcal{A} as a subalgebra of $\mathcal{R}_{[\rho]}$ and moreover it shows that (9.1) is an isomorphism of right \mathcal{A} -modules.

Second, notice that $\Lambda = \tilde{L}/\tilde{L}^\circ$ acts on $\text{End}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ})$ and the action factors through the finite quotient $\tilde{L}/Z(\tilde{L})\tilde{L}^\circ$. Moreover the action is diagonalizable and the eigenvalues (i.e., characters of Λ) that appear are, by definition, exactly the elements of \mathcal{G}_ρ . We deduce a canonical isomorphism of vector spaces

$$\text{End}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ}) \simeq \mathbb{C}[\mathcal{G}_\rho].$$

If $b_\chi, b_\mu \in \text{End}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ})$ correspond to $\chi, \mu \in \mathcal{G}_\rho$ through the previous isomorphism, then their product belongs to the eigenspace corresponding to $\chi\mu$, i.e., we have $b_\chi b_\mu = c(\chi, \mu)b_{\chi\mu}$ for some non-zero $c(\chi, \mu) \in \mathbb{C}^\times$. Associativity plus unity implies that $c(-, -)$ is a two cocycle of \mathcal{G}_ρ with values in \mathbb{C}^\times . This allows us to construct a twisted group algebra $\mathbb{C}[\mathcal{G}_\rho, c]$ and in turn provides us with an isomorphism of algebras

$$\text{End}_{\tilde{L}^\circ}(\rho|_{\tilde{L}^\circ}) \simeq \mathbb{C}[\mathcal{G}_\rho, c].$$

The last step consists in showing how the subalgebra \mathcal{A} and $\mathbb{C}[\mathcal{G}_\rho, c]$ interact inside $\mathcal{R}_{[\rho]}$. This is done by writing down explicitly the isomorphism in (9.1). \square

Recall (see [Art99, Proposition III.12.1]) that an Azumaya algebra R over a commutative ring Z with identity is a Z -algebra such that for some faithfully flat extension of commutative rings $Z \hookrightarrow Z'$ we have $R \otimes_Z Z' \simeq M_n(Z')$. Equivalently, R is a projective module of constant rank over Z and the multiplication map $R \otimes_Z R^{\text{op}} \rightarrow \text{End}_Z(R)$ is an isomorphism of rings.

In particular, one deduces in a straightforward way the following proposition from the above presentation of $\mathcal{R}_{[\rho]}$.

Proposition 9.4. *The natural inclusion $\mathcal{A}^{\mathcal{G}_\rho} \hookrightarrow \mathcal{R}_{[\rho]}$ identifies $\mathcal{A}^{\mathcal{G}_\rho}$ with the center $Z(\mathcal{R}_{[\rho]})$. In particular $Z(\mathcal{R}_{[\rho]})$ is isomorphic to a Laurent polynomial algebra. Moreover $\mathcal{R}_{[\rho]}$ is an Azumaya algebra over its center.*

For a different proof, see [BH03, Proposition 8.1]. Note that our algebra $\mathcal{R}_{[\rho]}$ is a matrix algebra over the algebra E_G of [BH03].

The next lemma is easy and is valid for any Azumaya algebra (see [Art99, Proposition IV.2.1]).

Lemma 9.5. *Let R be an Azumaya algebra over its center Z which is a noetherian ring. Then the trace map $\text{tr}: R \rightarrow Z$ provides an isomorphism of R -bimodules*

$$R \rightarrow \text{Hom}_Z(R, Z).$$

Proof. First, this is obvious for $R = M_n(Z)$. Using faithfully-flat descent, one first defines the trace map unambiguously for any Azumaya algebra R and then sees that it provides a non-degenerate symmetric pairing $R \otimes_Z R \rightarrow Z$ which leads to the isomorphism asserted in the statement of the lemma. \square

Applying this to our algebra $\mathcal{R}_{[\rho]}$ with center $\mathcal{Z}_{[\rho]} := Z(\mathcal{R}_{[\rho]})$ which is a Laurent polynomial ring, we get the following corollary (see (2.8)):

Corollary 9.6. *There is an isomorphism of $\mathcal{R}_{[\rho]}$ -bimodules*

$$\mathcal{R}_{[\rho]} \simeq \text{Hom}_{\mathcal{Z}_{[\rho]}}(\mathcal{R}_{[\rho]}, \mathcal{Z}_{[\rho]}).$$

In other words, $\mathcal{R}_{[\rho]}$ satisfies condition (FsG).

Let us now apply what we have learned about $\mathcal{R}_{[\rho]}$ to prove the following proposition which plays a crucial role in the study of the homological duality functor (see Theorem 10.2).

Recall that for ρ a cuspidal representation of \tilde{L} , we defined $d(\rho)$ to be the split rank of the center of L (which is a Levi subgroup of G).

Proposition 9.7. *For V , a $\mathcal{R}_{[\rho]}$ -module which is finite dimensional over \mathbb{C} , we have*

$$\mathrm{Ext}_{\mathcal{R}_{[\rho]}}^i(V, \mathcal{R}_{[\rho]}) = 0 \text{ for all } i \neq d(\rho).$$

Proof. For simplicity we will put $\mathcal{R} = \mathcal{R}_{[\rho]}$ and $\mathcal{Z} := Z(\mathcal{R}_{[\rho]})$ in this proof. The forgetful functor

$$F: \mathbf{mod}\text{-}\mathcal{R} \rightarrow \mathbf{mod}\text{-}\mathcal{Z}$$

can also be written as $F = - \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{Z}}$ where ${}_{\mathcal{R}}\mathcal{R}_{\mathcal{Z}}$ is viewed as an \mathcal{R} - \mathcal{Z} -bimodule. As such, F has a right adjoint $F' := \mathrm{Hom}_{\mathcal{Z}}(\mathcal{R}_{\mathcal{Z}}, -)$.

Since \mathcal{R} is an Azumaya algebra (Proposition 9.4), it is a projective \mathcal{Z} -module, hence both F and F' are exact functors. It follows that the adjunction extends to Ext groups. Applying Corollary 9.6 we therefore get

$$\mathrm{Ext}_{\mathcal{R}}^i(V, \mathcal{R}) = \mathrm{Ext}_{\mathcal{R}}^i(V, F'(\mathcal{Z})) = \mathrm{Ext}_{\mathcal{Z}}^i(V, \mathcal{Z}),$$

and now we conclude the proof of the proposition by a well-known result in commutative algebra since \mathcal{Z} is a Laurent polynomial algebra (by Proposition 9.4) of Krull dimension $d(\rho)$ and V is a \mathcal{Z} -module which is a finite dimensional vector space over \mathbb{C} . \square

Remark 9.8. The statement holds with the same proof if we only assume that $\mathcal{R}_{[\rho]} \simeq \mathrm{Hom}_{\mathcal{Z}_{[\rho]}}(\mathcal{R}_{[\rho]}, \mathcal{Z}_{[\rho]})$ as right $\mathcal{R}_{[\rho]}$ -modules.

