Pure and Applied Mathematics Quarterly
Volume 18, Number 6, 2339–2353, 2022

Eigenvalue sum estimates for lattice subgraphs

Frank Bauer and Gabor Lippner

Abstract: We prove sharp upper and lower bounds, up to an error term, on the sum of the first k Dirichlet eigenvalues of induced subgraphs of the d-dimensional lattice.

1. Introduction

Let Ω be a bounded open subset in $\mathbb{R}^n$ and consider the Dirichlet eigenvalue problem

$$\Delta\phi + \lambda\phi = 0 \text{ in } \Omega$$

and

$$\phi|_{\partial\Omega} = 0.$$

We order the eigenvalues of the Dirichlet Laplacian $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \ldots \leq \lambda_k(\Omega) \leq \ldots$ monotonically. Already in 1912, Weyl [12] obtained an asymptotic formula for the Dirichlet eigenvalues. Weyl's asymptotic formula states that as $k \to \infty$

$$\lambda_k(\Omega) \sim C_n \left(\frac{k}{V(\Omega)}\right)^{2/n},$$

where $V(\Omega)$ is the volume of Ω and $C_n = (2\pi)^2 V_n^{-2/n}$ is the Weyl constant with V_n being the volume of the unit ball in $\mathbb{R}^n$.

In 1961, Pólya [10] proved that, for plane domains Ω that tile $\mathbb{R}^2$,

$$\lambda_k(\Omega) \geq C_n \left(\frac{k}{V(\Omega)}\right)^{2/n} \tag{1}$$

for all $k \geq 1$. Pólya's proof also works in $\mathbb{R}^n$ and he conjectured that (1) is true for all domains in $\mathbb{R}^n$. A first important step towards the Pólya conjecture was made by Lieb [8] who proved that

$$\lambda_k(\Omega) \geq D_n \left(\frac{k}{V(\Omega)}\right)^{2/n},$$

Received June 16, 2021.

where $D_n < C_n$ is a constant that is proportional to C_n. In 1983, Li and Yau [7] obtained, for general domains Ω, the sharp estimate for the average of the first k Dirichlet eigenvalues

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega) \geq \frac{n}{n+2}C_n\left(\frac{k}{V(\Omega)}\right)^{2/n}$$

which implies

$$\lambda_k(\Omega) \geq \frac{n}{n+2}C_n\left(\frac{k}{V(\Omega)}\right)^{2/n}.$$

In [6] it was shown that that this result can be obtained as a corollary of an earlier result by Berezin [1] using the Legendre transformation. Later on, Li and Yau's estimate on the sum of the eigenvalues was improved by Melas [9] whose result was further improved in dimension two in [3]. However, up to today, Pólya's conjecture is still open.

Along another line of research, Li and Yau's estimate was complemented by Kröger [5] who proved an upper bound of the sum of the first k Dirichlet eigenvalues, which also involved an additional term that depends on the geometry of Ω. Moreover, in the same spirit, Kröger [4, 5] obtained upper and lower bounds for the sum of the first k Neumann eigenvalues. Finally, Strichartz [11] developed a general framework that allows to obtain upper and lower bounds for the sum of the first k Dirichlet and Neumann eigenvalues in a clear way.

1.1. Results

In this paper, we find appropriate discrete analogues of these estimates. Let $\mathbb{Z}^d$ be the infinite d-dimensional integer lattice, viewed as a graph. Let $\Omega \subset \mathbb{Z}^d$ be a finite induced subgraph. We prove upper bounds in the spirit of Kröger [5] and lower bounds in the spirit of Berezin [1] and Li-Yau [7] for the sum of the first k Dirichlet eigenvalues of Ω.

That is, we consider the Dirichlet eigenvalue problem

$$\Delta_\Omega^{\mathcal{D}}\phi = -\lambda\phi$$

where $\Delta_\Omega^{\mathcal{D}}$ is the Laplacian with Dirichlet boundary conditions. (For the precise definition see Section 2.2.) There are $|\Omega|$ eigenvalues (with multiplicities) of the Dirichlet problem, which are all real and positive. Let us denote them by

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|\Omega|}.$$

Our bounds are summarized in the following theorem.

