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The road to GGP

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The initials GGP stand for a collection of conjectures of Gan, Gross, and Prasad, which bridge representation theory and number theory. In this note, I will recall how Gan, Prasad, and I came to formulate these conjectures. For more details on the conjectures themselves see [17], as well as the surveys [16] and [31].

I want to thank my collaborators, Wee Teck Gan and Dipendra Prasad, for their help. In particular, Wee Teck helped me bring the subject up to date by sending me a draft of the Epilogue, and Dipendra allowed me to publish some of our correspondence. I also want to thank Michael Harris, Jean-Loup Waldspurger, and Wei Zhang for their comments.

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1. Durham

In July of 1983, I attended a conference on Modular Forms of One and Several Variables, which was held at the University of Durham. Don Zagier was one of the original invited speakers. Zagier and I had finished the proof of our limit formula [35], relating the heights of Heegner divisors on modular curves to the first derivatives of Rankin L-series, at $s = 1$ in the fall of 1982. Don wrote to the organizers – Bryan Birch and Robert Rankin – to ask if I could give one of his talks. I had been corresponding with Birch about this work [9], and he arranged for both of us to speak on the first day of the meeting.

Don spoke for an hour, sketching the analytic computations necessary to obtain a formula for the derivative, and I followed with an hour lecture summarizing the calculation of local heights. After my talk, we were summoned to the blackboard by an impromptu committee consisting of the two organizers and Jean-Pierre Serre. Would we each be willing to lecture every day of the meeting, giving all the details of the argument? Suffice it to say that it was an exhausting week, giving lectures by day and preparing lecture notes and a paper for Comptes Rendus by night [36].

At the end of the conference, there was an afternoon set aside for short talks. I planned to skip these lectures, to rest up for the trip home. But Marie-France Vignéras persuaded me to attend the talk of one of her recent doctoral students.

That student was Jean-Loup Waldspurger.

2. Local epsilon factors and quaternion algebras

Waldspurger summarized the local and global results that would appear in his great paper [64]. What was immediately apparent was that he was studying the same Rankin L-functions that Zagier and I considered, at the same central critical point. (We had normalized the L-function so that the central point for the functional equation is $s = 1$, but in this paper I will follow his normalization, where the central point is $s = \frac{1}{2}$.) Waldspurger was studying the special value at $s = \frac{1}{2}$ when the sign in the functional equation is $+1$, so the order of vanishing is even. We were studying the first derivative when the sign in the functional equation is -1, so the order of vanishing is odd.

What was completely new to me was the **interpretation** Waldspurger had for the central value, using ideas from representation theory and automorphic forms. We considered the Rankin L-function of the tensor product $f \otimes g$ of two holomorphic modular forms: f was a newform of weight 2 for the group $\Gamma_0(N)$ and g was a newform of weight 1 for the group $\Gamma_1(D)$, induced from an unramified ring class character of an imaginary quadratic field of discriminant D. To construct Heegner points on the curve $X_0(N)$, we needed the additional hypothesis that all primes dividing N are split in the quadratic field $K = \mathbb{Q}(\sqrt{D})$ [24]. Waldspurger worked in much greater generality, replacing the holomorphic form f by an irreducible automorphic cuspidal representation π of the group PGL_2 over an arbitrary number field k, and the ring class character by an irreducible automorphic representation χ of the maximal torus T in PGL_2 associated to an étale quadratic extension K of k, with rational points $T(k) = K^*/k^*$. Let $\pi(\chi)$ be the associated automorphic representation of $\mathrm{GL}_2(k)$; this has central character α, the quadratic character of K/k. Waldspurger's study of the special value $L(\pi \otimes \pi(\chi), \frac{1}{2})$ involved a study of the restriction of the representation π to the torus T.

He first considered this restriction problem in the local case. Let k_v be a local field and let K_v an étale quadratic extension of k_v, which corresponds to the character $\alpha_v : k_v^* \to \{\pm 1\}$ by local class field theory. Let π_v be an infinite dimensional, irreducible representation of the group $\mathrm{PGL}_2(k_v)$ and let χ_v be a character of the maximal torus $T(k_v) = K_v^*/k_v^*$. Then reinterpreting some results of Tunnell [62], Waldspurger showed that the complex vector space of $T(k_v)$-invariant continuous linear maps

$$\mathrm{Hom}_{T(k_v)}(\pi_v \otimes \chi_v, \mathbb{C})$$

has dimension zero or one. The dimension is one if and only if

$$\epsilon_v(\pi_v \otimes \pi(\chi_v)).\alpha_v(-1) = +1$$

where ϵ_v is the local epsilon factor associated to the tensor product of the two representations of $\mathrm{GL}_2(k_v)$ by Jacquet [45]. Moreover, when

$$\epsilon_v(\pi_v \otimes \pi(\chi_v)).\alpha_v(-1) = -1$$

the representation π_v is in the discrete series for PGL_2 and the quadratic algebra K_v is a field.

When π_v is in the discrete series, there is a quaternion division algebra D_v over k_v, which is unique up to isomorphism. The compact group D_v^*/k_v^* is an inner form of $\mathrm{PGL}_2(k_v)$, and Jacquet and Langlands [46] showed that there was a finite dimensional irreducible representation π_v^* of this inner form which corresponds to the infinite dimensional representation π_v in the following sense. Let s be a regular semi-simple conjugacy class in the group D_v^*/k_v^*. Then s lifts to a regular semi-simple conjugacy class in $\mathrm{PGL}_2(k_v)$, which has a well-defined trace on the representation π_v. The representations correspond in the sense that $\mathrm{Tr}(s|\pi_v) + \mathrm{Tr}(s|\pi_v^*) = 0$.

When K_v is a field, the torus $T(k_v) = K_v^*/k_v^*$ embeds as a maximal torus in the compact group D_v^*/k_v^*. In this case, one can consider the complex vector space

$$\mathrm{Hom}_{T(k_v)}(\pi_v^* \otimes \chi_v, \mathbb{C}).$$

Waldspurger showed that its dimension is either zero or one, and is one if and only if

$$\epsilon_v(\pi_v \otimes \pi(\chi_v)).\alpha_v(-1) = -1.$$

Hence the sign of the expression made from the local epsilon factor determines whether the character χ_v^{-1} of $T(k_v)$ occurs in the representation π_v or π_v^*. By Jacquet's work [45], $\epsilon_v(\pi_v \otimes \pi(\chi_v))$ is equal to the epsilon factor of the four dimensional symplectic representation $M \otimes N$, where M is the two dimensional symplectic representation of the Weil-Deligne group of k_v which is the Langlands parameter of π_v and N is the two dimensional orthogonal representation which is the Langlands parameter of the representation $\pi(\chi_v)$, with $\det N = \alpha_v$.