9.2. Non-cuspidal blocks: a projective generator

Moving on to the block in \tilde{G} corresponding to $\mathfrak{s} = [\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$, let us define the representation

$$\Pi_{\mathfrak{s}} := \bigoplus_{\tilde{P}} \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(\Pi_{[\rho]}) = \bigoplus_{\tilde{P}} \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(\mathrm{ind}_{\tilde{L}^{\circ}}^{\tilde{L}}(\rho|_{\tilde{L}^{\circ}}))$$

where the sum ranges over all parabolics \tilde{P} with Levi \tilde{L} (it is a finite sum).

Remark 9.9. Actually it would be enough to consider $\mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}}(\Pi_{[\rho]})$ for a single parabolic as this module turns out to be (non-canonically) independent of the parabolic containing \tilde{L} . See [Ber92, p. 96] for a proof as well as [Ren10, VI.10.1] for some more details. However, we will not need it. One advantage of taking the direct sum instead of a single parabolic induction is that it makes the isomorphism from Proposition 10.5 canonical.

Lemma 9.10. *The module $\Pi_{\mathfrak{s}}$ is a finite type projective-generator of $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$.*

Proof. The projectivity follows at once as a consequence of the second-adjointness Corollary 8.16 together with Proposition 9.1: the functor $\mathbf{i}_{L, \tilde{P}}^{\tilde{G}}$ has an exact right adjoint, hence it preserves projectives and $\Pi_{[\rho]}$ is projective. Since parabolic induction preserves finite type (see Theorem 7.16) we have that $\Pi_{\mathfrak{s}}$ is also finitely generated.

Let $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ be a non-zero representation of \tilde{G} . We need to show that $\text{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, \pi)$ is non zero and for that, since $\Pi_{\mathfrak{s}}$ is projective, it is enough to consider the case π irreducible. By definition of the block $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$, $\mathbf{r}_{L, \tilde{P}^-}^{\tilde{G}}(\pi)$ is non-zero in the block $\mathcal{M}(\tilde{L})_{[\rho]}$. The second-adjointness Theorem 8.15 and the fact that $\Pi_{[\rho]}$ is a generator of $\mathcal{M}(\tilde{L})_{[\rho]}$ (see Proposition 9.1) imply that

$$\text{Hom}_{\tilde{G}}(\mathbf{i}_{L, \tilde{P}}^{\tilde{G}}(\Pi_{[\rho]}), \pi) = \text{Hom}_{\tilde{L}}(\Pi_{[\rho]}, \mathbf{r}_{L, \tilde{P}^-}^{\tilde{G}}(\pi)) \neq 0$$

and hence $\text{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, \pi) \neq 0$. □

An abelian category admitting all coproducts and having a finitely generated (compact) progenerator is equivalent to the category of right modules over its endomorphism ring (see Proposition 2.8). Putting $\mathcal{R}_{\mathfrak{s}} := \text{End}_{\tilde{G}}(\Pi_{\mathfrak{s}})$, we thus deduce the following:

Proposition 9.11. *The functor*

$$\begin{aligned} \mathcal{M}(\tilde{G})_{\mathfrak{s}} &\rightarrow \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}} \\ V &\mapsto \text{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, V) \end{aligned}$$

is an equivalence of categories sending finite length representations of \tilde{G} in $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ to $\mathcal{R}_{\mathfrak{s}}$ -modules which are finite dimensional over \mathbb{C} .

9.3. Center of $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$

This section plays no role in this work but we record it for completeness. We give the description of the center of the algebra $\mathcal{R}_{\mathfrak{s}}$ for $\mathfrak{s} = [\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$. In turn this determines the center of the block $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ (see Proposition 9.11, Remark 2.4 and Example 2.5).

Recall that $\mathcal{X}(\tilde{L})$ acts on the irreducible cuspidals $\text{Irr}(\tilde{L})_{\text{cs}}$ and we denoted by \mathcal{G}_{ρ} the stabilizer of ρ . Denote by $W_{[\rho]} := \text{Stab}_{W_{\tilde{L}, \tilde{G}}}([\rho])$, the stabilizer of the inertia class $[\rho]$ in the relative Weyl group $W_{\tilde{L}, \tilde{G}}$.

Recall also the \mathbb{Z} -lattice $\Lambda(\tilde{L}) = \tilde{L}/\tilde{L}^{\circ} = \Lambda(L) = L/L^{\circ}$. The characters of $\Lambda(\tilde{L})$ form a torus whose ring of regular functions is identified with the group

algebra of $\Lambda(\tilde{L})$:

$$\mathbb{C}[\Lambda(\tilde{L})] = \mathcal{O}(\mathcal{X}(\tilde{L})).$$

The linear case of the following theorem is due to [Ber84, 2.12] but the proof works also for finite central extensions. Another reference is [Ren10, VI.10.4].

Theorem 9.12. *The algebra $\mathcal{R}_{\mathfrak{s}}$ contains the Laurent polynomial algebra $\mathcal{O}(\mathcal{X}(\tilde{L})/\mathcal{G}_{\rho})$ as a subalgebra and it is finite as a left (or right)-module over it. Moreover, the center of $\mathcal{R}_{\mathfrak{s}}$ is nothing but*

$$\mathcal{Z}(\mathcal{R}_{\mathfrak{s}}) = \mathcal{O}(\mathcal{X}(\tilde{L})/\mathcal{G}_{\rho})^{W_{[\rho]}}.$$

Since $\mathcal{Z}(\mathcal{R}_{\mathfrak{s}})$ has no non-trivial idempotents, we immediately get the following corollary.

Corollary 9.13. *The category $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ is indecomposable.*

9.4. Revisiting Howe's theorem

Since Bernstein's center plays a prominent role in the theory, it seems worthwhile mentioning the following applications whose proofs are parallel to the linear case. This section is only for completeness, it is not used in the sequel.

We say that a smooth representation V of \tilde{G} is $\mathcal{Z}(\tilde{G})$ -admissible (where $\mathcal{Z}(\tilde{G})$ is the Bernstein center of the category of smooth representations of \tilde{G}) if for any open compact $\tilde{K} \leq \tilde{G}$, the $\mathcal{Z}(\tilde{G})$ -module $V^{\tilde{K}}$ is finitely generated. We have

Theorem 9.14 ([Ren10, VI.10.8]). *A smooth representation V of \tilde{G} is of finite type if and only if it is $\mathcal{Z}(\tilde{G})$ -admissible and it has only finitely many non-zero Bernstein components.*

We say that a representation is $\mathcal{Z}(\tilde{G})$ -finite if the action of $\mathcal{Z}(\tilde{G})$ on it factors through an ideal of finite codimension in $\mathcal{Z}(\tilde{G})$.

The following is a generalization of Howe's theorem.

