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Spectral extremal results on the α-index of graphs without minors and star forests*

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Dedicated to Professor Fan Chung, with admiration and thanks

Abstract: Let G be a graph of order n, and let $A(G)$ and $D(G)$ be the adjacency matrix and the degree matrix of G respectively. Define the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ by $A_\alpha(G) = \alpha D(G)+(1-\alpha)A(G)$ for any real number $0 \leq \alpha \leq 1$. The *α-index* of G is the largest eigenvalue of $A_\alpha(G)$. In this paper, using some new eigenvector techniques introduced by Tait and coworkers, we determine the maximum α-index and characterize all extremal graphs for K_r minor-free graphs, $K_{s,t}$ minor-free graphs, and star-forest-free graphs for any $0 < \alpha < 1$ respectively.

Keywords: Spectral radius, α-index, extremal graphs, star forests, minors.

1. Introduction

Let G be an undirected simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$, where n is called the order of G. The *adjacency matrix* $A(G)$ of G is the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise. The *spectral radius* of G, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. The *signless Laplacian spectral radius* of G, denoted by $q(G)$, is the largest eigenvalue of $Q(G)$, where $Q(G) = A(G) + D(G)$ and $D(G)$ is the degree diagonal matrix. For $v \in V(G)$, the *degree* $d_G(v)$ of v is the number of

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vertices adjacent to v in G. We write $d(v)$ for $d_G(v)$ if there is no ambiguity. Denote by $\Delta(G)$ the maximum degree of G and $\overline{G}$ the complement graph of G. Let S_{n-1} be a star of order n. The *center* of a star is the vertex of maximum degree in the star. A *star forest* is a forest whose components are stars.

The *centers* of a star forest are the centers of the stars in the star forest. A graph G is H*-free* if it does not contain H as a subgraph. A graph H is called a *minor* of a graph G if it can be obtained from G by deleting edges, contracting edges or deleting vertices. A graph G is H *minor-free* if it does not contain H as a minor. For $X, Y \subseteq V(G)$, $e(X)$ denotes the number of edges in G with two ends in X and $e(X, Y)$ denotes the number of edges in G with one end in X and the other in Y. For two vertex disjoint graphs G and H, we denote by $G \cup H$ and $G \nabla H$ the *union* of G and H, and the *join* of G and H which is obtained by joining every vertex of G to every vertex of H, respectively. Denote by kG the union of k disjoint copies of G. For graph notation and terminology undefined here, readers are referred to [1].

To track the gradual change of $A(G)$ into $Q(G)$, Nikiforov [10] proposed and studied the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

for any real number $0 \leq \alpha \leq 1$. Note that $A_0(G) = A(G)$, $2A_{1/2}(G) = Q(G)$, and $A_1(G) = D(G)$. The α*-index* of G is the largest eigenvalue of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$. Clearly, $\rho_0(G) = \rho(G)$ and $2\rho_{1/2}(G) = q(G)$.

Let $\mathbf{x} = (x_u)_{u \in V(G)}$ be an eigenvector to $\rho_\alpha(G)$. By eigenequations of $A_\alpha(G)$ on any vertex $u \in V(G)$,

$$\rho_\alpha(G)x_u = \alpha d(u)x_u + (1-\alpha)\sum_{uv \in E(G)} x_v.$$

Since $A_\alpha(G)$ is a real symmetric matrix, Rayleigh's principle implies that

$$\rho_\alpha(G) = \max_{||\mathbf{x}||_2=1} \sum_{uv \in E(G)} (\alpha x_u^2 + 2(1-\alpha)x_u + \alpha x_v^2),$$

also see [10]. Note that $A_\alpha(G)$ is nonnegative. By Perron-Frobenius theory of nonnegative matrices, if G is connected then $A_\alpha(G)$ has a positive eigenvector corresponding to $\rho_\alpha(G)$; also see [10]. In addition, if G is connected and H is a proper subgraph of G, then

$$\rho_\alpha(G) > \rho_\alpha(H).$$

In spectral extremal graph theory, one of the central problems, which is called spectral Turán problem, is to find the maximum $\rho(G)$ or $q(G)$ of a graph G of order n without H as a subgraph or as a minor. This problem is intensively investigated in the literature for many classes of graphs.

For example, Tait [12] determined the maximum spectral radius for K_r minor-free graphs and $K_{s,t}$ minor-free graphs by using eigenvector, i.e.,

Theorem 1.1 ([12]). *Let $r \geq 3$ and G be a K_r minor-free graph of sufficiently large order n. Then*

$$\rho(G) \leq \rho(K_{r-2}\nabla K_{n-r+2})$$

with equality if and only if $G = K_{r-2}\nabla K_{n-r+2}$.

Theorem 1.2 ([12]). *Let $t \geq s \geq 2$ and G be a $K_{s,t}$ minor-free graph of sufficiently large order n. Then*

$$\rho(G) \leq \frac{s+t-3+\sqrt{(s+t-3)^2+4((s-1)(n-s+1)-(s-2)(t-1))}}{2}$$

with equality if and only if $n-s+1 = pt$ and $G = K_{s-1}\nabla pK_t$.

Furthermore, he pointed out the extremal graphs for maximizing the number of edges and spectral radius are the same for small values of r and s and then differed significantly. Chen, Liu and Zhang [2, 3] determined the maximum (signless Laplacian) spectral radius for kP_3-free graphs. They [4] also determined the maximum signless Laplacian spectral radius for $K_{2,t}$ minor-free graphs. In addition, Nikiforov [9] gave an excellent survey on this topic. For more results, see [6, 11, 12, 13, 15].

Motivated by above results, we investigate the maximum α-index for K_r minor-free graphs, $K_{s,t}$ minor-free graphs, and star-forest-free graphs. Following some new techniques introduced by Tait and coworkers, we show the extremal graphs of K_r minor-free graphs and $K_{s,t}$ minor-free graphs for maximizing α-index for any $0 < \alpha < 1$ and sufficiently large n. Furthermore, we determine the maximum α-index and characterize all extremal graphs for star-forest-free graphs for any $0 < \alpha < 1$. The main results of this paper are stated as follows.

Theorem 1.3. *Let $r \geq 3$ and G be a K_r minor-free graph of sufficiently large order n. Then for any $0 < \alpha < 1$,*

$$\rho_\alpha(G) \leq \rho_\alpha(K_{r-2}\nabla \overline{K}_{n-r+2})$$

with equality if and only if $G = K_{r-2}\nabla \overline{K}_{n-r+2}$.