The fact that this question in representation theory – whether the character χ_v^{-1} of the torus occurs in the restriction of π_v or π_v^* – was settled by the symplectic epsilon factor $\epsilon_v(M \otimes N)$ was of great interest to me. Deligne (cf. [61]) had generalized the results in Tate's thesis to define local root numbers for higher dimensional representations W of the Weil-Deligne group, and proved that they satisfied $\epsilon(W)\epsilon(W^\vee) = \det W(-1)$. Hence when the representation W is self-dual and has trivial determinant, $\epsilon(W) = \pm 1$. In the orthogonal case, Deligne had given an interpretation of the sign $\epsilon(W)$ in terms of the possible lifting of the representation from $\mathrm{SO}(W)$ to $\mathrm{Spin}(W)$ [14]. But in the symplectic case, the sign $\epsilon(W)$ remained mysterious. Here was an interesting interpretation, when $W = M \otimes N$ was four dimensional, and was the tensor product of a symplectic representation M and an orthogonal representation N, both of dimension 2.

Waldspurger then considered the global L-function $L(\pi \otimes \pi(\chi), s)$ in the automorphic case. Let S be the finite set of places v of k where the local factor $\epsilon_v(\pi_v \otimes \pi(\chi_v)).\alpha_v(-1)$ is equal to -1. Note that the set S contains no complex places. Since $\prod_v \alpha_v(-1) = +1$ by global class field theory, the sign in the functional equation of the global L-function is equal to

$$\epsilon(\pi \otimes \pi(\chi)) = \prod_v \epsilon_v(\pi_v \otimes \pi(\chi_v)) = \prod_v \epsilon_v(\pi_v \otimes \pi(\chi_v)).\alpha_v(-1) = (-1)^{\#S}.$$

This sign determines the parity of the order of vanishing at the central point $s = \frac{1}{2}$, so the only case when $L(\pi \otimes \pi(\chi), \frac{1}{2})$ can be non-zero is when S has even cardinality. In that case, there is a quaternion algebra D over k

which ramifies precisely at the places of S: the local algebra $D_v = D \otimes k_v$ is a division algebra when v is in S and is a matrix algebra elsewhere. The quaternion algebra D gives an inner form G^* of the group PGL_2 over the number field k, with rational points $G^*(k) = D^*/k^*$. Jacquet and Langlands [46] proved that there is an automorphic representation π^* of the adelic points $G(\mathbb{A})$ with local components π_v at places not in S and local components π_v^* for places v in S. The adelic points $T(\mathbb{A})$ of the torus embed in $G^*(\mathbb{A})$, and by the local theory, the complex vector space

$$\mathrm{Hom}_{T(\mathbb{A})}(\pi^* \otimes \pi(\chi), \mathbb{C})$$

has dimension one. Waldspurger observed that for functions ϕ on $G^*(\mathbb{A})/G^*(k)$ in the automorphic realization of π^*, the integral

$$\int_{T(\mathbb{A})/T(k)} \phi(t)\ \chi(t)\ dt$$

defines an element in this complex vector space. He called this explicit linear form a toric period. The main global result in this paper is that the toric period of χ defines a non-zero $T(\mathbb{A})$-invariant linear form if and only if the special value $L(\pi \otimes \pi(\chi), \frac{1}{2})$ is non-zero [64, Thm 2]. In fact, Waldspurger established an explicit formula, relating the product of the toric period of χ and the toric period of the contragredient representation χ^{-1} to this special value [64, Prop 7].

3. Shimura curves

Reconsidering our limit formula for the first derivative in the light of Waldspurger's results, I realized that we were treating the global L-function in the special case when $k = \mathbb{Q}$, the quadratic extension K is imaginary quadratic, and the finite set of places determined by the sign of the local factors is $S = \{\infty\}$. Indeed, the hypothesis that all primes p dividing the level N of f are split in K, which is necessary to construct Heegner points on the curve $X_0(N)$, forces $\epsilon_p(\pi_p \otimes \pi(\chi_p)).\alpha_p(-1) = +1$ [24], whereas the fact that π_∞ is a discrete series of weight 2 for $\mathrm{PGL}_2(\mathbb{R})$ and $\chi_\infty = 1$ for $\mathbb{C}^*/\mathbb{R}^*$ implies that the local factor $\epsilon_\infty(\pi_\infty \otimes \pi(\chi_\infty)).\alpha_\infty(-1) = -1$. The local representation π_∞^* is just the trivial representation of the compact group $\mathbb{H}^*/\mathbb{R}^* = \mathrm{SO}(3)$, where $\mathbb{H}$ is Hamilton's quaternion algebra, and for ring class characters χ of K, $\chi_\infty = 1$.

Thus, we had treated the simplest case of the first derivative, when $\#S$ is odd and the L-function vanishes to odd order at $s = \frac{1}{2}$. What about the general case? When $\#S$ is odd there is no global quaternion algebra over k which ramifies precisely at S. But when S **contains all infinite places**, there is a reasonable arithmetic object to consider. This simple additional hypothesis has a number of interesting implications. Since S contains no complex places, the number field k must be totally real. Moreover the local representation π_v at each real place is in the discrete series of $\mathrm{PGL}_2(k_v) = \mathrm{PGL}_2(\mathbb{R})$ and the étale quadratic extension K_v of $k_v = \mathbb{R}$ is a field. Hence $K_v = \mathbb{C}$ and K is a CM field. If π_v is the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of weight $2k_v \geq 2$, then the character χ_v of $\mathbb{C}^*/\mathbb{R}^*$ has the form $\chi_v(z) = (z/\overline{z})^{m_v}$, where m_v is an integer whose absolute value is less than k_v. The automorphic representation π corresponds to a holomorphic Hilbert modular form f of weight $(2k_1, 2k_2, \ldots, 2k_d)$, where d is the degree of k over $\mathbb{Q}$, and χ is an algebraic Hecke character of K with infinity type $(m_1, m_2, \ldots, m_d)$. The simplest case is when f has weight $(2, 2, \ldots, 2)$. Then $m_i = 0$ for all i and the character χ has finite order.