Theorem 9.15. *Let V be a smooth representation of \tilde{G} . Any two of the following properties implies the third, and then V is of finite length:*

1. V is of finite type,
2. V is admissible,
3. V is $\mathcal{Z}(\tilde{G})$ -finite.

Proof. Howe's theorem says that (1)+(2) imply finite length and hence by Schur's lemma we have (3).

Assume (1)+(3). Theorem 9.14 implies that V is $\mathcal{Z}(\tilde{G})$ -admissible. Hence for any open compact $\tilde{K} \leq \tilde{G}$ the invariants $V^{\tilde{K}}$ are finitely generated over $\mathcal{Z}(\tilde{G})$ and now (3) implies that $V^{\tilde{K}}$ is finite dimensional.

Assume (2)+(3). Clearly V is $\mathcal{Z}(\tilde{G})$ admissible and since the action of $\mathcal{Z}(\tilde{G})$, a unitary ring, factorizes through a finite codimension ideal we get that V has only finitely many non-zero Bernstein components. We conclude by Theorem 9.14. □

10. Homological duality

10.1. Vanishing

Here we prove the vanishing part of our main result, namely Theorem 1.3(1), following the strategy in [Ber92].

The main ingredients are Section 7 on Bernstein’s decomposition, Bernstein’s second adjoint Theorem 8.15, and the vanishing result from Proposition 9.7.

Consider $\mathcal{M}(\tilde{G})_{fg}$ the full subcategory of smooth representations of \tilde{G} which are finitely generated. It is still an abelian category and the parabolic induction and restriction functors preserve it. The homological duality is defined as the following functor between derived categories

$$(10.1) \quad \begin{aligned} D_h: \mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G})) &\rightarrow \mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G}))^{op} \\ \pi &\mapsto \mathrm{RHom}_{\tilde{G}}(\pi, \mathcal{H}(\tilde{G})) \end{aligned}$$

where $\mathcal{H}(\tilde{G})$ denotes the Hecke algebra of \tilde{G} . The structure of left \tilde{G} representation is through the right action of $\mathcal{H}(\tilde{G})$ on itself and through the involution $f \mapsto \check{f}: \mathcal{H}(\tilde{G}) \rightarrow \mathcal{H}(\tilde{G})$ defined by $\check{f}(g) := f(g^{-1})$.

Remark 10.1.

1. Notice that D_h lands indeed in the bounded derived category because the category of smooth representations $\mathcal{M}(\tilde{G})$ has finite global dimension (Section 3.4).
2. One could also define D_h without the assumption on finitely generated but then D_h would land outside smooth modules, indeed in the category of all $\mathcal{H}(\tilde{G})$ -modules. We could come back to smooth modules simply by taking the smooth part (see the paragraph after Proposition 8.9).

The following theorem is the most important property of the homological duality functor and is due to Bernstein. Fix $\mathfrak{s} = [\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$ a cuspidal datum.

Theorem 10.2. *If $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ is of finite length, then $D_h(\pi)$ has cohomology only in degree $d(\mathfrak{s})$.*

Proof (following [Ber92, Theorem 31]). We need to show that $\mathrm{Ext}_{\tilde{G}}^i(\pi, \mathcal{H}) = 0$ for $i \neq d(\mathfrak{s})$, where we put for short $\mathcal{H} = \mathcal{H}(\tilde{G})$.

Decompose the representation $\mathcal{H}(\tilde{G})$ according to Bernstein's decomposition, Theorem 7.14. Clearly only the component $\mathcal{H}(\tilde{G})_{\mathfrak{s}}$ is important to us. Moreover, since $\mathcal{H}(\tilde{G})$ is a projective representation (Remark 3.3), and $\Pi_{\mathfrak{s}}$ is a projective generator of $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ (see Lemma 9.10), it is enough to show:

$$(10.2) \quad \mathrm{Ext}_{\tilde{G}}^i(\pi, \Pi_{\mathfrak{s}}) = 0, \text{ for all } i \neq d(\mathfrak{s}).$$

Recall from §9.2 that $\Pi_{\mathfrak{s}} = \mathbf{i}_{L, \tilde{P}}^{\tilde{G}}(\Pi_{[\rho]})$. Since parabolic induction and restriction are exact functors, the Frobenius adjunction gives

$$(10.3) \quad \mathrm{Ext}_{\tilde{G}}^i(\pi, \Pi_{\mathfrak{s}}) = \mathrm{Ext}_{\tilde{L}}^i(\mathbf{r}_{L, \tilde{P}}^{\tilde{G}}(\pi), \Pi_{[\rho]}).$$

The representation π is of finite length, hence by Corollary 6.12 the same is true of $\mathbf{r}_{L, \tilde{P}}^{\tilde{G}}(\pi)$. Through the categorical equivalence $\mathcal{M}(\tilde{L})_{[\rho]} \simeq \mathbf{mod}\text{-}\mathcal{R}_{[\rho]}$ (see Proposition 9.11), the vanishing of (10.3) follows from

$$\mathrm{Ext}_{\mathcal{R}_{[\rho]}}^i(V, \mathcal{R}_{[\rho]}) = 0, \text{ for all } i \neq d(\rho) = d(\mathfrak{s}),$$

for all finite dimensional $\mathcal{R}_{[\rho]}$ -modules V . This was proved in Proposition 9.7. \square

Given $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\mathrm{fl}}$ a finite length representation in a given block, we will denote by $\mathbb{D}_h(\pi)$ the representation $H^{d(\mathfrak{s})}(D_h(\pi))$.

10.2. Interaction with induction and restriction

The objective of this section is to investigate how does the (full, derived) homological duality commute with parabolic induction and restriction. We follow the proof of the linear case from [Ber92, Theorem 31(4,5)].

Proposition 10.3. *Let $\tilde{P} = \tilde{L}N$ be a parabolic with Levi decomposition in \tilde{G} . We have the following natural isomorphisms of functors when restricted to the bounded derived category of finitely generated smooth \tilde{G} -modules $\mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G}))$:*

1. $D_h \mathbf{i}_{L, \tilde{P}}^{\tilde{G}} \simeq \mathbf{i}_{L, \tilde{P}^-}^{\tilde{G}} D_h,$

$$2. D_h \mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}} \simeq \mathbf{r}_{\tilde{L}, \tilde{P}}^{\tilde{G}} D_h.$$

Proof. To shorten the notation, we write $\mathbf{i}_{\tilde{P}}$ and $\mathbf{r}_{\tilde{P}}$ for the parabolic induction and restriction functors.