Theorem 1.4. *Let $t \geq s \geq 2$ and G be a $K_{s,t}$ minor-free graph of sufficiently large order n. Then for any $0 < \alpha < 1$, $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$, and equality holds if and only if $n - s + 1 = pt$ and $G = K_{s-1}\nabla pK_t$, where*

$$f_\alpha(x) = x^2 - \Big(\alpha n + s + t - 3\Big)x + \Big(\alpha(n - s + 1) + s - 2\Big)\Big(\alpha s - \alpha + t - 1\Big) - (1-\alpha)^2(s-1)(n-s+1).$$

Theorem 1.5. *Let $F = \cup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 1$. If G is an F-free graph of order $n \geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^3}$ for any $0 < \alpha < 1$, then $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$, and equality holds if and only if $G = K_{k-1}\nabla H$ and H is a $(d_k - 1)$-regular graph of order $n - k + 1$, where*

$$\begin{aligned} f_\alpha(x) &= x^2 - \Big(\alpha n + k + d_k - 3\Big)x + \Big(\alpha(n - k + 1) + k - 2\Big) \times \\ &\quad \Big(\alpha(k-1) + d_k - 1\Big) - (1-\alpha)^2(k-1)(n-k+1). \end{aligned}$$

The rest of this paper is organized as follows. In Section 2, some technical lemmas are presented. In Section 3, we present the proofs of Theorems 1.3 and 1.4. In Section 4, we give the proof of Theorem 1.5 and some corollaries.

2. Preliminary

Lemma 2.1. *Let $0 < \alpha < 1$, $k \geq 2$, and $n \geq k - 1$. If $G = K_{k-1}\nabla\overline{K}_{n-k+1}$, then $\rho_\alpha(G) \geq \alpha(n-1) + (1-\alpha)(k-2)$. In particular, if $n \geq \frac{(2k-3)^2}{2\alpha^2} - \frac{8k^2-18k+9}{2\alpha} + 2k(k-1)$, then $\rho_\alpha(G) \geq \alpha n + \frac{2k-3-(2k-1)\alpha}{2\alpha}$.*

Proof. Set for short $\rho_\alpha = \rho_\alpha(G)$ and let $\mathbf{x}_\alpha = (x_v)_{v\in V(G)}$ be a positive eigenvector to ρ_α. By symmetry, all vertices corresponding to K_{k-1} in the representation $G := K_{k-1}\nabla\overline{K}_{n-k+1}$ have the same eigenvector entries, denoted by x_1. Similarly, all remaining vertices have the same eigenvector entries, denoted by x_2. By eigenequations of $A_\alpha(G)$, we have

$$\begin{aligned} (\rho_\alpha - \alpha(n-1) - (1-\alpha)(k-2))x_1 &= (1-\alpha)(n-k+1)x_2 \\ (\rho_\alpha - \alpha(k-1))x_2 &= (1-\alpha)(k-1)x_1. \end{aligned}$$

Then $\rho_\alpha(G)$ is the largest root of $g(x) = 0$, where

$$g(x) = x^2 - (\alpha n + k - 2)x + (k-1)(2\alpha - 1)n + (k-1)(k - k\alpha - 1) = 0.$$

Clearly,

$$\begin{aligned}\rho_\alpha(G) &= \frac{\alpha n+k-2+\sqrt{(\alpha n+k-2)^2-4(k-1)[(2\alpha-1)n+k-k\alpha-1]}}{2} \\ &\geq \frac{(\alpha n+k-2)+(\alpha n+k-2-(k-1)\alpha)}{2} \\ &= \alpha(n-1)+(1-\alpha)(k-2).\end{aligned}$$

In addition,

$$\begin{aligned}& g\Big(\alpha n+\frac{2k-3-(2k-1)\alpha}{2\alpha}\Big) \\ = \;& -\frac{(1-\alpha)}{2}\Big(n-\frac{(2k-3)^2}{2\alpha^2}+\frac{8k^2-18k+9}{2\alpha}-2k(k-1)\Big) \\ \leq \;& 0,\end{aligned}$$

we have

$$\rho_\alpha(G)\geq \alpha n+\frac{2k-3-(2k-1)\alpha}{2\alpha}.$$

$\square$

Next we compare two lower bounds of $\rho_\alpha(G)$ in Lemma 2.1.

Remark. Note that $\alpha n+\frac{2k-3-(2k-1)\alpha}{2\alpha}-(\alpha(n-1)+(1-\alpha)(k-2)) = \frac{((2k-2)\alpha-(2k-3))(\alpha-1)}{2\alpha}$. If $0<\alpha\leq\frac{2k-3}{2k-2}$, then

$$\alpha n+\frac{2k-3-(2k-1)\alpha}{2\alpha}\geq \alpha(n-1)+(1-\alpha)(k-2).$$

If $\frac{2k-3}{2k-2}<\alpha<1$, then

$$\alpha n+\frac{2k-3-(2k-1)\alpha}{2\alpha}< \alpha(n-1)+(1-\alpha)(k-2).$$

Lemma 2.2. *Let $0<\alpha<1$, $d\geq 2$, $k\geq 1$, $n\geq\max\{k-1,2k-2+\frac{d-k+1}{\alpha}\}$, and H be a graph of order $n-k+1$. If $G=K_{k-1}\nabla H$ and $\Delta(H)\leq d-1$, then $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x)=0$, and equality holds if and only if H is a $(d-1)$-regular graph, where*

$$\begin{aligned}& f_\alpha(x)=x^2-\Big(\alpha n+k+d-3\Big)x+ \\ & \Big(\alpha(n-k+1)+k-2\Big)\Big(\alpha(k-1)+d-1\Big)-(1-\alpha)^2(k-1)(n-k+1).\end{aligned}$$

Proof. Let $u_1, u_2, \cdots, u_{k-1}$ be the vertices of G corresponding to K_{k-1} in the representation $G := K_{k-1}\nabla H$. Set for short $\rho_\alpha = \rho_\alpha(G)$ and let $\mathbf{x}_\alpha = (x_v)_{v\in V(G)}$ be a positive eigenvector to ρ_α. By symmetry, $x_{u_1} = \cdots = x_{u_{k-1}}$. Choose a vertex $v \in V(H)$ such that

$$x_v = \max_{z\in V(H)} x_z.$$

Since $\Delta(H) \leq d-1$ and $G = K_{n-1}\nabla H$, we have $d(v) \leq k-1+d-1 = k+d-2$. By eigenequations of $A_\alpha(G)$ on u_1 and v, we have

$$\begin{aligned}(\rho_\alpha - \alpha(n-1))x_{u_1} &= (1-\alpha)(k-2)x_{u_1} + (1-\alpha)\sum_{uu_1\in E(H)} x_u \\ &\leq (1-\alpha)(k-2)x_{u_1} + (1-\alpha)(n-k+1)x_v\end{aligned} \tag{1}$$

$$\begin{aligned}(\rho_\alpha - \alpha(k+d-2))x_v &\leq (\rho_\alpha - \alpha d(v))x_v \\ &= (1-\alpha)(k-1)x_{u_1} + (1-\alpha)\sum_{uv\in E(H)} x_u \\ &\leq (1-\alpha)(k-1)x_{u_1} + (1-\alpha)(d-1)x_v,\end{aligned} \tag{2}$$

which implies that

$$\begin{aligned}(\rho_\alpha - \alpha(n-1) - (1-\alpha)(k-2))x_{u_1} &\leq (1-\alpha)(n-k+1)x_v \\ (\rho_\alpha - \alpha(k+d-2) - (1-\alpha)(d-1))x_v &\leq (1-\alpha)(k-1)x_{u_1}.\end{aligned}$$