Theorem 1.1. *For* $1 \leq k \leq |\Omega| \min(1, V_d)$

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j \leq \frac{4\pi^2 d}{d+2}\left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} + \frac{|\partial\Omega|}{|\Omega|}$$

while for $1 \leq k \leq |\Omega| \min(1, V_d(\sqrt{6}/(2\pi))^d)$

$$\frac{4\pi^2 d}{d+2}\left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} - \frac{\pi^4 d}{3(d+4)}\left(\frac{k}{|\Omega|V_d}\right)^{\frac{4}{d}} \leq \frac{1}{k}\sum_{j=1}^{k} \lambda_j$$

As a consequence of the methods we also obtain a non-trivial upper bound on individual eigenvalues.

2. Preliminaries

2.1. Notation

We will use the standard notation for graphs. If G is a graph, $V(G)$ and $E(G)$ will denote the vertex and edge sets of G, though when it causes no confusion we will write $x \in G$ instead of $x \in V(G)$. For a finite graph G we denote by

$$C(G) = \{f : V(G) \to \mathbb{C}\}$$

the finite dimensional complex vector space of functions on $V(G)$, and equip it with the natural Hermitian scalar product

$$\langle f, g\rangle_G = \sum_{x\in V(G)} f(x)\overline{g(x)} : f, g \in C(G).$$

For any edge $(xy) \in E(G)$ we define the following gradient function:

$$\nabla_{xy} f = f(y) - f(x) : f \in C(G),$$

and then the Laplace operator $\Delta_G : C(G) \to C(G)$ acts via

$$\Delta_G f(x) = \sum_{y\sim x} \nabla_{xy} f = \sum_{y\sim x} f(y) - f(x).$$

It is standard that this operator is self-adjoint with respect to the above scalar product.

We consider $\mathbb{Z}^d$ as a group with the standard symmetric generating set $S = \{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$. We also think of $\mathbb{Z}^d$ as the Cayley graph obtained by this generating set. By a slight abuse of the gradient notation we are also going to consider, for any $s \in S$, the operator $\nabla_s : C(\mathbb{Z}^d) \to C(\mathbb{Z}^d)$ defined for a function $f \in C(\mathbb{Z}^d)$ as

$$\nabla_s f(x) = f(x+s) - f(x) : x \in \mathbb{Z}^d.$$

2.2. Dirichlet-Laplacian, Green's formula

Let Ω be a finite induced subgraph of $\mathbb{Z}^d$. We will now recall the notions of the closure $\overline{\Omega}$, the boundary $\partial\Omega$, and the edge boundary $\partial_e\Omega$ of the subgraph Ω:

$$\begin{aligned}\overline{\Omega} &= \Omega \cup \{x \in \mathbb{Z}^d : (\exists y \in \Omega : x \sim y)\} \\ \partial\Omega &= \overline{\Omega} \setminus \Omega \\ \partial_e\Omega &= \{(xy) \in E(\mathbb{Z}^d) : x \in \Omega, y \in \partial\Omega\}\end{aligned}$$

There is an embedding $C(\Omega) \to C(\overline{\Omega})$ obtained by extending functions to be 0 on $\partial\Omega$. We denote this by $f \mapsto \bar{f}$.

The Dirichlet-Laplace operator acts on $C(\Omega)$ in the following way.

Definition 2.1. For $f \in C(\Omega)$ let

$$\Delta_\Omega^{\mathcal{D}} f = (\Delta_{\overline{\Omega}} \bar{f})|_\Omega$$

Proposition 2.2. $\Delta_\Omega^{\mathcal{D}}$ *is self-adjoint with respect to the Hermitian inner product* $\langle \cdot, \cdot \rangle_\Omega$*.*

Proof. Let $f, g \in C(\Omega)$. The inner product on Ω extends naturally to $\overline{\Omega}$. Then

$$\begin{aligned}\langle f, \Delta_\Omega^{\mathcal{D}} g\rangle_\Omega = \langle f, (\Delta_{\overline{\Omega}} \bar{g})|_\Omega\rangle_\Omega &\overset{(a)}{=} \langle \bar{f}, \Delta_{\overline{\Omega}} \bar{g}\rangle_{\overline{\Omega}} \overset{(b)}{=} \\ &= \langle \Delta_{\overline{\Omega}} \bar{f}, \bar{g}\rangle_{\overline{\Omega}} \overset{(a)}{=} \langle (\Delta_{\overline{\Omega}} \bar{f})|_\Omega, \bar{g}\rangle_\Omega = \langle \Delta_\Omega^{\mathcal{D}} f, g\rangle_\Omega,\end{aligned}$$

where (a) follows since $\bar{f}$ and $\bar{g}$ both vanish outside of Ω, while (b) is simply the self-adjointness of $\Delta_{\overline{\Omega}}$. ☐

Let us recall the following standard result for the Laplace operator on finite graphs (without boundary).