Since all objects in $\mathcal{M}(\tilde{G})_{fg}$ admit a finite resolution by finitely generated projective objects, it is enough to define and prove the required isomorphisms for finitely generated projective objects. Therefore, using the second adjointness and the Frobenius reciprocity, we see that the isomorphisms in part (1) and in part (2) of the proposition are equivalent to proving natural isomorphisms

$$(10.4) \quad \mathbf{i}_{\tilde{P}^-} \text{Hom}_{\tilde{L}}(V, \mathcal{H}(\tilde{L})) \simeq \text{Hom}_{\tilde{L}}(V, \mathbf{r}_{\tilde{P}^-}(\mathcal{H}(\tilde{G}))),$$

$$(10.5) \quad \mathbf{r}_{\tilde{P}} \text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G})) \simeq \text{Hom}_{\tilde{G}}(W, \mathbf{i}_{\tilde{P}}(\mathcal{H}(\tilde{L}))),$$

for $V \in \mathcal{M}(\tilde{L})$ and $W \in \mathcal{M}(\tilde{G})$, both projective, and finitely generated representations, and where we have considered $\mathcal{H}(\tilde{L})$ as an $\tilde{L} \times \tilde{L}$ module and $\mathcal{H}(\tilde{G})$ as a $\tilde{G} \times \tilde{G}$ -module.

For the proof of these isomorphisms, we begin by noticing the following identifications of $\tilde{L} \times \tilde{G}$ -modules:

$$(10.6)$$

$$(\text{id} \times \mathbf{i}_{\tilde{P}})(\mathcal{H}(\tilde{L})) \simeq \text{Ind}_{\tilde{L} \times \tilde{P}}^{\tilde{L} \times \tilde{G}}(\mathcal{H}(\tilde{L})\delta^{-1/2}) \simeq \delta^{1/2} \otimes \mathcal{C}_c^\infty(\tilde{G}/N) \simeq (\text{id} \times \mathbf{r}_{\tilde{P}})(\mathcal{H}(\tilde{G})),$$

where in the first appearance, δ is a character of $\tilde{L} \times \tilde{P}$ trivial on \tilde{L} , whereas in the second appearance, δ is a character of $\tilde{L} \times \tilde{G}$ trivial on \tilde{G} ; this also explains why the power of delta changes sign since $\mathcal{H}(\tilde{L}) = \text{ind}_{\Delta(\tilde{L})}^{\tilde{L} \times \tilde{L}} \mathbb{C}$.

The following is an easy but crucial observation valid for any finitely generated smooth representation V of \tilde{L} (we will apply this to finitely generated projective modules as noted earlier):

$$(10.7) \quad \mathbf{i}_{\tilde{P}} \text{Hom}_{\tilde{L}}(V, \mathcal{H}(\tilde{L})) \simeq \text{Hom}_{\tilde{L}}(V, (\text{id} \times \mathbf{i}_{\tilde{P}})(\mathcal{H}(\tilde{L}))),$$

as representations of \tilde{G} where in $\text{Hom}_{\tilde{L}}(V, \mathcal{H}(\tilde{L}))$, both V and $\mathcal{H}(\tilde{L})$ are considered as left \tilde{L} -modules, thus $\text{Hom}_{\tilde{L}}(V, \mathcal{H}(\tilde{L}))$ is a right \tilde{L} -module through the right \tilde{L} -action on $\mathcal{H}(\tilde{L})$; the $\tilde{L} \times \tilde{G}$ -representation $(\text{id} \times \mathbf{i}_{\tilde{P}})(\mathcal{H}(\tilde{L})) = \mathbf{i}_{\tilde{P}}^{\tilde{G}}(\mathcal{H}(\tilde{L}))$ has \tilde{L} action through the left action of \tilde{L} on $\mathcal{H}(\tilde{L})$, and a right \tilde{G} action. To see the isomorphism in equation (10.7), note that by the definition of an induced representation, both sides of (10.7) give rise to functions

$F: \tilde{G} \times V \rightarrow \mathcal{H}(\tilde{L})$ which are linear maps $V \rightarrow \mathcal{H}(\tilde{L})$ when restricted to any $g \in \tilde{G}$, and satisfy:

1. $F(g, \ell v) = \ell F(g, v)$, for all $g \in \tilde{G}, \ell \in \tilde{L}, v \in V$.
2. $F(gp, v) = F(g, v) \cdot \ell$ where $p \in \tilde{P} = \tilde{L}N$ has the form $p = \ell n$, with $\ell \in \tilde{L}$, and $n \in N$.

Further, such an F arises from the left hand side of the isomorphism in (10.7) if and only if the corresponding map $\tilde{G} \rightarrow \text{Hom}(V, \mathcal{H}(\tilde{L}))$ is locally constant on \tilde{G} , whereas such an F arises from the right hand side of the equality in eq. (10.7) if and only if for each $v \in V$, $F(-, v)$ is a locally constant map from \tilde{G} to $\mathcal{H}(\tilde{L})$. Thus the functions F which arise from the left hand side of equation (10.7) are always contained in those which arise from the right hand side, and the converse holds if V is finitely generated over \tilde{L} , say by v_1, \dots, v_n . Then assuming that each of the functions $F(-, v_i)$ are constant in a neighborhood $U(g_0)$ of a fixed $g_0 \in \tilde{G}$, then by equation 1. above, $F(g, \ell v_i) = \ell F(g, v_i) = \ell F(g_0, v_i)$ for all $g \in U(g_0), \ell \in \tilde{L}$, hence $F(g, v) = F(g_0, v)$ for all $g \in U(g_0), v \in V$. Now the isomorphism (10.7) (applied to \tilde{P}^- in place of \tilde{P}) together with (10.6) proves the isomorphism (10.4), and hence the isomorphism in part 1. of the proposition.

Similarly, for any finitely generated smooth, projective, representation W of \tilde{G} we have

$$\mathbf{r}_{\tilde{P}} \text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G})) \simeq \text{Hom}_{\tilde{G}}(W, (\text{id} \times \mathbf{r}_{\tilde{P}})(\mathcal{H}(\tilde{G}))),$$

as \tilde{L} -modules and naturally in W . This assertion is equivalent to proving that the normalized Jacquet module of $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))$ (considered as a right G -module) is the same as $\delta^{1/2} \text{Hom}_{\tilde{G}}(W, \mathcal{C}_c^\infty(\tilde{G}/N))$ which is equivalent to proving that:

1. the natural map from $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))$ to $\text{Hom}_{\tilde{G}}(W, \mathcal{C}_c^\infty(\tilde{G}/N))$ is surjective. This is a consequence of W being projective.
2. The kernel of the natural map $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G})) \rightarrow \text{Hom}_{\tilde{G}}(W, \mathcal{C}_c^\infty(\tilde{G}/N))$ consists of $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))[N]$ where for any smooth representation \mathcal{W} of N ,

$$\mathcal{W}[N] = \{n \cdot w - w | w \in \mathcal{W}\} = \{v \in \mathcal{W} | \int_{N_i} n \cdot v = 0\}$$

where N_i is some compact open subgroup of N depending on $v \in \mathcal{W}$. It follows that

$$\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))[N] = \text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))[N],$$

using that $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))$ is a smooth representation of N which is the case as W is a finitely generated \tilde{G} -module.