Note that $K_{k-1}\nabla\overline{K}_{n-k+1}$ is a subgraph of G. By Lemma 2.1, we have

$$\rho_\alpha \geq \rho_\alpha(K_{k-1}\nabla\overline{K}_{n-k+1}) \geq \alpha(n-1)+(1-\alpha)(k-2) \geq \alpha(k+d-2)+(1-\alpha)(d-1).$$

Let

$$\begin{aligned}&f_\alpha(x) = x^2 - \Big(\alpha n + k + d - 3\Big)x + \\ &\Big(\alpha(n-k+1)+k-2)\Big)\Big(\alpha(k-1)+d-1\Big) - (1-\alpha)^2(k-1)(n-k+1).\end{aligned}$$

Then ρ_α is no more than the largest root of $f_\alpha(x) = 0$. If ρ_α is equal to the largest root of $f_\alpha(x) = 0$, then all equalities in (1) and (2) hold. So $d(v) = k+d-2$ and $x_z = x_v$ for any vertex $z \in V(H)$. Since for any $z \in V(H)$,

$$(\rho_\alpha - \alpha d(z))x_z = (1-\alpha)(k-1)x_{u_1} + (1-\alpha)\sum_{uz\in E(H)} x_u$$

$$\begin{aligned} &\leq (1-\alpha)(k-1)x_{u_1} + (1-\alpha)(d-1)x_v \\ &= (\rho_\alpha - \alpha d(v))x_v, \end{aligned}$$

we have $d(z) = d(v) = d + k - 2$. So H is $(d-1)$-regular. □

3. Graphs without minors

We first present some structural lemmas for K_r minor-free graphs and $K_{s,t}$ minor-free graphs respectively.

Lemma 3.1 ([12])**.** *Let $r \geq 3$ and G be a bipartite K_r minor-free graph of order n with vertex partition K and T. Let $|K| = k$ and $|T| = n-k$. Then there is an absolute constant C depending only on r such that*

$$e(G) \leq Ck + (r-2)n.$$

In particular, if $|K| = o(n)$, then $e(G) \leq (r-2+o(1))n$.

Lemma 3.2 ([12])**.** *Let G be a K_r minor-free graph of order n. Assume that $(1-2\delta)n > r$, and $(1-\delta)n > \binom{r-2}{2} + 2$, and that there is a set K with $|K| = r-2$ and a set T with $|T| = (1-\delta)n$ such that every vertex in K is adjacent to every vertex in T. Then we may add edges to K to make it a clique and the resulting graph is still K_r minor-free.*

Lemma 3.3 ([12, 14])**.** *Let $t \geq s \geq 2$ and G be a bipartite $K_{s,t}$ minor-free graph of order n with vertex partition K and T. Let $|K| = k$ and $|T| = n-k$. Then there is an absolute constant C depending only on s and t such that*

$$e(G) \leq Ck + (s-1)n.$$

In particular, if $|K| = o(n)$, then $e(G) \leq (s-1+o(1))n$.

Lemma 3.4 ([7])**.** *For any graph H, there is a constant C such that if G is an H minor-free graph of order n then*

$$e(G) \leq Cn.$$

Proof of Theorem 1.3. Let G be a K_r minor-free graph of sufficiently large order n with the maximum α-index.

Claim 1. G is connected.

If G is not connected, then we can add an edge to two components of G to get a K_r-minor free graph with larger α-index, a contradiction. This proves Claim 1.

Next let $\rho_\alpha = \rho_\alpha(G)$ and $\mathbf{x} = (x_v)_{v\in V(G)}$ with the maximum entry 1 be a positive eigenvector to ρ_α. Choose an arbitrary $w \in V(G)$ with

$$x_w = \max\{x_v : v \in V(G)\} = 1.$$

Set $L = \{v \in V(G) : x_v > \epsilon\}$ and $S = \{v \in V(G) : x_v \leq \epsilon\}$, where ϵ will be chosen later.

Since $K_{r-2}\nabla\overline{K}_{n-r+2}$ is K_r-minor free, by Lemma 2.1,

$$\rho_\alpha \geq \rho_\alpha(K_{r-2}\nabla\overline{K}_{n-r+2}) \geq \max\left\{\alpha n + \frac{2r-5(2r-3)\alpha}{2\alpha}, \alpha(n-1)\right\}. \tag{3}$$

By Lemma 3.4, there is a constant C_1 such that

$$2e(S) \leq 2e(G) \leq C_1 n. \tag{4}$$

Claim 2. There exists a constant C_2 such that

$$|L| \leq \frac{C_2(1-\alpha+\alpha\epsilon)}{\epsilon}.$$

In addition, ϵ can be chosen small enough that

$$e(L,S) \leq (k-1+\epsilon)n.$$

By eigenequations of A_α on any vertex $u \in L$, we have

$$(\rho_\alpha - \alpha d(u))\epsilon < (\rho_\alpha - \alpha d(u))x_u = (1-\alpha)\sum_{uv\in E(G)} x_v \leq (1-\alpha)d(u),$$

which implies that

$$d(u) > \frac{\rho_\alpha \epsilon}{1-\alpha+\alpha\epsilon}.$$

Thus

$$2e(G) = \sum_{u\in V(G)} d(u) \geq \sum_{u\in L} d(u) \geq \frac{|L|\rho_\alpha\epsilon}{1-\alpha+\alpha\epsilon},$$

which implies that

$$|L| \leq \frac{2e(G)(1-\alpha+\alpha\epsilon)}{\rho_\alpha\epsilon}. \tag{5}$$

For sufficiently large n, there is a constant C_2 such that

$$C_2 \geq \frac{2\alpha C_1}{2\alpha^2 + \frac{2r-5-(2r-3)\alpha}{n}}.$$

Hence by (3)–(5),

$$\begin{aligned} |L| &\leq \frac{2e(G)}{\rho_\alpha} \cdot \frac{(1-\alpha+\alpha\epsilon)}{\epsilon} \leq \frac{C_1 n}{\alpha n + \frac{2r-5-(2r-3)\alpha}{2\alpha}} \cdot \frac{(1-\alpha+\alpha\epsilon)}{\epsilon} \\ &= \frac{2\alpha C_1}{2\alpha^2 + \frac{2r-5-(2r-3)\alpha}{n}} \cdot \frac{1-\alpha+\alpha\epsilon}{\epsilon} \leq \frac{C_2(1-\alpha+\alpha\epsilon)}{\epsilon}. \end{aligned}$$

Choose ϵ small enough such that $|L| \leq \epsilon n$. By Lemma 3.1, $e(L,S) \leq (r-2+\epsilon)n$. This proves Claim 2.