Theorem 2.3 (Green's formula)**.** *Let G be a finite graph. Then for all functions $f, g \in C(G)$*

$$-\langle \Delta_G f, g\rangle_G = \sum_{(x,y) \in E(G)} (\nabla_{xy} f)(\overline{\nabla_{xy} g})$$

where the sum on the right hand side counts every edge once, with an arbitrary order of its nodes.

Applying this to the special case of $G = \overline{\Omega}$ and noting that

$$\langle \Delta_\Omega^{\mathcal{D}} f, g \rangle_\Omega = \langle \Delta_{\overline{\Omega}} \bar{f}, \bar{g} \rangle_{\overline{\Omega}}$$

and $\bar{f}|_{\partial\Omega} = 0$, we get a Green's formula for the Dirichlet-Laplace operator.

Corollary 2.4. For all functions $f, g \in C(\Omega)$:

$$-\langle \Delta_\Omega^{\mathcal{D}} f, g \rangle_\Omega = \sum_{(x,y)\in E(\Omega)} (\nabla_{xy} f)(\overline{\nabla_{xy} g}) + \sum_{(xy)\in \partial_e \Omega} f(x)\overline{g(x)}$$

In the following, we consider the Dirichlet eigenvalue problem

$$\Delta_\Omega^{\mathcal{D}} \phi = -\lambda \phi.$$

By Proposition 2.2 and Corollary 2.4 there are $|\Omega|$ eigenvalues (with multiplicities) and they are all positive real numbers. We label them in increasing order

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|\Omega|}.$$

The corresponding eigenfunctions are going to be denoted ϕ_i ($i = 1 \ldots |\Omega|$) respectively.

2.3. Fourier transform on $\mathbb{Z}^d$

We are going to use methods inspired by the Fourier transform to prove our main results. In this section, we collect some technical results about the Fourier transform on $\mathbb{Z}^d$ and Ω. To this end, let us define $h_z : \mathbb{Z}^d \to \mathbb{C}$ by

$$h_z(x) = e^{i\langle x,z\rangle}$$

for all $x \in \mathbb{Z}^d$ and all $z \in [-\pi, \pi]^d$. Here the inner product is just the usual inner product on $\mathbb{R}^d$, that is $\langle x, z\rangle = \sum_{j=1}^d x_j z_j$. By abuse of notation the restriction of h_z to Ω will also be denoted by h_z.

Let us write

$$\Phi(z) = \sum_{i=1}^{d} 2 - 2\cos z_i.$$

Lemma 2.5. *The h_z functions have the following properties:*

$$\langle h_z, h_z\rangle_\Omega = |\Omega| \tag{2}$$

$$\langle h_z, -\Delta_\Omega^{\mathcal{D}} h_z\rangle_\Omega \leq |\Omega|\Phi(z) + |\partial_e\Omega|. \tag{3}$$

Proof. The first property is simple, since

$$\langle h_z, h_z\rangle_\Omega = \sum_{x\in\Omega} h_z(x)\overline{h_z(x)} = \sum_{x\in\Omega} 1 = |\Omega|.$$

For the other property, observe using Corollary 2.4 that

$$\langle h_z, -\Delta_\Omega^{\mathcal{D}} h_z\rangle_\Omega = \sum_{(x,y)\in E(\Omega)} |\nabla_{xy} h_z|^2 + \sum_{(xy)\in\partial_e\Omega)} |h_z(x)|^2 \leq$$
$$\leq \frac{1}{2}\sum_{x\in\Omega}\sum_{s\in S} |e^{i\langle x,z\rangle} - e^{i\langle x+s,z\rangle}|^2 + |\partial_e\Omega|.$$

Here the only reason for the inequality is that the right hand sum picks up certain edges that aren't originally in $E(\Omega)$. Thus it suffices to prove

$$\frac{1}{2}\sum_{x\in\Omega}\sum_{s\in S} |e^{i\langle x,z\rangle} - e^{i\langle x+s,z\rangle}|^2 = |\Omega|\Phi(z).$$