Applying the functor $\text{Hom}_{\tilde{G}}(W, -)$ to the exact sequence

$$0 \rightarrow \mathcal{H}(\tilde{G})[N] \rightarrow \mathcal{H}(\tilde{G}) \rightarrow \mathcal{C}_c^\infty(\tilde{G}/N) \rightarrow 0,$$

and noting that W is projective, we have the exact sequence

$$0 \rightarrow \text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G})[N]) \rightarrow \text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G})) \rightarrow \text{Hom}_{\tilde{G}}(W, \mathcal{C}_c^\infty(\tilde{G}/N)) \rightarrow 0,$$

proving the assertion that the Jacquet module of $\text{Hom}_{\tilde{G}}(W, \mathcal{H}(\tilde{G}))$, considered as a right \tilde{G} -module, is the same as $\text{Hom}_{\tilde{G}}(W, \mathcal{C}_c^\infty(\tilde{G}/N))$. This completes the proof of the isomorphism in (10.5), and hence part 2. of the proposition. \square

10.3. On projective generators

In order to better understand the homological duality functor D_h , we will compute its value on the projective generators $\Pi_{[\rho]} \in \mathcal{M}(\tilde{G})_{[\rho]}$ and $\Pi_{\mathfrak{s}} \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ from Section 9. The answer is as nice as it could possibly be. Then we will use this computation to describe the functors $(-)^{\vee}$ and D_h on the $\mathcal{R}_{\mathfrak{s}}$ -side: we obtain the contragredient, resp. homological, duality for $\mathcal{R}_{\mathfrak{s}}$.

Recall that if $\rho \in \mathcal{M}(\tilde{G})$ is irreducible cuspidal, then Π_{ρ} was defined to be $\text{ind}_{\tilde{G}^{\circ}}^{\tilde{G}}(\rho|_{\tilde{G}^{\circ}})$. Since a cuspidal representation is compact modulo center by Harish-Chandra’s theorem, the following is a restatement of Corollary 4.20:

Lemma 10.4. *If $\rho \in \mathcal{M}(\tilde{G})$ is irreducible cuspidal, then*

$$D_h(\Pi_{[\rho]}) \simeq \Pi_{[\rho^{\vee}]}.$$

Let $\mathfrak{s} = (\tilde{L}, \rho) \in \mathcal{B}(\tilde{G})$ be a cuspidal datum (up to inertia and conjugation). The projective generator $\Pi_{\mathfrak{s}} \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ was defined in Section 9.2 as

$$\Pi_{\mathfrak{s}} = \bigoplus_{\tilde{P}} \mathbf{i}_{\tilde{L}, \tilde{P}}^{\tilde{G}} \Pi_{[\rho]}.$$

We denote by $\mathfrak{s}^{\vee} = (\tilde{L}, \rho^{\vee})$ the contragredient cuspidal datum (it could be the same as \mathfrak{s} but in general it is not).

Proposition 10.5. *With the above notation we have $D_h(\Pi_{\mathfrak{s}}) \simeq \Pi_{\mathfrak{s}^{\vee}}$.*

Proof. Use the previous lemma together with $D_h \mathbf{i}_{L, \tilde{P}}^{\tilde{G}} \simeq \mathbf{i}_{L, \tilde{P}^-}^{\tilde{G}} D_h$ (see Proposition 10.3) to get the following isomorphisms

$$\begin{aligned} D_h(\Pi_{\mathfrak{s}}) &= \oplus_{\tilde{P}} D_h(\mathbf{i}_{L, \tilde{P}}^{\tilde{G}}(\Pi_{[\rho]})) \\ &\simeq \oplus_{\tilde{P}} \mathbf{i}_{L, \tilde{P}^-}^{\tilde{G}} D_h(\Pi_{[\rho]}) \\ &\simeq \oplus_{\tilde{P}} \mathbf{i}_{L, \tilde{P}^-}^{\tilde{G}} \Pi_{[\rho^\vee]} \\ &= \Pi_{\mathfrak{s}^\vee}. \end{aligned} \quad \square$$

Corollary 10.6. *Homological duality restricts to a functor*

$$D_h: \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}}) \rightarrow \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee})^{op}$$

which is moreover involutive when restricted to finitely generated modules.

Proof. The functor D_h sends the projective generator $\Pi_{\mathfrak{s}}$ of $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ to the projective generator $\Pi_{\mathfrak{s}^\vee}$ of $\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee}$. The duality statement for finitely generated follows from Corollary 2.25. \square

Thanks to the vanishing result for finite length modules (Theorem 10.2), we can deduce

Corollary 10.7. *The functor D_h restricts to an involution*

$$\mathbb{D}_h: \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}} \rightarrow (\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee}^{\text{fl}})^{op}.$$

Corollary 10.8. *The homological duality induces an anti-isomorphism of algebras*

$$\mathcal{R}_{\mathfrak{s}} \simeq \mathcal{R}_{\mathfrak{s}^\vee},$$

or equivalently, an isomorphism of algebras $\mathcal{R}_{\mathfrak{s}}^{op} \simeq \mathcal{R}_{\mathfrak{s}^\vee}$.

Proof. Since $D_h^2 \simeq \text{Id}$ (see Corollary 2.25) we see that in particular D_h is an equivalence of categories (contravariant). Hence it induces an anti-isomorphism of algebras:

$$\mathcal{R}_{\mathfrak{s}} = \text{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}) \simeq \text{Hom}_{\tilde{G}}(D_h \Pi_{\mathfrak{s}}, D_h \Pi_{\mathfrak{s}}) = \mathcal{R}_{\mathfrak{s}^\vee}$$

where in the last equality we used Proposition 10.5. \square

The above corollary means that we have an equivalence of categories $\mathcal{R}_{\mathfrak{s}}\text{-mod} \simeq \text{mod-}\mathcal{R}_{\mathfrak{s}^\vee}$. In other words, given a right module $V \in \text{mod-}\mathcal{R}_{\mathfrak{s}}$ its dual $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is naturally a left $\mathcal{R}_{\mathfrak{s}^\vee}$ -module, hence we can view it

as a right $\mathcal{R}_{\mathfrak{s}^\vee}$ -module. Put shortly, taking the dual vector spaces gives us a functor

$$(-)^* : \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}} \rightarrow \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}^\vee}.$$

We are now ready to describe the contragredient on the $\mathcal{R}_{\mathfrak{s}}$ -module side. The answer is not surprising at all:

Proposition 10.9. *There is a commutative square of functors*

$$\begin{array}{ccc} \mathcal{M}(\tilde{G})_{\mathfrak{s}} & \xrightarrow{\sim} & \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}} \\ (-)^\vee \downarrow & & \downarrow (-)^* \\ (\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee})^{op} & \xrightarrow{\sim} & (\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}^\vee})^{op}. \end{array}$$

Proof. The fact that the vertical right arrow makes sense was discussed above as a consequence of Corollary 10.8

Note that $\Pi_{\mathfrak{s}}$ being projective and finitely generated, together with the calculation from Proposition 10.5, implies that we have natural isomorphisms of functors

$$\mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, -) \simeq \mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, \mathcal{H}(\tilde{G})) \otimes_{\mathcal{H}(\tilde{G})} - \simeq \Pi_{\mathfrak{s}^\vee} \otimes_{\mathcal{H}(\tilde{G})} -.$$