By Claim 2, we can choose ϵ small enough such that

$$2e(L) \leq C_1|L| \leq \frac{C_1 C_2(1-\alpha+\alpha\epsilon)}{\epsilon} \leq \epsilon n.$$

Claim 3. Let $u \in L$. Then for any $u \in L$, there is a constant C_3 such that

$$d(u) \geq (1 - C_3(1 - x_u + \epsilon))n.$$

Since

$$\begin{aligned} & \rho_\alpha \sum_{v \in V(G)} x_v \\ = & \sum_{v \in V(G)} \rho_\alpha x_v = \sum_{v \in V(G)} \left(\alpha d(v) x_v + (1-\alpha) \sum_{vz \in E(G)} x_z \right) \\ = & \alpha \sum_{v \in V(G)} d(v) x_v + (1-\alpha) \sum_{vz \in E(G)} (x_v + x_z) \\ = & \alpha \left(\sum_{v \in L} d(v) x_v + \sum_{v \in S} d(v) x_v \right) + (1-\alpha) \left(\sum_{vz \in E(L)} (x_v + x_z) + \right. \\ & \left. \sum_{vz \in E(L,S)} (x_v + x_z) + \sum_{vz \in E(S)} (x_v + x_z) \right) \\ \leq & \alpha(2e(L) + e(L,S)) + \alpha\epsilon(2e(S) + e(L,S)) + (1-\alpha)(2e(L) + \\ & (1+\epsilon)e(L,S) + 2\epsilon e(S)) \\ = & 2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S), \end{aligned}$$

we have

$$
(6)\qquad
\begin{aligned}
\sum_{v\in V(G)} x_v &\le \frac{2e(L)+2\epsilon e(S)+(1+\epsilon)e(L,S)}{\rho_\alpha}\\
&\le \frac{\epsilon n+\epsilon C_1 n+(1+\epsilon)(r-2+\epsilon)n}{\rho_\alpha}\\
&= \frac{((1+C_1)\epsilon+(1+\epsilon)(r-2+\epsilon))n}{\rho_\alpha}.
\end{aligned}
$$

By eigenequations of A_α on u, we have

$$
(7)\qquad (\rho_\alpha-\alpha d(u))x_u \;=\; (1-\alpha)\sum_{uv\in E(G)} x_v \le (1-\alpha)\sum_{v\in V(G)} x_v.
$$

By (3), (6), and (7), we have

$$
\begin{aligned}
d(u) &\ge \frac{\rho_\alpha}{\alpha}-\frac{(1-\alpha)\sum_{v\in V(G)}x_v}{\alpha x_u}\\
&\ge \frac{\rho_\alpha}{\alpha}-\frac{(1-\alpha)((1+C_1)\epsilon+(1+\epsilon)(r-2+\epsilon))n}{\rho_\alpha\alpha x_u}\\
&\ge n-1-\frac{(1-\alpha)((1+C_1)\epsilon+(1+\epsilon)(r-2+\epsilon)}{\alpha^2(1-\frac{1}{n})x_u}.
\end{aligned}
$$

Since n is sufficiently large and ϵ is small enough, there is a constant C_3 such that

$$
d(u)\ge n-1-\frac{(1-\alpha)((1+C_1)\epsilon+(1+\epsilon)(r-2+\epsilon)}{\alpha^2(1-\frac{1}{n})x_u}\ge (1-C_3(1-x_u+\epsilon))n.
$$

This proves Claim 3.

Claim 4. Let $1\le s<r-2$. Suppose that there is a set X of s vertices such that $X=\{v\in V(G): x_v\ge 1-\eta \text{ and } d(v)\ge(1-\eta)n\}$, where η is much smaller than 1. Then there is a constant C_4 and a vertex $v\in L\backslash X$ such that $x_v\ge 1-C_4(\eta+\epsilon)$ and $d(v)\ge(1-C_4(\eta+\epsilon))n$.

By eigenequations of A_α on w, we have

$$
\rho_\alpha-\alpha d(w)=(\rho_\alpha-\alpha d(w))x_w=(1-\alpha)\sum_{vw\in E(G)}x_v.
$$

Multiplying both sides of the above inequality by ρ_α, we have

$$
\rho_\alpha(\rho_\alpha-\alpha d(w))
$$

$$
\begin{aligned}
&= (1-\alpha)\sum_{vw\in E(G)} \rho_\alpha x_v \\
&= (1-\alpha)\sum_{vw\in E(G)} \left(\alpha d(v)x_v + (1-\alpha)\sum_{uv\in E(G)} x_u\right) \\
&= (1-\alpha)\sum_{vw\in E(G)} \alpha d(v)x_v + (1-\alpha)^2 \sum_{vw\in E(G)}\sum_{uv\in E(G)} x_u \\
&\leq (1-\alpha)\Big(\sum_{v\in V(G)} \alpha d(v)x_v - \alpha d(w)\Big) + (1-\alpha)^2\sum_{uv\in E(G)}(x_u+x_v) - \\
&\quad (1-\alpha)^2\sum_{vw\in E(G)} x_v \\
&= \alpha(1-\alpha)\sum_{uv\in E(G)}(x_u+x_v) - \alpha(1-\alpha)d(w) + \\
&\quad (1-\alpha)^2\sum_{uv\in E(G)}(x_u+x_v) - (1-\alpha)(\rho_\alpha - \alpha d(w)) \\
&= (1-\alpha)\sum_{uv\in E(G)}(x_u+x_v) - (1-\alpha)\rho_\alpha,
\end{aligned}
$$

which implies that

$$\sum_{uv\in E(G)}(x_u+x_v) \geq \frac{\rho_\alpha(\rho_\alpha + 1 - \alpha - \alpha d(w))}{1-\alpha}.$$

On the other hand,

$$
\begin{aligned}
&\sum_{uv\in E(G)}(x_u+x_v) \\
&= \sum_{uv\in E(L,S)}(x_u+x_v) + \sum_{uv\in E(S)}(x_u+x_v) + \sum_{uv\in E(L)}(x_u+x_v) \\
&\leq \sum_{uv\in E(L,S)}(x_u+x_v) + 2\epsilon e(S) + 2e(L) \\
&\leq \epsilon e(L,S) + \sum_{\substack{uv\in E(L\setminus X,S)\\ u\in L\setminus X}} x_u + \sum_{\substack{uv\in E(L\cap X,S)\\ u\in L\cap X}} x_u + 2\epsilon e(S) + 2e(L).
\end{aligned}
$$