The right arithmetic object is a Shimura curve X over k, and its special points over abelian extensions of K. Shimura had defined these curves starting with a quaternion algebra $D(v)$ over k which is split at one real place v and ramified at all others. Let G be the algebraic group over k with rational points $D(v)^*/k^*$, and let M be an open compact subgroup of the finite adelic points $G(\mathbb{A}^f)$. The real group $G(k_v) \cong \mathrm{PGL}_2(\mathbb{R})$ acts on the upper and lower half planes $\mathscr{H}^{\pm}$, and the orbit space

$$X_M(\mathbb{C}) = G(k)\backslash\mathscr{H}^{\pm} \times G(\mathbb{A}^f)/M$$

is a Riemann surface with a finite number of connected components. The Riemann surface $X_M(\mathbb{C})$ is compact unless $k = \mathbb{Q}$ and $D = M_2(\mathbb{Q})$, when it can be compactified with a finite number of cusps.

Shimura proved that the components of $X_M(\mathbb{C})$ descend canonically to algebraic curves defined over abelian extensions of k [59], and Deligne interpreted his results to show that the complex curve X_M descends canonically to an algebraic curve over k, embedded in $\mathbb{C}$ via the place v [13]. If one embeds k into $\mathbb{C}$ by another real place w, the curve X_M is uniformized by arithmetic subgroups of the group $D(w)^*/k^*$, where $D(w)$ is the quaternion algebra ramified at w, split at v, and otherwise locally isomorphic to $D(v)$ [15]. Hence the curve X_M over k does not correspond to a single quaternion algebra, but rather to the **odd set S of places** consisting of all the real places of k and the finite places ramified in either $D(v)$ or $D(w)$. The projective limit $X = \lim_M X_M$ defines a pro-curve over k which has an action of the group $G(\mathbb{A}^f)$ and depends only on S. It is this curve, or rather the Mordell-Weil group of its Jacobian, that replaces the space of automorphic forms in Waldspurger's argument. Indeed, the CM extension K of k gives a collection of special points on the curve X, which are rational over abelian extensions of K

which are "dihedral" over k. Using the character χ, viewed as a character of the Galois group of the maximal abelian extension of K by class field theory, one can construct zero cycles on X. These cycles have non-trivial class in the Mordell-Weil group precisely when the first derivative of the Rankin L-function is non-zero at the central point $s = \frac{1}{2}$, and the height pairing against these special cycles is the analog of Waldspurger's automorphic period. I proposed this generalization of our limit formula in [26]; the full result was proved by Shou-Wu Zhang and his students Xinyi Yuan and Wei Zhang [73].

4. Triple products

I arrived at Harvard in the fall of 1985 and gave a graduate course on quaternion algebras and their quadratic subfields, covering much of the above material. One of my students, Dipendra Prasad, had just arrived from India with a strong background in representation theory. He began to study an analog of Tunnell's and Waldspurger's local results, relating the restriction of irreducible representations to local epsilon factors. The case Prasad focused on was triple products of irreducible representations of PGL_2.

Let k_v be a local field and let π_1, π_2, and π_3 be three infinite dimensional irreducible representations of the group $\mathrm{PGL}_2(k_v)$. Then the tensor product $\pi_1 \otimes \pi_2 \otimes \pi_3$ is an irreducible representation of the group $\mathrm{PGL}_2(k_v)^3$. Restricting this representation to the subgroup $\mathrm{PGL}_2(k_v)$ diagonally embedded in the triple product, Prasad showed that the complex vector space

$$\mathrm{Hom}_{\mathrm{PGL}_2(k_v)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$$

has dimension zero or one. Let M_i be the two dimensional representation of the Weil-Deligne group which is the Langlands parameter of π_i. Then $M_1 \otimes M_2 \otimes M_3$ is an eight dimensional symplectic representation, so has local epsilon factor equal to $+1$ or -1. Prasad showed that the dimension of the vector space of invariant trilinear forms is equal to one if and only if

$$\epsilon(M_1 \otimes M_2 \otimes M_3) = +1.$$

If the local epsilon factor of the triple product is -1, then Prasad showed that all three representations π_i lie in the discrete series for $\mathrm{PGL}_2(k_v)$. Hence they correspond to finite dimensional irreducible representations π_i^* of the compact group D_v^*/k_v^*, where D_v is the quaternion division algebra over k_v. For three irreducible representations of this compact group, Prasad showed that the complex vector space

$$\mathrm{Hom}_{D_v^*/k_v^*}(\pi_1^* \otimes \pi_2^* \otimes \pi_3^*, \mathbb{C})$$

also has dimension zero or one. It has dimension one if and only if

$$\epsilon(M_1 \otimes M_2 \otimes M_3) = -1.$$

Hence the epsilon factor of the symplectic representation $M_1 \otimes M_2 \otimes M_3$ determines whether there is a trilinear form invariant under PGL_2 or its compact inner form [55, Thm. 1.4].

Michael Harris and Steve Kudla obtained a result analogous to Waldspurger's theorem on toric periods in this case [38], which had been conjectured by Jacquet. Let k be a number field and let π_1, π_2 and π_3 be three cuspidal automorphic representations of the adelic group $\mathrm{PGL}_2(\mathbb{A})$. The triple product L-function $L(\pi_1 \otimes \pi_2 \otimes \pi_3, s)$ has an analytic continuation and satisfies a functional equation, via an integral representation found by Paul Garrett [21]. Let S denote the finite set of places v of k where the local epsilon factor of this triple product is equal to -1, so the sign in this functional equation is equal to $(-1)^{\#S}$. When $\#S$ is even, so the order of vanishing at the central critical point $s = \frac{1}{2}$ is even, Jacquet conjectured that the central critical value detected the non-vanishing of certain period integrals. Specifically, let D be the quaternion algebra over k which is ramified at the places in S, and let π_1^*, π_2^*, and π_3^* be the automorphic representations of the inner form G of PGL_2 with rational points D^*/k^*, corresponding to π_1, π_2 and π_3 by the results of Jacquet and Langlands. By Prasad's local results, the vector space

$$\mathrm{Hom}_{G(\mathbb{A})}(\pi_1^* \otimes \pi_2^* \otimes \pi_3^*, \mathbb{C})$$

has dimension one. Harris and Kudla show that the invariant linear form mapping automorphic functions on $(G(\mathbb{A})/G(k))^3$ which lie in these representations to the integral

$$\int_{G(\mathbb{A})/G(k)} \phi_1(g)\phi_2(g)\phi_3(g)dg$$

is non-zero if and only if the central value $L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})$ is non-zero. Some years later, Atsushi Ichino gave an explicit formula for the special value, in terms of these period integrals [43].