Let $V \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ and use tensor-hom adjunction to obtain natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, V)^* &\simeq \mathrm{Hom}(\Pi_{\mathfrak{s}^\vee} \otimes_{\mathcal{H}(\tilde{G})} V, \mathbb{C}) \\ &\simeq \mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}^\vee}, V^*) \\ &\simeq \mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}^\vee}, V^\vee) \end{aligned}$$

where the last equality follows because the image of a smooth module is smooth. □

As a final computation, we show that the homological duality on the $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ side goes to homological duality on the $\mathcal{R}_{\mathfrak{s}}$ -module side:

Proposition 10.10. *There is a commutative square of functors*

$$\begin{array}{ccc} \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}}) & \xrightarrow{\sim} & \mathcal{D}^b(\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}) \\ D_h \downarrow & & \downarrow D_h \\ \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee})^{op} & \xrightarrow{\sim} & \mathcal{D}^b(\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}^\vee})^{op} \end{array}$$

where the D_h on the right is $\mathrm{RHom}_{\mathcal{R}_{\mathfrak{s}}}(-, \mathcal{R}_{\mathfrak{s}})$ and we again use the algebra anti-isomorphism $\mathcal{R}_{\mathfrak{s}} \simeq \mathcal{R}_{\mathfrak{s}^\vee}$ from Corollary 10.8.

Proof. We deal first with the left-bottom composition. Since D_h is involutive (see Corollary 2.25), it induces an isomorphism on Hom spaces. In particular, using Proposition 10.5, we get an isomorphism of left $\mathcal{R}_{\mathfrak{s}}$ -modules

$$\mathrm{RHom}_{\tilde{\mathcal{G}}}(\Pi_{\mathfrak{s}^\vee}, D_h(V)) = \mathrm{RHom}_{\tilde{\mathcal{G}}}(V, \Pi_{\mathfrak{s}})$$

for all $V \in \mathcal{D}^b(\mathcal{M}(\tilde{\mathcal{G}})_{\mathfrak{s}})$.

On the other hand, let us recall that the equivalence $\mathcal{M}(\tilde{\mathcal{G}})_{\mathfrak{s}} \rightarrow \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ was given by the functor $\mathrm{Hom}_{\tilde{\mathcal{G}}}(\Pi_{\mathfrak{s}}, -)$. The tensor-Hom adjunction gives its left adjoint as $- \otimes_{\mathcal{R}_{\mathfrak{s}}} \Pi_{\mathfrak{s}}$. Being an equivalence implies the adjunction is a bi-adjunction, i.e., we also have that $\mathrm{Hom}_{\tilde{\mathcal{G}}}(\Pi_{\mathfrak{s}}, -)$ is left adjoint to $- \otimes_{\mathcal{R}_{\mathfrak{s}}} \Pi_{\mathfrak{s}}$.

We can now compute the right-top composition and conclude the proof:

$$\begin{aligned} \mathrm{RHom}_{\mathcal{R}_{\mathfrak{s}}}(\mathrm{RHom}_{\tilde{\mathcal{G}}}(\Pi_{\mathfrak{s}}, V), \mathcal{R}_{\mathfrak{s}}) &\simeq \mathrm{RHom}_{\tilde{\mathcal{G}}}(V, \mathcal{R}_{\mathfrak{s}} \otimes_{\mathcal{R}_{\mathfrak{s}}} \Pi_{\mathfrak{s}}) \\ &\simeq \mathrm{RHom}_{\tilde{\mathcal{G}}}(V, \Pi_{\mathfrak{s}}), \end{aligned}$$

natural isomorphisms of left $\mathcal{R}_{\mathfrak{s}}$ -modules. □

Remark 10.11. Using a similar argument and Morita theory (see for example [Par70, 4.11 Theorem 2, Corollary 2]) one shows that under an equivalence of module categories $R\text{-mod} \simeq S\text{-mod}$, the R -bimodule R is sent to the S -bimodule S , and as a consequence, the homological duality for R corresponds to the homological duality for S .

As a consequence of this section, in order to study homological duality on $\mathcal{M}(\tilde{\mathcal{G}})$ we can study it on $\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ for all the cuspidal data \mathfrak{s} . This will play an important role in Section 12 where we show that homological duality on finite length cuspidals is isomorphic to the contragredient (up to a shift).

11. The duality theorem of Schneider–Stuhler

We will use the duality theorem from Section 2.3, applied to the Hecke algebra $\mathcal{H}(\tilde{\mathcal{G}})$, to deduce a theorem of Schneider and Stuhler [SS97, Duality theorem]. In loc.cit., the result is stated and proved in the subcategory of modules with central character but this restriction was removed in [NP20]. Our proof is independent both of [SS97] and of [NP20], and it provides moreover a generalization of the main result of [NP20] from irreducible modules to modules of finite length. The approach that we take was suggested in [BBK18, §3.4].

Let $\mathfrak{s} = [\rho, \tilde{L}] \in \mathcal{B}(\tilde{G})$ be a cuspidal datum and put $d = d(\mathfrak{s})$ the split rank of the center of \tilde{L} . We denote the contragredient cuspidal datum by $\mathfrak{s}^\vee = [\rho^\vee, \tilde{L}]$ for which we have $d(\mathfrak{s}^\vee) = d$.

Recall that in Theorem 10.2 it was proved that, for $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}}$, we have $H^i(D_h(\pi)) \neq 0$ only for $i = d$. We put $\mathbb{D}_h(\pi) = H^d(D_h(\pi))$.

Theorem 11.1. *Let $\pi \in \mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}}$ be a finite length representation in the Bernstein block \mathfrak{s} and let $\pi' \in \mathcal{M}(\tilde{G})$ be any smooth representation. Then the natural pairing*

$$\text{Ext}_{\tilde{G}}^i(\pi, \pi') \times \text{Ext}_{\tilde{G}}^{d-i}(\pi', \mathbb{D}_h(\pi)^\vee) \rightarrow \text{Ext}_{\tilde{G}}^d(\pi, \mathbb{D}_h(\pi)^\vee) \rightarrow \mathbb{C}$$

provides an isomorphism

$$\text{Ext}_{\tilde{G}}^i(\pi, \pi')^* \simeq \text{Ext}_{\tilde{G}}^{d-i}(\pi', \mathbb{D}_h(\pi)^\vee).$$

If π is, moreover, irreducible, then $\text{Ext}_{\tilde{G}}^d(\pi, \mathbb{D}_h(\pi)^\vee) \simeq \mathbb{C}$.