Let $t = |L\cap X|$. Combining with (3), we have

$$\sum_{\substack{uv\in E(L\setminus X,S)\\ u\in L\setminus X}} x_u$$

$$\begin{aligned}
&\geq \frac{\rho_\alpha(\rho_\alpha + 1 - \alpha - \alpha d(w))}{1-\alpha} - 2\epsilon e(S) - 2e(L) - \epsilon e(L,S) - \sum_{\substack{uv \in E(L\cap X, S)\\ u \in L\cap X}} x_u \\
&\geq \left(\frac{\alpha n}{1-\alpha} + \frac{2r-5-(2r-3)\alpha}{2\alpha(1-\alpha)}\right)\left(\frac{2r-5-(2r-3)\alpha}{2\alpha} + 1\right) - \\
&\quad \epsilon C_1 n - \epsilon n - \epsilon(r-2+\epsilon)n - tn \\
&= \left(r - \frac{5}{2} - t - \epsilon(C_1 + \epsilon + r - 1)\right)n + \frac{(2r-5)^2 - (2r-3)(2r-5)\alpha}{4\alpha^2} \\
&\geq \left(r - \frac{5}{2} - t - \epsilon(C_1 + \epsilon + r)\right)n
\end{aligned}$$

In addition,

$$\begin{aligned}
e(L\backslash X, S) &= e(L,S) - e(L\cap X, S) \\
&\leq (r-2+\epsilon)n - t(1-\eta)n + t(t-1) + t(|L| - t) \\
&\leq (r-2+\epsilon)n - t(1-\eta)n + t(t-1) + t(\epsilon n - t) \\
&\leq (r - 2 + 2\epsilon - t(1-\eta-\epsilon))n
\end{aligned}$$

Note that for any $\eta > 0$, there exists a constant C_4' such that $C_4'\eta \geq \frac{1}{2}$. Then there is a vertex $v \in L\backslash X$ such that

$$\begin{aligned}
x_v &\geq \frac{\sum_{\substack{uv\in E(L\backslash X,S)\\ u\in L\backslash X}} x_u}{e(L\backslash X, S)} \\
&\geq \frac{(r - \frac{5}{2} - t - \epsilon(C_1+\epsilon+r))n}{(r-2+\epsilon - t(1-\eta-\epsilon))n} \\
&= 1 - \frac{\frac{1}{2} + t\eta + \epsilon(C_1 + \epsilon + t + 1)}{r - 2 + \epsilon - t(1-\eta-\epsilon)} \\
&\geq 1 - \frac{\frac{1}{2} + (r-3)\eta + \epsilon(C_1 + \epsilon + r - 2)}{r-2+\epsilon-(r-3)(1-\eta-\epsilon)} \\
&= 1 - \frac{\frac{1}{2} + (r-3)\eta + \epsilon(C_1+\epsilon+r-2)}{1 + (r-2)\epsilon + (r-3)\eta} \\
&\geq 1 - \frac{\max\{C_1 + \epsilon + r - 2, C_4' + r - 3\}}{1 + (r-2)\epsilon + (r-3)\eta}(\eta + \epsilon)
\end{aligned}$$

By Claim 3, Claim 4 follows directly.

If we start with w and iteratively apply Claim 4, then for any $\delta > 0$, we can choose ϵ small enough that G contains a set X with $r-2$ vertices such that their common neighborhood of size is at least $(1-\delta)n$ and each eigenvector entry is at least $1-\delta$. From now on, denote by K the set X with $r-2$ vertices mentioned above. Let T be the common neighborhood of K and $R = V(G)\backslash(K \cup T)$. Clearly, $|K| = r-2$, $|T| \geq (1-\delta)n$, and $|R| \leq \delta n$.

Claim 5. *K induces a clique and T is an independent set.*

If K does not induce a clique, then we can add all possible edges to make it a clique. By Lemma 3.2, the resulting graph G' is still K_r minor-free. Since G is connected, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. Hence K induces a clique. If there is an edge in T, then there is a K_r minor in G, a contradiction. Thus T is an independent set. This proves Claim 5.

Claim 6. *For any $v \in V(G)\backslash K$, we have $x_v \leq \frac{\sqrt{\alpha}}{C_1}$, where C_1 is the constant in (4).*

Since G is K_r-minor free, any vertex in R can be adjacent to at most one vertex in T. By the definition of R, every vertex in R can be adjacent to at most $r-3$ vertices in K. In addition, by Claim 5, T is an independent set and thus any vertex in T has at most $r-2+|R|$ neighbors. Hence for any vertex $v \in V(G)\backslash K$,

$$d(v) \leq r-2+|R| \leq r-2+\delta n. \tag{8}$$

Since R is also K_r-minor free, we have

$$2e(R) \leq C_1|R| \leq C_1\delta n.$$

By eigenequations of $A_\alpha(G)$, we have

$$\begin{aligned}
\alpha(n-1)\sum_{u\in R} x_u &\leq \rho_\alpha \sum_{u\in R} x_u = \sum_{u\in R}\Big(\alpha d(u)x_u + (1-\alpha)\sum_{uv\in E(G)} x_v\Big)\\
&\leq \sum_{u\in R}(\alpha d(u) + (1-\alpha)d(u)) = \sum_{u\in R} d(u)\\
&\leq 2e(R) + (r-2)|R| \leq C_1\delta n + (r-2)\delta n\\
&= (C_1+r-2)\delta n,
\end{aligned}$$

which implies that

$$\sum_{u\in R} x_u \leq \frac{(C_1+r-2)\delta n}{\alpha(n-1)}. \tag{9}$$

By eigenequations of $A_\alpha(G)$ on any vertex $v \in V(G)\backslash K$, we have

$$(\rho_\alpha - \alpha d(v))x_v = (1-\alpha)\sum_{uv\in E(G)} x_u \le (1-\alpha)\Big(r-2+\sum_{u\in R} x_u\Big). \tag{10}$$

By Lemma 3.2 and (8)–(10), we have

$$\begin{aligned} x_v &\le \frac{(1-\alpha)(r-2+\sum_{u\in R} x_u)}{\rho_\alpha - \alpha d(v)} \\ &\le \frac{(1-\alpha)\Big(r-2+\frac{(C_1+r-2)\delta n}{\alpha(n-1)}\Big)}{\alpha(n-1)-\alpha(r-2+\delta n)} \\ &= \frac{(1-\alpha)\Big(r-2+\frac{(C_1+r-2)\delta}{\alpha(1-\frac{1}{n})}\Big)}{\alpha((1-\delta)n-r+1)} \end{aligned}$$

Then we can choose ϵ small enough to make δ small enough to get the result. This proves Claim 6.

Claim 7. R is empty.

If R is not empty, then there exists a vertex $v \in R$ such that v has at most C_1 neighbors in R. Let H be a graph obtained from G by removing all edges incident with v and then connecting v to each vertex in K. Since K induces a clique, H is still K_r minor-free. Let $u \in K$ be the vertex not adjacent to v. Then

$$\begin{aligned} &\rho_\alpha(H) - \rho_\alpha \\ \ge\ & \frac{\mathbf{x}^T A_\alpha(H)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} - \frac{\mathbf{x}^T A_\alpha \mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ \ge\ & \frac{1}{\mathbf{x}^T\mathbf{x}}\Big(\alpha x_u^2 + 2(1-\alpha)x_u x_v + \alpha x_v^2 - \sum_{\substack{vz\in E(G)\\ z\notin K}} (\alpha x_z^2 + 2(1-\alpha)x_v x_z + \alpha x_z^2)\Big) \\ \ge\ & \frac{1}{\mathbf{x}^T\mathbf{x}}\Big(2\alpha(1-\delta)^2) - \frac{\alpha(C_1+1)(\alpha+2(1-\alpha)+\alpha)}{C_1^2}\Big) \\ =\ & \frac{2\alpha}{\mathbf{x}^T\mathbf{x}}\Big((1-\delta)^2 - \frac{C_1+1}{C_1^2}\Big) \end{aligned}$$

Choose ϵ small enough so that $(1-\delta)^2 > \frac{C_1+1}{C_1^2}$. Then $\rho_\alpha(H) > \rho_\alpha$, a contradiction. This proves Claim 7.