This is easy to see, since for any $x \in \mathbb{Z}^d$ we have $|e^{i\langle x,z\rangle} - e^{i\langle x+e_j,z\rangle}|^2 = |1 - e^{iz_j}|^2 = 2 - 2\cos z_j$. □

Lemma 2.6. *For any $f \in C(\Omega)$ the following hold:*

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} |\langle f, h_z\rangle_\Omega|^2 dz = \langle f, f\rangle_\Omega. \tag{4}$$

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \Phi(z)|\langle f, h_z\rangle_\Omega|^2 dz = -\langle f, \Delta_\Omega^{\mathcal{D}} f\rangle_\Omega \tag{5}$$

Proof. The first equation is simply the L^2 invariance of Fourier transform. The second equation can be thought of as the analogue of how the Fourier transform of the derivative is just z times the Fourier transform. To prove it, let us extend f to all of $\mathbb{Z}^d$ by 0 outside of Ω. Then we can compute:

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} (2 - 2\cos z_j)|\langle f, h_z\rangle_\Omega|^2 dz = \tag{6}$$
$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} |e^{-i\langle e_j,z\rangle} - 1|^2 \cdot |\langle f, h_z\rangle_\Omega|^2 dz =$$

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}|\langle f,(e^{-i\langle e_j,z\rangle}-1)\cdot h_z\rangle_\Omega|^2dz =$$
$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}|\langle f,e^{i\langle x-e_j,z\rangle}-e^{i\langle x,z\rangle}\rangle_\Omega|^2dz =$$
$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}|\langle f,e^{i\langle x-e_j,z\rangle}-e^{i\langle x,z\rangle}\rangle_{\mathbb{Z}^d}|^2dz \overset{(*)}{=}$$
$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}|\langle f(x+e_j)-f(x),e^{i\langle x,z\rangle}\rangle_{\mathbb{Z}^d}|^2dz =$$
$$\langle\nabla_{e_j}f,\nabla_{e_j}f\rangle_{\mathbb{Z}^d},$$

where $(*)$ follows from the general "integration by parts" identity $\langle f,\nabla_{-e_j}g\rangle_{\mathbb{Z}^d} = -\langle\nabla_{-e_j}f,g\rangle_{\mathbb{Z}^d} = \langle\nabla_{e_j}f,g\rangle_{\mathbb{Z}^d}$. By summing over all coordinate directions we get:

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\Phi(z)|\langle f,h_z\rangle_\Omega|^2dz = \sum_{j=1}^d\langle\nabla_{e_j}f,\nabla_{e_j}f\rangle_{\mathbb{Z}^d}$$

In this last expression $|\nabla_{xy}f|^2$ is counted once for every edge where either x or y is in Ω. The rest can be ignored because f is zero outside of Ω. Thus by Corollary 2.4 this is equal to $-\langle f,\Delta_\Omega^{\mathcal{D}}f\rangle_\Omega$. □

3. Proof of Theorem 1.1

Let Ω be a finite induced subgraph of $\mathbb{Z}^d$. We are going to prove upper and lower bounds on the sum of the first k eigenvalues of the Dirichlet-Laplace operator on Ω.

3.1. Upper bound

We start with a general lemma about eigenspaces.

Lemma 3.1. *Let L be a self-adjoint, positive semidefinite operator on a finite dimensional, Hermitian, complex vector space W with Hermitian inner product $\langle\,,\rangle$. Let $0\le\gamma_1\le\ldots\gamma_s$ denote its eigenvalues, and let us choose an orthonormal basis of eigenfunctions $f_i : i=1,\ldots,s$ where f_i corresponds to γ_i. Then for any $1\le k\le s$ and any vector $g\in W$ one has*

$$\gamma_{k+1}\langle g,g\rangle \le \langle g,Lg\rangle + \sum_{j=1}^k(\gamma_{k+1}-\gamma_j)|\langle g,f_j\rangle|^2. \tag{7}$$

Proof. Let $V \subset W$ denote the subspace spanned by $f_1, \dots, f_k$, and let P be the orthogonal projection to V. By the Rayleigh quotient, we have

$$\gamma_{k+1} = \inf_{0 \neq w \perp V} \frac{\langle w, Lw \rangle}{\langle w, w \rangle}.$$