Proof. The first part is just a reformulation of Corollary 2.23. The second part comes from Corollary 2.21 by noticing that $D_{\text{Nak}}(\pi) = \mathbb{D}_h(\pi)^\vee[d]$. \square

Remark 11.2. Note that $(\cdot)^\vee \circ \mathbb{D}_h$ is an endofunctor of the category $\mathcal{M}(\tilde{G})_{\mathfrak{s}}^{\text{fl}}$ of finite length modules. We would like to think that \mathbb{D}_h commutes with the contragredient on finite length modules but we are not able to prove it. We do not know this even for finite length representations of linear groups. However, we can treat the cuspidal case; see Corollary 12.4. More generally, D_h should commute with Grothendieck–Serre duality (over the Bernstein center) for all finitely generated \tilde{G} -modules.⁹ This will be investigated in future work.

12. Dualities on finite length representations

In this section, we prove that the homological duality restricted to finite length cuspidal representations is nothing else but contragredient duality up to a shift (see Corollary 12.4). This is due to the fact that the condition (FsG) is satisfied on a cuspidal block (see Corollary 9.6). Along the way we will show the well-known fact that Grothendieck–Serre duality over the Bernstein center, restricted to finite length modules, is just the contragredient duality (see Corollary 12.2). We make use of the generalities developed in Section 2.4.

Recall the Bernstein decomposition (Theorem 7.14): the category $\mathcal{M}(\tilde{G})$ of smooth representations of \tilde{G} decomposes into blocks $\mathcal{M}(\tilde{G}) \simeq \prod_{\mathfrak{s}} \mathcal{M}(\tilde{G})_{\mathfrak{s}}$ and, for each block, we have $\mathcal{M}(\tilde{G})_{\mathfrak{s}} \simeq \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ (see Proposition 9.11).

⁹This was suggested to us by Roman Bezrukavnikov.

Moreover, we have shown (see Proposition 10.9 and Proposition 10.10) that under the equivalence $\mathcal{M}(\tilde{G})_{\mathfrak{s}} \simeq \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$, the contragredient on $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ corresponds to taking dual vector spaces on $\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$ and that homological duality on $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ corresponds to homological duality on $\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$.

Recall from Section 9.3 that the center of the block $\mathcal{M}(\tilde{G})_{\mathfrak{s}}$ is $\mathcal{Z}_{\mathfrak{s}} := \mathcal{Z}(\mathcal{R}_{\mathfrak{s}})$ and that $\mathcal{R}_{\mathfrak{s}}$ is finite over $\mathcal{Z}_{\mathfrak{s}}$. From the block decomposition, the (Bernstein) center of the category $\mathcal{M}(\tilde{G})$ satisfies

$$\mathcal{Z} = \prod_{\mathfrak{s}} \mathcal{Z}_{\mathfrak{s}}.$$

Moreover, we know that each $\mathcal{Z}_{\mathfrak{s}}$ is the ring of invariants of a finite group acting on a Laurent polynomial ring (see Theorem 9.12), therefore it is a Cohen-Macaulay ring, and has a dualizing complex $\omega_{\mathcal{Z}_{\mathfrak{s}}}^{\circ}$. The dualizing complex for the ring \mathcal{Z} is then

$$\omega_{\mathcal{Z}}^{\circ} = \prod_{\mathfrak{s}} \omega_{\mathcal{Z}_{\mathfrak{s}}}^{\circ}.$$

The following is the relative version of Definition 2.34 and the bound- edness of the derived categories is justified by the fact that $\omega_{\mathcal{Z}}^{\circ}$ has finite injective dimension (see Remark 2.31):

Definition 12.1. The Grothendieck–Serre duality relative to the center is defined by

$$D_{GS/\mathcal{Z}} = \mathrm{RHom}_{\mathcal{Z}}(-, \omega_{\mathcal{Z}}^{\circ})^{\mathrm{sm}} : \mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G})) \rightarrow \mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G}))^{op}$$

where the superscript “sm” signifies taking smooth vectors.

Corollary 12.2. *The functor $D_{GS/\mathcal{Z}}$ restricted to admissible modules is isomorphic to the contragredient functor. In particular, this holds for all finite length \tilde{G} -modules.*

Proof. Let V be an admissible module for \tilde{G} . For any open compact subgroup $\tilde{K} \subset \tilde{G}$, $V^{\tilde{K}}$ is a finite dimensional module over the \tilde{K} -biinvariant Hecke algebra $\mathcal{H}(\tilde{G} // \tilde{K})$ and it is also a module over the center \mathcal{Z} .

We have the following natural isomorphisms:

$$\begin{aligned} \mathrm{RHom}_{\mathcal{Z}}(V, \omega_{\mathcal{Z}}^{\circ}) &= \mathrm{RHom}_{\mathcal{Z}}\left(\bigcup_{\tilde{K}} V^{\tilde{K}}, \omega_{\mathcal{Z}}^{\circ}\right) \\ &= \varprojlim_{\tilde{K}} \mathrm{RHom}_{\mathcal{Z}}(V^{\tilde{K}}, \omega_{\mathcal{Z}}^{\circ}) \end{aligned}$$

$$\begin{aligned}
 &= \varprojlim_{\tilde{K}} (V^{\tilde{K}})^* && \text{by Corollary 2.41} \\
 &= V^*,
 \end{aligned}$$

and we conclude by taking smooth vectors. □

Grothendieck–Serre duality relative to the center behaves well under equivalences of categories. We record the following for completeness as an analogue of Proposition 10.9 and Proposition 10.10.

Proposition 12.3. *Let $\mathfrak{s} = [\tilde{L}, \rho] \in \mathcal{B}(\tilde{G})$ be a cuspidal datum. Then, under the equivalence $\mathcal{M}(\tilde{G})_{\mathfrak{s}} \simeq \mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}$, we have the following commutative square of functors*

$$\begin{array}{ccc}
 \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}}) & \xrightarrow{\sim} & \mathcal{D}^b(\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}}) \\
 D_{GS} \downarrow & & \downarrow D_{GS} \\
 \mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}^\vee})^{op} & \xrightarrow{\sim} & \mathcal{D}^b(\mathbf{mod}\text{-}\mathcal{R}_{\mathfrak{s}^\vee})^{op}.
 \end{array}$$

Proof. Under an equivalence of categories, the centers are isomorphic. In this proof we put $Z := \mathcal{Z}_{\mathfrak{s}}$. So the two D_{GS} functors in the diagram are both $\mathrm{RHom}_Z(-, \omega_Z^\circ)$.

Corollary 12.2 ensures that the target of the left vertical arrow is correct. Similarly, use Corollary 2.41 together with Corollary 10.8 for the right vertical arrow.

We have to show that, for $V \in \mathcal{M}(\tilde{G})$, there are natural isomorphisms

$$(12.1) \quad \mathrm{RHom}_Z(\mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, V), \omega_Z^\circ) \simeq \mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}^\vee}, \mathrm{RHom}_Z(V, \omega_Z^\circ)).$$

Notice that $\Pi_{\mathfrak{s}}$ is projective, so we don't need to derive the Hom space from it.