By Claims 6 and 7, $G = K_{r-2}\nabla\overline{K}_{n-r+2}$. This completes the proof. □

Proof of Theorem 1.4. Let G be a $K_{s,t}$ minor-free graph of order n with the maximum α-index.

Similarly to the proof of Claim 1 in Theoren 1.3, G is connected. Next let $\rho_\alpha = \rho_\alpha(G)$ and $\mathbf{x} = (x_v)_{v\in V(G)}$ be a positive eigenvector to ρ_α such that $w \in V(G)$ and

$$x_w = \max\{x_u : u \in V(G)\} = 1.$$

Set $L = \{v \in V(G) : x_v > \epsilon\}$ and $S = \{v \in V(G) : x_v \le \epsilon\}$, where ϵ will be chosen later.

Claim 1. For any $\delta > 0$, if we choose ϵ small enough, then G contains a set K with $s-1$ vertices such that their common neighborhood of size is at least $(1-\delta)n$ and each eigenvector entry is at least $1-\delta$.

We omit the proof of Claim 1 as it is similar to the proofs of of Claims 2-4 in Theorem 1.3.

Let T be the common neighborhood of K and $R = V(G)\backslash(K \cup T)$.

Claim 2. R is empty.

Noting any vertex in $R \cup T$ has at most $t-1$ neighbors in $R \cup T$ as G is $K_{s,t}$ minor-free. In addition, noting the graph obtained from G by adding a vertex adjacent to every vertex in K is still $K_{s,t}$ minor-free. The proof of Claim 2 is similar to the proofs of Claims 6 and 7. Hence it is omitted here.

Now $|K| = s-1$ and $|T| = n-s+1$. Let H be the subgraph of G induced by T. Now $G = G[K]\nabla H$. Since G is $K_{s,t}$ minor-free, $\Delta(H) \le t-1$.

First suppose that K induces a clique. By Lemma 2.2, $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$, where $f_\alpha(x) = x^2 - (\alpha n + s + t - 3)x + (\alpha(n-s+1)+s-2))(\alpha(s-1)+t-1) - (1-\alpha)^2(s-1)(n-s+1)$ and equality holds if and only if $G = K_{s-1}\nabla H$, where H is a $(t-1)$-regular graph. It suffices to prove that equality can hold if and only if $G = K_{s-1}\nabla pK_t$, where $n-s+1 = pt$. Suppose that H has a connected component H_1 that is not isomorphic to K_t and set $h := |V(H_1)|$. Clearly H_1 is a $(t-1)$-regular graph of order $h \ge t+1$. If $h = t+1$, then any two nonadjacent vertices in H have $t-1$ common neighbors, which combining with clique K_{s-1} yields $K_{s,t}$, a contradiction. Thus $h \ge t+2$. Note that G is $K_{s,t}$ minor-free, we have H_1 is $K_{1,t}$ minor-free. Hence

$$e(H_1) \le h + \frac{t(t-3)}{2},$$

see [5]. However, since H_1 is a $(t-1)$-regular graph of order h, we have

$$e(H_1) = \frac{h(t-1)}{2} > h + \frac{t(t-3)}{2},$$

a contradiction. Hence H is the union of disjoint complete graphs of order t, i.e., $G = K_{s-1}\nabla pK_t$, where $n - s + 1 = pt$.

Next suppose that K does not induce a clique. Let G' be the graph obtained from G by adding edges to K to make it a clique. Then $\rho_\alpha(G) < \rho_\alpha(G')$. By Lemma 2.2, $\rho_\alpha(G')$ is no more than the largest root of $f_\alpha(x) = 0$, and thus $\rho_\alpha(G)$ is less than the largest root of $f_\alpha(x) = 0$. This completes the proof. □

Let $\alpha = \frac{1}{2}$. It is easy to get the following corollary for $q(G)$.

Corollary 3.5. *Let $t \geq s \geq 2$ and G be a $K_{s,t}$ minor-free graph of sufficiently large order n. Then*

$$q(G) \leq \frac{n + 2s + 2t - 6 + \sqrt{(n + 2s - 2t - 2)^2 + 8(s-1)(t-s+1)}}{2}$$

with equality if and only if $n - s + 1 = pt$ and $G = K_{s-1}\nabla pK_t$.

4. Graphs without star forests

In this section, we present the proof of Theorem 1.5 and some corollaries.

Lemma 4.1. *Let $F = \cup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 1$. If G is an F-free graph of order $n \geq \sum_{i=1}^{k} d_i + k$, then*

$$e(G) \leq \frac{1}{2}\Big(\sum_{i=1}^{k} d_i + 2k - 3\Big)n - \frac{1}{2}(k-1)\Big(\sum_{i=1}^{k} d_i + k - 1\Big).$$

Proof. Let $C = \{v \in V(G) : d(v) \geq \sum_{i=1}^{k} d_i + k - 1\}$. Since G is F-free, $|C| \leq k-1$, otherwise we can embed an F in G by the definition of C. Hence

$$\begin{aligned}
2e(G) &= \sum_{v\in C} d(v) + \sum_{v \in V(G)\setminus C} d(v) \\
&\leq (n-1)|C| + (n - |C|)\Big(\sum_{i=1}^{k} d_i + k - 2\Big) \\
&= \Big(n - \sum_{i=1}^{k} d_i - k + 1\Big)|C| + \Big(\sum_{i=1}^{k} d_i + k - 2\Big)n
\end{aligned}$$

$$\begin{aligned} &\le (k-1)\Big(n-\sum_{i=1}^{k} d_i - k + 1\Big) + \Big(\sum_{i=1}^{k} d_i + k - 2\Big)n \\ &= \Big(\sum_{i=1}^{k} d_i + 2k - 3\Big)n - (k-1)\Big(\sum_{i=1}^{k} d_i + k - 1\Big) \end{aligned}$$

This completes the proof. □

Next we prove the following result for star-forest-free connected graphs, which plays an important role in the proof of Theorem 1.5.