Since for any $g \in W$ we have $g - Pg \perp V$, this gives us

$$\gamma_{k+1} \langle g - Pg, g - Pg \rangle \leq \langle g - Pg, L(g - Pg) \rangle.$$

By the Pythagorean theorem, the left hand side is equal to

$$\gamma_{k+1}(\langle g, g \rangle - \langle Pg, Pg \rangle) = \gamma_{k+1} \left(\langle g, g \rangle - \sum_{j=1}^{k} |\langle g, f_j \rangle|^2 \right).$$

Since V is invariant under L and $g - Pg \perp LPg$ as well as $L(g - Pg) \perp Pg$, the right hand side is equal to

$$\langle g, Lg \rangle - \langle Pg, LPg \rangle = \langle g, Lg \rangle - \sum_{j=1}^{k} \gamma_j |\langle g, f_j \rangle|^2.$$

Putting the two sides together we get

$$\gamma_{k+1} \langle g, g \rangle \leq \langle g, Lg \rangle + \sum_{j=0}^{k} (\gamma_{k+1} - \gamma_j) |\langle g, f_j \rangle|^2,$$

and this is what we wanted to prove. □

We are going to use this lemma by "averaging" it over a set of carefully chosen g's.

Lemma 3.2. *Let Ω be as before. Fix $1 \leq k \leq |\Omega|$, and let $B \subset [-\pi, \pi]^d$ be a measurable subset. Then*

$$\lambda_{k+1}(|\Omega||B| - k(2\pi)^d) \leq |\Omega| \int_{z \in B} \Phi(z) - (2\pi)^d \sum_{j=1}^{k} \lambda_j + |B| \cdot |\partial_e \Omega|.$$

Proof. We use Lemma 3.1 with the choice of $W = \mathbb{C}^\Omega$, $L = -\Delta_\Omega^{\mathcal{D}}$ and $g = h_z$, as defined in the previous section. In this case $\gamma_j = \lambda_j$ and $f_j = \phi_j$. Let us integrate over $z \in B$. This gives

$$\lambda_{k+1} \int_{z\in B} \langle h_z, h_z\rangle_\Omega \le \int_{z\in B} \langle h_z, -\Delta_\Omega^{\mathcal{D}} h_z\rangle_\Omega + \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j) \int_{z\in B} |\langle h_z, \phi_j\rangle_\Omega|^2.$$

By Lemma 2.5 the left hand side is $\lambda_{k+1}|B||\Omega|$, and the first term on the right hand side is at most $|B||\partial_e\Omega| + |\Omega| \int_{z\in B} \Phi(z)$. Finally, since $\lambda_{k+1} \ge \lambda_j$, the second term on the right hand side cannot decrease if we increase the domain of integration from B to the whole $[-\pi,\pi]^d$. Then again, using Lemma 2.6, and observing that $\langle \phi_j, \phi_j\rangle = 1$, we obtain

$$\lambda_{k+1}|B||\Omega| \le |B||\partial_e\Omega| + |\Omega| \int_{z\in B} \Phi(z) + \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)(2\pi)^d.$$

Finally, grouping all the terms containing λ_{k+1} we get the claimed inequality. $\square$

Theorem 3.3. *Let Ω be a finite induced subgraph of $\mathbb{Z}^d$. Then for any $k \le V_d|\Omega|$ the eigenvalues of the Dirichlet-Laplace satisfy*

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j \le (2\pi)^2 \frac{d}{d+2} \left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} + \frac{|\partial\Omega|}{|\Omega|},$$
$$\lambda_{k+1} \le (2\pi)^2 \frac{d\cdot 2^{\frac{d+2}{d}}}{d+2} \left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} + \frac{|\partial\Omega|}{|\Omega|}$$

Proof. Let us choose B to be a ball centered around the origin of radius

$$R = 2\pi \left(\frac{k}{V_d|\Omega|}\right)^{\frac{1}{d}}.$$

This R has been chosen such that $|\Omega||B| = k(2\pi)^d$ holds. Now the left hand side of Lemma 3.2 becomes 0, so we get that the right hand side is non-negative:

$$(2\pi)^d \sum_{j=1}^{k} \lambda_j \le |B||\partial_e\Omega| + |\Omega| \int_{z\in B} \Phi(z).$$