The case when $\#S$ is odd is more difficult. Under the additional assumption that S contains all infinite places of k, Kudla and I conjectured [28] that the first derivative $L'(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})$ was related to the height pairing of the modified diagonal cycle [32] on the triple product of the Shimura curve determined by S. Shouwu Zhang has investigated this conjecture, and shown that it has surprising applications to the construction of k-rational points on elliptic curves, when $\pi_1 = \pi_2$ [77, §5.3].

5. A letter from India

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Dear Professor Dick Gross,

19/10/90

I am writing now to confirm the conjecture that I was making some time back about the decomposition of a representation of $Sp(2)$ restricted to $SL_2 \times SL_2$. The precise statements are as follows.

Let D be a discrete series representation of $Sp(2,\mathbb{R})$ say with trivial central character (i.e. trivial on $\pm 1 \in Sp(2,\mathbb{R})$). There are 4-representations in the L-packet of D. Let V be the sum of these representations. The decomposition of V restricted to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ follows from Harris-Kudla in the case of non-holomorphic discrete series reps., and from general methods for holomorphic and anti-holomorphic discrete series representations (see e.g. S. Martin's thesis, a student of W. Schmid, published in PNAS, -73). Let V' be the representation of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ such that a discrete series rep. $D_n \otimes D_m$ appears in V' iff $D_n \otimes D_{-m}$ appears in V. It follows from the decomposition of V that V and V' do not have any rep. in common, and that $W_1 = V \oplus V'$ can be extended to a representation of $\frac{SL_2(\mathbb{R})^{\pm} \times SL_2(\mathbb{R})^{\pm}}{(-1,-1)} \cong O(2,2)$ (where $SL_2(\mathbb{R})^{\pm} = \{g \in GL_2(\mathbb{R}) \mid \det g = \pm 1\}$)

Now $\underset{\pm}{Sp(2,\mathbb{R})} = SO(3,3)$ has $SO(4,1)$ and $SO(5)$ as its inner forms, and the L-packet of D also "lives" on these inner forms. The corresponding L-packet on $SO(4,1)$ has 2 elt and has one element for $SO(5)$. Restricting the sum of these 2 rep. of $SO(4,1)$ and the rep. of $SO(5)$ to $SO(4)$, we get a rep. W_2 of $SO(4)$. One gets the decomposition of a rep. of $SO(4,1)$ restricted to $SO(4)$ from the paper of Dixmier, Bulletin Society de France, -63,

This is where matters stood, when in the fall of 1990 I received a short letter from Prasad. This letter was hand-written, and took three weeks to travel from India to Boston. (I have included the first page in Figure 1, to give some idea of how mathematicians communicated thirty years ago.) It concerned some recent results of Harris and Kudla [39] on the restriction of discrete series representations from $\mathrm{Sp}_4(\mathbb{R})$ to the subgroup $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ fixing a decomposition of the symplectic spaces into a direct sum of two hyperbolic planes. The restriction of holomorphic and anti-holomorphic discrete series was already known. Harris and Kudla used the theta correspondence to completely work out which discrete series for $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ occurred as quotients in the restriction of a generic discrete series for $\mathrm{Sp}_4(\mathbb{R})$.

Prasad reformulated these results, for discrete series with trivial central character, as a restriction problem for orthogonal groups. He observed that the quotient $\mathrm{Sp}_4(\mathbb{R})/\langle \pm 1 \rangle$ maps to a subgroup of index two in $\mathrm{SO}(3,2)$ via the second exterior power representation, and that the quotient of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ by the diagonally embedded $\langle \pm 1 \rangle$ maps to a subgroup of index two in $\mathrm{SO}(2,2)$ via the tensor product of the two standard representations. The latter orthogonal group is the subgroup of $\mathrm{SO}(3,2)$ fixing a vector with positive inner product. There are two discrete series representations of $\mathrm{SO}(3,2)$ with infinitesimal character $\alpha_1 > \alpha_2 > 0$ where the α lie in $\frac{1}{2}\mathbb{Z} - \mathbb{Z}$, and two discrete series of $\mathrm{SO}(2,2)$ with infinitesimal character $\beta_1 > |\beta_2|$, where the β lie in $\mathbb{Z}$. The holomorphic discrete series of $\mathrm{SO}(3,2)$ with infinitesimal character $\alpha_1 > \alpha_2 > 0$ restricts to a Hilbert direct sum of holomorphic discrete series for $\mathrm{SO}(2,2)$ with infinitesimal characters satisfying the inequalities $\beta_1 > \alpha_1 > \alpha_2 > |\beta_2|$. The generic discrete series does not have a discrete decomposition when restricted to $\mathrm{SO}(2,2)$, but this restriction has discrete series quotients whose infinitesimal characters satisfy the inequalities $\beta_1 > |\beta_2| > \alpha_1 > \alpha_2 > 0$ and $\alpha_1 > \alpha_2 > \beta_1 > |\beta_2|$.

Prasad compared these restrictions with the restrictions from orthogonal groups with the same rank and discriminant, which were not split. There is a unique discrete series representation of the group $\mathrm{SO}(1,4)$ with infinitesimal character $\alpha_1 > \alpha_2 > 0$, and this restricts to the Hilbert direct sum of finite dimensional representations of the compact subgroup $\mathrm{SO}(0,4)$ whose infinitesimal characters satisfy $\beta_1 > \alpha_1 > |\beta_2| > \alpha_2 > 0$. Finally, the finite dimensional representation of the compact group $\mathrm{SO}(5,0)$ with infinitesimal character $\alpha_1 > \alpha_2 > 0$ restricts to a direct sum of the finite dimensional representations of $\mathrm{SO}(4,0)$ whose infinitesimal characters satisfy $\alpha_1 > \beta_1 > \alpha_2 > |\beta_2|$. The latter is the classical branching formula for restriction of representations of compact orthogonal groups.

These restriction results become much clearer if one puts the discrete series for all the different orthogonal groups with a given infinitesimal character together, as David Vogan had suggested [63]. These collections of representations – of orthogonal groups $\mathrm{SO}(V)$, as V ranges through the orthogonal spaces with a fixed dimension and discriminant – are now called a Vogan L-packet. Considering the inequalities above, we can conclude that for each pair of infinitesimal characters $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > |\beta_2|$ there is a unique discrete series representation π of an orthogonal group $G = \mathrm{SO}(V_5)$ in the Vogan L-packet for $\mathrm{SO}(3,2)$ and a unique discrete series representation σ of an orthogonal group $H = \mathrm{SO}(W_4)$ in the Vogan L-packet for $\mathrm{SO}(2,2)$ such that W_4 embeds as a subspace of V_5 and the complex vector space $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C})$ has dimension one. For all other pairs (π^*, σ^*) in the L-packets, either W_4 does not embed in V_5 or the vector space $\mathrm{Hom}_H(\pi^* \otimes \sigma^*, \mathbb{C})$ has dimension zero.