Theorem 4.2. *Let $F = \cup_{i=1}^{k} S_{d_i}$ be a star forest with $k \ge 2$ and $d_1 \ge \cdots \ge d_k \ge 2$. If G is an F-free connected graph of order n for any $0 < \alpha < 1$, where $n \ge \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2}$, then $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$ and equality holds if and only if $G = K_{k-1}\nabla H$ and H is a $(d_k - 1)$-regular graph of order $n - k + 1$, where*

$$\begin{aligned} &f_\alpha(x) = x^2 - \Big(\alpha n + k + d_k - 3\Big)x + \\ &\Big(\alpha(n-k+1) + k - 2\Big)\Big(\alpha(k-1) + d_k - 1\Big) - (1-\alpha)^2(k-1)(n-k+1). \end{aligned}$$

Proof. Let G be an F-free connected graph of order n with the maximum α-index. Set for short $A_\alpha = A_\alpha(G)$ and $\rho_\alpha = \rho_\alpha(G)$. Let $\mathbf{x}_\alpha = (x_v)_{v \in V(G)}$ be a positive eigenvector to ρ_α such that $w \in V(G)$ and

$$x_w = \max\{x_u : u \in V\} = 1.$$

Since $K_{k-1}\nabla \overline{K}_{n-k+1}$ is F-free, it follows from Lemma 2.1 that

$$\rho_\alpha \ge \rho_\alpha(K_{k-1}\nabla \overline{K}_{n-k+1}) \ge \alpha n + \frac{2k - 3 - (2k-1)\alpha}{2\alpha}.$$

Let $L = \{u \in V(G) : x_u > \epsilon\}$ and $S = \{u \in V(G) : x_u \le \epsilon\}$, where

$$\epsilon = \frac{1}{4(\sum_{i=1}^{k} d_i + 3k - 5)}.$$

Claim. $|L| = k - 1$.

First suppose that $|L| \ge k$. By eigenequations of A_α on any vertex $u \in L$, we have

$$(\rho_\alpha - \alpha d(u))\epsilon < (\rho_\alpha - \alpha d(u))x_u = (1-\alpha)\sum_{uv \in E(G)} x_v \le (1-\alpha)d(u),$$

which implies that

$$d(u) > \frac{\rho_\alpha \epsilon}{1-\alpha+\alpha\epsilon} \geq \left(\alpha n + \frac{2k-3-(2k-1)\alpha}{2\alpha}\right)\frac{\epsilon}{1-\alpha+\alpha\epsilon} \geq \sum_{i=1}^{k} d_i + k - 2,$$

where the last inequality holds as $\epsilon \geq \frac{2\alpha(1-\alpha)(\sum_{i=1}^{k} d_i+k-2)}{2\alpha^2(n-\sum_{i=1}^{k} d_i-k+2)-(2k-1)\alpha+2k-3}$. Thus

$$d(u) \geq \sum_{i=1}^{k} d_i + k - 1.$$

Then we can embed an F in G with all centers in L, a contradiction.

Next suppose that $|L| \leq k-2$. Then

$$e(L) \leq \binom{|L|}{2} \leq \frac{1}{2}(k-2)(k-3)$$

and

$$e(L,S) \leq (k-2)(n-k+2).$$

In addition, by Lemma 4.1,

$$e(S) \leq e(G) \leq \frac{1}{2}\left(\sum_{i=1}^{k} d_i + 2k - 3\right)n.$$

By eigenequations of A_α on w, we have

$$\rho_\alpha - \alpha d(w) = (\rho_\alpha - \alpha d(w))x_w = (1-\alpha)\sum_{vw\in E(G)} x_v.$$

Multiplying both sides of the above equality by ρ_α, we have

$$\begin{aligned} & \rho_\alpha(\rho_\alpha - \alpha d(w)) \\ = \; & (1-\alpha)\sum_{vw\in E(G)} \rho_\alpha x_v \\ = \; & (1-\alpha)\sum_{vw\in E(G)} \left(\alpha d(v)x_v + (1-\alpha)\sum_{uv\in E(G)} x_u\right) \\ = \; & (1-\alpha)\sum_{vw\in E(G)} \alpha d(v)x_v + (1-\alpha)^2 \sum_{vw\in E(G)}\sum_{uv\in E(G)} x_u \end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha)\Big(\sum_{v\in V(G)} \alpha d(v)x_v - \alpha d(w)\Big) + (1-\alpha)^2 \sum_{uv\in E(G)} (x_u+x_v) - \\
&\quad (1-\alpha)^2 \sum_{vw\in E(G)} x_v \\
&= \alpha(1-\alpha)\sum_{uv\in E(G)} (x_u+x_v) - \alpha(1-\alpha)d(w) + \\
&\quad (1-\alpha)^2 \sum_{uv\in E(G)} (x_u+x_v) - (1-\alpha)(\rho_\alpha - \alpha d(w)) \\
&= (1-\alpha)\sum_{uv\in E(G)} (x_u+x_v) - (1-\alpha)\rho_\alpha,
\end{aligned}$$

which implies that

$$\sum_{uv\in E(G)} (x_u+x_v) \geq \frac{\rho_\alpha(\rho_\alpha + 1 - \alpha - \alpha d(w))}{1-\alpha}.$$

On the other hand,

$$\begin{aligned}
&\sum_{uv\in E(G)} (x_u+x_v) \\
&= \sum_{uv\in E(L,S)} (x_u+x_v) + \sum_{uv\in E(S)} (x_u+x_v) + \sum_{uv\in E(L)} (x_u+x_v) \\
&\leq \sum_{uv\in E(L,S)} (x_u+x_v) + 2\epsilon e(S) + 2e(L) \\
&\leq \sum_{uv\in E(L,S)} (x_u+x_v) + \epsilon\Big(\sum_{i=1}^{k} d_i + 2k-3\Big)n + (k-2)(k-3).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{uv\in E(L,S)} (x_u+x_v) \\
&\geq \frac{\rho_\alpha(\rho_\alpha + 1 - \alpha - \alpha d(w))}{1-\alpha} - \epsilon\Big(\sum_{i=1}^{k} d_i + 2k-3\Big)n - (k-2)(k-3) \\
&\geq \Big(\frac{\alpha n}{1-\alpha} + \frac{2k-3-(2k-1)\alpha}{2\alpha(1-\alpha)}\Big)\Big(\frac{2k-3-(2k-1)\alpha}{2\alpha} + 1\Big) - \\
&\quad \epsilon\Big(\sum_{i=1}^{k} d_i + 2k-3\Big)n - (k-2)(k-3)
\end{aligned}$$

$$= \Bigg(k-\frac{3}{2}-\epsilon\Big(\sum_{i=1}^{k} d_i+2k-3\Big)\Bigg)n+\frac{(2k-3)^2-(2k-3)(2k-1)\alpha}{4\alpha^2}-$$
$$(k-2)(k-3),$$

where the second inequality holds as $d(w) \le n-1$. On the other hand, by the definition of L and S, we have

$$\sum_{uv\in E(L,S)} (x_u+x_v) \le (1+\epsilon)e(L,S) \le (1+\epsilon)(k-2)(n-k+2).$$