Since $2 - 2\cos x \le x^2$ for all x, we get $\Phi(z) \le ||z||^2$, giving

$$\int_{z\in B} \Phi(z) \le \int_{z\in B} |z|^2\, dz = dV_d \int_{r=0}^{R} r^{d+1} dr = \frac{dV_d}{d+2} R^{d+2}.$$

Plugging this back into the previous inequality, we obtain

$$\sum_{j=1}^{k} \lambda_j \leq \frac{d}{d+2} \frac{V_d|\Omega|}{(2\pi)^d} R^{d+2} + \frac{k|\partial_e \Omega|}{|\Omega|}$$

Substituting our choice of R, we get

$$\sum_{j=1}^{k} \lambda_j \leq (2\pi)^2 \frac{d}{d+2} \cdot k \left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} + \frac{k|\partial_e \Omega|}{|\Omega|}$$

as was claimed.

This method also allows to get an upper bound on a single eigenvalue. To this end, let us use Lemma 3.2 with B' a ball of radius $R' = R \cdot 2^{1/d}$ centered around the origin. Then $|\Omega||B'| = 2k(2\pi)^d$ and thus $|\Omega||B'| - k(2\pi)^d = k(2\pi)^d$, and ignore the $\sum_1^k \lambda_k$ term on the right hand side. Thus we get

$$\lambda_{k+1} \leq \frac{d}{d+2} \frac{V_d|\Omega|}{k(2\pi)^d} R^{d+2} 2^{\frac{d+2}{d}} + \frac{|\partial\Omega|}{|\Omega|} = (2\pi)^2 \frac{d \cdot 2^{\frac{d+2}{d}}}{d+2} \left(\frac{k}{V_d|\Omega|}\right)^{\frac{2}{d}} + \frac{|\partial\Omega|}{|\Omega|}. \quad \square$$

Remark 3.4. For the eigenvalue problem on Ω as a graph without boundary, a similar result was obtained by Harrell and Stubbe [2]. They however choose B to be a cube $[-c, c]^d$ which allows k to be any integer between 1 and Ω, however yields a slightly worse constant than our choice of B, whereas we obtain the sharp constant compared to the lower bound presented in the next section.

3.2. Lower bound

We are going to use an adaptation of Li and Yau's method [7] that involves expressing the eigenvalue sum as an integral, and then use bounds on the integrand to get an estimate for the sum. The integral we get in the discrete case is slightly different from the one in the original continuous version, so first we need to prove a modified version of the lemma that enables them to derive such lower bounds.

Lemma 3.5 (modification of Lemma 1 from [7])**.** *Let F be a real-valued function on $\mathbb{R}^d$ such that $0 \leq F \leq M$ and*

$$\int_{[-\pi,\pi]^d} F(z)\, dz \geq K.$$

Assume

$$\left(\frac{K}{MV_d}\right)^{\frac{1}{d}} \leq \sqrt{6} < \pi,$$

where V_d is the volume of the unit ball in $\mathbb{R}^d$. Then

$$\int_{[-\pi,\pi]^d} \Phi(z)F(z)dz \geq \frac{d \cdot K}{d+2}\left(\frac{K}{MV_d}\right)^{\frac{2}{d}} - \frac{d \cdot K}{12(d+4)}\left(\frac{K}{MV_d}\right)^{\frac{4}{d}},$$

where $\Phi(z) = \sum_i 2 - 2\cos z_i$ as before.

Proof. We may assume $\int_{[-\pi,\pi]^d} F(z)dz = K$ by decreasing F as necessary. As a first step we find a suitable radially symmetric function $\varphi(z)$ such that $0 \leq \varphi(z) \leq \Phi(z)$ for all $z \in [-\pi,\pi]^d$. Observe, that $2 - 2\cos x \geq x^2 - x^4/12$ for all x. Thus,

$$\sum_{i=1}^{d} 2 - 2\cos z_i \geq \sum_{i=1}^{d} z_i^2 - z_i^4/12 \geq \sum_{i=1}^{d} z_i^2 - \frac{1}{12}\left(\sum_{i=1}^{d} z_i^2\right)^2 = |z|^2 - \frac{1}{12}|z|^4.$$