What was even better was that, in the real case, the signs of the local epsilon factors determine the branching of the infinitesimal characters. Hence Prasad viewed these results on restriction of discrete series as a generalization of the results of Tunnell on restriction from $\mathrm{PGL}_2 = \mathrm{SO}(V_3)$ to a maximal torus $T = \mathrm{SO}(W_2)$, and of his own results on invariant trilinear forms for PGL_2, which involves restriction from $\mathrm{PGL}_2 \times \mathrm{PGL}_2 = \mathrm{SO}(V_4)/\langle \pm 1 \rangle$ to $\mathrm{PGL}_2 = \mathrm{SO}(W_3)$. He suggested that there might be a similar result on restriction from $\mathrm{SO}(V)$ to $\mathrm{SO}(W)$, where W was an orthogonal space of codimension one in the orthogonal space V over a local field k. There were already a number of results in the literature suggesting that for any irreducible representation π of $G = \mathrm{SO}(V)$ and any irreducible representation σ of $H = \mathrm{SO}(W)$, the complex vector space $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C})$ has dimension less than or equal to one. We learned about these results from Joseph Bernstein, at the time I wrote a general paper on Gelfand pairs [27]. Eventually, the full result on multiplicities was established by his students Aizenbud and Gourevitch, and by Rallis and Schiffmann in the p-adic case [1]. The real and complex cases were settled by Sun and Zhu [60].

Dipendra's letter suggested that the multiplicity might be equal to one if one summed over a Vogan L-packet for the two groups, and the representations π and σ in the L-packet where $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C})$ has dimension one could somehow be distinguished by local epsilon factors.

6. The conjecture for local orthogonal groups

To formulate a precise conjecture I went over to MIT to speak with David Vogan, to learn more about how he parametrized the different irreducible representations in an L-packet. David had worked out the theory completely for real and complex groups, and knew what to expect for p-adic groups [63]. This was all based on the fundamental conjecture of Langlands, that irreducible complex representations of a reductive group $G(k)$ over a local field k should be parametrized by equivalence classes of homomorphisms from the

Weil group $W(k)$ (or the Weil-Deligne group $WD(k)$ in the non-Archimedean case) to the L-group of G. The L-group is a semi-direct product of the dual group by the Galois group of a finite extension of k, and the equivalence is up to conjugation by the dual group. Such homomorphisms ϕ are called Langlands parameters. There are a number of conditions such a parameter must satisfy; for an introduction to the local Langlands conjecture, see [31].

In the case where $G = \mathrm{SO}(V)$ with $\dim V = 2n+1$, the L-group is the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$. Therefore, a Langlands parameter is a symplectic representation

$$\phi : WD(k) \to \mathrm{Sp}(M)$$

where M has dimension $2n$ over $\mathbb{C}$. In the case where $G = \mathrm{SO}(V)$ with $\dim V = 2n$ and $\operatorname{disc} V = d \in k^*/k^{*2}$, the L-group is either the special orthogonal group $\mathrm{SO}_{2n}(\mathbb{C})$ or the full orthogonal group $O_{2n}(\mathbb{C})$. Therefore, a Langlands parameter gives an orthogonal representation, up to conjugation by the subgroup $\mathrm{SO}(N)$

$$\phi : WD(k) \to O(N)$$

where N has dimension $2n$ over $\mathbb{C}$ and the determinant of ϕ is the quadratic character of the extension $k(\sqrt{d})$. In each case, Langlands proposed that there should be a finite number of irreducible representations of $G(k)$ associated to each parameter.

Since the parameters are the same for all orthogonal spaces of the same rank and discriminant, Vogan realized that it was more convenient to put the irreducible representations (of the different orthogonal groups of these spaces) together in the same L-packet. That gave him a simpler conjectural description of the irreducible representations associated to a single parameter ϕ. Let C_ϕ be the centralizer in the dual group of the image of ϕ in the L-group. This group is well-defined up to conjugacy; let $A_\phi = C_\phi/C_\phi^0$ be its finite component group. Then David conjectured that the individual representations in a Vogan L-packet of ϕ should be parametrized by the irreducible complex representations ρ of A_ϕ. There were many extremely attractive features of this correspondence. For example, the restriction of ρ to the center of the L-group gives information on the specific group acting on the representation [29]. And the idea of putting the representations of the groups of different orthogonal spaces in the same packet fitted our restriction problem perfectly.

In the special case when G is a special orthogonal group, the centralizer C_ϕ of ϕ is a product of orthogonal, symplectic, and general linear groups [29]. Hence the component group A_ϕ is an elementary abelian 2-group and an irreducible representation of A_ϕ is a quadratic character χ. To identify which irreducible representation in the Vogan L packet had a non-trivial invariant linear form, I needed to construct such a character from the Langlands parameters.

The Langlands parameters consists of a symplectic representation M of dimension $2n$ and an orthogonal representation N of dimension $2n$ or $2n+2$, with a fixed determinant. In the cases when $n = 1$ and $n = 2$, Tunnell and Prasad had used the local sign $\epsilon(M \otimes N). \det N(-1)$ to say which group had a non-trivial restriction. That suggested using similar epsilon factors to define the character χ. I was encouraged to do this by Michael Harris, who noted that there are more symplectic epsilon factors to consider when either M or N is reducible. After some experimentation, I came up with the following recipe. Let (a, b) be a pair of elements, where a is in the centralizer of the symplectic representation M and b is in the centralizer in $\mathrm{SO}(N)$ of the orthogonal representation N. Let M^a be the subspace of M where $a = -1$ and let N^b be the subspace of N where $b = -1$. Since a and b centralize the image of the Weil-Deligne group, M^a is a symplectic representation and N^b is an orthogonal representation of even dimension. Define

$$\chi(a,1) = \epsilon(M^a \otimes N) \det N(-1)^{\dim M^a/2}$$
$$\chi(1,b) = \epsilon(M \otimes N^b) \det N^b(-1)^{\dim M/2}.$$

Using some general properties of epsilon factors, I could see that these signs depend only on the image of the pair (a, b) in the component groups $A_M \times A_N$ of the parameters. Even better, the resulting function on the component groups is a quadratic character! That was so remarkable that it was easy to conjecture that this character determines the distinguished representations π and σ. On the centers of the L-groups, we have

$$\chi(-1,1) = \chi(1,-1) = \epsilon(M \otimes N) \det N(-1)^{\dim M/2}.$$

and this sign determines the Hasse-Witt invariant of the two quadratic spaces involved, showing that W is indeed a subspace of V.