Thus

$$\begin{aligned} & (1+\epsilon)(k-2)(n-k+2) \\ \ge\ & \Bigg(k-\frac{3}{2}-\epsilon\Big(\sum_{i=1}^{k} d_i+2k-3\Big)\Bigg)n+\frac{(2k-3)^2-(2k-3)(2k-1)\alpha}{4\alpha^2}- \\ & (k-2)(k-3), \end{aligned}$$

which implies that $\Big(-\frac{1}{2}+\epsilon\Big(\sum_{i=1}^{k} d_i+3k-5\Big)\Big)n \ge \frac{(2k-3)^2-(2k-3)(2k-1)\alpha}{4\alpha^2}+(k-2)(1+\epsilon(k-2))$. Since $\epsilon=\frac{1}{4(\sum_{i=1}^{k} d_i+3k-5)}$, we have

$$\begin{aligned} n &\le -\frac{(2k-3)^2-(2k-3)(2k-1)\alpha}{\alpha^2}-4(k-2)\Big(1+\frac{k-2}{4(\sum_{i=1}^{k} d_i+3k-5)}\Big) \\ &< \frac{(2k-3)(2k-1)}{\alpha} \\ &< \frac{4(\sum_{i=1}^{k} d_i+k-2)(\sum_{i=1}^{k} d_i+3k-5)}{\alpha^2}, \end{aligned}$$

a contradiction. This proves the Claim.

By Claim, $|L|=k-1$ and thus $|S|=n-k+1$. Then the subgraph H induced by S in G is S_{d_k}-free. Otherwise, we can embed an F in G with $k-1$ centers in L and a center in S as $d(u) \ge \sum_{i=1}^{k} d_i+k-1$ for any $u \in L$, a contradiction. Now $\Delta(H) \le d_k-1$. Note that the resulting graph obtained from G by adding all edges in L and all edges with one end in L and the other in S is also F-free and its spectral radius increase strictly. By the extremality of G, we have $G = K_{k-1}\nabla H$. By Lemma 2.2 and the extremality of G, ρ_α is no more than largest root of $f_\alpha(x)=0$, and ρ_α is equal to the largest root of $f_\alpha(x)=0$ if and only if H is a (d_k-1)-regular graph of order $n-k+1$, where

$$f_\alpha(x) = x^2 - \Big(\alpha n + k + d_k - 3\Big)x + \\ \Big(\alpha(n-k+1)+k-2\Big)\Big(\alpha(k-1)+d_k-1\Big) - (1-\alpha)^2(k-1)(n-k+1).$$

This completes the proof. $\square$

Proof of Theorem 1.5. Let G be an F-free graph of order n with the maximum α-index.

If G is connected, then the result follows directly from Theorem 4.2. Next we suppose that G is not connected. Since $K_{n-1}\nabla\overline{K}_{n-k+1}$ is F-free,

$$\rho_\alpha(G) \geq \rho_\alpha(K_{n-1}\nabla\overline{K}_{n-k+1}) \geq \alpha n + \frac{2k-3-(2k-1)\alpha}{2\alpha}.$$

Let G_1 be a component of G such that $\rho_\alpha(G_1) = \rho_\alpha(G)$. Set $n_1 = |V(G_1)|$. Then

$$\begin{aligned} n_1 - 1 &\geq \rho_\alpha(G_1) = \rho_\alpha(G) \geq \alpha n + \frac{2k-3-(2k-1)\alpha}{2\alpha} \\ &\geq \frac{4(\sum_{i=1}^k d_i + k - 2)(\sum_{i=1}^k d_i + 3k - 5)}{\alpha^2} + \frac{2k-3-(2k-1)\alpha}{2\alpha}, \end{aligned}$$

which implies that

$$\begin{aligned} n_1 &\geq \frac{4(\sum_{i=1}^k d_i + k - 2)(\sum_{i=1}^k d_i + 3k - 5)}{\alpha^2} + \frac{(2k-3)(1-\alpha)}{2\alpha} \\ &> \frac{4(\sum_{i=1}^k d_i + k - 2)(\sum_{i=1}^k d_i + 3k - 5)}{\alpha^2} \end{aligned}$$

By Theorem 4.2 again, $\rho_\alpha(G_1)$ is no more than the largest root of $x^2-(\alpha n_1+k+d_k-3)x+\big(\alpha(n_1-k+1)+k-2\big)\big(\alpha(k-1)+d_k-1\big)-(1-\alpha)^2(k-1)(n_1-k+1) = 0$. Hence $\rho_\alpha(G_1)$ is less than the largest root of $x^2-(\alpha n+k+d_k-3)x+\big(\alpha(n-k+1)+k-2\big)\big(\alpha(k-1)+d_k-1\big)-(1-\alpha)^2(k-1)(n-k+1) = 0$. This completes the proof. $\square$

Let $F_{n,k} = K_{k-1}\nabla(pK_2 \cup qK_1)$, where $n-(k-1) = 2p+q$ and $0 \leq q < 2$.

Corollary 4.3. *Let $F = \cup_{i=1}^k S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k = 2$. If G is an F-free graph of order $n \geq \frac{4(\sum_{i=1}^k d_i+k-2)(\sum_{i=1}^k d_i+3k-5)}{\alpha^3}$, then*

$$\rho_\alpha(G) \leq \rho_\alpha(F_{n,k})$$

with equality if and only if $G = F_{n,k}$.

Proof. Let G be a graph having the maximum α-index among all F-free graphs of order n. It suffices to show that $G = F_{n,k}$. If G is connected, then by the proof of Theorem 4.2, we have $G = K_{k-1}\nabla H$, where H is a graph of order $n-k+1$ with $\Delta(H) \leq 1$. So H is the union of some edges and isolated vertices. Hence $G = F_{n,k}$. If G is not connected, then by the similar proof of Theorem 1.5, there is a component G_1 of G such that $|V(G_1)| \geq \frac{4(\sum_{i=1}^{k} d_i+k-2)(\sum_{i=1}^{k} d_i+3k-5)}{\alpha^2}$ and $\rho(G) = \rho(G_1)$. By above case,

$$\rho_\alpha(G) = \rho_\alpha(G_1) = \rho_\alpha(F_{n_1,k}) < \rho_\alpha(F_{n,k}).$$

Hence the result follows. $\square$

Let $\alpha = \frac{1}{2}$. By Theorem 1.5 and Corollary 4.3, we have the following corollary.

Corollary 4.4. *Let $F = \cup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 2$ and G be an F-free graph of order $n \geq 32(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)$.*
(i) If $d_k = 2$, then

$$q(G) \leq q(F_{n,k})$$

with equality if and only if $G = F_{n,k}$.
(ii) If $d_k \geq 3$, then

$$q(G) \leq \frac{n + 2k + 2d_k - 6 + \sqrt{(n + 2k - 2d_k - 2)^2 + 8(k-1)(d_k - k + 1)}}{2}$$

with equality if and only if $G = K_{k-1}\nabla H$, where H is a $(d_k - 1)$-regular graph of order $n - k + 1$.

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