Note that $x^2 - x^4/12$ is monotone increasing on $[0,\sqrt{6}]$ where it has a local maximum. Also, even though $\Phi(z)$ is not radially symmetric, it is clearly monotone increasing in $[-\pi,\pi]^d$ along each half line starting at 0. Let us define

$$\varphi(z) = \begin{cases} |z|^2 - |z|^4/12 & (|z| \leq \sqrt{6}) \\ 3 & (|z| > \sqrt{6}) \end{cases}.$$

Then by the radial monotonicity of Φ we get that $\Phi(z) \geq \varphi(z)$ for all $z \in [-\pi,\pi]^d$. Since $F(z) \geq 0$, we get

$$\int_{z\in[-\pi,\pi]^d} \Phi(z)F(z)dz \geq \int_{z\in[-\pi,\pi]^d} \varphi(z)F(z)dz.$$

Let R be such that $MR^dV_d = K$, and let

$$\tilde{F}(z) = \begin{cases} M & (|z| \leq R) \\ 0 & (|z| > R) \end{cases}.$$

This radius is chosen such that $\int \tilde{F} = \int F$, and since φ is radially symmetric and monotonic, it is intuitively clear that $\tilde{F}$ minimizes $\int \varphi F$ among all F that satisfy the assumptions as long as $R \leq \pi$. We can make this intuition precise:

$$\int_{z\in[-\pi,\pi]^d} \varphi(z)(F(z) - \tilde{F}(z))dz =$$

$$= \int_{z\in[-\pi,\pi]^d\setminus B_R} \varphi(z)F(z)dz - \int_{B_R} \varphi(z)(M-F(z))dz \geq$$
$$\geq \varphi(R)\int_{z\in[-\pi,\pi]^d\setminus B_R} F(z)dz - \varphi(R)\int_{B_R} M - F(z)dz =$$
$$= \varphi(R)\left(\int_{[-\pi,\pi]^d} F(z)dz - \int_{B_R} M\right) = 0,$$

where $\varphi(R)$ denotes the value $\varphi(z)$ for any z such that $|z| = R$. In the inequality we used $0 \leq F(z) \leq M$ and that $\varphi(z)$ is radially monotonic. Thus we have

$$\int_{z\in[-\pi,\pi]^d} \Phi(z)F(z)dz \geq \int_{z\in[-\pi,\pi]^d} \varphi(z)F(z)dz \geq M\int_{B_R} \varphi(z)dz.$$

If $R \leq \sqrt{6}$ then the last expression is simply

$$M\int_0^R \left(r^2 - \frac{r^4}{12}\right) d\cdot r^{d-1}V_d dr = d\cdot MV_d\left(\frac{R^{d+2}}{d+2} - \frac{R^{d+4}}{12(d+4)}\right).$$

Since R was chosen such that $MR^dV_d = K$, this can be written as

$$K\left(\frac{d}{d+2}R^2 - \frac{d}{12(d+4)}R^4\right) = \frac{d\cdot K}{d+2}\left(\frac{K}{MV_d}\right)^{\frac{2}{d}} - \frac{d\cdot K}{12(d+4)}\left(\frac{K}{MV_d}\right)^{\frac{4}{d}},$$

and this is what we wanted to prove. □

Theorem 3.6. *Let Ω be a finite induced subgraph of $\mathbb{Z}^d$ and let k be an integer such that*

$$\frac{k}{|\Omega|} \leq \left(\frac{\sqrt{6}}{2\pi}\right)^d V_d \tag{8}$$

holds. Then the Dirichlet-Laplace eigenvalues of Ω satisfy

$$\frac{1}{k}\sum_{j=1}^k \lambda_j \geq (2\pi)^2\cdot\frac{d}{d+2}\left(\frac{k}{|\Omega|V_d}\right)^{\frac{2}{d}} - (2\pi)^4\frac{d}{12(d+4)}\left(\frac{k}{|\Omega|V_d}\right)^{\frac{4}{d}}.$$