I was pleased to find this formula for a character, but knew that it was only going to solve the restriction problem in certain cases. Even for PGL_2, one had to assume that the representation π was infinite dimensional. In the general case, I guessed that the restriction problem would only have a simple answer when the two Vogan L-packets were **generic**. By this we mean that the packet contains a generic representation (one of maximal Gelfand-Kirillov dimension) of a quasi-split group. Vogan explained his recipe for generic L-packets in the real and complex cases, and that led me to a conjecture in the general case – that a Vogan L-packet is generic precisely when the adjoint L-function of the Langlands parameter is regular at the point $s = 1$. In the case of orthogonal groups associated to spaces of dimensions $2n+1$ and $2n$, the adjoint L-function is associated to the orthogonal representation $\mathrm{Sym}^2 M \oplus \wedge^2 N$ of dimension $4n^2$ of the Weil-Deligne group. It is interesting to note that this is the same dimension as the dimension of the symplectic representation $M \otimes N$.

7. The conjecture for global orthogonal groups

Prasad and I also proposed a conjecture in the global case, although we had almost no evidence beyond the analogy with the low dimensional cases which were already known. Assume that k is a number field, with completions k_v and ring of adeles $\mathbb{A} = \prod'_v k_v$, and let V be a split orthogonal space of dimension $2n + 1$ over k, and W a quasi-split orthogonal space of dimension $2n$ which is the orthogonal complement of a non-isotropic vector in V. Let $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ and let H be the subgroup $\mathrm{SO}(W)$ embedded diagonally in G. Finally, let $\pi \otimes \sigma$ be a tempered automorphic representation of the group $G(\mathbb{A})$ which occurs in the space of cusp forms on $G(k)\backslash G(\mathbb{A})$.

Such an automorphic representation was conjectured to have a global Langlands parameter, which is a homomorphism of the global Langlands group to the L-group $\mathrm{Sp}(M) \times O(N)$ and restricts to the local Langlands parameters (M_v, N_v) of $\pi_v \otimes \sigma_v$ for all places v. It should also have a global L-function $L(M \otimes N, s)$ for the tensor product representation, which is defined in a half plane by the infinite product of local L-functions. At the time, there was no method known to obtain the analytic continuation and functional equation of this L-function beyond the two cases of low dimension, so our conjecture about its central critical value was purely formal.

Since the local parameters are tempered, they are generic and determine generic Vogan L-packets. By the local conjecture, there is a unique representation $\pi_v^* \otimes \sigma_v^*$ in each local packet such that the complex vector space $\mathrm{Hom}_{H_v^*}(\pi_v^* \otimes \sigma_v^*, \mathbb{C})$ is one dimensional. This is a representation of the group G_v^*, associated to two local orthogonal spaces $W_v^* \subset V_v^*$ of the same dimension and discriminant as W and V. Almost all of these representations are unramified (by the local theory), so we can make the tensor product representation of the adelic group $G_{\mathbb{A}}^* = \prod'_v G_v^*$. When

$$\epsilon(M \otimes N) = \prod_v \epsilon_v(M_v \otimes N_v) = \prod_v \epsilon_v(M_v \otimes N_v) \det N_v(-1)^{\dim M_v/2} = +1$$

these local orthogonal spaces are the components of global spaces $W^* \subset V^*$. This gives a discrete subgroup $G^*(k)$ of $G^*(\mathbb{A})$, and we can ask if the tensor product representation defined above is automorphic. We conjectured that this would be the case when

$$\epsilon(M^a \otimes N) = \epsilon(M \otimes N^b) = +1$$

for all (a, b) in the global centralizer. If this is the case, the embedding into the space of automorphic forms should be unique up to scaling, by Arthur's multiplicity formula [2].

If these global epsilon factors are all $+1$, we get an invariant linear form on the automorphic representation via integration over the diagonal subgroup. $H^*(\mathbb{A})$. Our global conjecture is that this period is non-zero if and only if the central L-value $L(M \otimes N, \frac{1}{2})$ is non-zero. This conjecture was later refined by Atsushi Ichino and Tamotsu Ikeda [43] [44], who gave a precise conjectural formula for the central L-value in terms of the product of the period and the period of the contragredient representation. This generalized the formula of Waldspurger involving toric periods. In fact, their precise formula involves the quotient of this central L-value by the value of the adjoint L-function at the point $s = 1$:

$$L(M \otimes N, 1/2)/L(\mathrm{Sym}^2 M \oplus \wedge^2 N, 1).$$

8. OSU and UCSD

Prasad and I worked this out, with more examples, in an exchange of letters in the fall of 1990. We started to write this up [29] when I was invited by Steve Rallis to speak at a conference on representation theory in Ohio State in 1991. At the time, Vogan's theory was not well known, so I spent the first half of my talk summarizing his ideas, and presented our conjecture on the restriction of generic packets from SO_n to SO_{n-1} at the end. When I finished, the response of the audience was complete silence. However, Jacquet came up afterwards to offer encouragement. And the next day Rallis showed me how our conjecture on the adjoint L-function at $s = 1$ was compatible with the formula of Casselman-Shalika [10] for unramified principle series, giving the value of the Whittaker functional on a spherical vector.

One point that puzzled me was that the formula for the character χ of the component group $A_M \times A_N$ worked equally well whenever M was a symplectic representation and N was an orthogonal representation of even dimension. There was no need to assume that $\dim N = \dim M$ or $\dim N = \dim M + 2$, as we were doing in restriction from $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$. Was there a more general restriction problem that involved this character of the component group? Again, Rallis pointed us in the right direction, suggesting that we look at Bessel models for orthogonal groups, and this led to the more general conjectures posed in [30].

In 1993, I took a sabbatical year and visited the University of California in San Diego. My plan was to work with Harold Stark, on his conjectures on Artin L-functions at $s = 0$. But when I arrived, Stark told me he had almost no free time – he was department chair and the UC system was in one of its periodic crises. So I decided to learn some more representation theory, and found an excellent teacher on the faculty – Nolan Wallach. I showed Wallach our conjecture for the discrete series of real orthogonal groups, where the symplectic epsilon factors could actually be calculated, and we decided to work on some simple cases, where the restriction was a direct sum of discrete series representations of the smaller orthogonal group [34]. This led us to the study of quaternionic discrete series and their continuations [33]. These special cases all checked out, and so did a number of cases with tamely ramified parameters on p-adic groups [31]. I was becoming more confident of the correctness of the local conjecture but couldn't make any general progress on it.