Using Lemma 2.16 and Proposition 10.5 we have $\mathrm{Hom}_{\tilde{G}}(\Pi_{\mathfrak{s}}, V) \simeq \Pi_{\mathfrak{s}^\vee} \otimes_{\mathcal{H}(\tilde{G})} V$. We can rewrite the left hand-side of (12.1) and use tensor-hom adjunction to conclude. □

Let $\rho \in \mathcal{M}(\tilde{G})$ be an irreducible cuspidal representation and put as usual $d = \mathrm{rank}_{\mathbb{Z}}(\tilde{G}/\tilde{G}^\circ)$. It has already been proved that homological duality restricts to a contravariant functor

$$\mathbb{D}_h := [d] \circ D_h : \mathcal{M}(\tilde{G})_{[\rho]}^{\mathrm{fl}} \rightarrow (\mathcal{M}(\tilde{G})_{[\rho^\vee]}^{\mathrm{fl}})^{op},$$

see Corollary 10.6 and Theorem 10.2. We can see that it is not new:

Corollary 12.4. *For $\rho \in \mathcal{M}(\tilde{G})$ an irreducible cuspidal representation of \tilde{G} , and $d = \text{rank}_{\mathbb{Z}}(\tilde{G}/\tilde{G}^\circ)$, the above functor \mathbb{D}_h is isomorphic to the contragredient.*

Proof. The block $\mathcal{M}(\tilde{G})_{[\rho]}$ is equivalent to the category $\mathbf{mod}\text{-}\mathcal{R}_{[\rho]}$ and this equivalence commutes with homological duality (see Proposition 10.10) as well as with contragredient (see Proposition 10.9). Moreover, it sends finite length \tilde{G} -modules to finite length $\mathcal{R}_{[\rho]}$ -modules. In order to prove the claim it is therefore enough to do it for $\mathbf{mod}\text{-}\mathcal{R}_{[\rho]}$.

The center of $\mathcal{R}_{[\rho]}$ is a Laurent polynomial algebra of dimension d (see Proposition 9.4) and moreover $\mathcal{R}_{[\rho]}$ satisfies the condition (2.8) (a particular case of (FsG)) by Corollary 9.6. Applying Corollary 2.48 we conclude. \square

The same argument gives the following corollary.

Corollary 12.5. *Let $\rho \in \mathcal{M}(\tilde{G})$ be an irreducible cuspidal representation and $d = \text{rank}_{\mathbb{Z}}(\tilde{G}/\tilde{G}^\circ)$. Then on the cuspidal block $\mathcal{D}_{fg}^b(\mathcal{M}(\tilde{G})_{[\rho]})$, we have a natural isomorphism of functors*

$$[d] \circ D_h \simeq D_{GS/\mathbb{Z}}.$$

Remark 12.6. If G is a torus then all smooth representations of a finite central extension \tilde{G} are cuspidal and therefore the above corollaries apply to *all* finite length smooth \tilde{G} -modules. Moreover, in this situation the Grothendieck–Serre duality and homological duality agree (up to shift) on all smooth modules.

More generally, the following gives a necessary and sufficient condition for homological duality to be isomorphic to Grothendieck–Serre duality:

Corollary 12.7. *On a block $\mathcal{D}^b(\mathcal{M}(\tilde{G})_{\mathfrak{s}})$, we have that $[d(\mathfrak{s})] \circ D_h \simeq D_{GS/\mathbb{Z}}$ if and only if $\mathcal{R}_{\mathfrak{s}}$ satisfies condition (FsG).*

Proof. Given Proposition 12.3, this is simply a restatement of Corollary 2.47. \square

Example 12.8. Let $\mathfrak{s} = [\tilde{L}, \rho]$ be a cuspidal datum such that the stabilizer of the inertia class $[\rho]$ in the relative Weyl group is trivial, i.e., there is no non-trivial $w \in W(\tilde{L}, \tilde{G})$ such that ${}^w\rho \simeq \rho\chi$ for some unramified character χ of L . Then by a result of Roche [Roc02, Theorem 3.1], parabolic induction induces an equivalence of blocks $\mathcal{M}(L)_{[\rho]} \rightarrow \mathcal{M}(G)_{\mathfrak{s}}$ and therefore a Morita equivalence of $\mathcal{R}_{[\rho]}$ and $\mathcal{R}_{\mathfrak{s}}$. As $\mathcal{R}_{[\rho]}$ is Azumaya by Proposition 9.4, the same is true of $\mathcal{R}_{\mathfrak{s}}$. By the general property of Azumaya algebras recalled in Lemma 9.5, we find that the algebra $\mathcal{R}_{\mathfrak{s}}$ satisfies condition (FsG).

Example 12.9. By Corollary 12.2, the Grothendieck–Serre duality on irreducible representations of \tilde{G} is simply the contragredient. On the other hand, as it is well known, the homological duality (same as Aubert–Zelevinsky duality for irreducible representations) interchanges the trivial representation and the Steinberg representation of $\mathrm{SL}_2(F)$. Therefore the Corollary above is not true for all representations of $\mathrm{SL}_2(F)$ which means that the condition (FsG) does not hold for the Iwahori block for $\mathrm{SL}_2(F)$.

Let us also note the particular case of a free abelian group $\Gamma \simeq \mathbb{Z}^d$. (The abelian category of representations of Γ is the Iwahori block of a split torus over any p -adic field, so in considering this example, we are not going outside the realm of p -adic groups!) The category of representations of Γ is the same as the category of representations of the group algebra $\mathbb{C}[\Gamma] \simeq \mathbb{C}[X_1^\pm, \dots, X_d^\pm]$. From Corollary 2.48 we get:

Corollary 12.10. *The Grothendieck–Serre duality, (shifted by d) homological duality and contragredient duality all agree on finite length Γ -modules:*

$$D_{GS} = [d] \circ D_h = (-)^* \text{ as functors on } \mathcal{M}(\Gamma)^{\mathrm{fl}}.$$

Equivalently, for a finite length module M over $A = \mathbb{C}[X_1^\pm, \dots, X_d^\pm]$,

$$\mathrm{Ext}_A^d(M, A) \cong \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}),$$

an isomorphism of A -modules, an assertion which can also be made for a finite length module M over $A = \mathbb{C}[X_1, \dots, X_d]$.

Acknowledgments

D.F. would like to thank IIT Mumbai, where this work has started, for their hospitality. The second author thanks SERB, India for its support through the JC Bose Fellowship, JBR/2020/000006. His work was also supported by a grant of the Government of the Russian Federation for the state support of scientific research carried out under the agreement 14.W03.31.0030 dated 15.02.2018. We would like to thank Roman Bezrukavnikov for some very useful remarks concerning homological duality and Grothendieck–Serre duality. The authors are grateful to the referees for a careful reading and helpful remarks.

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Dragoş Frăţilă
Université de Strasbourg, IRMA
7 rue René Descartes
67000 Strasbourg
France
E-mail: fratila@math.unistra.fr
url: <https://irma.math.unistra.fr/~fratila/>

Dipendra Prasad

Indian Institute of Technology Bombay

Mumbai

India

E-mail: prasad.dipendra@gmail.com

url: <https://sites.google.com/view/dipendra-prasad/home>