Proof. Let us denote

$$F_j(z) = \langle P_jh_z, P_jh_z\rangle_\Omega = ||P_jh_z||^2$$

where $||\cdot||$ denotes the L_2 norm and we use it to avoid excessive scalar products in our formulas, P_j is the projection on the space spanned by the

j-th Dirichlet eigenfunction ϕ_j, and $h_z(x) = e^{i<x,z>}$ for all $x \in \Omega$ as before. Following Li and Yau we consider the function

$$F(z) = \sum_{j=1}^{k} F_j(z) = \sum_{j=1}^{k} ||P_j h_z||^2 = ||Ph_z||^2$$

where P is the projection on the space spanned by the first k Dirichlet eigenfunctions $\phi_1, \ldots, \phi_k$. Because the eigenfunctions $\{\phi_j\}_{j=1}^{|\Omega|}$ form an orthonormal basis of $C(\Omega)$ we observe that

$$F(z) = \sum_{x\in\Omega} |Ph_z(x)|^2 = \sum_{j=1}^{k} |\langle \phi_j, h_z\rangle|^2. \tag{9}$$

In order to apply Lemma 3.5 to the function $F(z)$ we need to bound $F, \int F$, and $\int \Phi F$. This is done using Lemma 2.6:

$$F(z) = ||Ph_z||^2 \leq ||h_z||^2 = |\Omega|;$$

$$\int_{z\in[-\pi,\pi]^d} F(z)dz = \sum_{j=1}^{k} \int_{z\in[-\pi,\pi]^d} |\langle \phi_j, h_z\rangle_\Omega|^2 dz =$$

$$= (2\pi)^d \sum_{j=1}^{k} ||\phi_j||^2 = (2\pi)^d k;$$

$$\int_{z\in[-\pi,\pi]^d} \Phi(z)F(z)dz = \sum_{j=1}^{k} \int_{z\in[-\pi,\pi]^d} \Phi(z)|\langle \phi_j, h_z\rangle_\Omega|^2 =$$

$$= (2\pi)^d \sum_{j=1}^{k} \langle \phi_j, -\Delta_\Omega^{\mathcal{D}} \phi_j\rangle_\Omega = (2\pi)^d \sum_{j=1}^{k} \lambda_j.$$

Here $\Phi(z) = \sum_{l=1}^{d} (2 - 2\cos z_l)$ as before.

Now we apply Lemma 3.5 to F with the choice of $M = |\Omega|$ and $K = (2\pi)^d k$. The assumption (8) guarantees that the lemma can indeed be applied. As a result, we obtain

$$\sum_{i=1}^{k} \lambda_i = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \Phi(z)F(z)dz \geq$$

$$\geq \frac{kd}{d+2} \left(\frac{k(2\pi)^d}{|\Omega|V_d}\right)^{\frac{2}{d}} - \frac{kd}{12(d+4)} \left(\frac{k(2\pi)^d}{|\Omega|V_d}\right)^{\frac{4}{d}},$$

proving the theorem. □

References

[1] F.A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 441–479. MR0350504

[2] E.M. Harrell II, J. Stubbe, *On sums of graph eigenvalues*, Lin. Alg. Appl. **455** (2014), 168–186. MR3217405

[3] H. Kovarik, S. Vugalter, and T. Weidl, *Two-dimensional Berezin-Li-Yau inequalities with a correction term*, Commun. Math. Phys. **287** (2009), no. 3, 959–981. MR2486669

[4] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. **106** (1992), no. 2, 353–357. MR1165859

[5] P. Kröger, *Estimates for sums of eigenvalues of the Laplacian*, J. Funct. Anal. **126** (1994), 217–227. MR1305068

[6] A. Laptev, T. Weidl, *Recent results on Lieb–Thirring inequalities*, Journeées Equations aux Dérivés Partielles (La Chapelle sur Erdre, **2000**), pp. Exp. No. XX, 14. University of Nantes, Nantes (2000). MR1775696

[7] P. Li, S.T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys. **88** (1983), no. 3, 309–318. MR0701919

[8] E. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, Proc. Sym. Pure Math. **36** (1980), 241–252. MR0573436

[9] A. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. **131** (2003), 631–636. MR1933356

[10] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc. **11** (1961), 419–433. MR0129219

[11] R.S. Strichartz, *Estimates for sums of eigenvalues for domains in homogeneous spaces*, J. Funct. Anal. **137** (1996), no. 1, 152–190. MR1383015

[12] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. **71** (1912), 1134–1167. MR1511670

Frank Bauer
E-mail: bauerf80@gmx.de

Gabor Lippner
Northeastern University
Boston, MA
USA
E-mail: g.lippner@northeastern.edu