It was another sabbatical at UCSD, in 2007, that got us started up again. This time it was Dipendra who was visiting, and Wee Teck Gan on the faculty was his host. Although Wee Teck was my PhD student, he had received a lot of guidance from Gordan Savin and had focused up to that point on exceptional groups. He had studied the restriction of Saito-Kurokawa representations of SO(5) to SO(4), but that was motivated by the question of constructing non-tempered Arthur packets on the exceptional group G_2. He began his transition to the world of classical groups when considering the local Langlands conjecture for the group $GSp(4)$, and corresponded frequently with Prasad about it. This may have persuaded Dipendra to take his sabbatical at UCSD.

During this visit, Wee Teck and Dipendra formulated a generalization of our conjectures on the restriction of representations of orthogonal groups to the restriction of representations of other classical groups, like unitary groups. I had discussed possible extensions of our conjectures to unitary groups with Michael Harris in the early 1990s. Together with Kudla and Sweet [40], Harris discovered a beautiful formula for the explicit theta correspondence for p-adic unitary groups using local epsilon factors. This was certainly encouraging, but a precise conjecture had eluded me.

As in the orthogonal case, where the two orthogonal spaces $W \subset V$ play a critical role, Dipendra and Wee Teck formulated the restriction problem for unitary groups using Hermitian spaces. Let K be an étale quadratic extension of the local field k, and let $W \subset V$ be non-degenerate Hermitian spaces over the K of dimensions $n - 1$ and n respectively. Let $G = U(W) \times U(V)$ and let H be the subgroup $U(W)$ embedded diagonally in G. When $K = k \times k$ is the split étale algebra, the group G is isomorphic to $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ and H is isomorphic to GL_{n-1}. This case is much easier, as the L-packets for G contain only one representation $\pi \otimes \sigma$, and for generic parameters a non-zero H invariant linear form on $\pi \otimes \sigma$ is known to exist. In what follows, we will assume that K is a field.

Let $\pi \otimes \sigma$ be an irreducible representation of $G(k) = U(W) \times U(V)$. Then the dimension of $\mathrm{Hom}_H(\pi \otimes \sigma, \mathbb{C})$ is again less than or equal to 1 (cf. [1]). The local conjecture predicts when the dimension is equal to 1 in terms of epsilon factors of the respective Langlands parameters. For representations of the local unitary group U_n, a Langlands parameter is a homomorphism from the Weil-Deligne group of k to a semi-direct product

$$\phi : WD(k) \to \mathrm{GL}_n(\mathbb{C}).\,\mathrm{Gal}(K/k) = \mathrm{GL}(M).\,\mathrm{Gal}(K/k),$$

Here the Galois group of K/k acts on $\mathrm{GL}_n(\mathbb{C})$ by a pinned outer automorphism. A surprising fact is that the restriction of ϕ to the subgroup $WD(K)$, which is a representation M of dimension n, completely determines the parameter. The representation M is not arbitrary: its conjugate is isomorphic to its dual, and the conjugate duality of M is orthogonal if n is odd and symplectic if n is even. For proofs, as well as the definition of the sign of a conjugate duality, see [17, Ch 3].

In our case, the representations π and σ have Langlands parameters M and N of dimensions n and $n-1$, so one is conjugate orthogonal and the other is conjugate symplectic. Hence the tensor product $M \otimes N$ is conjugate symplectic, and we can use its epsilon factors to define a quadratic character χ of the component group, just as in the orthogonal case. The local conjecture is that the dimension of the H-homomorphisms is 1 when one sums over a generic L-packet, and the pair of representations (π, σ) in the L-packet where the dimension is equal to 1 is given by the distinguished character. The global conjecture, involving periods of automorphic representations and the central value of the tensor product L-function, is similar to the orthogonal case. However, a tremendous advantage in the Hermitian case is that the global L-function is just the Rankin L-function over K of the representation $M \otimes N$. In this case, both the analytic continuation and the functional equation of the L-function were known, so the global conjecture could actually be tested.

9. Paris

Beyond the Hermitian case, and its generalization to Bessel models, we realized that we could formulate a conjecture on restriction in four general cases [17]:

- Orthogonal spaces $W \subset V$ of odd codimension,
- Hermitian spaces $W \subset V$ of odd codimension,
- Symplectic spaces $W \subset V$ of even codimension,
- Anti-Hermitian spaces $W \subset V$ of even codimension.

The latter two cases involve the Weil representation. They are reflections of the former two, through the mirror of the theta correspondence. Each case has its own distinguishing features, but the local conjecture always involves the signs of symplectic epsilon factors and the global case the central value of the corresponding tensor product L-function, whose non-vanishing is related to the non-triviality of certain automorphic periods. A precise formula for this central value was given in a refined conjecture, which in the orthogonal case is due to Ichino and Ikeda [44] and in the Hermitian case is due to Neal Harris [41]. Automorphic periods had previously been studied by Ginzburg, Jiang, and Rallis [22, 23], in a slightly different context. Our conjectures encompassed almost all of the restriction problems in the literature, where the multiplicity was less than or equal to one. We had a number of examples in low dimension where the conjectures worked out [18], and there were some encouraging partial results for real groups (cf. [49] [50]), but we had no idea how to proceed in general.

In some orthogonal and Hermitian cases, when the sign in the functional equation is -1, we also made an arithmetic conjecture for the first derivative of the tensor product L-function at the central point. This involved the heights of diagonal cycles on Shimura varieties, generalizing the limit formula I found with Zagier. For the first derivative, we needed an additional assumption. Namely, at all Archimedean places of the number field k, the group acting on the distinguished representations π and σ in the L-packet should be **compact**. This implies that k is totally real and, in the Hermitian case, that K is a CM field. Moreover, the codimension of W in V must be equal to 1. Yifeng Liu [51] formulated a conjecture for the first derivative in the anti-Hermitian case, with the same hypothesis on the Archimedean places, which forces k to be totally real, K to be CM, and $W = V$.

I was visiting Paris to give some talks on this material in July of 2008. After one of my lectures, Waldspurger invited me into his office and reported that he had found a proof of the full p-adic conjecture in the orthogonal case! When he sketched out his miraculous argument I was so astonished that I began to question my command of French. In the proof, he had to assume some standard conjectures on Vogan L-packets, but these would soon be established for classical groups by Jim Arthur [2].

Waldspurger's ideas opened a flood of research in the subject (some of the subsequent developments are described in an epilogue). Since his lecture at Durham had been my entry point into the subject, this seems like a good place to end the story.

10. Epilogue

The fundamental work of Waldspurger [65, 66, 67, 68] (and Moeglin-Waldspurger [54]) is based on an interpretation of the restriction multiplicity in terms of the characters of the representations involved, proved via a local trace formula. It quickly led to a great number of developments, to the extent that the GGP conjecture for Hermitian spaces is now essentially proved. Here are some of the highlights – for more details see the Bourbaki report [6].

- (Local Hermitian case) Adapting Waldspurger's local trace formula techniques to the Hermitian case, Raphaël Beuzart-Plessis initially established the local GGP conjecture for unitary groups over p-adic fields [3, 4] and later settled the archimedean case [5]. His results were for tempered L-packets. They were extended to generic L-packets by Gan-Ichino [20], who also settled the local skew-Hermitian case via the theta correspondence. In the real case, Hongyu He [42] found a compact proof of the GGP conjecture for discrete series L-packets, and Hang Xue treated the general Archimedean case [70, 71].
- (Global Hermitian case) The main tool for attacking the GGP conjecture in the global case is a relative trace formula, which was developed by Jacquet and Rallis [47] around 2010. The strategy involves comparing the relative trace formula for the pair of unitary groups (U_n, U_{n-1}) with the relative trace formula for the pair $(\mathrm{GL}_n, \mathrm{GL}_{n-1})$. As with any application of the trace formula, it is necessary to prove an analog of the fundamental lemma and the smooth transfer of test functions. The fundamental lemma was established by Zhiwei Yun [74] over local function fields using geometric arguments and the result was transported to p-adic fields by Julia Gordon. A major breakthrough was made by Wei Zhang [79], who established the smooth transfer of test functions at non-archimedean places. This allowed him to prove the global GGP conjecture [79] and also Ichino-Ikeda's precise conjectural formula [80] under some simplifying local hypotheses. Some of these local hypotheses were weakened through the work of Beuzart-Plessis and Xue [69].
  There was a general belief that to obtain the full global conjecture, it would be necessary to work out all the intricacies of the Jacquet-Rallis trace formula, rather than using a simple version of it. It was thus a surprise when in 2019, Beuzart-Plessis, Yifeng Liu, Wei Zhang

and Xinwen Zhu [8] were able to remove the last local assumptions in the global GGP conjecture for stable L-packets of unitary groups. In the meantime, Pierre-Henri Chaudouard and his student Michal Zydor [83] tackled the Jacquet-Rallis trace formula [12] in full. In a recent paper [7], Beuzart-Plessis, Chaudouard and Zydor have combined their improved understanding to establish the full global GGP conjecture for unitary groups and its refinement by Ichino-Ikeda and Harris!

With the Hermitian case settled (and the skew-Hermitian case likely to follow from work on the relative trace formula [52] [72]), what remains to be done? Here are three active areas of research.

- (Global Orthogonal case) The original conjectures of [29, 30] remain open in the local archimedean setting and the global setting. A recent paper [53] of Zhilin Luo began the adaptation of Waldspurger's method to the archimedean case. In the global situation, there is no known analog of the Jacquet-Rallis relative trace formula. On the other hand, a different approach to the GGP conjecture was pioneered by David Ginzburg, Dihua Jiang, and Rallis [22, 23]. Jiang and Lei Zhang [48] extended this approach and successfully proved one implication of the global GGP conjecture in the orthogonal and Hermitian case – the nonvanishing of the global period integral implies the nonvanishing of the central L-value. They also proved the converse in some situations.
- (Arithmetic case) In the case where the sign of the global functional equation is -1 one expects a formula for the first derivative in terms of the heights of cycles on Shimura varieties. Wei Zhang has proposed a general program to attack this case. As a first step, he formulated an arithmetic fundamental lemma [78], which is an identity relating the first derivative of an orbital integral to certain local height pairings. There has been some recent progress in this direction [56, 57], and a proof in the Hermitian case [82], some of which is surveyed in [81].
  Following up on some of these ideas, Zhiwei Yun and Wei Zhang revisited the original case of torus periods on PGL_2 over function fields. They established some astonishing results [75, 76] giving an interpretation of the coefficients in the **entire Taylor expansion** of the Rankin L-function at the central critical point, in terms of intersection numbers of special cycles on the moduli spaces of Drinfeld's shtukas. This generalizes both Waldspurger's formula for the special value and my formula with Zagier for the first derivative! It was totally unexpected and opens an exciting approach to the conjecture of Birch and Swinnerton-Dyer for elliptic curves over function fields.

- (Nontempered case) After the completion of our papers [17, 18], Gan and Prasad spent a semester at MSRI in the fall of 2014 working out an extension of our restriction conjectures to certain nontempered representations arising from Arthur parameters [19]. A local Arthur parameter is a homomorphism from the product $WD \times \mathrm{SL}_2(\mathbb{C})$ to the complex L-group, which has bounded image when restricted to the Weil-Deligne group WD. When the image of $\mathrm{SL}_2(\mathbb{C})$ is trivial, these are the tempered Langlands parameters. When the image of SL_2 is non-trivial, there is a standard method to convert the Arthur parameter into a (non-generic) Langlands parameter, and this determines an L-packet of non-tempered irreducible representations.

  Now consider Arthur parameters M for SO_{2n+1} and N for SO_{2n}. Decomposing for the action of $\mathrm{SL}_2(\mathbb{C})$ we may write $M = \sum_n \mathrm{Sym}^n(\mathbb{C}^2) \otimes M_n$ and $N = \sum_n \mathrm{Sym}^n(\mathbb{C}^2) \otimes N_n$ where M_n is a symplectic representation of the Weil-Deligne group when n is even and is an orthogonal representation when n is odd. Similarly, N_n is orthogonal when n is even and symplectic when n is odd. We say the pair (M, N) of Arthur parameters is **relevant** if these representations are "within distance one" of each other. More precisely, a pair is relevant when we have decompositions $M_n = M_n^+ + M_n^-$ and $N_n^+ + N_n^-$ such that for $n \geq 0$ we have $M_n^+ = N_{n+1}^-$ and for $n \geq 1$ we have $M_n^- = N_{n-1}^+$. In this case, we predict that the restriction problem, for representations in the associated local L-packets, behaves exactly as in the tempered case. When the Arthur parameters are not relevant we predict that the sum of multiplicities over the local L-packets is zero. A similar recipe works for all pairs of classical groups. The fact that relevance is necessary for the pair $(\mathrm{GL}_n, \mathrm{GL}_{n-1})$ was proved by M. Gurevich [37], and the full conjecture in this case was proved for p-adic groups by K.Y Chan [11].

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